Equality, Efficiency and the Skill Gap

Joël Hellier and Nathalie Chusseau

EQUIPPE, University of Lille 1 and LEMNA, University of Nantes

28 rue de Sévigné 75004 Paris, FRANCE
joel.hellier@wanadoo.fr

Abstract: *A model is constructed in which (i) an individual's human capital depends on her parents' human capital and her schooling time, and (ii) both skilled and unskilled labour are necessary for production. The model exhibits a continuum of Pareto-optimal steady states, one of which is efficient in terms of net income per capita. When the proportion of skilled workers at the steady state is below this efficient value (Skill Gap), a skill-enhancing educational policy can increase both equality and efficiency. This policy requires that the less educated families receive more educational services from the State than the most educated ones.*

Key words: Efficiency, Education, Inequality, Skill gap.

JEL classification: J24, J31, J62.

Corresponding author:

Pr. Joël Hellier
EQUIPPE, University of Lille 1 and LEMNA, University of Nantes
28 rue de Sévigné 75004 Paris, France
+33 1 42 77 57 40
joel.hellier@wanadoo.fr

1 Introduction

The relation between inequality and efficiency has been analysed for long in the economic literature. Most of the related analyses have focused on the interactions between inequality, growth, and the income per capita. In this respect, three strands of approaches can be distinguished.

Firstly, following Kuznets' seminal article (1955), a number of theoretical and empirical works have attempted to explain and verify the inverted-U shaped relationship between inequality and the development process[1].

A second strand of literature is centred on the impact of pro-equality policies, and particularly redistribution, on production and growth. In pure competition, redistribution reduces production because both levies and public transfers reduce labour supply. In addition taxes on both capital and capital income lower saving, investment and growth. Redistribution is thus bad for production and growth. However, within a political economy framework, this result can be used to show that before tax inequality reduces growth because the higher inequality, the more redistribution is enforced by the median voter (Alesina and Rodrick, 1994; Persson and Tabellini, 1994). When inequality is harmful to growth (Galor and Zeira, 1993; Maoz and Moav, 1999; Glomm and Kaganovich, 2008), redistribution can in contrast foster growth.

A third set of literature, often related to the previous, has focused on the issue: is inequality good for growth and income per capita? Centred on physical capital accumulation, a first series of analyses answered 'yes' to this question. As a matter of fact, if capital accumulation is positively related to saving and if the rich have a higher marginal saving rate than the poor (Kaldor, 1955-56), then a transfer of income from the later to the former boosts capital accumulation and growth. However, this positive relationship has been questioned from the empirical evidence that egalitarian countries (Asian NICs) have experienced higher growth than non egalitarian countries (South America and Africa). A number of empirical works have shown that growth was negatively related to inequality (e.g., Persson and Tabellini, 1994, Alesina and Rodrik, 1994, Deininger and Squire, 1998). In actual fact, when economic development is not essentially based on capital accumulation, there are several channels through which inequality tends to reduce growth. Firstly, inequality can make the poor move from productive to appropriative strategies (Grossman, 1991, 1994) such as strikes, revolts, revolutions and criminal activities, which jeopardise production and growth (Alesina and

[1] see Anand and Ravi Kambur (1993) for a review.

Perotti, 1996; Sala-i-Martin, 1997). Secondly, inequality creates low mobility traps (Piketty, 2000) and poverty traps in human capital accumulation through a number of different channels: credit constraints (Loury, 1981; Galor and Zeira, 1993; Barham et al., 1995), a fixed cost of education (Galor and Zeira, 1993), a S-shaped education function (Galor and Tsiddon, 1997), a neighbourhood effect resulting from local externalities (Benabou, 1993, 1996a, 1996b; Durlauf 1994, 1996), limited parental altruism (Das, 2007). Galor and Moav (2004) came to the conclusion that inequality is good for growth at the early stage of development when growth is driven by physical capital accumulation, and harmful for growth at the later stage when growth essentially depends on human capital accumulation.

The existence of non-convexities is the usual condition for the emergence of poverty traps[2] and these reduce growth when human capital accumulation is its driving force, either directly or indirectly through R&D. Most of the approaches to under-education traps are constructed within frameworks where higher human capital always results in higher wages. There is however another simple reason why certain individuals remain unskilled, which is that unskilled workers are necessary to produce (Bauduin and Hellier, 2006). Assuming this, the wage of unskilled workers tends towards infinite when the utilisation of unskilled labour tends towards zero. If the quest for higher earnings is the reason for education, then there will always be a number of individuals who will choose to stay unskilled, even without being constrained in their educational choice.

In this article, an intergenerational model of human capital is constructed that assumes that both skilled and unskilled labours are necessary to produce. To be in the most favourable situation for education, we also suppose that the time spent for studying is the only cost of education. In particular, the market for credit is perfect, the interest rate is assumed to be nil and education expenditures are publicly funded. We then show that there is a continuum of Pareto-optimal steady states over an interval of proportions of skilled workers in the population. However, only one of these steady states is efficient in terms of net income per capita. There is then a *skill gap* when the steady state is characterised by a share of skilled workers that stands beneath its efficient value. To reach the efficient steady state, it is possible to support education during the transitional dynamics. In this respect, it is shown that the policy maker must provide the less skilled families with more educational services than the highly skilled families.

---

[2] Except in Das (2007).

Section 2 presents the production technology and the related factor prices (wages). The individuals' educational strategies and the possible steady states are examined in Section 3. The efficient steady state is then determined and the case of skill gap is defined (Section 4). In Section 5, the transitional dynamics and the educational policies tailored to reach the efficient steady state are analysed and simulated. The conclusions are presented in Section 6.

2 Production and wages

2.1. Technology and production

There are two factors of production, unskilled labour *L* and skilled labour *H*.

L consists of simple occupations for which no skill is required. In *L*, one unit of working time represents one unit of labour, whatever the individual’s human capital.

H consists of skilled occupations of different complexity and perfectly substitutable. Individuals’ occupation complexity is proportional to their human capital, and the contribution to *H* of one individual’s unit of time is equal to her human capital.

This definition of skilled and unskilled occupations is suggested by Atkinson (2001)[3].

The economy produces one good, the price of which is 1, with the Cobb-Douglas technology:

$$Y_{jt} = H_{jt}^{\alpha} L_{jt}^{1-\alpha} \tag{1}$$

where Y_{jt}, H_{jt} and L_{jt} respectively denote firm *j*’ s production and use of skilled and unskilled labour at time *t*.

Firms are in perfect competition on both the market for goods and the markets for skilled and unskilled labour.

We finally suppose that skilled and unskilled labours do not need to be utilised at the same time in the production process. In a model with successive generation, this assumption is necessary to ensure that the unskilled can work while the future skilled workers are in education.

[3] Atkinson (2001) assumes two types of industry, one in which output is proportional to individuals' human capital, and the other in which all workers are equally productive, whatever their skill.

2.2. Wages

The firm's profit maximisation determines the factor demands, and thus wages and the unit skill premium at the macroeconomic equilibrium:

$$w_{Lt} = (1-\alpha)\left(H_t / L_t\right)^{\alpha} \tag{2}$$

$$w_{Ht} = \alpha\left(L_t / H_t\right)^{1-\alpha} \tag{3}$$

$$\frac{w_{Ht}}{w_{Lt}} = \frac{\alpha}{1-\alpha}\frac{L_t}{H_t} \tag{4}$$

w_{Ht} is the wage *per unit of human capital*, w_{Lt} the wage per unit of unskilled labour, and L_t and H_t respectively the amounts of unskilled and skilled labour used in production at time *t*.

Let us consider an individual provided with human capital *h* at the end of her education time. We suppose that there is a proportional income tax the rate of which is τ. The individual perceives the after-tax wage $\omega_L = (1-\tau)w_L$ per unit of working time when she works in an unskilled position, and wage $\omega_H = (1-\tau)w_H \times h$ per unit of working time if she works in a skilled position. Individuals may choose between using her human capital to work in a skilled position and working in an unskilled position in which her human capital is useless.

Lemma 1: At time *t*, all workers whose human capital is lower (higher) than w_{Lt} / w_{Ht} are employed in unskilled (skilled) occupations.

Proof: An individual with human capital *h* at time *t* only decides to fill a skilled position if $(1-\tau)w_{Ht}h \geq (1-\tau)w_{Lt}$, i.e. $h \geq w_{Lt} / w_{Ht}$. In contrast, she decides to work in an unskilled position if $h < w_{Lt} / w_{Ht}$.

Definition 1: The *unit skill premium* at time *t* is the ratio $w_t \equiv w_{Ht} / w_{Lt}$ of the wage of one unit of human capital working one unit of time on the wage of one unit of time in an unskilled occupation.

Note that the skill premium of a skilled worker provided with human capital *h* is then $w_{Ht}h / w_{Lt} > 1$ since, as she has a skilled occupation, $h > w_{Lt} / w_{Ht}$.

3 Individuals and Education

We consider a succession of generations with the same number M of individuals. The successive generations linked by a parent-child relationship form a dynasty. The individual of dynasty j and belonging to generation t is called 'individual (j,t)'.

During their childhood, individuals receive directly from their parents, e.g. inside their family, a basic education that depends on their parents' human capital.

Being adult, individuals live one period of time they can divide between schooling and working. The government provides free education to the individuals who decide to go to education. Pursuing education is a choice of the individual who takes her decision by comparing the related income benefit and cost. It is also assumed that the market for credit is perfect and that the interest rate is nil. These assumptions are tailored so as to place individuals in the most favourable situation for their educational choice.

