Multidimensional inequality comparisons

Jean-Yves Duclos[*], David E. Sahn [†] and Stephen D. Younger [‡]

February 5, 2009

Abstract

Keywords: .

JEL Classification: .

[*] Institut d'Anàlisi Econòmica (CSIC), Barcelona, Spain, and Département d'économique and CIRPÉE, Université Laval, Québec, Canada

[†] Cornell University, Ithaca, USA

[‡] Cornell University, Ithaca, USA

1 Introduction

Weymark (2004), p. 29: "Although much has already been learned about multidimensional normative inequality indices, much more remains to be discovered. Compared to the theory of univariate inequality measurement, the analysis of multidimensional inequality is in its infancy."

We use the framework developed by Duclos, Sahn, and Younger (2006) to make robust multidimensional comparisons.

In doing this, we develop and implement procedures for making multidimensional inequality comparisons that are valid for a broad class of aggregation rules. This is in the spirit of the dominance approach to making inequality comparisons, as developed for instance by Atkinson (1970), Foster and Shorrocks (1988) and Formby, Smith, and Zheng (1998) in the unidimensional context, and Atkinson and Bourguignon (1982) in the multidimensional context. One advantage of this approach is that it is capable of generating inequality orderings that are robust over broad classes of inequality indices and over broad classes of aggregation rules used across dimensions of well-being.

In contrast to earlier work on multidimensional inequality comparisons, our orderings also enable us to focus on "downside" inequality aversion by thinking of inequality indices as being a function of the relative position of those at the bottom of the welfare distribution. This is done by focussing our comparisons on those below a multidimensional inequality "frontier". This is also analogous to thinking of inequality as a special case of relative poverty, a case in which everyone's relative position in the welfare distribution can have an impact on inequality indices and for which the inequality frontier is above everyone. We show how our orderings can also be considered to be "frontier-robust".

For most of the paper, we limit ourselves to the case of two measures of well-being, though we do ??? provide examples of a three-dimensional comparisons.

It is of course possible to think of making multidimensional inequality comparisons by performing univariate inequality comparisons independently for each dimension of well-being. We see how this might lead us to conclude that inequality in A is lower than poverty in B but that a multivariate inequality analysis might conclude the opposite, and vice-versa. We argue indeed that a reasonable multidimensional inequality index should allow the level of welfare in one dimension to affect our assessment of how the other dimension can affect overall inequality. In practice, populations exhibiting higher correlations between measures of well-being will be more unequal than those that do not, relative to what one would expect from making univariate comparisons alone.

2 Multiple indicators of relative welfare

Let x and y be two indicators of individual well-being *relative* to some norm. These indicators could be, for instance, income, expenditures, health status, caloric consumption, life expectancy, height, cognitive ability, the extent of personal safety and freedom, *etc.*, relative to what is deemed to be enjoyed by a representative individual in a society.

Denote by

$$\lambda(x,y) : \Re^2 \to \Re \left| \frac{\partial \lambda(x,y)}{\partial x} \leq 0, \frac{\partial \lambda(x,y)}{\partial y} \leq 0 \right. \tag{1}$$

a summary indicator of the degree of relative deprivation of an individual. Note that the derivative conditions in (1) mean that different indicators can each contribute to decreasing overall deprivation. We make the differentiability assumptions for expositional simplicity, but they are not strictly necessary so long as $\lambda(x,y)$ is non-decreasing over x and y.

One alternative to thinking of x and y as *two* indicators of relative individual well-being could be to define a aggregate function $U(x,y)$ of overall individual welfare, and think of the distance between this and a norm defined in units os overall welfare. This, however, would require specifying a particular definition of $U(x,y)$, something that we wish to avoid since we prefer to think that the overall individual welfare function is unknown.

Let the distribution of these two indicators in the population be given by an $n \times 2$ matrix denoted as X, where n is the number of individuals. Let the domain of admissible distributions be denoted as Ξ.

We will represent inequality indices by P_X for inequality in X. For all $X_1, X_2 \in \Xi$, we will therefore say that X_1 is more unequal than X_2 if and only if $P_{X_1} \geq P_{X_2}$.

