REDISTRIBUTION AS A PUBLIC GOOD: AN EVOLUTIVE MODEL OF VOLUNTARY REDISTRIBUTION

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Abstract

Our model is built on Rege (2004), Holländer (1990) and the stocasthic learning models of Bush-Mosteller (1955), Cross (1973) and Borgers-Sarin(1997). The first two conceive the production and consumption of social approval to be the consequence of social interaction. I consider a society with three groups: two groups composed by potential donors and a third group composed by those who benefit from the transfers from the first two. These transfers constitute the public good. I establish four stable equilibriums: first, one in which everybody is a donor (1,1); second, one in which nobody is a donor, (0,0); and two asymmetric ones (0,1 and 1,0), in which one group makes transfers while the other behaves as a free-rider. I present two replicator equations which form a system of non linear differential equations. I study the conditions under which the trajectories point to the stable equilibrium (1,1) in which the entire population of potential donors actually makes transfers. The results show that the crowding out of the private sector by the government is stronger than in the Bergstrom, Blume, and Varian (1986) traditional model. I see warm-glow as a limit case of social interaction. It explains the attitude of the donors: to give is a relational good as defined by Uhlaner (1989). As a final conclusion, I see the theory of public goods in a totally different way and arrive to radically different policy implications. Instead of using it to show that the private sector must be completely crowded out in order to reach a Pareto optimum in the Samuelson sense, I argue that it shows that the government policy must be crowd out in order to allow the community to produce, via social interaction, the relational good and reach an even greater Pareto optimum.

KEYWORDS: PHILANTROPY, PUBLIC GOODS, VOLUNTARISM

But as of yet, there is no well-developed theory of how other-regardingness is channeled by civil society or by government[1].

David Beito

1. Introduction

We argue that income distribution could spontaneously appear as a product of social interaction[2]. So, it is natural to treat it within the processes of self-organization. The characteristic of these processes are evolutive, so it is necessary to use the theory of evolutionary games. Evolution demands to be placed in a dynamic context, in which the main question is related to the conditions of convergence towards a stable equilibrium, an attractor. In the evolutive dynamic, the players are defined as replicators of social behaviors or norms, who can chose an optimal strategy of adaptation for the people in which they are present. From these ideas we derive a differential equation called dynamic replicator, with which the possible long term trajectories are calculated. The theory of stochastic learning defines a process in which each player develops a habit or behavior norm depending on the stimulus or payment received. Using this theory we deduce the equation of the dynamic replicator in continuous time used in the evolutive models[3].

Our model is built on the Rege and Hollander models of private provision of public goods. The first one is evolutive and uses a dynamic replicator. The second is a classic optimization model, with focus in the study of welfare. Both have in common the assumption of social interaction with the purpose of producing and consuming social

[1] Beito, and others (2002), p. 3.

[2] As David Beito, Robert Sugden also ask for an elaboration of a philanthropic theory: "Whichever of the assumptions of the public good theory is dropped, the conventional argument that private philanthropy leads to the under-supply of charitable activities cannot be sustained. It may be that the same conclusion can be arrived at by using a more acceptable theory of philanthropy; but this remains to be shown." Sugden (1982), p. 350.

[3] For the dynamic replicator see Binmore (1994). For the stochastic learning theory see Bush and Mosteller (1955). For the learning model see Cross (1973) and Borgers and Sarin (1997).

approval. We use elements of both models to build ours. This one arrives to a set of different equilibriums for a society formed by three groups: two groups formed by rich people potentially givers to the public good, defined as the well being of the third, the poor group. We present a phase diagram of the replicator dynamics for different combinations of the parameters of the model. We are particularly interested in the comparison with the alternative of government provision of the public good.

Section 2 points out the importance of the concepts of self-organization and learning in the generation of social norms. Section 3 discusses the models of Rege and Hollander. Section 4 develop our model. In section 5 I discuss the comparative static results of our model with no government intervention. Section 6 presents the treatment of government intervention. Section 7 discusses the welfare implications of the different equilibriums. Finally, section 8 presents a summary and some concluding remarks.

2. Self-organization and learning[4]

Some processes, biological or social, among which we will place the Cross learning model, have a characteristic in common. When they are "away from equilibrium, they are characterized by the appearance of new properties which did not exist in its precedent states"[5]. This characteristic is called self-organization. It refers to changes in open systems that evolve progressively, from less to more complex states. The stationary state of the open system can be reached from different initial positions and in different ways, and is solely determined by the parameters of the system.

When the open system that we study is the human society, self-organization means two special properties of the dynamic interaction among people.[6] The first property is the generation of endogenous structural changes in the society. The second property is the production of new facts, not foreseen in the initial plans of the people. In section 4 we show the evolutive model of voluntary distribution. This model presents the possibility of an endogenous structural change in the way people behave, the elimination of the free rider behavior, and the appearance of a new fact not foreseen

[4] See Rubio de Urquía and others (2003). In which follows we use ideas from Rafael Rubio de Urquía, "Estructura fundamental de los procesos de autoorganización mediante modelos teórico-económicos", and Luis Morales de la Paz, "Autoorganización y orden espontáneo: ¿dos caras de la misma moneda?".
[5] Rodríguez García-Brazales and Vara Crespo (2003), p. 195.
[6] Rubio de Urquía and others (2003), p. 61.

initially in personal plans: the production of the relational good. By self-organization we understand a wider concept referred to structures of organization that appear spontaneously in the global dynamic of the society which converge to a stationary order or state. That is to say, what is spontaneous is not the order but what generates the process of self-organization: endogenous and structural changes in the plans of the people.

In the theory of evolutive games, the players have limited rationality, it could even be said that they do not arrive to think. Rationality in this case, is not observed in players individually but in how they behave during the process, making adjustments, copying others' behaviors, changing in such a way that the process finally finds its equilibrium. It seems that this final equilibrium is achieved spontaneously, that is why we talk of spontaneous order or invisible hand.

Coleman (1990) defines the social rule as a behavior rule imposed by means of social sanctions. Homans (1961) describes the sociological theory of social exchange. Societies value certain behavior when they generate norms of behavior. So, a norm is a statement made by a group that establishes how they ought to behave in a certain circumstance. The group values that its members behave in accordance to the rule, and in consequence, the approval of the group originates a satisfaction in each member. Although each person establishes its own values and it is not easy to say which are these values in a given moment, the fact of belonging to a group has the consequence of adopting the norm that the group conforms to, that everybody value and, in consequence, in the case of no compliance he would have to pay the cost of disapproval of his behavior from the other members of the group.

Private provision of public goods can be considered within our definition of social norm. How do we arrive to the norm? How does it appear within the social group? The answer is: by evolution.

3. The models of Rege and Hollander

3.1 Rege

She considers a big society where each person $i \in [0,1]$ can decide between contributing or not to public goods. Each voluntary contributor gives a fixed value, $g_i = 1$, while those who do not contribute chooses $g_i = 0$. The evolution of the proportion of contributors is identified by x. Each person decides to dedicate his income to the consumption of a private good, c_i, which acts as numeraire, or to contribute to the financing of the public good whose price is p. The budget restriction is:

$$I = c_i + pg_i \qquad (1)$$

The contribution to the public good produces a social approval q. All people have the same preferences, represented by the following quasi linear function, linear in c_i and in q_i:

$$U_i = c_i + w(\overline{g}) + q_i \qquad (2)$$

Where $w' > 0$ and $w'' < 0$. $\overline{g}$ is the average of the contribution to public good in the whole population, considered always fixed by each individual, as we assume a continuum of person in the interval $[0,1]$.

After a person decides whether to contribute or not, he observes the reaction of the other members of his social group, composed by m_i people. They all belong to groups of the same size. A contributor will experiment social approval if he sees that another contributor has observed it. On the contrary, a non contributor will perceive social disapproval is he observes that a contributor is watching him. But it is supposed that in front of a non contributor a person can behave indistinctively contributing or not, since he does not perceive anything.

Social approval q_i is produced and consumed by a person when contributing, so an individual has a production function ("household production function") of the Stigler –Becker (1977) type.

Rege presents the following production function for social approval:

$$q_i = \lambda \cdot (g_i - \overline{g}) \frac{1}{m} \sum_{j \in m_i} E_i(g_j)$$

Where $E_i(g_j)$ is the i expectation that j be a contributor and λ measures the utility for one person to be part of a society where everyone contributes.

Measured in terms of the numeraire, λ is equal to:

$$\lambda = w(1) - p \qquad \text{with } \lambda > 0$$

λ reflects the concept of Homans social interaction. The greater the value that the activity of a person has for society, the greater is the social approval that this person receives from those that share that value.

The factor $(g_i - \overline{g})$ shows that the standard behavior of the society makes the norm. The increase of social approval that a person achieves when contributing to public goods, as it increases the standard, reduces the social approval of the other members of the society. This negative externality will be called status externality.

People are related randomly, independently of the action that they take, so whether they contribute or not, they will expect to find a group of people in which a proportion x will be contributors. Here Rege makes a reasonable assumption in sociological terms establishing that people are related more frequently to those with whom they have similar behavior. This fact, called viscosity in biological studies was introduced in the game theory by Myerson and others (1991). Each player has a probability $k > 0$ of opposing an opponent that comes from their own group and a probability $(1-k)$ of opposing one taken randomly from the total population. This means that even when the group that uses a certain strategy is a small fraction of the population, the people that use this strategy has a positive probability of finding each other. The fraction of people in the population that observes the behavior of the person i is called z_i, they are their "observers", the components of this fraction will be different depending on the strategy adopted by i:

if $g_i = 1 \quad z_i = k + (1-k)x$

if $g_i = 0 \quad z_i = (1-k)x$

The proportion of contributors in the society, x, is equivalent to the average contribution $\overline{g}$. So, the social approval expected by each person depends on x. The expectancy for i that j be a contributor, $E_i(g_j)$, is equal to z_i. The expected social approval for i is finally equal to:

If $g_i = 1 \quad q_i(x) = \lambda(1-x)\left(k+(1-k)x\right)$ (3)

if $g_i = 0 \quad q_i(x) = -\lambda x(1-k)x$ (4)

The increase in social approval when contributing will be equal to:

$$\Delta q_i(x) = \lambda(1-x)\left(k+(1-k)x\right) + \lambda x(1-k)x$$

Then,

$$\Delta q_i(x) = \lambda\left(k+(1-2k)x\right) \quad (5)$$

The bigger the proportion of people that accept a norm of contributing voluntarily to public good, the greater will be the social approval of a person that decides to contribute, since we suppose that $\lambda > 0$. Notice that the average contribution to the public good equals the proportion of contributors in the population, $\bar{g} = x$, and that its value is not modified when a person decides to be a contributor. So, this decision is solely based on the utility that gives him social approval.