Each parent gives to her child a proportion a of her lifetime income as a bequest. This corresponds to a lifetime utility function $v = (1-a)\log c + a\log b$, where c is the individual's consumption and b her bequest to her child, coefficient a denoting the parent's altruism. We assume that this bequest is given as goods and that these are always sufficient to ensure all children's consumption during the schooling time whatever its utilisation (to be consumed or to be sold). These assumptions have no impact on the steady states. They just guarantee that all the children can either receive, or find on the market the goods they need for consumption during the education time. Their waiving would imply to introduce overlapping generations so that the economy can produce goods when skilled individual are still educating themselves (because skilled labour is necessary for production), which would not modify the steady states but would make the transitional dynamics far more complex and difficult to analyse.

We finally suppose that the contribution of any individual to the total amount of unskilled or skilled labour is negligible so that she does not account for the impact of her decision on wages when making her educational choice.

3.1. The education function

The intra-family externality produces individual *j*'s basic human capital according to the following function:

$$\underline{h}_j = \underline{\delta}\left(h_j(-1)\right)^{\eta} \tag{5}$$

$\underline{h}_j$ is the human capital acquired by the individual within her family, and $h_j(-1)$ her parent's human capital. We suppose $0 < \eta < 1$, which indicates that the marginal impact of the intra-family externality is decreasing.

The education function is $h_j = \max\left\{\delta e_j^{\varepsilon} \underline{h}_j, \underline{h}_j\right\}$, with δ being the productivity in education that is determined by public expenses in education (exogenous for the individual), e_j individual *j*'s schooling time, and $0 < \varepsilon < 1$. This function shows that the individual's human capital cannot be lower than $\underline{h}_j$ (intra-family education) whatever her education time e_j. When effective ($\delta e_j^{\varepsilon} \underline{h}_j > \underline{h}_j$), education thus depends (i) on productivity δ, (ii) on the time spend to study e_j with decreasing returns, and (iii) on the intra-family acquired education $\underline{h}_j$. In addition, there is a minimum schooling time for education to be effective, i.e. $e_j > \delta^{-1/\varepsilon}$ (because $\delta e_j^{\varepsilon} \underline{h}_j > \underline{h}_j$), which creates a non-convexity in education. We show hereafter that this condition has no impact on the individual's decision.

Because of (5), the education function can thus be written:

$$h_j = \begin{cases} \delta\underline{\delta} e_j^{\varepsilon}\left(h_j(-1)\right)^{\eta} & \text{iif} \quad e_j > \delta^{-1/\varepsilon} \\ \underline{h}_j & \text{otherwise} \end{cases} \tag{6}$$

Definition 2: the *lifetime skill premium* of individual *j* who is endowed with human capital h_j and works as a skilled worker is the ratio $s \equiv w_H(1-e_j)h_j / w_L$ of her lifetime labour income on the lifetime labour income of an unskilled worker.

3.2. The choice for education

Individuals pursue education so as to improve their lifetime incomes. We denote $\hat{e}$ the optimal education time when the individual goes to education[4] and $\hat{h}_j$ the related human capital: $\hat{e} = \underset{e_j}{\arg\max}\, I_j = \omega_H \delta\underline{\delta}(1-e_j)e_j^{\varepsilon}\left(h_j(-1)\right)^{\eta}$, and $\hat{h}_j = \delta\underline{\delta}\hat{e}^{\varepsilon}\left(h_j(-1)\right)^{\eta}$.

The individual's net income is $\hat{I}_j = \omega_H(1-\hat{e})\hat{h}_j$ if she pursues education, and $\max\left\{\omega_L, \omega_H \underline{h}_j\right\}$ if she directly joins the labour market.

For individual *j* to pursue education, the related income must be higher than the income without education: $\omega_H(1-\hat{e})\hat{h}_j > \max\left\{\omega_L, \omega_H \underline{h}_j\right\}$. When this is the case, the condition on human capital $\hat{h}_j > \underline{h}_j$ (the human capital with further education must be higher than her basic human capital) is always fulfilled.

By inserting $\hat{h}_j = \delta\underline{\delta}\hat{e}^{\varepsilon}\left(h_j(-1)\right)^{\eta}$ and $\underline{h}_j = \underline{\delta}\left(h_j(-1)\right)^{\eta}$ into $w_H(1-\hat{e})\hat{h}_j > \max\left\{w_L, w_H \underline{h}_j\right\}$, we obtain after rearranging the following conditions to pursue higher education:

$$h_j(-1) > \left(\frac{w_L / w_H}{\delta\underline{\delta}(1-\hat{e})\hat{e}^{\varepsilon}}\right)^{1/\eta} \quad \text{if} \quad w_L > w_H \underline{h}_j \Leftrightarrow h_j(-1) < \left(w_L / \underline{\delta} w_H\right)^{1/\eta} \tag{7}$$

$$\delta > \frac{1}{(1-\hat{e})\hat{e}^{\varepsilon}} \quad \text{if} \quad w_L < w_H \underline{h}_j \Leftrightarrow h_j(-1) > \left(w_L / \underline{\delta} w_H\right)^{1/\eta} \tag{8}$$

Feature (7) provides the condition for *j* to go to education when her intra-family acquired human capital would make her select an unskilled position at the end of childhood (i.e., without education), and feature (8) the same condition when this human capital would make her choose a skilled position.

Lemma 2: Nobody pursues education when productivity δ is lower than $\left((1-\hat{e})\hat{e}^{\varepsilon}\right)^{-1}$.

Proof: To prove lemma 2, it must be shown that condition (7) is not fulfilled when condition (8) is not fulfilled. Condition (8) is not fulfilled when $1/\delta(1-\hat{e})\hat{e}^{\varepsilon} > 1$ and condition (7)

[4] Subscript *j* indicating the individual is omitted because, as shown hereafter, $\hat{e}$ is the same for all individuals.

applies when $w_L > w_H \underline{h}_j$, i.e. $w_L / w_H > \underline{\delta}\left(h_j(-1)\right)^{\eta}$. By putting together both these inequalities we obtain $h_j(-1) < \left(\dfrac{w_L / w_H}{\delta\underline{\delta}(1-\hat{e})\hat{e}^{\varepsilon}}\right)^{1/\eta}$, which shows that condition (7) is not met[5].

Let us now suppose that condition $\delta > 1/(1-\hat{e})\hat{e}^{\varepsilon}$ is fulfilled. As a consequence, individual j such that $h_j(-1) > \left(w_L / \underline{\delta} w_H\right)^{1/\eta}$ goes to education (from condition 8). In addition, $\delta > \dfrac{1}{(1-\hat{e})\hat{e}^{\varepsilon}}$ can be written after rearranging[6]: $\left(\dfrac{w_L / w_H}{\delta\underline{\delta}(1-\hat{e})\hat{e}^{\varepsilon}}\right)^{1/\eta} < \left(\dfrac{w_L / w_H}{\underline{\delta}}\right)^{1/\eta}$. Consequently, all the individuals such as $\left(\dfrac{w_L / w_H}{\delta\underline{\delta}(1-\hat{e})\hat{e}^{\varepsilon}}\right)^{1/\eta} < h_j(-1) < \left(w_L / \underline{\delta} w_H\right)^{1/\eta}$ educate themselves (condition 7) whereas the individuals such as $h_j(-1) < \left(\dfrac{w_L / w_H}{\delta\underline{\delta}(1-\hat{e})\hat{e}^{\varepsilon}}\right)^{1/\eta}$ do not. The necessary and sufficient condition for an individual to go to education is thus: $h_j(-1) > \left(\dfrac{w_L / w_H}{\delta\underline{\delta}(1-\hat{e})\hat{e}^{\varepsilon}}\right)^{1/\eta}$. The individual's education time e is thus determined by the following programme:

$$e = \begin{cases} \hat{e} = \arg\max\limits_{e}\left(\omega_H \delta\underline{\delta}(1-e)e^{\varepsilon}\left(h_j(-1)\right)^{\eta}\right) & \text{iif } h_j(-1) > \left(\dfrac{w_L / w_H}{\delta\underline{\delta}(1-\hat{e})\hat{e}^{\varepsilon}}\right)^{1/\eta} \\ 0 & \text{otherwise} \end{cases}$$

The corresponding value of $\hat{e}$ is[7]:

$$\hat{e} = \frac{\varepsilon}{1+\varepsilon} \tag{9}$$

And condition $\delta > 1/(1-\hat{e})\hat{e}^{\varepsilon}$ is:

$$\delta > (1+\varepsilon)^{1+\varepsilon} / \varepsilon^{\varepsilon} \tag{10}$$

---

5 $\delta < \dfrac{1}{(1-\hat{e})\hat{e}^{\varepsilon}} \Leftrightarrow \dfrac{1}{\delta\underline{\delta}(1-\hat{e})\hat{e}^{\varepsilon}} > \dfrac{1}{\underline{\delta}}$ and $w_L > w_H \underline{h}_j \Leftrightarrow \left(\dfrac{1}{\underline{\delta}}\dfrac{w_L}{w_H}\right)^{1/\eta} > h_j(-1)$ entail $h_j(-1) < \left(\dfrac{w_L / w_H}{\delta\underline{\delta}(1-\hat{e})\hat{e}^{\varepsilon}}\right)^{1/\eta}$.

6 $\delta > \left((1-\hat{e})\hat{e}^{\varepsilon}\right)^{-1} \Leftrightarrow \left(\delta(1-\hat{e})\hat{e}^{\varepsilon}\right)^{-1/\eta} < 1$, and multiplying both sides of this inequality by $\left(w_L / \underline{\delta} w_H\right)^{1/\eta}$.