Definition of a strongly relative inequality index P: For all $X_1, X_2 \in \Xi$ and all $q \times q$ diagonal matrices Γ with elements $\gamma_{ii} > 0$ for all $i = 1, ..., q$, $P_{X_1} \geq P_{X_2}$ if and only if $P_{X_1\Gamma} \geq P_{X_2\Gamma}$.

Let $\mathbf{1}$ be a distribution matrix whose entries are all equal to 1.

Definition of a strongly translatable inequality index P: For all $X_1, X_2 \in \Xi$ and all $q \times q$ diagonal matrices for which $X_1 + \mathbf{1}\Gamma$ and $X_2 + \mathbf{1}\Gamma$ are both members of Ξ, $P_{X_1} \geq P_{X_2}$ if and only if $P_{X_1+\mathbf{1}\Gamma} \geq P_{X_2+\mathbf{1}\Gamma}$

These definitions were proposed by Tsui (1995) — see also Weymark (2004). They may not be uniformly acceptable. For instance, it may well be that, if education is doubled for instance, then the contribution of other indicators (such as health or income) to overall inequality should be affected.

The above nevertheless suggests that we can apply two types of normalizations to each indicator of welfare.

We can use gaps between indicators and their mean (absolute inequality) or those same gaps but normalized by the mean (relative inequality). Let

$$x_\gamma = \gamma \left(\frac{x - \mu_x}{\mu_x} \right) + (1 - \gamma) (x - \mu_x) \tag{2}$$

and

$$F_x^\gamma(z) = F_x \left(\frac{\mu z + (1-\gamma)\mu^2 + \gamma\mu}{(1-\gamma)\mu + \gamma} \right) \tag{3}$$

where $F_x(z)$ is the marginal distribution of x (and similarly for y). Then, we can compare absolute inequality in x by using x_0 and relative inequality by using x_1 (and similarly for y). This will make the indices strongly relative and strongly translatable in x and in y, respectively. For expositional simplicity, we drop from now on the indices γ from x_γ and y_γ,

We then assume that we wish to compute an aggregate index of inequality based on the distribution of x and y, and that we wish to focus on those with the greatest degree of relative deprivation. This can be done by drawing an inequality frontier separating those with lower and those with greater relative deprivation. We can think of this frontier as a series of points at which overall relative deprivation is kept constant. This frontier is assumed to be defined implicitly by a locus of the form $\lambda(x, y) = 0$, and is analogous to the usual downward-sloping indifference curves on the (x, y) space. The set of those over whom we want to aggregate relative deprivation is then obtained as:

$$\Lambda(\lambda) = \{(x, y) \,|(\lambda(x, y) \geq 0\} . \tag{4}$$

Consider Figure 1 with thresholds z_x and z_y in dimensions of relative well-being x and y. $\lambda_1(x, y)$ gives an "intersection" frontier: it considers someone to be relatively deprived only if he is deprived in *both* of the two dimensions of x and y, and therefore if he lies within the dashed rectangle of Figure 1. $\lambda_2(x, y)$ (the L-shaped, dotted line) gives a "union" frontier: it considers someone to be relatively deprived if he is deprived in *either* of the two dimensions, and therefore if he lies below or to the left of the dotted line. Finally, $\lambda_3(x, y)$ provides an intermediate approach. Someone can be relatively deprived even if $y > z_y$, if his x value is sufficiently low to lie to the left of $\lambda_3(x, y) = 0$.

To define multidimensional inequality indices more precisely, let the joint distribution of x and y be denoted by $F(x, y)$. For analytical simplicity, we focus on classes of inequality indices that are additive across individuals. An additive inequality index that combines the two dimensions of well-being can be defined generally as $P(\lambda)$,

$$P(\lambda) \;=\; \int\int_{\Lambda(\lambda)} \pi(x, y; \lambda)\, dF(x, y), \tag{5}$$

where $\pi(x, y; \lambda)$ is the contribution to inequality of an individual with relative well-being indicators x and y. By the definition of the inequality frontier, we have that

$$\pi(x, y; \lambda) \begin{cases} \geq 0 \text{ if } \lambda(x, y) \geq 0 \\ = 0 \text{ otherwise.} \end{cases} \tag{6}$$

The π function in equation (6) is thus the weight that the inequality measure attaches to someone who is inside the inequality frontier. That weight could be 1 (for a count of how many are relatively deprived), but it could take on many other values as well, depending on the inequality measure of interest.