Replacing in the utility function (2), c_i by its corresponding expression in the budget constraint (1), it results:

$$U_i = I - pg_i + w(x) + q_i$$

If the individual i is a contributor (C), his utility will be equal to:

$$U_i^C = I - pg_i + w(x) + q_i(x)$$

And, replacing by (3),

$$U_i^C = I - p + w(x) + \lambda(1-x)\left(k+(1-k)x\right)$$

If the individual i is a non contributor (NC), his utility will be equal to:

$$U_i^{NC} = I + w(x) + q_i(x)$$

And, replacing by (4),

$$U_i^{NC} = I + w(x) - \lambda x(1-k)x$$

Using (5), the increase in utility between contributing and not contributing will be equal to:

$$\Delta U_i(x) = U_i^C - U_i^{NC} = \lambda\left(k+(1-2k)x\right) - p \quad (6)$$

If the benefit of contributing is higher than the cost, people will want to contribute, increasing in consequence the number of contributors. When both values are equal $\Delta U_i(x) = 0$, the proportion of contributors, x', will remain constant and equal to:

$$x' = \left(\frac{p}{\lambda} - k\right)\frac{1}{1-2k}$$

For given values of p and λ, this proportion depends on the value of the viscosity.

$\Delta U_i(x) > 0$ and $\frac{p}{\lambda} \le 1-k$ means that $x' \le x \le 1$. So, in this case, $g_i = 1$ for all i will be a Nash equilibrium with ESS. On the other hand, $\Delta U_i(x) < 0$ and $\frac{p}{\lambda} \ge k$ means that $0 \le x \le x'$. The condition now for $g_i = 0$ for all i is also a Nash equilibrium with ESS. Finally the mixed strategy (α^*, α^*) where $\alpha^* = (x, 1-x)$ tell us to play $g_i = 1$ with the probability x (proportion of donors) and $g_i = 0$ with the probability $1-x$, is the third Nash equilibrium. In this last case, $\Delta U_i(x) = 0$ and $k < \frac{p}{\lambda} < 1-k$ means that $0 \le x = x' \le 1$. This equilibrium in mixed strategies is not ESS since any point close to it diverges following a trajectory towards one of the first two equilibriums in pure strategies.

The Rege model will be a coordination model if $k < 1/2$, so as to comply with what we mentioned in the above paragraph.

Rege uses the dynamic replicator to characterize the process of apprentice that conducts to the choice of the strategy that maximizes the utility or payment to each player. Following the previous notation, the equation of the replicator is:

$$\boxed{\frac{dx}{dt} = x\left(U_i^C(x) - \bar{U}(x)\right)} \qquad (7)$$

Where $\bar{U}(x) = xU_i^C(x) + (1-x)U_i^{NC}(x)$. Substituting in (7) and using the expression (6) we have the following expression of the replicator:

$$\boxed{\frac{dx}{dt} = x(1-x)\Delta U_i(x)} \qquad (8)$$

In figure 1 we show the trajectory of the dynamic replicator, under the condition that $k \le \frac{p}{\lambda} \le 1-k$. There we observe that the game has an asymptotical stationary state, $x = 1$, where all people contribute, towards which points the process, if the initial proportion of contributors is over the $\max\{0,x'\}$. The other asymptotical state, $x=0$, where nobody contributes, is towards which points the process when the initial proportion of contributors is under the $\min\{1,x'\}$. The unstable stationary state is $x=x'$.

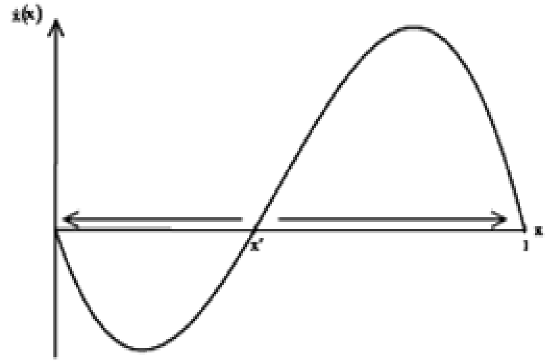

Figure 1: Rege model. Evolution of the social rule of voluntary contribution to a public good[7].

The existence of two asymptotical states is related to the social approval received by the person that voluntarily complies with the norm and is positively correlated to the average of persons that conforms to the norm. In both asymptotical states, none of them perceives social approval. In the state corresponding to $x=0$ since nobody can disapprove the behavior of the non contributor. In the state corresponding to $x=1$ as everyone contributes, the contribution of one person in particular is equal to the average, and the value of $q_i=0$. However, the utility of any particular person is bigger in the stationary state where the social norm of voluntarily contributing is complied by everyone, since we have established, in terms of private consumption, that there is a positive difference of individual utility equal to $\lambda = w(1)-p$. In other words, the lower utility due to a reduction in private consumption in order to finance the public good is more than compensated with the increase of utility resulting from the joint consumption of the public good. So the change from the non contributing state towards the state where everybody contributes is an improvement in the Pareto sense. We cannot demonstrate that this last state, although everybody contributes, is optimal in the Pareto

[7] Rege (1990), p. 71.

sense, since it has appeared as a result of an evolutive process and not as a result of an optimization program.

3.2 Hollander

Hollander applies the same Rege's scheme. He studies the voluntary cooperation in the financing of a public good, supposing that the voluntary contribution does not have significant effect on his offer. It is the classical situation of the Prisoners Dilemma, with a rational incentive to act as free-rider. The solution of the dilemma comes from social approval following Homans theory. But the difference with Rege is that he does not adopt an evolutive dynamic. He remains in the traditional frame of the optimization program.

The family production function of Hollander is:

$$a = w(b - \sigma c)$$

Where a is the value of social approval, w is the subjective value of a contribution unit, b is the contribution to public good and $0 \le \sigma \le 1$ is the coefficient that shows the importance or force of the negative externality or status that arises from the average contribution of c. When $\sigma = 1$, this status effect is maximum, so the contribution has value only if it is above the average. When $\sigma = 0$, the negative externality has no importance. To compare with the familiar production function of Rege, we must take into account the following equivalences: $\lambda = \upsilon = w$, $b = g_i$, $c = \overline{g}$, $\sigma = 1$.

The first important change is that w in Hollander is no longer an exogenous but and endogenous parameter that arises from equilibrium, and its value is calculated from the first order condition of the utility maximization problem of one person subject to budget constraint (where $\pi = income$) and from the marginal substitution rate between the private good ($\pi - b$) and the public good in the utility function. This rate is called social approval rate. It is the hypothetical benefit, measured in terms of private good, that a person could obtain if not only the marginally observed person increases the contribution but also all the persons that form the society except himself:

$$\upsilon = \frac{u_c'(c) - \sigma w u_a'\left[w(b - \sigma c)\right]}{u_p'(\pi - b)} \qquad (9)$$

where $u_c'(c)$ is the marginal utility of the public good, $u_a'\left[w(b-\sigma c)\right]$ is the marginal utility of social approval and $u_p'(\pi-b)$ is the marginal utility of the private good.

So, in the equilibrium, if $b=c$ $\upsilon=w$ y $c>0$, we have that:

$$w=\frac{u_c'(c)}{u_p'(\pi-c)}-\sigma$$

A second change is that in Hollander the importance of the status externality, although is always an exogenous variable, is not constant and equal to one as in Rege, it varies between zero and one, then we can carry out an study of comparative static with different values of σ.

Hollander also demonstrates the existence of social equilibriums in pure strategies, with $c^*>0$, where the equilibrium values of w and c are established. The standard norm of behavior or media contribution to public good, c^*, is associated to a "normal" quantity of social approval $a^*=(1-\sigma)w^*c^*$, since the equilibrium condition $b=c$, as in Rege. But, differing from Rege, where it is always zero, in the Hollander equilibrium, we see that "normal" social approval is positive if $\sigma<1$. The other equilibrium is with $c^*=0$, obtained when the approval rate is too low, and it is not enough to sustain any positive level of cooperation.

Hollander stresses, when studying the problems related to welfare, the interdependence between the individual functions of production of social approval, since the average contribution affects the generation of approval implying the characterization of three externalities. First, the already mentioned status effect, for $\sigma>0$, whose sign is negative and is the only externality mentioned by Rege. The second externality in consumption is positive, and it is the classic externality implied in the definition of public good. A third one, is formed by the two mentioned above, and affects the rate of individual approval, υ. The first part of this rate equation is the marginal utility of public good, while the second is the status externality, the addition of these two can be positive or negative. In summary, Hollander demands that these externalities do not operate to ensure a Pareto efficient allocation. That is the reason why he says that people should adopt a Kantian rule obliging everybody to make the contribution that he wishes all others to make.

4. The evolutive model of voluntary redistribution

Our aim is to answer the questions that David Beito and Robert Sugden express in the epigraph of this paper and in the note 2 respectively. We consider that our theory represented by the evolutive model of voluntary redistribution is a more complete theory of philanthropy than the classic public good theory.

4.1 Social groups and public good

We consider a population composed by three groups of individuals. Two groups correspond to people who are over the poverty line, called rich people type 1, composed by I persons, and type 2, composed by J persons, indexed by i and j respectively. They differ in the utility function that characterizes them, having different consumption preferences from those represented by the individuals of the third group, which is under the poverty line, and whose consumption constitutes the public good. In fact, an infinite number of agents would be necessary (a continuous as in the Rege model), but this assumption can be relaxed if we consider, as Hollander does, that each rich group is composed by a number of individuals large enough so that the effect of an individual contribution over the average quantity of the public good produced can be dismissed.

4.2 Utility function and budget constraint

The utility function that represents the preferences of any rich person has three arguments: 1) The consumption of a private good c, which is used as numeraire. 2) The consumption of a public good, assuming constant returns of scale. Utility is derived from the average quantity produced of the public good by the two groups, $\overline{g}$ [8]. We use the average quantity instead of the total quantity because in this way we can compare the individual behavior with the total population. We also assume that the contribution to the public good, g, is only posible in a quantity that is normalized to one. The set of

[8] For the whole population the public good is composed by the consumption of the entire group of poor people. Any rich, regardless of his particular group, observes the same $\overline{g}$.

strategies for any person is reduced, then to $\{0,1\}$[9]. 3) The consumption of social approval, q.