7 $\partial I_j / \partial e = \omega_H \delta'\left(h_j(-1)\right)^{\eta}(\varepsilon e^{\varepsilon-1} - (1+\varepsilon)e^{\varepsilon}) = 0 \Rightarrow \hat{e} = \varepsilon/(1+\varepsilon)$

It can be noted that the condition for the efficiency of education $e > \delta^{-1/\varepsilon}$ is always fulfilled when $e = \hat{e} = \dfrac{\varepsilon}{1+\varepsilon}$ and $\delta > (1+\varepsilon)^{1+\varepsilon} / \varepsilon^{\varepsilon}$ because $e > \delta^{-1/\varepsilon} \Leftrightarrow \delta > \left((1+\varepsilon)/\varepsilon\right)^{\varepsilon}$.

We can consequently state the following two propositions:

Proposition 2: *Nobody pursues education when the productivity* δ *is lower than* $(1+\varepsilon)^{1+\varepsilon} / \varepsilon^{\varepsilon}$.

Proposition 3: *Assume that* $\delta > (1+\varepsilon)^{1+\varepsilon} / \varepsilon^{\varepsilon}$. *Then, at any time t, there is a threshold value of their parents' human capital* $\underline{h}_t = \left(\dfrac{(1+\varepsilon)^{1+\varepsilon}}{\delta \underline{\delta} \varepsilon^{\varepsilon}} \dfrac{w_{Lt}}{w_{Ht}} \right)^{1/\eta}$ *below which individuals do not go to education and above which they allow time* $\hat{e} = \varepsilon(1-\varepsilon)^{-1}$ *to education.*

The values of h_j and I_j corresponding to $\hat{e}$ are:

$$\hat{h} = \delta \underline{\delta} \left(\frac{\varepsilon}{1+\varepsilon} \right)^{\varepsilon} \left(h_j(-1) \right)^{\eta} \tag{11}$$

$$\hat{I}_j = \omega_H \delta \underline{\delta} \frac{\varepsilon^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}} \left(h_j(-1) \right)^{\eta} \tag{12}$$

Remark: if the interest rate was not nil, the optimal education time would be the value of *e* that maximises the lifetime earnings $I_t = \int_e^1 w \exp[-r\theta] h_t(s) d\theta$ with $h_t = \delta \underline{\delta} e^{\varepsilon} (h_{t-1})^{\eta}$. This is the unique solution of equation $z(e) = (\varepsilon - er)\exp[r(1-e)] - \varepsilon = 0$, and this optimum is decreasing with the interest rate ($\partial \hat{e} / \partial r < 0$) and $\hat{e} = \varepsilon / (1+\varepsilon)$ for *r* = 0.

3.3. Public education

Education is freely provided by the government and funded by a tax at rate τ on the preceding generation's income. The educational expenditure is thus totally defined by the tax rate τ and it is assumed for simplicity that the produced good can be utilised for the educational activity.

The educational policy τ determines the efficiency in education δ according to the following functions:

$$\delta = \overline{\delta} \left(\tau y_{-1} / m_H \right)^{\beta} \tag{13}$$

Subscript (-1) indicates the parents generation; $y_{-1} = Y_{-1} / M$ is the real total income per household; $0 < \beta < 1$ indicates that the marginal efficiency of public education expenditures is decreasing; m_H denotes the proportion of skilled households in the population *M*, and $m_L = 1 - m_H$ the proportion of unskilled households.

Equation (13) can be rewritten $\delta = \bar{\delta}\left(\tau Y_{-1} / m_H M\right)^{\beta}$. Thus, the efficiency of the educational policy depends on the educational spending per student ($\tau Y_{-1} / m_H M$).

We also suppose that $\eta + \alpha\beta < 1$, i.e. that the marginal impact of public expenditures on the income per head is decreasing.

It can firstly be noted that δ typically changes over time since it depends on y_{-1}.

Secondly, by inserting Relation (13) into $\underline{h}_t = \left(\frac{(1+\varepsilon)^{1+\varepsilon}}{\delta \underline{\delta} \varepsilon^{\varepsilon}} \frac{w_{Lt}}{w_{Ht}} \right)^{1/\eta}$ we obtain:

$$\underline{h}_t = \left(\frac{(1+\varepsilon)^{1+\varepsilon} m_{Ht}{}^{\beta}}{\bar{\delta}\underline{\delta}\tau^{\beta}\varepsilon^{\varepsilon} y_{t-1}{}^{\beta}} \frac{w_{Lt}}{w_{Ht}} \right)^{1/\eta} \tag{14}$$

For a given tax rate τ, the value of threshold $\underline{h}_t$ varies over time and depends on m_{Ht}, y_{-1} and w_{Lt} / w_{Ht}.

Thirdly, condition $\delta > (1+\varepsilon)^{1+\varepsilon} / \varepsilon^{\varepsilon}$ now becomes:

$$\tau > \left(\frac{(1+\varepsilon)^{1+\varepsilon}}{\bar{\delta}\varepsilon^{\varepsilon}} \right)^{1/\beta} \frac{m_H}{y_{-1}} \tag{15}$$

Relation (15) shows that the only situation in which nobody goes to education due to condition $\delta > \frac{1}{(1-\hat{e})\hat{e}^{\varepsilon}}$ is when $\tau < \left(\frac{(1+\varepsilon)^{1+\varepsilon}}{\bar{\delta}\varepsilon^{\varepsilon}} \right)^{1/\beta} \frac{1}{Y_{-1}}$ (because 1/*M* is the lowest possible value of m_H).

It must finally be noted that, since δ is the same for everyone, all the individuals who go to education receive the same amount of educational service from the government.

4 Steady states, equality and efficiency

4.1. The steady states

The steady state values are depicted by a star (*).

Lemma 3: If the successive generations of a dynasty do not go to education, this dynasty tends towards the human capital $\underline{h}^*$ with:

$$\underline{h}^* = \underline{\delta}^{\frac{1}{1-\eta}} \tag{15}$$

Proof: $\underline{h}^* = \underline{\delta}^{\frac{1}{1-\eta}}$ is the stable steady state of dynamics $h_j(t) = \underline{\delta}\left(h_j(t-1)\right)^{\eta}$.

Lemma 4: If the successive generations of a dynasty pursue higher education, this dynasty tends towards the human capital $\overline{h}^*$ with:

$$\overline{h}^* = (\overline{\delta}\underline{\delta})^{\frac{1}{1-\eta}} \left(\frac{\tau y^*}{m_H{}^*}\right)^{\frac{\beta}{1-\eta}} \left(\frac{\varepsilon}{1+\varepsilon}\right)^{\frac{\varepsilon}{1-\eta}} \tag{16}$$

Proof: The steady state of dynamics $h_j(t) = \delta\underline{\delta}\left(\frac{\varepsilon}{1+\varepsilon}\right)^{\varepsilon}\left(h_j(t-1)\right)^{\eta}$ is $\overline{h}^* = (\delta\underline{\delta})^{\frac{1}{1-\eta}}\left(\frac{\varepsilon}{1+\varepsilon}\right)^{\frac{\varepsilon}{1-\eta}}$. As $\delta = \overline{\delta}\left(\tau y^* / m_H{}^*\right)^{\beta}$ at the steady state, then $\overline{h}^* = (\overline{\delta}\underline{\delta})^{\frac{1}{1-\eta}}\left(\tau y^* / m_H{}^*\right)^{\frac{\beta}{1-\eta}}\left(\frac{\varepsilon}{1+\varepsilon}\right)^{\frac{\varepsilon}{1-\eta}}$.

Let $w_L{}^*$ and $w_H{}^*$ be the level of the unit wages at the steady state (we consider here one particular steady state since there is an infinite number of them as shown hereafter). Then, the existence of an under-education trap is conditioned by the fact that the steady state without higher education $\underline{h}^* = \underline{\delta}^{1/(1-\eta)}$ is lower than the threshold under which individuals decide not to pursue higher education $\underline{h} = \left(w_L{}^* / \underline{\delta} w_H{}^*\right)^{1/\eta}$: $\underline{h}^* < \underline{h} \Leftrightarrow \frac{w_H{}^*}{w_L{}^*} < \underline{\delta}^{-1/(1-\eta)}$.

Proposition 4: *The tax rate* τ *being given, the economy exhibits a continuum of steady states corresponding to all the proportions of skilled household* m_H* *belonging to a certain interval* $\left[\underline{m}_H*,\overline{m}_H*\right]$ *and characterised by the following features:*

1) Production per household is $y*\equiv\dfrac{Y*}{M}=\left(\overline{\delta}\underline{\delta}\tau^{\beta}\varepsilon^{\varepsilon}\left(\dfrac{1}{1+\varepsilon}\right)^{1+\varepsilon-\eta}(m_H*)^{-\beta}\right)^{\frac{\alpha}{1-\eta-\alpha\beta}}\left((m_H*)^{\alpha}\left(m_L*\right)^{1-\alpha}\right)^{\frac{1-\eta}{1-\eta-\alpha\beta}}$

2) All skilled workers possess human capital $\overline{h}*=\left(\overline{\delta}\underline{\delta}\tau^{\beta}\left(\dfrac{\varepsilon}{1+\varepsilon}\right)^{\varepsilon}(1+\varepsilon)^{-\alpha\beta}\left(\dfrac{m_L*}{m_H*}\right)^{(1-\alpha)\beta}\right)^{\frac{1}{1-\eta-\alpha\beta}}$ *and all unskilled workers human capital* $\underline{h}*=\underline{\delta}^{1/(1-\eta)}$.

3) Skilled labour is $H*=m_H*M\left(\overline{\delta}\underline{\delta}\tau^{\beta}\varepsilon^{\varepsilon}\left(1+\varepsilon\right)^{-(1+\varepsilon-\eta)}\left(\dfrac{m_L*}{m_H*}\right)^{(1-\alpha)\beta}\right)^{\frac{1}{1-\eta-\alpha\beta}}$ *and unskilled labour* $\overline{L}*=m_L*M$.