A *bi-dimensional dominance surface* can be defined as:

$$P^{\alpha_x,\alpha_y}(z_x, z_y) = \int_0^{z_y} \int_0^{z_x} (z_x - x)^{\alpha_x} (z_y - y)^{\alpha_y}\, dF(x, y) \tag{7}$$

for integers $\alpha_x \geq 0$ and $\alpha_y \geq 0$. We generate the dominance surface by varying the values of z_x and z_y over an appropriately chosen domain, with the height of the surface determined by equation 7. This function is a two-dimensional generalization of the FGT index defined over indicators of relative well-being. $P^{0,0}(z_x, z_y)$ generates a bivariate cumulative density function. An important feature of the dominance surface is that it is influenced by the covariance between x and y, the two measures of well-being, because the integrand is multiplicative. Rewriting (7), we find indeed that

$$\begin{aligned} P^{\alpha_x,\alpha_y}(z_x, z_y) = & \int_0^{z_y} (z_x - x)^{\alpha_x}\, dF(x) \int_0^{z_x} (z_y - y)^{\alpha_y}\, dF(y) \\ & + \operatorname{cov}\left((z_x - x)^{\alpha_x}, (z_y - y)^{\alpha_y}\right). \end{aligned} \tag{8}$$

The height of the dominance surface is therefore the product of the two unidimensional curves plus the covariance in the "poverty gaps" in the two dimensions.

Thus, the higher the correlation between x and y, the higher the dominance surfaces, other things equal.

Our poverty comparisons make use of orders of dominance, s_x in the x and s_y in the y dimensions, which will correspond respectively to $s_x = \alpha_x + 1$ and $s_y = \alpha_y + 1$. The parameters α_x and α_y also capture the aversion to inequality in the x and in the y dimensions, respectively.

3 Dominance conditions

To describe the class of inequality measures for which the dominance surfaces defined in equation (7) are sufficient to establish multidimensional inequality orderings, assume first that π in (5) is left differentiable[1] with respect to x and y over the set $\Lambda(\lambda)$, up to the relevant orders of dominance, s_x for derivatives with respect to x and s_y for derivatives with respect to y. Denote by π^x the first derivative[2] of $\pi(x, y; \lambda)$ with respect to x; by π^y the first derivative of $\pi(x, y; \lambda)$ with respect to y; by π^{xy} the derivative of $\pi(x, y; \lambda)$ with respect to x and to y; and treat similar expressions accordingly.

We can then define the following classes of bidimensional inequality indices:

$$\Pi^{1,1}(\lambda^+) = \left\{ P(\lambda) \;\middle|\; \begin{array}{l} \Lambda(\lambda) \subset \Lambda(\lambda^+) \\ \pi(x, y; \lambda) = 0, \text{ whenever } \lambda(x, y) = 0 \\ \pi^x(x, y; \lambda) \leq 0 \text{ and } \pi^y(x, y; \lambda) \leq 0 \; \forall x, y \\ \pi^{xy}(x, y; \lambda) \geq 0, \; \forall x, y. \end{array} \right\} \tag{9}$$

$$\Pi^{2,1}(\lambda^+) = \left\{ P(\lambda) \;\middle|\; \begin{array}{l} P(\lambda) \in \Pi^{1,1}(\lambda^+) \\ \pi^x(x, y; \lambda) = 0 \text{ whenever } \lambda(x, y) = 0 \\ \pi^{xx}(x, y; \lambda) \geq 0 \; \forall x, y \\ \text{and } \pi^{xxy}(x, y; \lambda) \leq 0, \; \forall x, y. \end{array} \right\} \tag{10}$$

[1] This differentiability assumption is made for expositional simplicity. It could be relaxed.