We use an additive utility function for each group of contributors:

$$U_i(c_i, q_i, \overline{g}) = u_i(c_i) + f_i(q_i) + y_i(\overline{g}) \tag{10}$$

$$U_j(c_j, q_j, \overline{g}) = u_j(c_j) + f_j(q_j) + y_j(\overline{g}) \tag{11}$$

We also assume, that the function is cuasilineal, that is to say lineal in c_i and q_i, and that the function y is potential, with $0 < a < 1$, and assuming, as Rege does, a strictly concave function for the public good´s utility:

$$y = b\overline{g}^a \tag{12}$$

The final expressions are:

$$U_i(c_i, q_i, \overline{g}) = c_i + q_i + b_1\overline{g}^{a_1} \tag{13}$$

$$U_j(c_j, q_j, \overline{g}) = c_j + q_j + b_2\overline{g}^{a_2} \tag{14}$$

Each person has an initial endowment of Y units of the private good, his monetary income, whose price is equal to 1, so we suppose that both groups of rich face the same price of the private good. The budget constraints of each individual are:

$$Y_i = c_i + p_1 g_i$$

$$Y_j = c_j + p_2 g_j \tag{15}$$

Where p is the price of the public good (strictly is the price of the contribution of each individual to the production of the public good) which can be the same or not for each group of rich. We suppose that there are no transaction costs, so one unity of numeraire deducted to the private consumption is transformed in a unity of the public good transferred to the group of poor. The price of the public good different from one reflects the possibility that a unit of numeraire could acquire variable quantities of the public good. This would show the presence of transaction costs, different returns or economies to scale, or government intervention resulting in different prices for goods

[9] In this dynamic it is not possible to enlarge the strategy space to a continuo. The strategy choices that emerge from the dynamic replicator do not allow the introduction of new strategies in the population. One kind of strategy could be superior to another but, if it is not present in the initial population, it can not be utilized in the future. Hofbauer and others (2005) demostrate that, with differrent dynamics, as the "Brown-von Neumann-Nash", with properties that allow the creation of new strategies, this problem can be resolve and make possible that the frecuency of new strategies with a pay-off superior to the average, used or not in the past, could increase in the population.

traded voluntarily. To introduce any of these conditions it would be necessary to use a production function for the public good, $\overline{g} = f\sum g_i$.

4.3 Control Group

We suppose that each person interacts with a number m of people called control group, who observe his behavior, of the same size for every people. If the individual does not contribute, he receives "social disapproval", and if he does contribute he receives "social approval", in both cases by the contributors within his control group. The members of his control group who do not contribute, do not express neither approval nor disapproval. The individual cannot identify which person within his control group is a contributor or not but he can built an expectation $E(g_h)$ [10] about the h individual within his group being a contributor or not.

4.4 The status parameter, σ

When an individual is observed by another, the quantity of approval or disapproval he receives is in relation to the average contribution of the whole population, which matches the average quantity of the public good. We include in the production function of social approval the Hollander "status" parameter, σ, which reflects the force of the negative externality that has the average contribution of the society in the production of social approval. When σ takes a value close to one, the individual receives approval only from the difference between his contribution (= 1) and the average of the society. When σ is close to 0, all his contribution is valued to produce social approval. Notice here the difference with the Rege model, which considers that $\sigma = 1$. In the Rege equilibriums (in the equilibrium where everybody contributes as well in that where nobody does) nobody receives social approval. In particular, when everybody contributes there is no difference between the contribution of an individual and the average, which is why the social approval is zero, the total utility is only derived by the

[10] Taking into account that $g_h = 1$ when the individual h within the control group is a contributor, and $g_h = 0$ when it is not, the $E(g_h)$ will indicate the expected probability that h is a contributor.

utility that comes from the consumption of private and public goods. This is not the case with $\sigma < 1$. In the equilibrium where everybody contributes, the utility has a component that results from social approval. This is of great importance for the characterization of the equilibriums in terms of Pareto optimality and their comparison under the Samuelson criteria.

4.5 The subjective rate, λ, and the quantity, q, of social approval

The difference between the contribution of the individual and the average contribution of the society, weighed by σ, is evaluated in the social approval function with the parameter λ, the subjective rate of social approval different for each group of rich, which, following Rege, reflects the difference in the level of individual utility, in terms of private good, between the situation in which everybody contributes and the situation in which nobody does. In other words, λ reflects the net utility that the individual will reach if the problem of the free-rider does not exist and everybody contributes[11].

The quantity of social approval perceived by an individual is measured, then, for each group as follows[12]:

$$q_i = \lambda_1 (\mathrm{g_i} - \sigma \overline{\mathrm{g}}) \frac{1}{m} \sum_{h=1}^{m} E_i(g_h)$$

$$q_j = \lambda_2 (\mathrm{g_j} - \sigma \overline{\mathrm{g}}) \frac{1}{m} \sum_{h=1}^{m} E_j(g_h) \qquad (16)$$

We will see the working of the model with an exogenous value of λ. This is calculated for each one of the rich groups. It is defined as the excess utility, defined in

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[11] From Rege model, $\lambda = w(1) - p$. In marginal terms that is equal to the Hollander approval rate, $\upsilon = \dfrac{u_c'(c) - \sigma w u_a' \left[w(b - \sigma c) \right]}{u_p'(\pi - b)}$.

[12] Social approval is an expected value where each expected probability related to the type of contributor h one faces is wheigted by the approval factor $q_i^h = \lambda_1 (\mathrm{g_i} - \sigma \overline{\mathrm{g}})$, the same for $\forall\ h$. Alternatively it could be write as $q_i = \dfrac{1}{m} \sum_{h=1}^{m} E_i(g_h)\ q_i^h$.

terms of the numeraire, that would be received if $g=1$, where the whole group contributes, over the cost of the public good represented by his price.

λ_1=y$_1(1)-p_1$

λ_2=y$_2(1)-p_2$

Using function(12), with $\overline{g}=1$, we arrive to the expressions:

$$\lambda_1 = b_1 - p_1 \qquad (17)$$

$$\lambda_2 = b_2 - p_2$$

4.6 Proportion of contributors and control group

We define x_1 and x_2, as the proportion of rich "contributors" of each type:

$$x_1 = \frac{1}{I}\sum_{i=1}^{I} g_i$$

$$x_2 = \frac{1}{J}\sum_{j=1}^{J} g_j$$

First, we suppose that each individual interacts only with people of his same type, that is to say that his control group comes from the rich type to which he belongs. Additionally, we include the viscosity parameter, k, by which the interaction is bigger between individuals adopting similar strategies: there is a greater possibility of being controlled by someone with the same level of contribution. We give a k probability to the fact that the observer comes from his own behavioral group (contributor or not contributor) and a probability $1-k$ if the observers come randomly from the total population of rich people of his type.

We define z_i and z_j, as the proportion of sanctioners that each person has if he makes contributions or not. Since the proportion of sanctioners depends on whether the individual is a contributor or not, for each set of rich people, the function z is defined in two steps. When g is one, there is a k probability that the observer is a contributor (in consequence “sanctioner”) and a $1-k$ probability that the contributor belongs to the same group of rich independently if it is a contributor or not, and in consequence, only an x proportion of them will be sanctioners. When the individual is non contributor, $g=0$, there is a k probability that the observer is non contributor, and as a result not a sanctioner and a probability $1-k$ that the observer belongs to the same group of rich but independently if it is contributor or not. In this way the proportion of sanctioners is defined for each rich group in relation to their contribution as:

Rich type 1

$$z_i = k + (1-k)x_1 \quad \text{if } g_i = 1 \text{ (contributor)},$$
$$z_i = (1-k)x_1 \quad \text{if } g_i = 0 \text{ (non contributor)}.$$

Rich type 2

$$z_j = k + (1-k)x_2 \quad \text{if } g_j = 1 \text{ (contributor)},$$
$$z_j = (1-k)x_2 \quad \text{if } g_j = 0 \text{ (non contributor)}.$$

Since g can only take 0 or 1 values, the functions in the two steps previously defined can be written equivalently as:

$$z_i(g_i) = k + (1 - k)x_1 - k(1-g_i), \tag{18}$$
$$z_j(g_j) = k + (1 - k)x_2 - k(1-g_j).$$

We shall call these z functions, "group sanctioners".

Alternatively, the control group can come from the total population instead of the rich group to which the observed person belongs. In this case the proportions of sanctioners are defined as:

$$z_i^G(g_i) = k + (1 - k)x - k(1-g_i), \tag{19}$$
$$z_j^G(g_j) = k + (1 - k)x - k(1-g_j),$$

where $x = x_1 n_1 + x_2(1-n_1)$, and $n_1 = \dfrac{I}{I+J}$.

We call these, z^G functions as "global sanctioners".

In order to calculate social approval each individual has an expectation over the control groups that he will face. People observe the average contribution, $\overline{g}$, from which, as the contribution is fixed, the proportion of contributors or sanctioners can be deducted. This formation of accurate expectations, as Rege says, determines that the z_i and z_j values match the value of sanctioner expected by any person. So, $E_i(g_h) = z_i$, and $E_j(g_h) = z_j$, for each h belonging to the control group of the individual i and of the individual j, respectively. We have to stress that there is not a process of revision of expectations related to the past, as time elapses. In this model of continuous time, each person has an accurate expectation in each moment t given the observation of the average contribution at that moment.

4.7 The utility function with group sanctioners

First, we calculate the social approval with groups sanctioners. We replace (17) and (18) in the production function of social approval (16):

$$q_i = (b_1 - p_1)(\mathrm{g_i} - \sigma \overline{\mathrm{g}})\left(k + (1 - k)x_1 - k(1 - g_i)\right) \quad (20)$$

$$q_j = (b_2 - p_2)(\mathrm{g_j} - \sigma \overline{\mathrm{g}})\left(k + (1 - k)x_2 - k(1 - g_j)\right) \quad (21)$$

Then for each group we replace (15), (20) and (21), in its utility function , (13) and (14), respectively:

$$U_i(c_i, \overline{g}) = Y_i - p_1 g_i + (b_1 - p_1)(g_i - \sigma \overline{g})\left(k + (1 - k)x_1 - k(1 - g_i)\right) + b_1 \overline{g}^{a_1} \quad (22)$$

$$U_j(c_j, \overline{g}) = Y_j - p_2 g_j + (b_2 - p_2)(g_j - \sigma \overline{g})\left(k + (1 - k)x_2 - k(1 - g_j)\right) + b_2 \overline{g}^{a_2} \quad (23)$$

To determine the public good contributing utility difference, given a certain level of contribution from the rest of the society, we define U_i^1 y U_j^1 as the level of expected utility obtained by each rich type when his contribution is positive $(= 1)$ and U_i^0 y U_j^0 as the level of expected utility obtained when they do not make any contribution. Given the assumption that the population is big, each individual takes the value of x_1 and x_2 as fixed.