4) The unit skill premium is $w*=\dfrac{\alpha}{1-\alpha}\left(\dfrac{\left(1+\varepsilon\right)^{1+\varepsilon-\eta}}{\overline{\delta}\underline{\delta}\tau^{\beta}\varepsilon^{\varepsilon}}\left(\dfrac{m_L*}{m_H*}\right)^{1-\eta-\beta}\right)^{\frac{1}{1-\eta-\alpha\beta}}$, *the skill premium* $\varpi*\equiv w*\overline{h}*=\left(1+\varepsilon\right)\dfrac{\alpha}{1-\alpha}\dfrac{m_L*}{m_H*}$, *and the lifetime skill premium* $s*=\dfrac{\alpha}{1-\alpha}\dfrac{m_L*}{m_H*}$.

5) All these equilibria are Pareto-optimal for a given value of the income tax rate τ.

Proof: Interval $\left[\underline{m}_H*,\overline{m}_H*\right]$ is built as follows: $\underline{m}_H*$ is the lowest value of m_H* such that children from unskilled families decide not to pursue higher education, and $\overline{m}_H*$ the highest m_H* such that children from skilled families decide to pursue higher education. The analytical determination of $\underline{m}_H*$ and $\overline{m}_H*$ is described in Appendix 1, as well as the proofs of the five features of proposition 4.

It can be noted that all the steady state values except $\underline{h}*$ depend on the educational policy because m_H* depends on τ (see Appendix 1).

4.2. The efficient steady state

Let $z=(1-\tau)y$ be the net income per capita (households), i.e. the income per capita net of educational expenditures.

The net income per capita at the steady state is $z^* = (1-\tau)y^*$, and thus:

$$z^* = (1-\tau)\left(\bar{\delta}\underline{\delta}\tau^{\beta}\varepsilon^{\varepsilon}\left(\frac{1}{1+\varepsilon}\right)^{1+\varepsilon-\eta}(m_H^*)^{1-\eta-\beta}\left(m_L^*\right)^{\frac{(1-\alpha)(1-\eta)}{\alpha}}\right)^{\frac{\alpha}{1-\eta-\alpha\beta}} \tag{17}$$

Proposition 5: *There is one efficient steady state in terms of net income per capita characterised by:*

1) The income tax rate $\hat{\tau} = \alpha\beta/(1-\eta)$.

2) The proportion of skilled workers in the working population $\hat{m}_H = \dfrac{\alpha(1-\eta-\beta)}{1-\eta-\alpha\beta}$.

Proof: The efficient steady state is determined by maximising the net income per capita. Conditions $\partial z^*/\partial\tau = 0$ and $\partial z^*/\partial m_H^* = 0$ respectively determine features 1) and 2) (see Appendix 2 for calculations).

The tax rate $\hat{\tau}$ may be seen as determining the 'golden rule' of the model because it maximises the net income (and thus consumption) per capita for a given share of skilled workers at the steady state m_H^*.

It is important to note that the optimal tax rate $\hat{\tau}$ is independent from the proportion of skilled workers at the steady state. Let $\hat{\underline{m}}_H$ and $\hat{\bar{m}}_H$ be respectively the values of $\underline{m}_H^*$ and $\bar{m}_H^*$ for $\tau = \hat{\tau}$ (the values of $\hat{\underline{m}}_H$ and $\hat{\bar{m}}_H$ are given in Appendix 3). Then all the shares m_H^* inside interval $\left[\hat{\underline{m}}_H, \hat{\bar{m}}_H\right]$, combined with the tax rate $\hat{\tau}$ are Pareto-optimal steady states (because of feature 5 of Proposition 4). Among these Pareto-optimal steady states, the one with $\hat{m}_H = \dfrac{\alpha(1-\eta-\beta)}{1-\eta-\alpha\beta}$ is efficient since it maximises the net income per capita.

Table 1 depicts the values of the variables corresponding to the efficient steady state

Table 1: The efficient Steady state

$\hat{\hat{h}}$	$\hat{w}$	$\hat{\varpi}$	$\hat{s}$
$\left(\frac{\bar{\delta}\underline{\delta}\beta^{\beta}\varepsilon^{\varepsilon}}{(1+\varepsilon)^{\varepsilon+\alpha\beta}}\left(\frac{1-\alpha}{1-\eta-\beta}\right)^{(1-\alpha)\beta}\left(\frac{\alpha}{1-\eta}\right)^{\alpha\beta}\right)^{\frac{1}{1-\eta-\alpha\beta}}$	$\left(\frac{(1+\varepsilon)^{1+\varepsilon-\eta}}{\beta^{\beta}\bar{\delta}\underline{\delta}\varepsilon^{\varepsilon}}\left(\frac{\alpha}{1-\alpha}\right)^{(1-\alpha)\beta}\frac{(1-\eta)^{1-\eta}}{(1-\eta-\beta)^{1-\eta-\beta}}\right)^{\frac{1}{1-\eta-\alpha\beta}}$	$\frac{(1+\varepsilon)(1-\eta)}{1-\eta-\beta}$	$\frac{1-\eta}{1-\eta-\beta}$

From Table 1, it can be seen that the efficient lifetime skill premium $\hat{s}$ is higher than 1, i.e., that skilled workers earn more than the unskilled over their lifetime at the efficient steady state, which is of course the condition for this steady state to be sustainable.

It must finally be underlined that $\hat{m}_H = \dfrac{\alpha(1-\eta-\beta)}{1-\eta-\alpha\beta}$ is a steady state only if it is located into interval $\left[\underline{\hat{m}}_H, \hat{\hat{m}}_H\right]$. If $\hat{m}_H < \underline{\hat{m}}_H$, then $\underline{\hat{m}}_H$ is the most efficient steady state. Similarly, $\hat{\hat{m}}_H$ is the most efficient steady state when $\hat{m}_H > \hat{\hat{m}}_H$.

4.3. The skill gap, equality and efficiency

Definition 3: There is a *Skill Gap* when the difference $g \equiv \hat{m}_H - m_H *$ is positive.

Thus, there is a skill gap when the steady state is characterised by a proportion of skilled workers lower than its efficient value $\hat{m}_H$.

Proposition 6: *In case of skill gap* ($m_H * < \hat{m}_H$), *higher equality and higher efficiency are complementary goals over a certain interval of proportions of skilled workers.*

Proof: This is straightforward because (i) $m_H * < \hat{m}_H$ and (ii) a higher $m_H *$ induces lower inequality. The lower limit of the corresponding interval is proportion $m_H *$, and its upper limit is the proportion above $\hat{m}_H$ that allows to reach the same income per capita as $m_H *$.

In a number of situations, the two objectives of higher equality and higher income per capita are thus complementary. In contrast, there is an equality-efficiency trade-off when the proportion $m_H *$ is higher than $\hat{m}_H$ at the steady state, i.e. in the case of *skill excess*.

Firstly suppose that the social planner has a single objective that is efficiency. In the case of skill gap, her search for efficiency will *ipso facto* induce more equity, i.e. less inequality. Suppose now that the social planner's goal is twofold, i.e. a weighted composition of efficiency and equality, the first being measured by the net income per capita and the second by the skill premium at the steady state. The maximisation of the social planner objective function determines one steady state $\tilde{m}_H$ that is optimal to her, and thus one income per capita $\tilde{y}$ and one skill premium $\tilde{\varpi}$. Let (y^*, ϖ^*) be the couple (income per capita, skill premium) that emerges without change in policy τ. Efficiency and equality are complementary when $y^* < \tilde{y}$ and $\varpi^* > \tilde{\varpi}$.

5 Transitional dynamics and educational policies

The initial distribution of human capital across dynasties and the educational policy τ determines (i) the initial distribution of individuals (generation 0) between two sets of workers, one skilled and the other unskilled, and (ii) the distribution of their children (generation 1) between these who go and those who do not go to education. If the M individuals are ranked in ascending order of human capital, individual f such that $h_f \geq \frac{w_{L0}}{w_{H0}}$ and $h_{f-1} < \frac{w_{L0}}{w_{H0}}$, with $\frac{w_{L0}}{w_{H0}} = \frac{(1-\alpha)}{\alpha(f-1)} \sum_{j=f}^{M} h_j$ (relation 4) separates the first two sets. This determines the initial product Y_0 and product per household y_0. From this initial situation, the successive generations of individuals will divide themselves between those who choose education and those who do not, and consequently between skilled and unskilled workers.

We firstly show that, for a given educational policy and under rather weak conditions, the initial distribution of dynasties that results from the initial decision to go or not to go to education determines the segmentation between skilled and unskilled workers over all the following periods. We subsequently assume a situation of skill gap and we analyse the educational policy patterns that make it possible to increase the number of skilled workers. A series of simulation are finally carried out so as to illustrate these findings.

5.1. Transitional dynamics

Let us assume that the following three features apply all over the transition to the steady state: (i) $\underline{h}_t > \underline{h}^*$, (ii) H_t does not decrease for a given m_H, and (iii) τ remains unchanged. Assumption (i) signifies that the threshold $\underline{h}_t$ above (under) which individuals choose to educate (not to educate) themselves is higher than the human capital of non educated individual at the steady state $\underline{h}^*$. This is always the case, at least after a number of periods, because otherwise no one would select to be unskilled, which is impossible. Assumption (ii) stipulates that when the proportion of individuals who choose to go to education remains constant, then the amount of skilled labour H_t does not decrease over time. This is normally the case, except in the very unlikely situation in which a majority of individuals initially possess a human capital higher than that of the skilled workers at the steady state. Assumption (iii) signifies that we consider a given and unchanged educational policy. Assuming these three features, the following proposition may be established:

Proposition 7: *All the descendants of the individuals who select to educate themselves also select to educate themselves, and all the descendants of those who select not to go to education do not go to education.*

Proof: See Appendix 4.