[2] The derivatives include the implicit effects of x and y on $\lambda(x, y)$.

and

$$\Pi^{2,2}(\lambda^+) = \left\{ P(\lambda) \;\middle|\; \begin{array}{l} P(\lambda) \in \Pi^{2,1}(\lambda^+) \\ \pi^{xx}(x,y;\lambda) = 0 \text{ whenever } \lambda(x,y) = 0 \\ \pi^{yy}(x,y;\lambda) = 0 \text{ whenever } \lambda(x,y) = 0 \\ \pi^{xy}(x,y;\lambda) = 0 \text{ whenever } \lambda(x,y) = 0 \\ \pi^{yy}(x,y;\lambda) \geq 0 \; \forall x,y \\ \pi^{xyy}(x,y;\lambda) \leq 0, \; \forall x,y \\ \text{and } \pi^{xxyy}(x,y;\lambda) \geq 0, \; \forall x,y. \end{array} \right\} \tag{11}$$

The classes $\Pi^{1,1}$ and $\Pi^{2,2}$ are reminiscent of the classes of welfare functions used by Atkinson and Bourguignon (1982); they allow for possibly different signs for $\pi^{xy}(x,y)$ and $\pi^{xxyy}(x,y)$ since they also consider the case of functions that show "complementarity" in indicators.

The conditions for membership in $\Pi^{2,2}(\lambda)$ require that the inequality indices be convex in both x and y, and that they therefore obey the principle of transfers in both of these dimensions. They also require that this principle be stronger in one dimension of relative well-being the lower the level of the other dimension of relative well-being. Finally, they also impose that the second-order derivative in one dimension of well-being be convex in the level of the other indicator of well-being.

Definition (from Weymark 2004) of bistochastic majorization as a multi-attribute version of the Pigou-Dalton transfer: For all $X_1, X_2 \in \Xi$ for which $X_1 \neq X_2$, X_1 is more unequal than X_2, denoted $P_{X_1} \geq P_{X_2}$, if $X_2 = BX_1$ for some $n \times n$ bistochastic matrix B that is not a permutation matrix.

As Savaglio (2006) writes, "this is a sort of *decomposability* property, which allows [orderings] to be coherent with an inequality measurement via an additive evaluation function" (p.90) — see also for instance Dardanoni (1995).

Figure 2 shows how a bi-stochastic transformation of can increase inequality in welfare as captured by U. The bi-stochastic transformation moves point A to B and point D to C. Overall welfare (or utility) was the same at U^0 initially; now it is different for the two individuals.

The conditions for membership in $\Pi^{2,2}(\lambda)$ do not imply the bistochastic majorization condition. They only imply that inequality should fall if, on Figure 3, if A and B are moved closer, or if A and C are moved closer, or if B and C are moved closer, or if B and C are moved closer, but not necessarily if A and D are moved closer. The conditions for membership in $\Pi^{2,2}(\lambda)$ also imply that the fall in inequality will be larger if A and B are moved closer than if B and D are moved

closer, and will also be larger if and C and D are moved closer than if A and B are moved closer.
This leads to the following type of dominance condition:

Proposition 1 **(Π^{s_x,s_y} poverty dominance)**

$$\begin{aligned} &\Delta P(\lambda) > 0,\ \forall P(\lambda) \in \Pi^{s_x,s_y}(\lambda^+) \\ \textit{iff}\ &\Delta P^{s_x-1,s_y-1}(x,y) > 0,\ \forall (x,y) \in \Lambda(\lambda^+). \end{aligned} \tag{12}$$

(1,1) inequality dominance: can only be "partial" since we have normalized by the mean ($\Lambda(\lambda^+)$ cannot include everyone)

(2,1) dominance: inequality in x is more important when it affects groups with lower y

Example: inequality within earlier periods of life (e.g., distinction between opportunities earlier in life and outcomes later in life); within underprivileged groups (immigrants, "blacks", women, children, less healthy, vulnerable, fewer assets, more investment producing, greater production or investment elasticity, facing greater uncertainty and capital market imperfections)