Case $g_i, g_j = 1$

$$U_i^1(c_i, \overline{g}) = Y_i - p_1 + (b_1 - p_1)(1 - \sigma \overline{g})\left(k + (1 - k)x_1\right) + b_1 \overline{g}^{a_1} \quad (24)$$

$$U_j^1(c_j, \overline{g}) = Y_j - p_2 + (b_2 - p_2)(1 - \sigma \overline{g})\left(k + (1 - k)x_2\right) + b_2 \overline{g}^{a_2} \quad (25)$$

Case $g_i, g_j = 0$

$$U_i^0(c_i, \overline{g}) = Y_i - \sigma \overline{g}(b_1 - p_1)(1 - k)x_1 + b_1 \overline{g}^{a_1} \quad (26)$$

$$U_j^0(c_j, \overline{g}) = Y_j - \sigma \overline{g}(b_2 - p_2)\left((1 - k)x_2\right) + b_2 \overline{g}^{a_2} \quad (27)$$

Now, we calculate the difference in utility between contributing or not, $\Delta U_i = U_i^1 - U_i^0$ y $\Delta U_j = U_j^1 - U_j^0$:

$$\begin{aligned}\Delta U_i(\bar{g}) &= -p_1 + (b_1 - p_1)\big((1-\sigma\bar{g})\big(k+(1-k)x_1\big)+\sigma\bar{g}(1-k)x_1\big) \\ &= (b_1 - p_1)\big(x_1 + k(1-x_1-\sigma\bar{g})\big) - p_1\end{aligned} \tag{28}$$

$$\begin{aligned}\Delta U_j(\bar{g}) &= -p_2 + (b_2 - p_2)\big((1-\sigma\bar{g})\big(k+(1-k)x_2\big)+\sigma\bar{g}(1-k)x_2\big) \\ &= (b_2 - p_2)\big(x_2 + k(1-x_2-\sigma\bar{g})\big) - p_2\end{aligned}$$

4.8 The function of utility with global sanctioners

In second place, we calculate the social approval with global sanctioners, following the same steps made in the first case, but now with the functions z^G obtained in (19), and taking into account that $x = x_1 n_1 + x_2(1-n_1)$:

$$\begin{aligned}q_i &= \lambda_1(g_i - \sigma\bar{g})\big(k+(1-k)x - k(1-g_i)\big) \\ &= (b_1 - p_1)(g_i - \bar{g}\sigma)\big(g_i k + (1-k)(n_1(x_1-x_2)+x_2)\big)\end{aligned} \tag{29}$$

The same for the group j :

$$q_j = (b_2 - p_2)(g_j - \bar{g}\sigma)\big(g_j k + (1-k)(n_1(x_1-x_2)+x_2)\big) \tag{30}$$

For each group we replace (29) y (30) in their utility function (13) and (14), using (15), respectively.

$$U_i^G(c_i,\bar{g}) = Y_i - p_1 g_i + (b_1 - p_1)(g_i - \bar{g}\sigma)\big(g_i k + (1-k)(n_1(x_1-x_2)+x_2)\big) + b_1\bar{g}^{a_1} \tag{31}$$

$$U_j^G(c_j,\bar{g}) = Y_j - p_2 g_j + (b_2 - p_2)(g_j - \bar{g}\sigma)\big(g_j k + (1-k)(n_1(x_1-x_2)+x_2)\big) + b_2\bar{g}^{a_2} \tag{32}$$

Then, we define U_i^{1G} y U_j^{1G} as the level of expected utility that each rich type obtains when his contribution is positive (=1) and U_i^{0G} y U_j^{0G} as the level of utility obtained when they do not make any contribution.

Case $g_i, g_j = 1$

$$U_i^{1G}(c_i,\bar{g}) = Y_i - p_1 + (b_1 - p_1)(1-\bar{g}\sigma)\big(k + (1-k)(n_1(x_1-x_2)+x_2)\big) + b_1\bar{g}^{a_1} \tag{33}$$

$$U_j^{1G}(c_j,\overline{g}) = Y_j - p_2 + (b_2 - p_2)(1-\overline{g}\sigma)\left(k+(1-k)(n_1(x_1-x_2)+x_2)\right) + b_2\overline{g}^{a_2} \quad (34)$$

Case $g_i, g_j = 0$

$$U_i^{0G}(c_i,\overline{g}) = Y_i - \overline{g}\sigma(b_1 - p_1)\left((1-k)(n_1(x_1-x_2)+x_2)\right) + b_1\overline{g}^{a_1} \quad (35)$$

$$U_j^{0G}(c_j,\overline{g}) = Y_j - \overline{g}\sigma(b_2 - p_2)\left((1-k)(n_1(x_1-x_2)+x_2)\right) + b_2\overline{g}^{a_2} \quad (36)$$

Now, we calculate the differences of utility between contributing or not contributing $\Delta U_i^G = U_i^{1G} - U_i^{0G}$ y $\Delta U_j^G = U_j^{1G} - U_j^{0G}$:

$$\begin{aligned} \Delta U_i^G(\overline{g}) &= (b_1 - p_1)\left(k+(1-k)(n_1(x_1-x_2)+x_2) - k\overline{g}\sigma\right) - p_1 \\ &= (b_1 - p_1)\left(n_1(x_1-x_2)+x_2 + k\left(1-n_1(x_1-x_2)-x_2-\overline{g}\sigma\right)\right) - p_1 \end{aligned} \quad (37)$$

$$\begin{aligned} \Delta U_j^G(\overline{g}) &= (b_2 - p_2)\left(k+(1-k)(n_1(x_1-x_2)+x_2) - k\overline{g}\sigma\right) - p_2 \\ &= (b_2 - p_2)\left(n_1(x_1-x_2)+x_2 + k\left(1-n_1(x_1-x_2)-x_2-\overline{g}\sigma\right)\right) - p_2 \end{aligned}$$

Comparing the expressions (28) and (37) shows that the assumption of group sanctioners is equivalent to that of global sanctioners, when in this last case there is only one group. Notice that in the case of global sanctioner, in the variation of utility appears as an argument the difference in the average contribution proportion of the two types, weighed by the relative size of the group, which increases the level of interdependence between both groups.

4.9 Replicator´dynamics

We have to define the dynamic for each population separately.

The key assumption is that the rate of contributors is reproduced proportionally to the difference between the average payment of the individual and that of his rich group, and not in respect to the population in general[13]. As in Rege, the players behave in function of a learning process, a strategy revision represented by equation (7).

[13] When we carry out the strategy revision process and the learning dynamic found in the replicator equation, we asume that the agents use as reference the average utlity within their same group of rich. The comparison with the utility including individuals of other groups would be meaningless, especially if the functions were different and measured in different scales.

When the values x_1 and x_2 are such that the payment of contributing or not are equal for every individual independently if they are contributors or not, no individual has incentives to change his strategy and in consequence the evolutionary equilibrium is conformed. When all individuals are not contributors, the probability of being sanctioned for not contributing is zero, and there are no incentives to contribute and nobody wants to change his strategy. Notice that as the percentage of the public good is not affected by the contribution of one individual, this one does not value marginally his own contribution, except for the effect from social approval. When the individuals are contributors, the probability of being sanctioned if they do not contribute is $1-k$ [14], so, the utility from social approval will diminish and for every player to change the strategy would be suboptimum if this loss would be greater than the profit from the private consumption (p) obtained when he deviates, and this strategy would not spread into the population.

The loss of social approval utility due to deviation from the equilibrium where everybody contributes, for the group of rich type 1, is calculated with the difference between the value of the social approval q_i, when the contribution is zero, compared to the value when the contribution is one. First we observe the situation with group controller. We start with the position where $\overline{g}=1$ and $x_1=1$. With the equation (20) we calculate the amount of social approval with $g_i=0$, and with $g_i=1$, and we make the difference, $\Delta deviation$:

With $g_i=0$ → $q_i=-\sigma(b_1-p_1)(1-k)$

With $g_i=1$ → $q_i=(b_1-p_1)(1-\sigma)$

$\Delta deviation_i=-\sigma(b_1-p_1)(1-k)-(b_1-p_1)(1-\sigma)$

$\Delta deviation_i=(b_1-p_1)(-1+\sigma k)$

[14] The k parameter measures the proportion of observers of individual i playing his same strategy. Rege does not say how she solves the continuity of function z when only one individual deviates and $0<k<1$, since if $k=0$ everybody control him, while if $k=1$ nobody controls him. A solution is that at least two individuals deviate. Another one would be that when i deviates in a group where everybody contribute he would be considered as the only member of his group of non contributors, as if he were facing his own concience, and will be observed by a $1-k$ proportion of individuals, all sanctioners. A third possibility is to fix a k probability of not being controled.

The pay-off due to bigger consumption of the private good resulting from this deviation is the price of the public good p_1, given the assumption that the utility is cuasilineal, so, an equilibrium where everybody contribute is given by the condition $-\Delta deviation_i - p_1 > 0$. Or, which is the same:

$$\frac{p_1}{b_1 - p_1} < 1 - \sigma k \tag{38}$$

Which is the equivalent condition to the one established by Rege (see section 3) for an homogenous population: $\frac{p}{\lambda} \leq 1 - k$. Note here that Rege considers $\sigma = 1$ and $\lambda = b_1 - p_1$.

A similar reasoning allows us to elaborate the condition of equilibrium where nobody contributes. The utility pay-off for social approval due to a deviation from the equilibrium where nobody contributes, for the rich type 1, is calculated making the difference between the value of the social approval, q_i, when the contribution is one, related to the value when the contribution is zero. We use as a comparison situation the position where $\overline{g} = 0$ and $x_1 = 0$. With equation (20) we calculate the quantity of social approval with $g_i = 1$, and with $g_i = 0$, and we make the difference, $\Delta deviation$:

With $g_i = 1 \rightarrow q_i = (b_1 - p_1)k$

With $g_i = 0 \rightarrow q_i = 0$

$\Delta deviation_i = (b_1 - p_1)k$

The loss of utility due to the reduced consumption of the private good, resulting from this deviation is the price of the public good p_1, so, an equilibrium where nobody contributes is given by the condition $\Delta deviation_i - p_1 < 0$, which is the same as:

$$\frac{p_1}{b_1 - p_1} > k \tag{39}$$

Which is the condition equivalent to the one established by Rege for an homogeneous populations: $\frac{p}{\lambda} \geq k$.

Similarly, with a symmetric result, we calculate the condition of equilibrium for the rich type 2.

In our model we have two other stable equilibriums, the (0,1) and (1,0). The division in two groups does not prevent that the game results in stable equilibriums where each group coordinates itself choosing different strategies. The condition of stability of each equilibrium is deduced following the same previous reasoning, except that it must be taken into account that $\bar{g}$ must equal the participation of the rich type i which in each equilibrium decides to coordinate to contribute.

The social approval utility pay-off due to deviation from the equilibrium where the rich type 1 contributes, the equilibrium (1,0), for this type, is calculated making the difference between the value of the social approval, q_i, when the contribution is zero, with respect to the value when the contribution is one, starting with the position where $\bar{g} = n_1$ and $x_1 = 1$. With the equation (20) we calculate the quantity of social approval with $g_i = 0$, and with $g_i = 1$, and we make the difference, $\Delta deviation$:

With $g_i = 0 \rightarrow q_i = -\sigma n_1(b_1 - p_1)(1-k)$

With $g_i = 1 \rightarrow q_i = (b_1 - p_1)(1-\sigma n_1)$

$\Delta deviation_i = -\sigma n_1(b_1 - p_1)\left(1-k\right) - (b_1 - p_1)(1-\sigma n_1)$

$\Delta deviation_i = (b_1 - p_1)(-1+\sigma n_1 k)$

The utility pay-off due to the greater consumption of private good resulting from this deviation is the price of public good p_1, so, an equilibrium where all the members of group 1 contribute, is given by the condition $-\Delta deviation_i - p_1 > 0$. Then, it results in:

$$\frac{p_1}{b_1 - p_1} < 1 - \sigma n_1 k \qquad (40)$$

Again if $\sigma, n_1 = 1$, the condition of Rege where everybody contribute is reproduced. In this equilibrium (1,0) we must also deduct the condition for the rich group 2 who decides not to contribute.