Proposition 7 shows that, under rather weak conditions, the initial distribution of individuals between these who go to education and those who do not, totally determines the segmentation of dynasties between skilled and unskilled workers for all the following periods as long as the educational policy τ remains unchanged.

From this result, it is possible to derive the following two lessons:

1. The initial distribution of human capital across dynasties and the educational policy τ determine the initial proportion m_H of dynasties who decide to pursue higher education, and this proportion does not change over time as long as the initial policy is maintained.

2. The only way to change proportion m_H is thus to modify the educational policy.

5.2. Skill-enhancing educational policies

Let us suppose that the initial proportion m_H of dynasties that opts for education is lower that the social planner goal $\tilde{m}_H$. The setting of a skill-enhancing educational policy can be seen as a means to increase m_H.

Equally distributed public education

If the educational policy takes the form defined by relation (13), i.e. when public education is equally provided to all the individuals who choose education, then an increase in the public expenditure for education, i.e. a rise in τ, is the only way to support education.

Proposition 8: *A policy that modifies the amount of public expenditure for education and provides everyone with the same education services has no impact on the distribution of individuals and dynasties between the educated and the non-educated, and thereby on the distribution of the working population between skilled and unskilled workers*.

Proof: see Appendix 5.

Proposition 8 reveals that a greater effort in public education has no impact on m_H when this education is the same for everyone. This is because a rise in τ has two opposite effects on threshold $\underline{h}_t = \left(\frac{(1+\varepsilon)^{1+\varepsilon} {m_{Ht}}^{\beta}}{\overline{\delta}\underline{\delta}\tau^{\beta}\varepsilon^{\varepsilon} {y_{t-1}}^{\beta}} \frac{w_{Lt}}{w_{Ht}} \right)^{1/\eta}$. On the one hand, it directly decreases $\underline{h}_t$ (through τ^{β}). On the other hand, since it raises $H_t = \overline{\delta}\underline{\delta}\left(\tau y_{-1} / m_H\right)^{\beta} \left(\frac{\varepsilon}{1+\varepsilon} \right)^{\varepsilon} \sum_{i=f}^{M} \left(h_i(-1) \right)^{\eta}$, it increases $\frac{w_{Lt}}{w_{Ht}} = \frac{1-\alpha}{\alpha} \frac{H_t}{L_t}$ by the same amount. These two effects offset each other.

It must finally be noted that Proposition 8 is only valid because we assume perfect expectations, i.e., that individuals perfectly anticipate the variation of w_{Lt} / w_{Ht} induced by the increase in human capital that derives from higher educational expenditures. With myopic expectations, individuals would not integrate the forthcoming increase in H_t / L_t in their skill premium expectations, and higher public expenditures for education would then reduce threshold $\underline{h}_t$ and cause a number of individuals to pass from non-education to education.

Specific action in favour of the poorest

The policy maker can however modulate her educational effort according to the position of the family on the human capital scale, and thereby on the parent's income. This consists in changing the educational pattern (13) and to replace it by a system that boosts the educational services received by the families with low human capital endowments. This specific action supposes that more educational services are given to the low skilled than to the high skilled families[8]. We can thus consider two public education functions, one that applies to relatively low skilled families (δ_1), and the other to the relatively high skilled (δ_2):

$$\delta_1 = \bar{\delta}\left(q\tau y_{-1}/lm_H\right)^{\beta} \tag{18}$$

$$\delta_2 = \bar{\delta}\left(\frac{(1-q)\tau y_{-1}}{(1-l)m_H}\right)^{\beta} \tag{19}$$

With q being the share of the total levies allocated to the low skilled families and l the share of the relatively low skilled families in the population that goes to education. Since $\delta = \bar{\delta}\left(\tau y_{-1}/m_H\right)^{\beta}$, equations (18) and (19) may respectively be written $\delta_1 = \left(q/l\right)^{\beta}\delta$ and $\delta_2 = \left(\frac{1-q}{1-l}\right)^{\beta}\delta$. Logically, $q > l$ is the condition for such a policy to favour the education of the children from low skilled families, i.e. for $\delta_1 > \delta_2$.

Several conditions must be met for such a policy to be implemented:

1. The social planner must determine the threshold that separates the relatively low skilled from the relatively high skilled families, i.e. a level of the parents' human capital under which the supply of educational services follows relation (18), and above which it follows relation (19). Note that this threshold must typically be higher than the value $\underline{h}_t$ corresponding to the universal education function (13). As a matter of fact, since there are two different regimes of public education, these define two different thresholds of the parents' human capital for choosing education: $\underline{h}_{1t} = \left(\frac{(1+\varepsilon)^{1+\varepsilon}}{\delta_1\underline{\delta}\varepsilon^{\varepsilon}}\frac{w_{Lt}}{w_{Ht}}\right)^{1/\eta}$ for the low skilled and $\underline{h}_{2t} = \left(\frac{(1+\varepsilon)^{1+\varepsilon}}{\delta_2\underline{\delta}\varepsilon^{\varepsilon}}\frac{w_{Lt}}{w_{Ht}}\right)^{1/\eta}$ for the high skilled, with $\underline{h}_{1t} < \underline{h}_t < \underline{h}_{2t}$ since $\delta_1 > \delta > \delta_2$ (see Figure 1). This shows that these dynasties that would have chosen to go to education with the universal system described by equation (13) and that stand in the vicinity of $\underline{h}_t$, will choose no education if the threshold from which the education system (19) (for high skilled families) applies is $\underline{h}_t$. To prevent this unwilling result, it is thus necessary to choose $\underline{h}_{2t}$ as the parents' human capital threshold under which the education system for the less skilled families applies.

[8] The term 'specific action' is preferred to 'affirmative action' because the latter usually relates to ethnic-oriented policies.

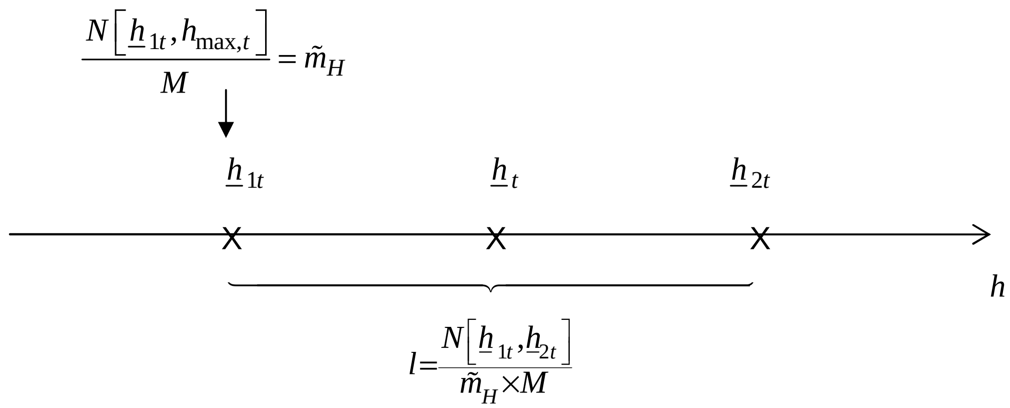

Figure 1: The pattern of affirmative action towards the less skilled

2. The choice of the proportion q that makes it possible to reach the objective $\tilde{m}_H$ is independent of the tax rate τ, provided that condition $\delta_2 > (1+\varepsilon)^{1+\varepsilon} / \varepsilon^{\varepsilon}$ is met[9]. Of course, the tax rate $\hat{\tau}$ must be chosen sooner or later if the social planner targets the efficient steady state. During the transition to the steady state, the social planner can however choose the tax rate according to her preference in terms of trade-off between the net income of the actual generation of parents and the speed of the transition process.

3. At any time t, there is a set of policies defined by the couples (q, τ) that make it possible to reach the social planner's goal $\tilde{m}_H$. These policies are such that (i) $N\left[\underline{h}_{1t}, h_{\max,t}\right] = \tilde{m}_H M$, (ii) $l = \dfrac{N\left[\underline{h}_{1t}, \underline{h}_{2t}\right]}{\tilde{m}_H \times M}$, (iii) $\dfrac{\underline{h}_{2t}}{\underline{h}_{1t}} = \left(\dfrac{q(1-l)}{l(1-q)}\right)^{\beta/\eta}$, (iv) $\delta_2 = \bar{\delta}\left(\dfrac{(1-q)\tau y_{-1}}{(1-l)\tilde{m}_H}\right)^{\beta} > \dfrac{(1+\varepsilon)^{1+\varepsilon}}{\varepsilon^{\varepsilon}}$, and (v) $0 < q < 1$. The parents being ranked in ascending order of human capital, condition (i) determines the parent's human capital $\underline{h}_{1t}$ such that the number of dynasties $N\left[\underline{h}_{1t}, h_{\max,t}\right]$ into the interval $\left[\underline{h}_{1t}, h_{\max,t}\right]$ ($h_{\max,t}$ being the highest parent's human capital) accounts for the proportion $\tilde{m}_H$ of the dynasties (Figure 1). Condition (ii) determines the proportion l of families the social planner must consider as less skilled and thereby integrate into the educational pattern δ_1 (Figure 1). This condition defines an increasing function between $\underline{h}_{2t}$ and l. Condition (iii) is determined by inserting relations (18) and (19) into

[9] As $\delta_1 > \delta_2$, then $\delta_2 > (1+\varepsilon)^{1+\varepsilon} / \varepsilon^{\varepsilon} \Rightarrow \delta_1 > (1+\varepsilon)^{1+\varepsilon} / \varepsilon^{\varepsilon}$.