Figure 1: Inequality frontiers

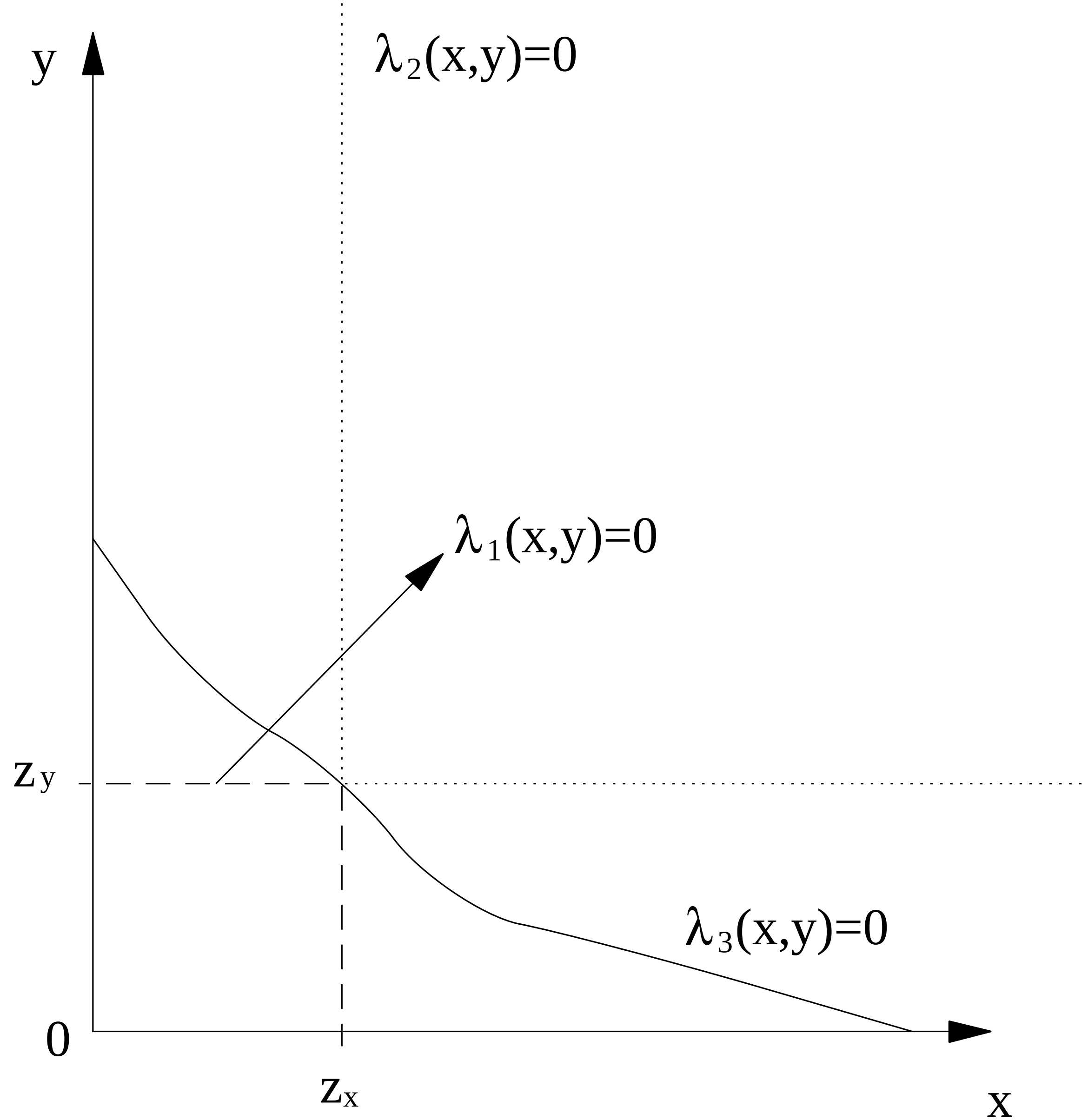