The utility pay-off for social approval due to a deviation from equilibrium where nobody contributes, for the group of rich type 2, when the group 1 does contribute, is calculated making the difference between the value of the social approval, q_j, when the contribution is one, compared to the value when the contribution is zero. The original assumption is $\bar{g} = n_1$, and now $x_2 = 0$. With equation (21) we calculate the quantity of social approval with $g_j = 1$, and with $g_j = 0$, and we make the difference, $\Delta deviation$:

With $g_j = 1$ → $q_j = (b_2 - p_2)(1 - \sigma n_1)k$

With $g_j = 0$ → $q_j = 0$

$\Delta deviation_j = (b_2 - p_2)(1 - \sigma n_1)k$

The loss of utility due to less consumption of the private good resulting from this deviation is the price of the public good p_2, so the equilibrium where nobody from the group 2 contributes, when the group 1 contributes, is given by the condition $\Delta deviation_j - p_2 < 0$. Or, the same result:

$$\frac{p_2}{(b_2 - p_2)(1 - \sigma n_1)} > k \tag{41}$$

For the equilibrium (0,1) we can deduce similar conditions to (40) y (41), with the only difference that now $\bar{g} = 1 - n_1$.

With global controllers we use the equation (29) and the equivalent results, for the equilibrium where everybody contribute corresponding to rich type 1 are:

With $g_i = 0$ → $q_i = (b_1 - p_1)(-\sigma)\big((1-k)(n_1(x_1 - x_2) + x_2)\big)$

With $g_i = 1$ → $q_i = (b_1 - p_1)(1-\sigma)\big(k + (1-k)(n_1(x_1 - x_2) + x_2)\big)$

$\Delta deviation_i = (b_1 - p_1)\big(k(\sigma - 1) - \big(n_1(x_1 - x_2) + x_2\big)(1-k)\big)$

And the Rege deviation condition is now established as:

$$\frac{p_1}{b_1 - p_1} < k(1-\sigma) + \big(n_1(x_1 - x_2) + x_2\big)(1-k) \tag{42}$$

Again note that under the assumption mentioned when deriving the equation (38), apart from $n_1 = 1$, we reproduce the original Rege condition. Following a similar reasoning we derive the conditions equivalent to (39), (40) and (41), using global controllers. Notice that the asymmetric equilibriums are now unstable (see figure 20).

4.10 Average contribution to the public good

The difference with the Rege model is that the public good average quantity now depends on both proportions of contributors (x_1 and x_2), so the system is interdependent. In the Rege model $\overline{g} = x$, the dynamic of the replicator was only in function of x. In our model we must redefine $\overline{g}$. The interaction between both types of rich results from the public good average, with a value of:

$$\overline{g} = n_1 x_1 + (1-n_1)x_2 \tag{43}$$

Now we can calculate the average utility for each group of rich: we must weigh the utility perceived by contributors, U_i^1, and non contributors, U_i^0, with the proportion of rich in each group. For type 1 rich group we call it U_i^M:

$$U_i^M = x_1 U_i^1 + (1-x_1)U_i^0$$

4.11 Equation of the replicator with group controllers

Finally the equation of the replicator for the proportion of contributors of the rich type 1, following the definition (7), is:

$$\frac{dx_1}{dt} = x_1(U_i^1 - U_i^M)$$

Assuming group controllers, we must use (24), (26) y (43). We define the average utility as $U_i^M = x_1 U_i^1 + (1-x_1)U_i^0$. So, we get the following expression of the replicator:

$$\frac{dx_1}{dt} = (-1+x_1)\, x_1 \left\{ p_1\left(1+x_1 - k(-1+x_1+x_2\sigma+n_1(x_1-x_2)\sigma)\right) + b_1\left(-x_1 + k\left(-1+x_1+x_2\sigma+n_1(x_1-x_2)\sigma\right)\right)\right\} \tag{44}$$

A similar expression can be deduced for the dynamic replicator of x_2, using (25), (27) and (43), and defining $U_j^M = x_2 U_j^1 + (1-x_2)U_j^0$:

$$\frac{dx_2}{dt} = x_2(U_j^1 - U_j^M)$$

$$\frac{dx_2}{dt} = (-1+x_2)x_2\left\{ p_2\left(1+x_2 - k\left(-1+x_2+x_2\sigma+n_1(x_1-x_2)\sigma\right)\right) - b_2\left(x_2 - k\left(-1+x_2+x_2\sigma+n_1(x_1-x_2)\sigma\right)\right)\right\} \tag{45}$$

These two equations of the replicator defined for x_1 and x_2, form a system of non lineal differential equations called GROUP system.

4.12 Equation of the replicator with global controllers

In this case we use, (33), (35) and (43) to derive the equation of the replicator of x_1, and (34), (36) and (43) for x_2.

$$\begin{aligned}\frac{dx_1}{dt} &= (-1+x_1)\,x_1(p_1(1+n_1x_1+x_2-n_1x_2-k(-1+x_2+x_2\sigma+n_1(x_1-x_2)(1+\sigma)))+\\ &b_1(-n_1x_1-x_2+n_1x_2+k(-1+x_2+x_2\sigma+n_1(x_1-x_2)(1+\sigma)))\end{aligned} \tag{46}$$

$$\begin{aligned}\frac{dx_2}{dt} &= (-1+x_2)\,x_2(p_2(1+n_1x_1+x_2-n_1x_2-k(-1+x_2+x_2\sigma+n_1(x_1-x_2)(1+\sigma)))+\\ &b_2(-n_1x_1-x_2+n_1x_2+k(-1+x_2+x_2\sigma+n_1(x_1-x_2)(1+\sigma)))\end{aligned} \tag{47}$$

This new system of two non linear differential equation is called GLOBAL.

4.13 GROUP system phase diagram

First, we shall study the phase diagrams, the qualitative properties of the temporal trajectories of each variable. We must deduce the non trivial phase curves. Each one of these shows the set of points (x_1, x_2) in which one of the differential equations of the system is in equilibrium, where the change in time of the variable equals zero.

The phase curve for x_1, making the expression (44) equal to cero, is:

$$x_2\big|_{\dot{x}_1} = \frac{(p_1+kp_1-b_1k)}{k(-1+n_1)(b_1-p_1)\sigma}+\frac{(k+kn_1\sigma-1)}{(k\ (-1+n_1)\sigma)}x_1 \tag{48}$$

And, for x_2, making the expression (45) equal to zero, we have:

$$x_2\big|_{\dot{x}_2} = \frac{(p_2+kp_2-b_2k)}{(b_2-p_2)\left(1+k(-1+(-1+n_1)\sigma)\right)}+\frac{kn_1\sigma}{1+k\left(-1+(-1+n_1)\sigma\right)}x_1 \tag{49}$$

For each rich type, there are two lines (for $x_1, x_2 = 1$ and $x_1, x_2 = 0$), where the derivatives equal zero. We arrive at them in a trivial form. They correspond to the cases in which the entire rich type follows the same strategy. The result is trivial because there is no other strategy to compare with.

To evaluate the behavior of the derivative of x_1 at both sides of the non trivial line in the phase diagram, we observe its sign for a small increment δ calculated from any point corresponding to the line. For $x_2 = x_2|_{\dot{x}_1}$ and $x_1 = x_1 + \delta$, $\frac{dx_1}{dt}$ is equal to:

$$\frac{dx_1}{dt} = (b_1 - p_1)\delta(-1 + x_1 + \delta)(x_1 + \delta)(-1 + k + kn_1\sigma)$$

In this expression, as $(b_1 - p_1)$, $(x_1 + \delta)$ and δ are always positive and $(-1 + x_1 + \delta)$ is always negative, the sign of the derivative corresponds to the last factor. If $(-1 + k + kn_1\sigma) > 0$, then $\frac{dx_1}{dt} < 0$. If $(-1 + k + kn_1\sigma) < 0$, then $\frac{dx_1}{dt} > 0$. When $n_1, \sigma = 1$, the condition is the same to the one established by Rege, and the sign of the derivative changes when k is greater or smaller than $1/2$.

Taking into account the conditions established before and the three lines where the derivative of x_1 with respect to time is zero, it is possible to graph the phase diagram of x_1, as seen in figure 2.

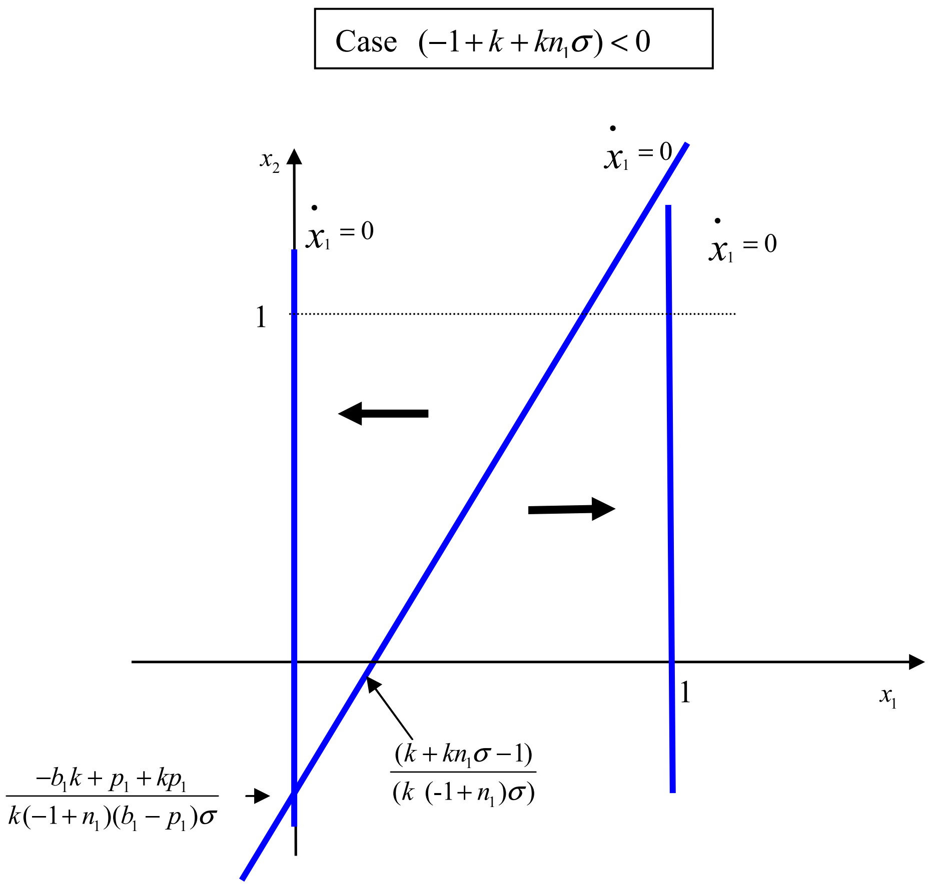

Figure 2. Phase diagram for $\frac{dx_1}{dt}$.

The direction of the arrows is established by the sign of $\frac{dx_1}{dt}$ and shows the behavior of x_1 when this variable has a value outside the equilibrium lines. It can be proved that when $\sigma = n_1 = 1$ the slope of the phase curve of x_1 takes an infinite value behaving in the same manner as in the Rege model (see Figure 1), since in this case x_1 takes independent values from x_2.