$\underline{h}_{1t} = \left(\frac{(1+\varepsilon)^{1+\varepsilon}}{\delta_1 \underline{\delta} \varepsilon^{\varepsilon}} \frac{w_{Lt}}{w_{Ht}} \right)^{1/\eta}$ and $\underline{h}_{2t} = \left(\frac{(1+\varepsilon)^{1+\varepsilon}}{\delta_2 \underline{\delta} \varepsilon^{\varepsilon}} \frac{w_{Lt}}{w_{Ht}} \right)^{1/\eta}$. Condition (iii) thus provides a relation between $\underline{h}_{2t}$, q and l. Combining conditions (ii) and (iii) defines l as an increasing function of q: $q = q(l)$ with $\partial q / \partial l > 0$ [10]. Finally, inserting $q = q(l)$ into inequality (iv) and assuming $0 < q < 1$ (inequality (v)) determine all the set of policies (τ, q) consistent with the social planner's goal $\tilde{m}_H$.

4. The sooner the policy is carried out, the easier it is. As a matter of fact, the later the policy is implemented, (i) the lower the unskilled dynasties descend on the human capital scale and (ii) the more threshold $\underline{h}_{1t}$ decrease, and the more intense must the educational policy be.

5.3. Simulations

A series of simulations are now implemented that utilises plausible values of the parameters to illustrate the model's findings. These simulations can in no way represent a realistic situation because, as already mentioned, the model's assumptions are chosen so as to be in the most favourable condition in terms of education costs, which tends to increase the number of individuals who go to education. They only show that cases of skill gap are likely outcomes even in this optimistic situation. We present the results provided by one set of plausible parameters, all the other simulations implemented having produced the same general outcomes (available from the author upon request).

Table 2 depicts the model coefficients and the induced schooling time $\hat{e}$, and Table 3 the related couple $(\hat{\tau}, \hat{m}_H)$ at the efficient steady state and the corresponding values of the model variables.

Table 2: The model parameters and the schooling time

α	η	ε	$\underline{\delta}$	$\overline{\delta}$	β	$\hat{e}$
0.6	0.5	0.2	1	4	0.2	0.167

[10] Condition (ii) yields $\underline{h}_{2t} = h_2(l)$ with $\partial h_2 / \partial l > 0$. Condition (iii) can thus be written $q = \frac{(h_2(l))^{\eta/\beta}}{(\underline{h}_{1t})^{-\eta/\beta}(l^{-1}-1) + (h_2(l))^{\eta/\beta}}$, which determines $q = q(l)$ with $\partial q / \partial l > 0$.

Table 3: The efficient steady state

$\hat{\tau}$	$\hat{m}_H$	$\underline{\hat{h}}$	$\hat{\bar{h}}$	$\hat{y}$	$\hat{w}$	$\hat{\varpi}$	$\hat{s}$
0.24	47.4	1	6.81	2.19	0.29	2	1.67

Coefficient α is such that skilled labour represents 60% of the total wage bill. In fact, we know that in the long term the 20% highest incomes represent 50% of total income for the US and the UK (Lindert, 2000). As the calculations for the selected parameters show that the share of skilled workers in the working population is of about 38%, and accounting for the fact that capital incomes are more unevenly distributed that labour income, a share of 60% of the income going to the skilled is an acceptable estimation. Coefficient η is consistent with a number of estimates of the elasticity of the children's education with respect to their parent's (see Solon, 1999). Coefficient ε is selected so as to make the education time represent between 15% and 20% of the lifetime.

Two sets of simulations are implemented. In the first, we suppose that the population is initially formed of 10000 groups that are uniformly distributed and ranked in ascending order over interval $\left[h_{\min} = 0, h_{\max} = \hat{\bar{h}} = 6.81\right]$. We also assume that each group comprises one dynasty only so that M = 10000. In the second series of simulations, we assume a more uneven initial distribution of human capital. To simplify, we start from an initial situation in which 10% of the population possess human capital $h_{\max} = \hat{\bar{h}} = 6.81$, 60% possess human capital $h_{\min} = \underline{\hat{h}} = 1$, and the remaining 30% are uniformly distributed between these 2 values, i.e. over interval $[1, 6.81]$. These proportions broadly reproduce the shares of the individuals with higher education, with primary education and beneath, and in between, in the population over 25 years old in the advanced countries (Australia, Canada, Japan, New Zealand, the US and Western Europe) in 1970 according to Barro and Lee (2000) calculations.

In both cases, we calculate (i) the distribution of the initial generation (generation 0) between skilled and unskilled occupations, (ii) the distribution of their children between these who go to education and those who do not, (iii) the corresponding steady state, and (iv) the related position of this steady state in relation to the efficient one.

Table 4 describes the proportion of skilled workers, the value of the net income per capita (efficiency index) and the value of the lifetime skill premium (inequality index) at the steady states corresponding to the two different initial situations in terms of human capital distribution, with $\tau = \hat{\tau}$ in both cases. These values are compared with these at the efficient steady state.

Table 4: Comparison of the different steady states

Configuration	m_H*	Skill Gap[a]	$y*$	$s*$
Uniform distribution of human capital	$m_{H1}* = 54.4$	-7	$y_1* = 1.37$	$s_1* = 2.46$
Non uniform distribution	$m_{H2}* = 0.379$	9.5	$y_2* = 1.37$	$s_2* = 2.46$
Efficient steady state	$\hat{m}_H = 0.474$	0	$\hat{y} = 1.40$	$\hat{s} = 1.67$

[a] *A negative value reveals a skill excess.*

5.4. Results and discussion

The different simulations implemented lead to the following results:

1. When the individuals are initially uniformly distributed over an interval of human capital $[h_{\min}, h_{\max}]$ with $h_{\min} < h_{\max}$ and $h_{\min} \geq 0$, then the spontaneous dynamics results in a skill excess, i.e., the proportion of skilled workers is higher than its efficient value. This result is consistent with the large number of simulations implemented with different plausible values of the parameters.

2. When the initial distribution of human capital across dynasties is more uneven, thereby in line with the distributions observed in the real world, then all the simulations reveal a skill gap. The comparisons between the attained steady states and the efficient steady states show that the efficiency gap (difference between the disposable income per capita in the two situations) is rather small whereas the difference in the lifetime skill premia are rather substantial. This reveals that correcting the skill gap by an adequate policy can only slightly improve efficiency, but it can significantly reduce inequality.

6 Conclusion

We have analysed the human capital dynamics and the distribution of families between the skilled and the unskilled when production requires the use of both skilled and unskilled labour. In such a framework, the division of dynasties between the skilled and the unskilled and the corresponding steady state totally depend on the initial distribution of human capital. We have shown that there is a continuum of Pareto optimal steady states one of which only is efficient. As a consequence, the economy can generate *skill gaps*, i.e., situations in which inefficiency results from a lack of skilled workers. In this case, efficiency and equality are complementary goals of the policy maker. In addition, an increase in public educational expenditures that provides everyone with the same amount of educational services does not permit to increase the number of dynasties that opt for education, and thereby the proportion of skilled workers. A specific action that allocates more education to the less skilled families only is necessary to achieve this objective.

The model is built within a very simple framework with perfect competition on the credit market, an interest rate that is nil, and education being publicly funded without impact of the families' incomes. These simplifying assumptions are chosen so as to minimise the factors that create inequalities between families in their educational decisions and to limit the cost of educating to the only time spent for education. The only factor of inequality persistence is thus the existence of an intra-family externality in the education function. Less constraining assumptions such as imperfections on the credit market, a fixed cost of education and/or local externalities *à la* Benabou (1993, 1996a and 1996b) etc. would obviously produce a more realistic picture. This would typically reinforce the findings of the model by increasing the number of individuals and dynasties that choose to remain non-educated. An interesting way of extending the model would also consist in distinguishing basic education from higher education. In the approach developed here, basic education is provided inside the family through externalities. A more accurate way of modelling would be to insert these externalities into a basic education function that could also depend on public spending. This would in particular permit to address the issue of the efficient distribution of public expenditures between basic and higher education.

References

Alesina A. and Perotti R. (1996), Income Distribution, Political Instability and Investment, *European Economic Review*, 40, 1203-1228

Alesina A. and Rodrik D. (1994), Distributive politics and economic growth, *Quarterly Journal of Economics*, 109, 465–490.

Anand S. and Ravi Kambur S.M. (1993), The Kuznets process and the inequality-development relationship, *Journal of Development Economics*, 40(1), 25-52.

Barham V., Boadway R., Marchand M. and Pestieau P. (1995), Education and poverty trap, *European Economic Review*, 39(7), 1257-1275.

Barro R. & J.W. Lee (2000) International date on educational attainment: updates and implications, With appendix data set downloadable from: http://www2.cid.harvard.edu/ciddata/, Centre for International Development, Harvard University.

Bauduin N. and Hellier J. (2006), Skill Dynamics, Inequality and Social Policies, *ECINEQ Working Paper Series* no34.

Benabou R. (1993), Workings of a city: location, education and production, *Quarterly Journal of Economics*, 108, 619-652.

Benabou R. (1996a), Heterogeneity, stratification and growth: macroeconomic implications of community structure and school finance, *American Economic Review*, 86, 584-609.

Benabou R. (1996b), Equity and efficiency in human capital investment: the local connection, *Review of Economic Studies*, 63, 237-264.

Das M. (2007), Persistent inequality: An explanation based on limited parental altruism, *Journal of Development Economics*, 84(1), 251-270.

Deininger K. and Squire L. (1998), New ways of looking at old issues: inequality and growth, *Journal of Development Economics*, 57, 259–287.

Durlauf S. (1994), Spillovers, stratification, and inequality, *European Economic Review*, 38(3-4), 836-845.

Durlauf S. (1996), A Theory of Persistent Income Inequality, *Journal of Economic Growth*, 1, 75-93.