Figure 2: Bistochastic transformations may increase inequality in welfare

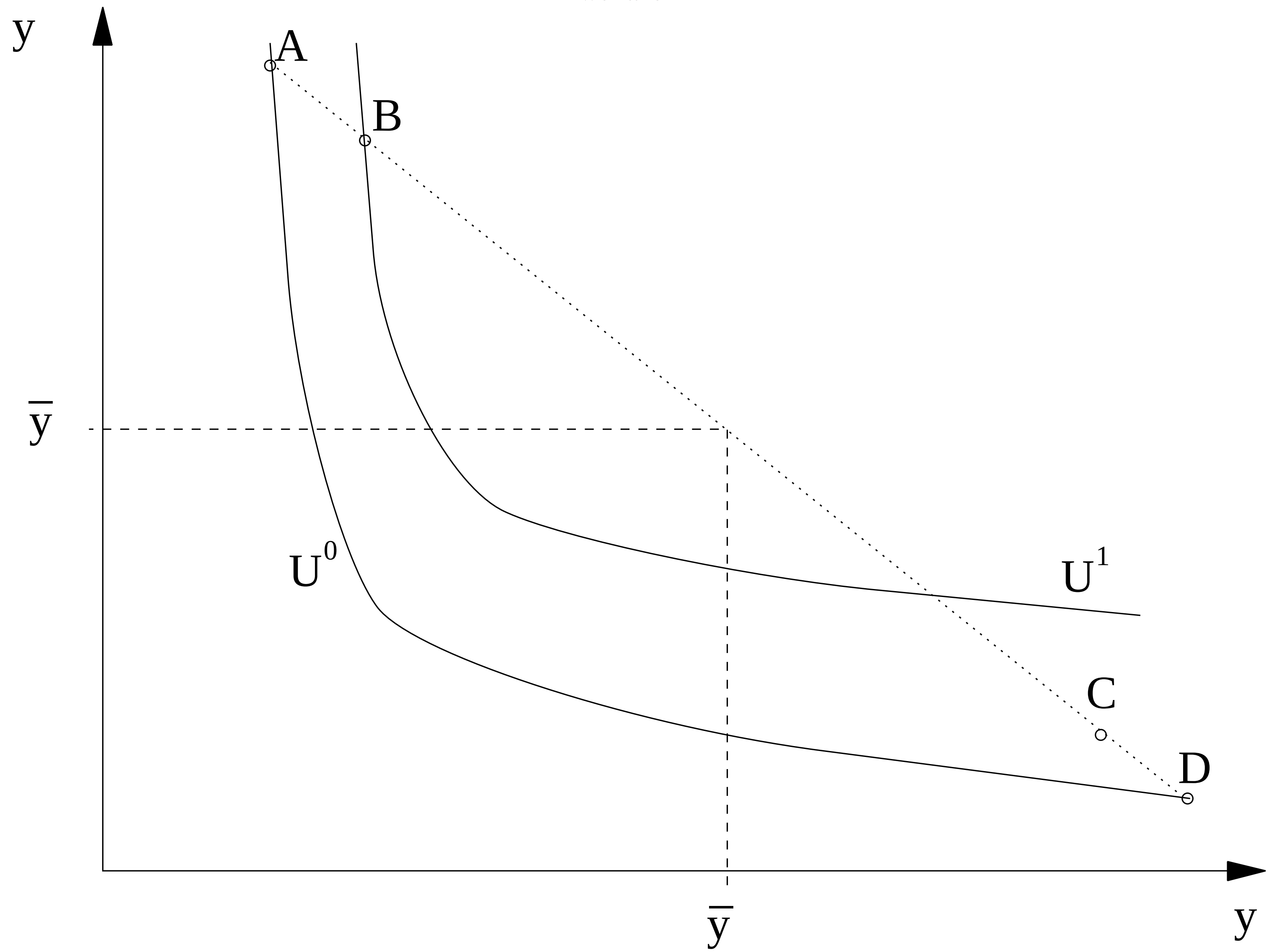
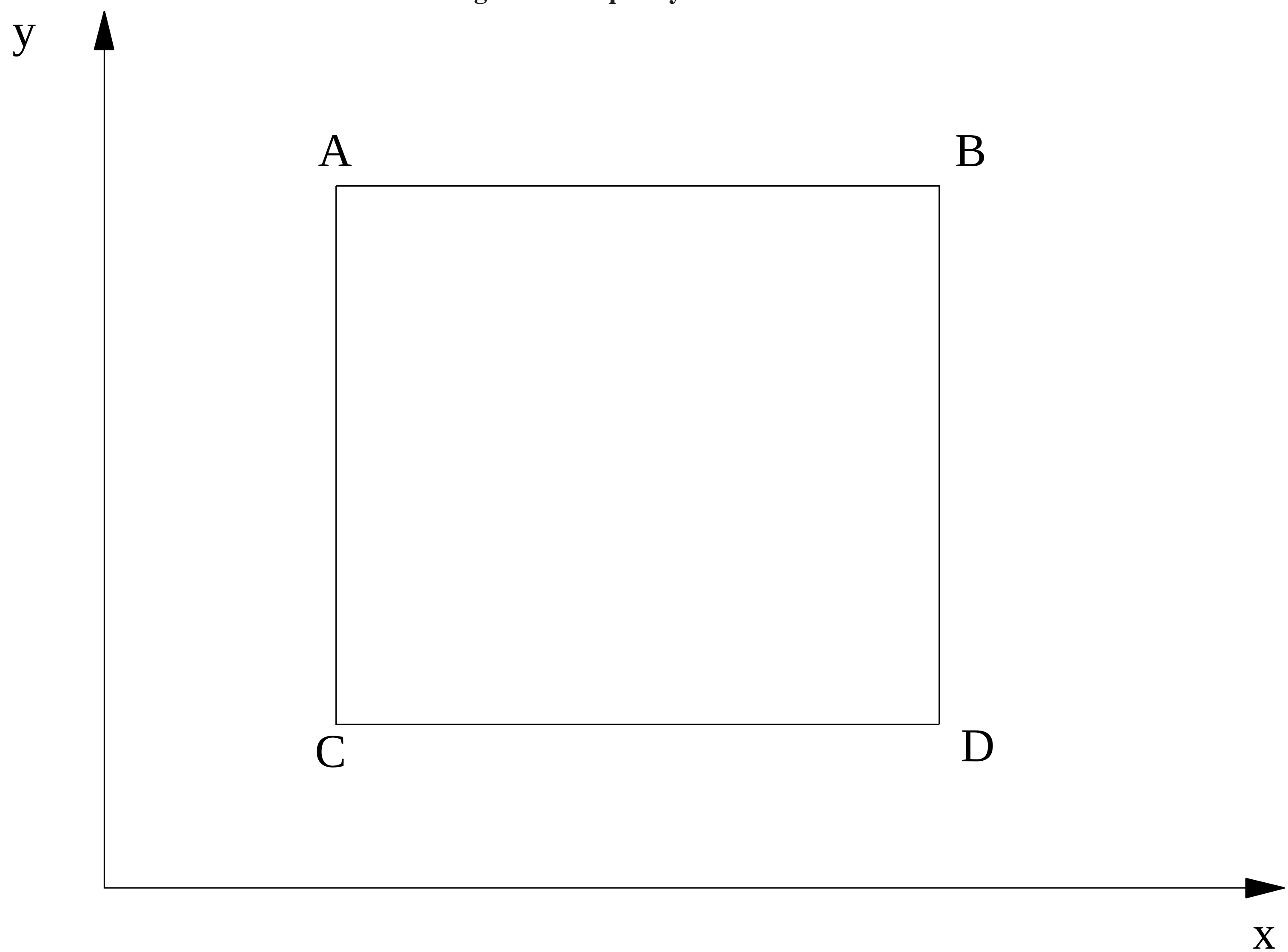