When $(-1+k+kn_1\sigma)>0$, the phase line changes to a negative slope and the game is no longer a coordination one. In figure 3 the arrows show that the trajectories point in opposite direction to coordination equilibriums, (0,0) y (1,1).

Case (-1 + k + k n1 ♦) > 0

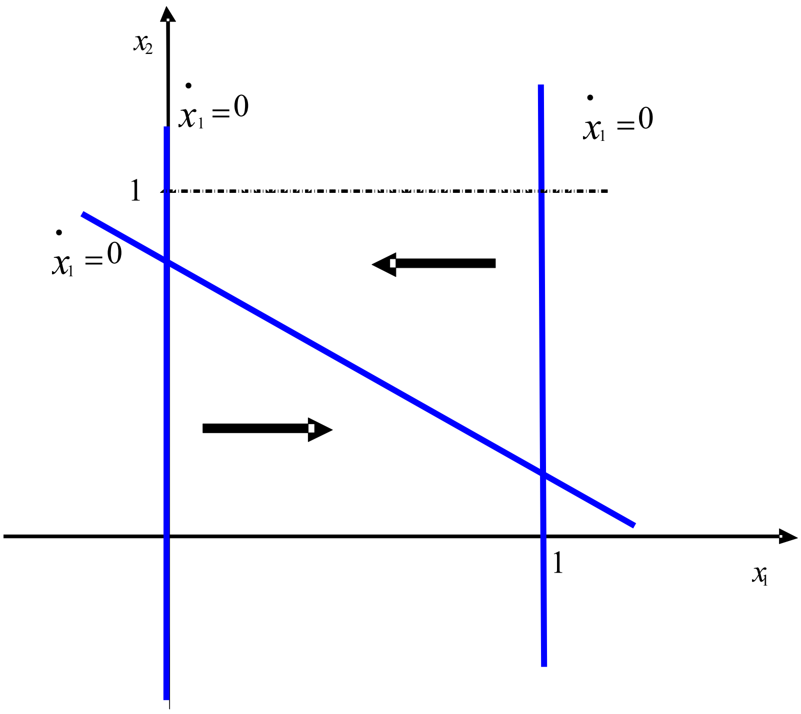

Figure 3. Phase diagram for $\frac{dx_1}{dt}$ in a non coordination game.

For x_2 the process is symmetric, arriving to similar diagrams. To evaluate the behavior of the x_2 derivatives at both sides of the $\dot{x}_2$ line, we fix $x_2 = x_2|_{\dot{x}_2}$ and $x_1 = x_1 + \delta$, the sign of $\frac{dx_2}{dt}$ will result from the same expression calculated before for x_1.

The global working of the system can be seen when both phase diagrams, x_1 and x_2, are taken together in figure 4 (the phase lines for x_2 are those in red). We only consider the case $(-1 + k + kn_1\sigma) < 0$, for it is equivalent to the Rege condition $k < 1/2$ for the existence of a coordination game[15].

[15] In a coordination game the players always receive a greater pay-off when they coordinate in a Nash equilibrium. In the Rege model if $k > 1/2$ (at the limit equal to one), a player that decides not to contribute will face a very small proportion of sanctioners (at the limit no sanctioners at all) then, although the proportion of contributors in the whole population is high, there is no incentives to

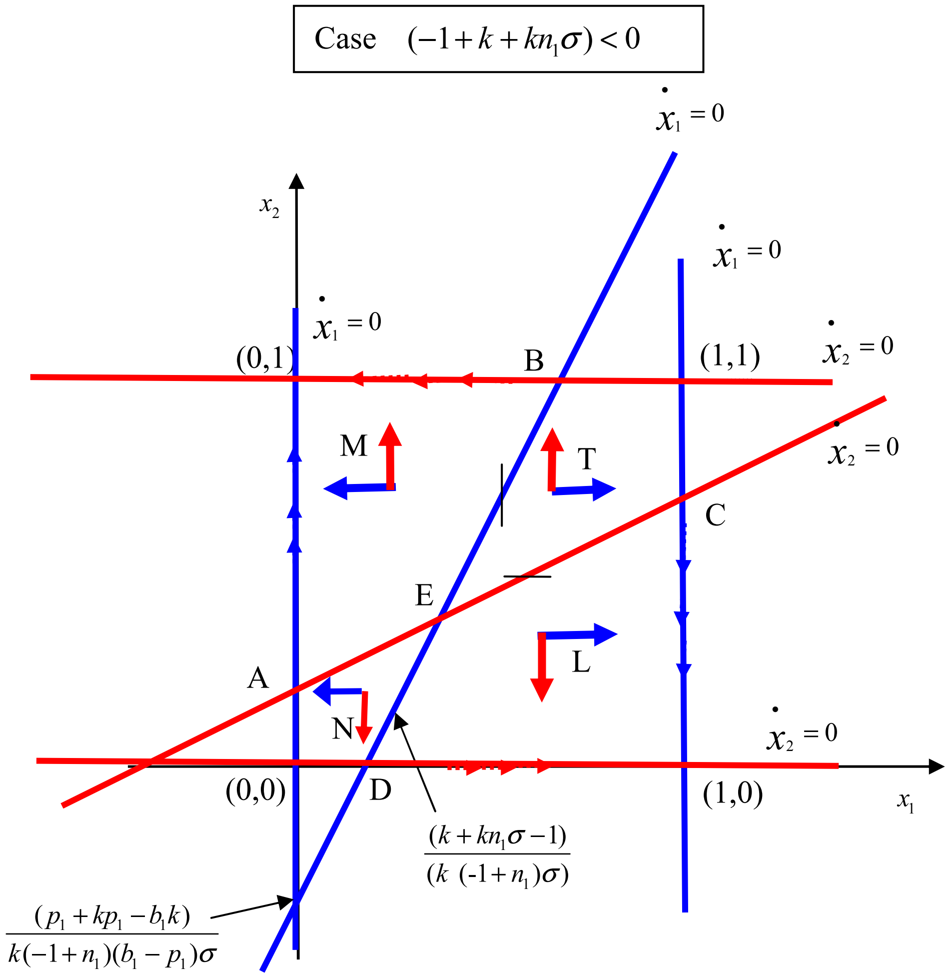

Figure 4. The evolution of social norms of voluntary contribution. The phase diagram.

Within the square $x_i \in [0,1], i=1,2$ there are four regions where we can establish the initial point of departure of the system trajectories. There are 9 stationary points, labelled (0,0), A, (0,1), B, (1,1), C, (1,0), D, and E, at the intersection of base curves. Five of them, those identified as A, B, C, D and E are clearly unstable. Zone T corresponds to points from where the trajectories could go towards equilibrium (1,1),

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coordinate his strategy with the majority of the population. The same reasoning is applied to the opposite situation: a player who decides to contribute, will do it independently of the proportion of contributors in the whole population and, at the limit, he will contribute although practically nobody contributes. In our model, the possibilities of coordination are greater than those of the Rege model. If n_1, σ $=1/2$,the game will continue to be a coordination game while $k<4/5$.

where every individual of both groups of rich are contributors. Zone N corresponds to points from where the trajectories could go towards equilibrium (0,0), where every individual of both groups of rich does not contribute. Finally the points in the M or L zones correspond to possible trajectories towards equilibriums (0,1) or(1,0) respectively, where only one group is a contributor and the other acts as free-rider.

We shall use the Wolfram Mathematic 6.0 program to observe the shape and direction of the trajectories corresponding to selected points in each one of the four mentioned zones. In particular, we want to show that the equilibrium E is unstable and that all the initial points in the T and N points could converge to the equilibriums (1,1) and (0,0) respectively.

4.14 Analysis by linearization

Since the system is nonlinear, to determine more precisely the behavior of the trajectories towards equilibrium, parting from an initial value, it is convenient to apply the Grobman-Hartman theorem[16] which says that the behavior of a nonlinear system, using its linearization around an equilibrium point is asymptotically similar to the behavior of the original system. In this way we can deduce the behavior of the trajectories in the neighborhood of equilibrium, analyzing the eigenvalues and eigenvectors of the matrix of the linearized system, evaluated at the equilibrium. A sufficient condition to make this study, for a stationary state, is that the determinant of the Jacobian matrix of the system evaluated in that point, be different from zero.[17]

We will concentrate in the four potential stationary stable points of (0,0), (0,1), (1,1) and (1,0) and in the unstable equilibrium E defined by the intersection of the phase curves. Note that the last one could be placed outside the square that contains the possible values for x_1 and x_2.

The values of the Jacobian in each of the stable equilibriums and of the eigenvalues have a correspondence with the conditions over k that we established in (38) to (41).

The value of the Jacobian in the equilibrium E is:

[16] See de la Fuente (2000), p. 487.
[17] Ibid., p. 488: theorem 3.4.

$$\frac{1}{16}(-1+k)(b_1 - p_i)(b_2 - p_j)(-1+k+k\sigma)$$

While $k < \frac{1}{1+\sigma}$, his sign will be different from zero. Supposing a coordination game, $(-1+k+k\sigma)<0$), the eigenvalues are positive, as well as the Jacobian, for any value of k, showing that it is an unstable node.

4.15 Phase Diagrams

The parameters that determine the position of the phase curves within the square are:

n_1 = Proportion of rich type 1 in the whole population of rich
k = viscosity coeficient
b_1 = coeficient of the public good utility function for group type 1
b_2 = coeficiente of the public good utility function for group type 2
p_1 = public good price for group type 1
p_2 = public good price for group type 2
σ = status effect

We use a Base Diagram with the following parameter values:

$n_1 = 0.5$
$b_1 = 3$
$b_2 = 3$
$p_1 = 1$
$p_2 = 1$
$\sigma = 1$
$k = 0.2$

With these values, point E, where the phase lines cross, is placed in (0.5, 0.5) (figure 5). In zone T, most of the trajectories converge towards equilibrium where everybody contribute. Besides, as we have foreseen before, some trajectories cross one phase line towards the asymmetric equilibriums. The same happens in zone N, where the majority of the trajectories converges to the equilibrium where nobody contributes, but some of them cross one phase line and go towards one of the asymmetric equilibriums. This is a very interesting behavior. The trajectories changing zones correspond to the initial points close to E and slightly unbalanced. In the case of zone T, the group of rich that start with a smaller proportion of contributors receives an incentive to stop contributing, since the public good utility is greater that the social disapproval and this makes them to behave as free-riders. In zone N, we observe the opposite situation. The group of rich x_2 starts the trajectory with a greater proportion of contributors and face the reduction in the contribution of the member of group x_1. This increases the social approval to group 2, who observing the free-rider behavior of group 1, instead of accepting the equilibrium without contribution, decides to contribute.