Galor, O. and Moav O. (2004), From physical to human capital accumulation: Inequality in the process of development, *Review of Economic Studies*, 71, 1001–1026.

Galor O. and Tsiddon D. (1997), The distribution of human capital and economic growth, *Journal of Economic Growth*, 2(1), 93-124.

Galor O. and Zeira J. (1993), Income distribution and macroeconomics, *Review of Economic Studies*, 60, 35-52.

Glomm G and Kaganovich M. (2008), Social security, public education and the growth–inequality relationship, *European Economic Review*, 52, 1009–1034.

Gradstein M, Justman M. and Meier V. (2005), *The political economy of education. Implications for growth and inequality*, The MIT Press, Cambridge and London.

Grossman H.L. (1991), A general equilibrium model of insurrections, *American Economic Review*, 81, 912-21.

Grossman H.L. (1994), Production, appropriation, and land reform, *American Economic Review*, 84, 705-712.

Kaldor N. (1955-56), Alternative theories of distribution, *Review of Economic Studies*, 23, 94-100.

Kuznets S. (1955), Economic Growth and Income Inequality, *American Economic Review*, 45(1), 1–28.

Lindert P.H. (2000), 'Three centuries of inequality in Britain and America', in A.B. Atkinson and F. Bourguignon (eds.), *Handbook of Income Distribution*, Elsevier, Chapter 3, 167-216.

Loury G.C. (1981), Intergenerational Transfers and the Distribution of Earnings, *Econometrica*, 49(4), 843-867.

Maoz, Y.D. and Moav, O. (1999), Intergenerational mobility and the process of development, *Economic Journal*, 109, 677–697.

Persson T. & Tabellini G (1994), Is inequality harmful for growth, *American Economic Review*, 84(3), 600-621.

Piketty T. (2000), Theories of Persistent Inequality and Intergenerational Mobility, in: A. Atkinson and F. Bourguignon (eds), *Handbook of Income Distribution*, Chapter 8, 429-476.
Sala-I-Martin X. (1997), Transfers, Social Safety Nets and Economic Growth, *IMF Staff Papers*, 44(1), 81-102.
Solon G. (1999), 'Intergenerational mobility in the labour market', in: Ashenfelter O. and Card D. Eds., *Handbook of Labor Economics*, vol. 3A, Ch. 29, 1761-1800.

Appendix 1: Proof of proposition 4

Determination of $\underline{m}_H*$ ***and*** $\overline{m}_H*$**:** We place ourselves at a steady state. $\underline{m}_H*$ is the lowest value of m_H* such that children from unskilled families decide not to pursue higher education, and $\overline{m}_H*$ the highest m_H* such that children from skilled families decide to pursue higher education.

The threshold $\underset{\sim}{h_t} = \left(\dfrac{(1+\varepsilon)^{1+\varepsilon} w_L}{\delta\underline{\delta}\varepsilon^{\varepsilon} w_H} \right)^{1/\eta}$ that separates individuals who go to education from those who do not, is at the steady state: $\underset{\sim}{h}* = \left(\dfrac{(1+\varepsilon)^{1+\varepsilon} w_L*}{\delta\underline{\delta}\varepsilon^{\varepsilon} w_H*} \right)^{1/\eta}$. We obtain after replacing w_L*, w_H* and δ by their values:

$$\underset{\sim}{h}* = \left(\frac{1-\alpha}{\alpha} \right)^{1/\eta} \left(\left(\frac{\overline{\delta}\underline{\delta}\varepsilon^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}} \right)^{\eta+\alpha\beta} (1+\varepsilon)^{\eta} \tau^{\beta} \left(\frac{m_H*}{1-m_H*} \right)^{1-\eta-\beta} \right)^{\frac{1/\eta}{1-\eta-\alpha\beta}}$$

The condition for children from unskilled families to decide not to pursue higher education is $\underline{h}* < \underset{\sim}{h}*$, which gives after rearranging:

$$m_H* > \underline{m}_H* = \frac{1}{1 + \left(\underline{\delta}^{\frac{\alpha\beta}{1-\eta}} \left(\frac{1-\alpha}{\alpha} \right)^{1-\eta-\alpha\beta} \left(\frac{\overline{\delta}\varepsilon^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}} \right)^{\eta+\alpha\beta} (1+\varepsilon)^{\eta} \tau^{\beta} \right)^{\frac{1}{1-\eta-\beta}}}$$

Secondly, the condition for children from skilled families to decide pursuing higher education is $\overline{h}* > \underset{\sim}{h}*$, which gives after rearranging:

$$m_H* < \overline{m}_H* = \frac{1}{1 + \left[\left(\frac{1-\alpha}{\alpha} \right)^{1-\eta-\alpha\beta} \left(\frac{\overline{\delta}\underline{\delta}\tau^{(1-\eta)/\alpha}\varepsilon^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon-\eta}} \right)^{\alpha\beta} \right]^{\frac{1}{1+(1-\alpha)\eta\beta-\eta-\beta}}}$$

Finally, inequality $\underline{m}_H^* < \bar{m}_H^*$ can be easily verified: **??**

***Proof of feature* 1:** By inserting $\bar{h}^* = (\bar{\delta}\underline{\delta})^{\frac{1}{1-\eta}}(\tau y^*/m_H^*)^{\frac{\beta}{1-\eta}}\left(\frac{\varepsilon}{1+\varepsilon}\right)^{\frac{\varepsilon}{1-\eta}}$ (relation 16) and $(1-\hat{e}) = (1+\varepsilon)^{-1}$ into $H^* = (1-\hat{e})m_H M\bar{h}^*$, we obtain:

$$H^* = (1+\varepsilon)^{-\frac{1-\eta+\varepsilon}{1-\eta}}(m_H^*)^{1-\frac{\beta}{1-\eta}}(\bar{\delta}\underline{\delta})^{\frac{1}{1-\eta}}\tau^{\frac{\beta}{1-\eta}}(y^*)^{\frac{\beta}{1-\eta}}\varepsilon^{\frac{\varepsilon}{1-\eta}}M \tag{A1}$$

Inserting this relation and $L^* = m_L M$ into $y^* = H^{*\alpha}L^{*1-\alpha}/M$, we determine after rearranging: $y^* = \left(\bar{\delta}\underline{\delta}\tau^{\beta}\varepsilon^{\varepsilon}(1+\varepsilon)^{-(1-\eta+\varepsilon)}(m_H^*)^{-\beta}\right)^{\frac{\alpha}{1-\eta-\alpha\beta}}\left((m_H^*)^{\alpha}(m_L^*)^{1-\alpha}\right)^{\frac{1-\eta}{1-\eta-\alpha\beta}}$.

***Proof of Feature* 2**: $\bar{h}^*$ is calculated by inserting y^* as defined above into relation (16), and $\underline{h}^* = \underline{\delta}^{1/(1-\eta)}$ is given by relation (15).

***Proof of Feature* 3**: $\bar{H}^*$ is calculated by inserting y^* as determined in Feature 1 into (A1). The determination of $\bar{L}^*$ is straightforward.

***Proof of Feature* 4**: by inserting $\bar{L}^*$ and $\bar{H}^*$ into (4), $\varpi^* \equiv w^*\bar{h}^*$ and $s \equiv w_H(1-e_j)h_j/w_L$.

***Proof of Feature* 5**: the higher m_H, the higher ω_L^* and the lower $\omega_H^*\bar{h}^*$ because (see the technical note) $\frac{\partial\omega_H^*\bar{h}^*}{\partial m_H^*} = -\frac{(1-\tau)\alpha(1-\alpha)\left(\delta'\varepsilon^{\varepsilon}\right)^{\frac{\alpha}{1-\eta}}(1+\varepsilon)^{\frac{(1-\alpha)(1-\eta)+\alpha\varepsilon}{1-\eta}}}{(1+\alpha\varepsilon)(m_L^*)^{\alpha}(m_H^*)^{2-\alpha}} < 0$ and $\frac{\partial w_L^*}{\partial m_H^*} = \frac{\alpha(1+\alpha)\left(\delta'\varepsilon^{\varepsilon}\right)^{\frac{\alpha}{1-\eta}}m_L^{-(1+\alpha)}m_H^{\alpha-1}}{\alpha(1+\varepsilon)^{\frac{\alpha\varepsilon+(1+\alpha)(1-\eta)}{1-\eta}}+(1+\alpha)} > 0$.