Figure 3: Inequality reductions



Proof of Theorem for $s_x = 2, s_y = 1$. ∎

Start with

$$
\begin{aligned}
P(z_x(y), z_y) &= -\int_0^{z_x(z_y)} \pi^x(x, z_y; \lambda^+)\, P^{0,0}(x, z_y)\, dx && (13)\\
&+ \int_0^{z_y} z_x^{(1)}(y)\, \pi^x(z_x(y), y; \lambda^+)\, P^{0,0}(z_x(y), y)\, dy && (14)\\
&+ \int_0^{z_y} \int_0^{z_x(y)} \pi^{xy}(x, y; \lambda^+)\, P^{0,0}(x, y)\, dx\, dy. && (15)
\end{aligned}
$$

and

$$
\Pi^{2,1}(\lambda^+) = \left\{ P(\lambda) \;\middle|\; \begin{array}{l} P(\lambda) \in \Pi^{1,1}(\lambda^+) \\ \pi^x(x, y; \lambda) = 0 \text{ whenever } \lambda(x, y) = 0 \\ \pi^{xx}(x, y; \lambda) \geq 0\ \forall x, \\ \text{and } \pi^{xxy}(x, y; \lambda) \leq 0,\ \forall x, y. \end{array} \right\} \quad (16)
$$

Integrating (15) once more by parts with respect to x, and imposing the continuity conditions characterizing the indices $\Pi^{2,1}(\lambda^+)$ in (16), we find:

$$
\begin{aligned}
P(\lambda^+) &= \int_0^{z_x(z_y)} \pi^{xx}(x, z_y; \lambda^+) P^{1,0}(x, z_y) dx \\
&+ \int_0^{z_y} \pi^{xy}(z_x(y), y; \lambda^+) P^{1,0}(z_x(y), y) dy \\
&- \int_0^{z_y} \int_0^{z_x(y)} \pi^{xxy}(x, y; \lambda^+) P^{1,0}(x, y) dx\, dy. && (17)
\end{aligned}
$$

Proof of Theorem for $s_x = 2, s_y = 2$. ∎

Integrating (17) once more by parts with respect to y, and imposing the continuity conditions characterizing the indices $\Pi^{2,2}(\lambda^+)$ in (11), we find:

$$
\begin{aligned}
P(z_x(y), z_y) &= -\int_0^{z_x(z_y)} \pi^{xxy}(x, z_y; \lambda^+)\, P^{1,1}(x, z_y)\, dx && (18)\\
&+ \int_0^{z_y} z_x^{(1)}(y)\, \pi^{xxy}(z_x(y), y; \lambda^+)\, P^{1,1}(z_x(y), y)\, dy && (19)\\
&+ \int_0^{z_y} \int_0^{z_x(y)} \pi^{xxyy}(x, y; \lambda^+)\, P^{1,1}(x, y)\, dx\, dy. && (20)
\end{aligned}
$$

The rest of the proof is as above.

References

ATKINSON, A. (1970): "On the Measurement of Inequality," *Journal of Economic Theory*, 2, 244–63.

ATKINSON, A. AND F. BOURGUIGNON (1982): "The Comparison of Multi-Dimensional Distributions of Economic Status," in *Social Justice and Public Policy*, ed. by L. Harvester Wheatsheaf, London, vol. chapter 2.

DARDANONI, V. (1995): "On multidimensional inequality measurement," in *Income distribution, social welfare, inequality and poverty*, ed. by C. Dagum and A. Lemmi, Stamford, CT: JAI Press, vol. 6 of *Research on economic inequality*, 201–207.

DUCLOS, J.-Y., D. E. SAHN, AND S. D. YOUNGER (2006): "Robust Multidimensional Poverty Comparison," *Economic Journal*, 113, 113, 943–968.

FORMBY, J., W. SMITH, AND B. ZHENG (1998): "Inequality Orderings, Stochastic Dominance and Statistical Inference," .

FOSTER, J. AND A. SHORROCKS (1988): "Inequality and Poverty Orderings," *European Economic Review*, 32, 654–661.

SAVAGLIO, E. (2006): "Multidimensional inequality with variable population size," *Economic Theory*, 28, 85–94.

TSUI, K. Y. (1995): "Multidimensional Generalizations of the Relative and Absolute Inequality Indices: The Atkinson-Kolm-Sen Approach," *Journal of Economic Theory*, 67, 251–65.

WEYMARK, J. A. (2004): "The normative approach to the measurement of multidimensional inequality," Working Paper 03-W14R, Vanderbilt University, Nashville.