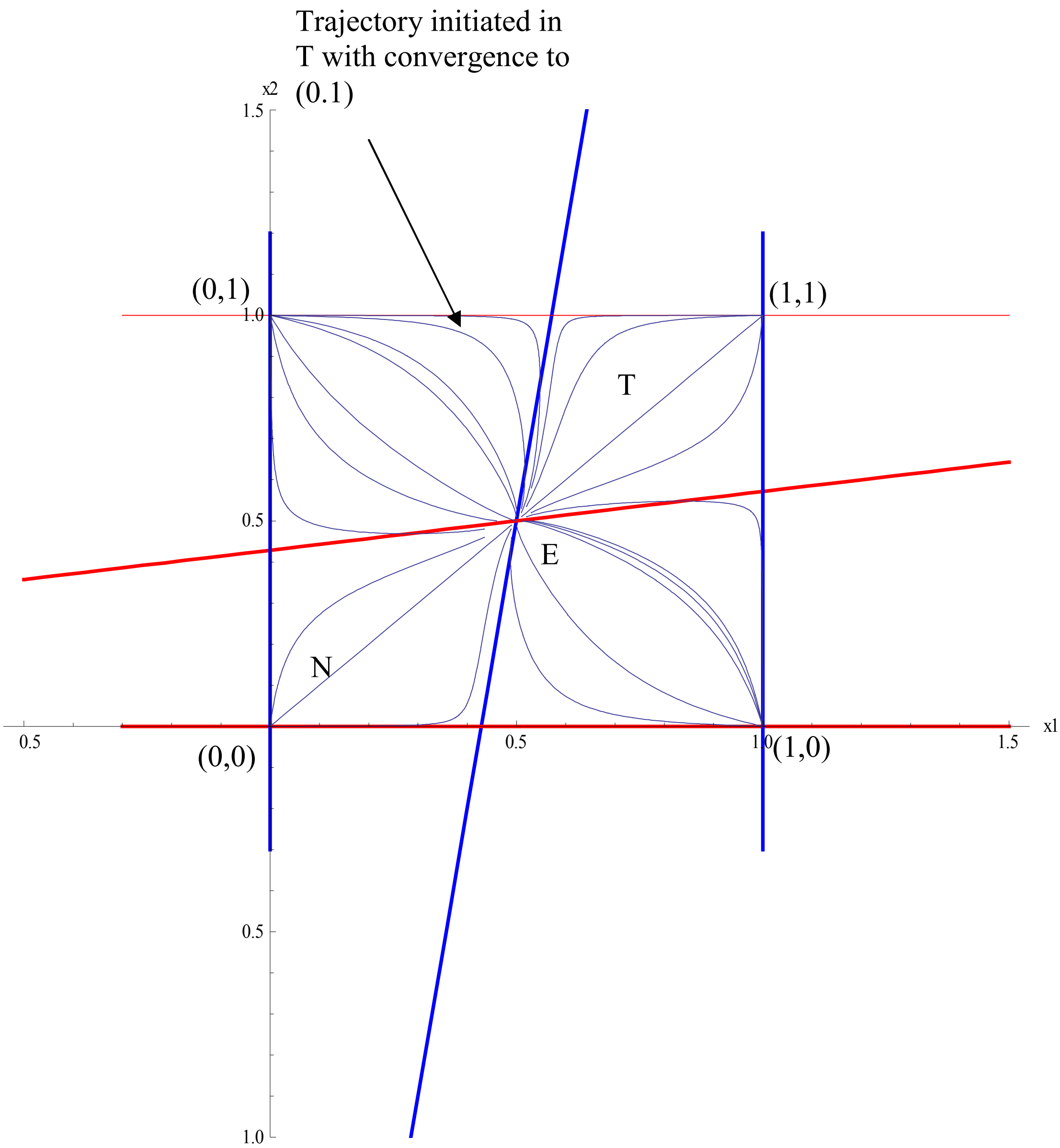

Figure 5. Trajectories in the Basic Diagram

5. Analysis of comparative static

Although we do not start from an optimal point defined as such by an optimization program, we use the concept of comparative static in an analogous way, to study the change in the convergence zones under different values of the parameters.

5.1 Changes in the status effect: σ

The reduction of σ produces the phase lines to open, tend to be parallels to the axes and move to the origin. When σ tends to zero point E tends to aprox. (0.38, 0.38).

Zone T is significantly widened as can be seen in figure 6. It remains a zone N, from where the trajectories go to equilibrium (0,0). Points of departure within this zone N show that, even without the status effect, a proportion of contributors smaller than 0.38 do not generate enough social approval utility to counteract the cost, in terms of private consumption utility of making a contribution.

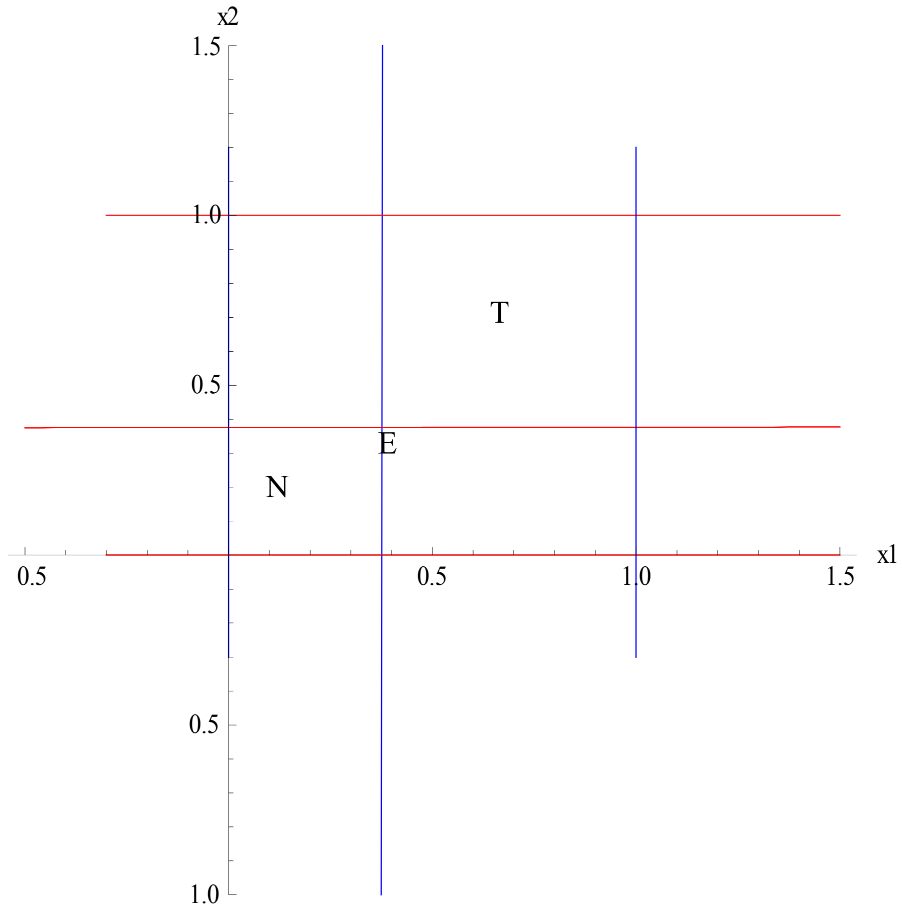

Figure 6. The phase diagram for $\sigma \to 0$.

5.2 Changes in the viscosity, k, and its relation with σ.

The viscosity coefficient measures the weight that has the part of the population as a whole using the same strategy than the individual. The greater the viscosity, within the limits imposed by the condition $(-1+k+kn_1\sigma)<0$, the greater the social approval that the contributor will receive. The increase of the viscosity to 0.5 moves the phase curves towards the left with point E at the coordinate's origin. The equilibrium (0,0) is now unstable. If the status effect is complete ($\sigma = 1$) both phase lines overlap, going through the origin and through point (1,1), showing the Rege result: the game no longer allows coordination, only remains the asymmetric equilibriums, from origin points in zones M and L (figure 7 shows only a red line, since it is overlapping the blue one). When σ is reduced, the lines open, increasing the T area. For $\sigma \to 0$ the phase lines are overlapped with the axes and the T area covers the entire square.

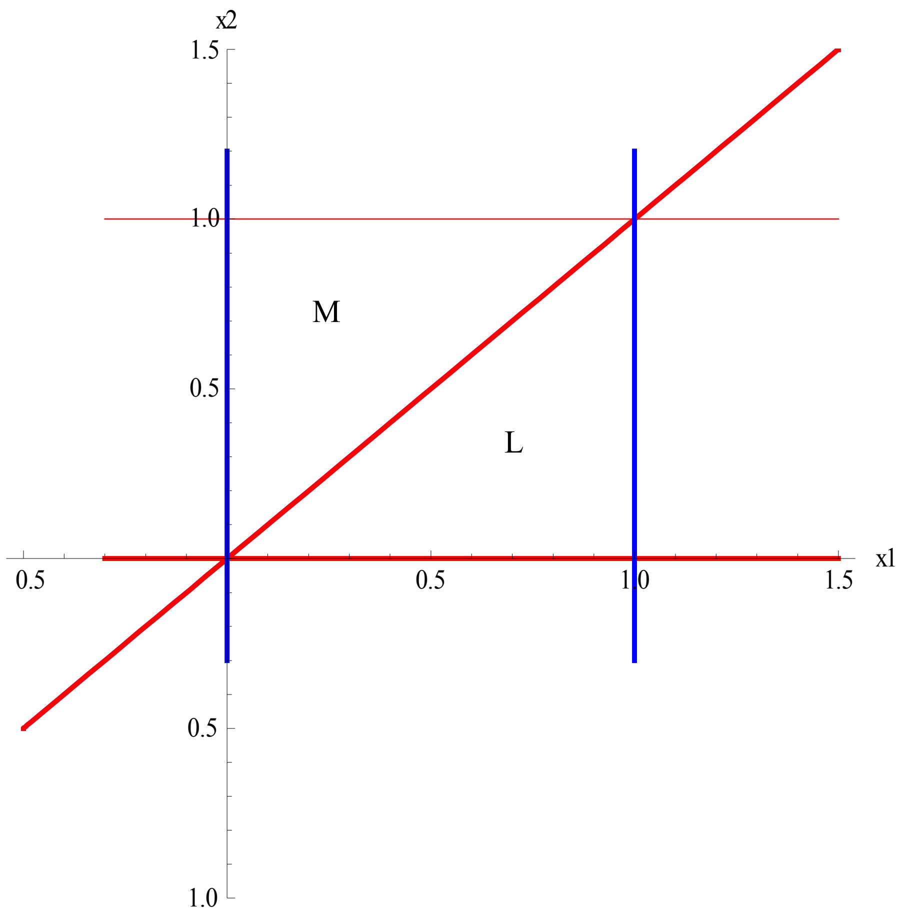

Figure 7. Phase diagram for $k = 0.5$ and $\sigma = 1$.

Figure 8 shows the opening of the phase lines for $\sigma = 0.2$.