Appendix 2: Proof of Proposition 5

The net income per capita at the steady state is:

$$z^* = (1-\tau)\left(\bar{\delta}\underline{\delta}\tau^{\beta}\varepsilon^{\varepsilon}\left(\frac{1}{1+\varepsilon}\right)^{1+\varepsilon-\eta}(m_H^*)^{-\beta}\right)^{\frac{\alpha}{1-\eta-\alpha\beta}}\left((m_H^*)^{\alpha}(m_L^*)^{1-\alpha}\right)^{\frac{1-\eta}{1-\eta-\alpha\beta}}$$

1) $z^* = C_\tau \times (1-\tau)\tau^{\frac{\alpha\beta}{1-\eta-\alpha\beta}}$, with $C_\tau = \left(\bar{\delta}\underline{\delta}\varepsilon^{\varepsilon}\left(\frac{1}{1+\varepsilon}\right)^{1+\varepsilon-\eta}(m_H^*)^{-\beta}\right)^{\frac{\alpha}{1-\eta-\alpha\beta}}\left((m_H^*)^{\alpha}(m_L^*)^{1-\alpha}\right)^{\frac{1-\eta}{1-\eta-\alpha\beta}}$

$$\frac{\partial z^*}{\partial \tau} = C_\tau \times \tau^{\frac{\alpha\beta}{1-\eta-\alpha\beta}} \left(\frac{\alpha\beta}{1-\eta-\alpha\beta} \frac{1-\tau}{\tau} - 1 \right) = 0 \Rightarrow \hat{\tau} = \frac{\alpha\beta}{1-\eta}$$

2) $$z^* = (1-\tau)\left(\bar{\delta}_H \delta_B \tau^\beta \varepsilon^\varepsilon \left(\frac{1}{1+\varepsilon} \right)^{1+\varepsilon-\eta} \right)^{\frac{\alpha}{1-\eta-\alpha\beta}} \left((m_H^*)^{\alpha(1-\eta-\beta)} \left(m_L^* \right)^{(1-\alpha)(1-\eta)} \right)^{\frac{1}{1-\eta-\alpha\beta}}$$

$$\frac{\partial \left((m_H^*)^{\alpha(1-\eta-\beta)} \left(1-m_H^* \right)^{(1-\alpha)(1-\eta)} \right)}{\partial m_H^*} = (m_H^*)^{\alpha(1-\eta-\beta)} \left(m_L^* \right)^{(1-\alpha)(1-\eta)} \left(\frac{\alpha(1-\eta-\beta)}{m_H^*} - \frac{(1-\alpha)(1-\eta)}{1-m_H^*} \right) = 0$$

$$\Rightarrow \hat{m}_H = \frac{\alpha(1-\eta-\beta)}{1-\eta-\alpha\beta}$$

Appendix 3: **The interval $\left[\underline{\hat{m}}_H , \hat{\hat{m}}_H \right]$ at the efficient steady state**

1) As $$\underline{m}_H^* = \frac{1}{1 + \left(\underline{\delta}^{\frac{\alpha\beta}{1-\eta}} \left(\frac{1-\alpha}{\alpha} \right)^{1-\eta-\alpha\beta} \left(\frac{\bar{\delta}\varepsilon^\varepsilon}{(1+\varepsilon)^{1+\varepsilon}} \right)^{\eta+\alpha\beta} (1+\varepsilon)^\eta \tau^\beta \right)^{\frac{1}{1-\eta-\beta}}}$$ and $\hat{\tau} = \dfrac{\alpha\beta}{1-\eta}$:

$$\underline{\hat{m}}_H = \frac{1}{1 + \left(\underline{\delta}^{\frac{\alpha\beta}{1-\eta}} \left(\frac{1-\alpha}{\alpha} \right)^{1-\eta-\alpha\beta} \left(\frac{\bar{\delta}\varepsilon^\varepsilon}{(1+\varepsilon)^{1+\varepsilon}} \right)^{\eta+\alpha\beta} (1+\varepsilon)^\eta \left(\frac{\alpha\beta}{1-\eta} \right)^\beta \right)^{\frac{1}{1-\eta-\beta}}}$$

2) As $$\bar{m}_H^* = \frac{1}{1 + \left[\left(\frac{1-\alpha}{\alpha} \right)^{1-\eta-\alpha\beta} \left(\frac{\bar{\delta}\underline{\delta}\tau^{(1-\eta)/\alpha}\varepsilon^\varepsilon}{(1+\varepsilon)^{1+\varepsilon-\eta}} \right)^{\alpha\beta} \right]^{\frac{1}{1+(1-\alpha)\eta\beta-\eta-\beta}}}$$ and $\hat{\tau} = \alpha\beta/(1-\eta)$:

$$\hat{\hat{m}}_H = \frac{1}{1 + \left[\left(\frac{1-\alpha}{\alpha} \right)^{1-\eta-\alpha\beta} \left(\frac{\bar{\delta}\underline{\delta}\varepsilon^\varepsilon}{(1+\varepsilon)^{1+\varepsilon-\eta}} \right)^{\alpha\beta} \left(\frac{\alpha\beta}{1-\eta} \right)^{\beta(1-\eta)} \right]^{\frac{1}{1+(1-\alpha)\eta\beta-\eta-\beta}}}$$

Appendix 4: Proof of proposition 7

By inserting $\dfrac{w_{Ht}}{w_{Lt}} = \dfrac{\alpha}{1-\alpha}\dfrac{L_t}{H_t}$ and $y_{t-1} = \dfrac{H_{t-1}{}^{\alpha} L_{t-1}{}^{1-\alpha}}{M}$ into $\underline{h}_t = \left(\dfrac{(1+\varepsilon)^{1+\varepsilon} m_H{}^{\beta}}{\bar{\delta}\underline{\delta}\tau^{\beta}\varepsilon^{\varepsilon}(y_{t-1})^{\beta}} \dfrac{w_{Lt}}{w_{Ht}} \right)^{1/\eta}$,

we obtain $\underline{h}_t = \left(\dfrac{(1+\varepsilon)^{1+\varepsilon}(m_H M)^{\beta}}{\bar{\delta}\underline{\delta}\tau^{\beta}\varepsilon^{\varepsilon}(m_L M)^{1+(1-\alpha)\beta}} \dfrac{1-\alpha}{\alpha} \dfrac{H_t}{H_{t-1}{}^{\alpha\beta}} \right)^{1/\eta}$ and thus: $\underline{h}_t = constant\ term \times \left(\dfrac{H_t}{H_{t-1}{}^{\alpha\beta}} \right)^{1/\eta}$

for a given m_H. Since H_t does not decrease over time, then $\underline{h}_t$ either increases or remains constant.

Let us firstly consider the dynasty, denoted *m*, that initially possesses the highest human capital among the non educated dynasties. Two cases are possible:

1. If the human capital of generation 0 $h_m(0)$ is lower than $\underline{h}^* = \underline{\delta}^{1/(1-\eta)}$, then $h_m(t)$ tends to $\underline{h}^*$ at the steady state, and since $\underline{h}_t > \underline{h}^*$, then $\underline{h}_t > \underline{h}^* \geq h_m(t)$. This shows that dynasty *m* remains non-educated as well as all the dynasties with a lower human capital. Thus, all the descendants of the individuals who initially do not go to education make the same choice.

2. If the human capital $h_m(0)$ is higher than $\underline{h}^*$, then $h_m(t)$ will decrease and tend to $\underline{h}^*$. In addition $h_m(0) < \underline{h}_1$ and $\underline{h}_t$ increases or remains constant over time. As a consequence, $h_m(t-1) < \underline{h}_t$, which shows that dynasty *m* remains non-educated and thereby unskilled from generations to generations, as well as all the dynasties with an initial human capital lower than $h_m(0)$.

This shows that all the dynasties that initially choose non- education will subsequently remain non-educated.

Let us secondly consider the dynasty that possesses the lowest human capital amongst the educated dynasties, denoted *l*. Since H_t does not decrease, $\underline{h}_1 < h_l(0) < \bar{h}^*(\tau)$. If we demonstrate that $\underline{h}_t < h_l(t-1)$ over all the subsequent periods, then dynasty *l* will remain educated forever and so will all the dynasties that were educated at period 1. Since $\underline{h}_1 < h_l(0)$, an increase in $h_j(t-1)/\underline{h}_t$ over time is a sufficient condition for $\underline{h}_t < h_l(t-1)$, $\forall t$. Because of (14), we can write: $\delta\underline{\delta}\left(\dfrac{\varepsilon}{1+\varepsilon}\right)^{\varepsilon} \dfrac{w_{Ht}}{w_{Lt}} \underline{h}_t{}^{\eta} = (1+\varepsilon)$. In addition, $h_l(t) = \delta\underline{\delta}\left(\dfrac{\varepsilon}{1+\varepsilon}\right)^{\varepsilon} \left(h_l(t-1)\right)^{\eta}$.

By putting together these two relations we obtain: $\dfrac{h_l(t-1)}{\underline{h}_t} = \left(\dfrac{w_{Lt}}{w_{Ht}} \times \dfrac{h_l(t)}{1+\varepsilon} \right)^{1/\eta}$. As w_L / w_H

increase over time (because H_t / L_t increases) as well as $h_l(t)$ (as long as it has not reached the steady state), then $h_j(t-1)/\underline{h}_t$ increases and $\underline{h}_t < h_l(t-1)$ is always true.

This shows that all the dynasties that initially choose not to go to education will subsequently remain educated.

Appendix 5: Proof of Proposition 8

The dynasties are arranged in ascending order of human capital. At time *t*, if $h_{f-1}(t) < \underline{h}_t \leq h_f(t)$ with threshold $\underline{h}_t = \left(\frac{(1+\varepsilon)^{1+\varepsilon}}{\delta\underline{\delta}\varepsilon^{\varepsilon}} \frac{w_{Lt}}{w_{Ht}} \right)^{1/\eta}$, then the individual from the *f*-th dynasty is the one with the lowest human capital among these who choose to educate (and thereby the individual from the (*f*-1)-th dynasty is the one with the highest human capital among those who choose not to educate). By inserting $\frac{w_{Lt}}{w_{Ht}} = \frac{1-\alpha}{\alpha} \frac{H_t}{L_t}$, $H_t = \sum_{i=f}^{M} h_i(t) = \delta\underline{\delta} \left(\frac{\varepsilon}{1+\varepsilon} \right)^{\varepsilon} \sum_{i=f}^{M} \left(h_i(-1) \right)^{\eta}$, $L_t = m_{Lt} M$ and $m_{Lt} = \frac{f-1}{M}$ into this expression, we obtain:

$$h_{f-1}(t) < \underline{h}_t = \left((1+\varepsilon) \frac{1-\alpha}{\alpha} \frac{\sum_{i=f}^{M} \left(h_i(t-1) \right)^{\eta}}{f-1} \right)^{1/\eta} \leq h_f(t)$$

As $\underline{h}_t$ decreases with *f*, this inequality that determines a unique *f*, and thereby one unique distribution of individuals (and dynasties) between the educated and the non-educated. This determination is independent from τ, which establishes Proposition 8.