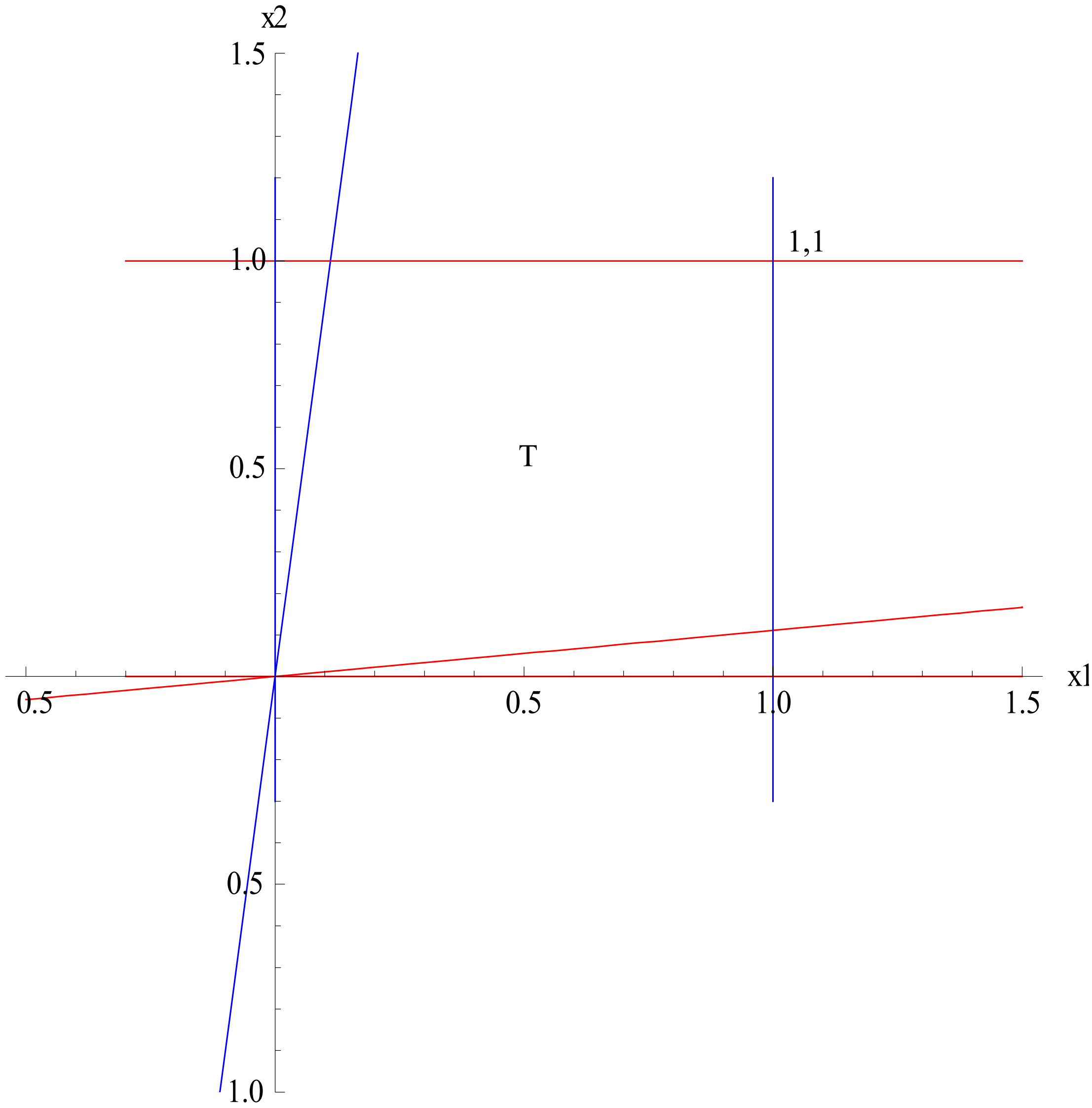

Figure 8. Phase diagram for $k = 0.5$ and $\sigma = 0.2$

For $\sigma = 0.5$, figure 9 shows the area enclosed between two thick line trajectories, in which all trajectories tend to equilibrium (1,1). This area covers most part of the T zone. In spite of this, for lower contributor proportions, small unbalances produce trajectories going to asymmetric equilibriums. Besides that, initial points with equal proportion of contributors, follows a straight line to full contribution equilibrium (1,1). Note that as the contributor proportions increase in both rich groups, the trajectories from unbalanced initial points towards equilibrium (1,1) grow.

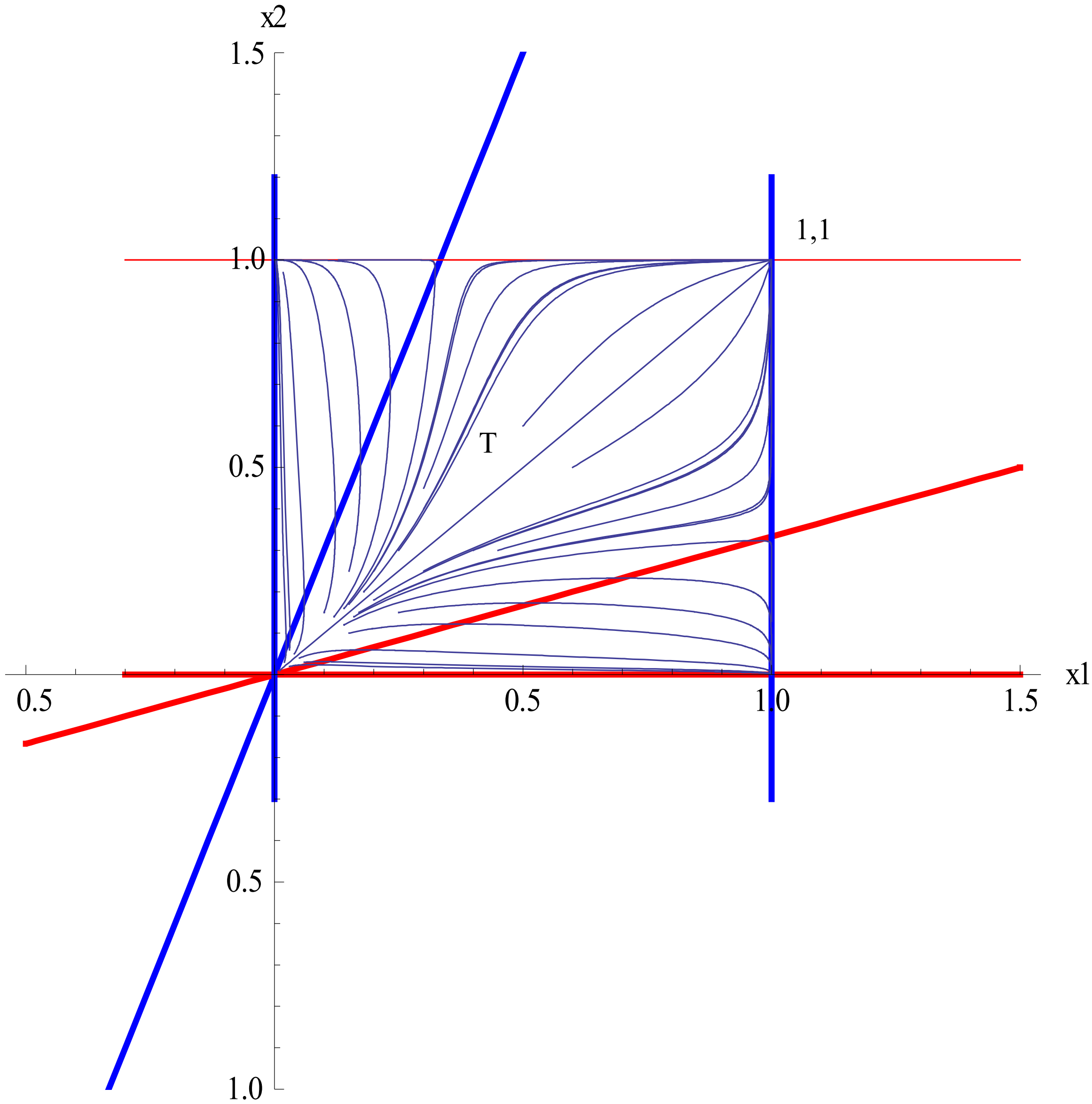

Figure 9. Phase diagram for $\sigma, k = 0.5$

In the Rege model, the game is no longer a coordination for $k \geq 1/2$. Our model, has a higher minimum value of k equal to 2/3, for $\sigma = 1$. And, this fraction increases when the value of σ is reduced, as it can be seen in figure 10 (see explanation in note 15).

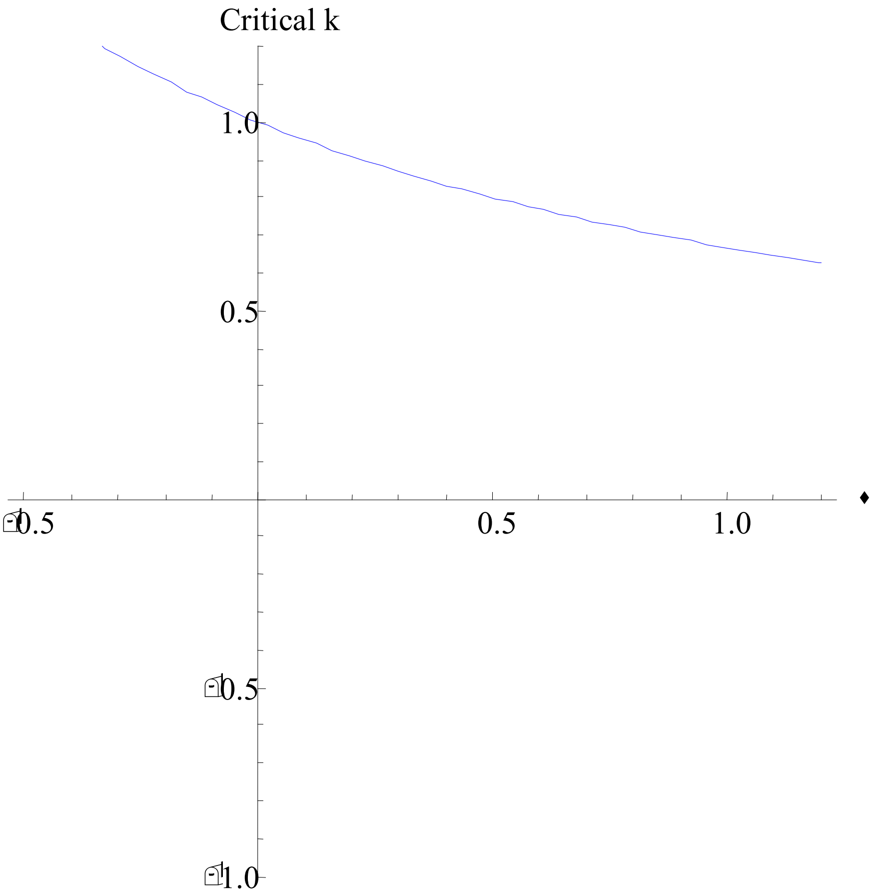

Figure 10. Relation between the critical k with σ

5.3 Changes in the proportion of the rich group type 1 in the total: n_1

The non equal proportion of rich in each group has three effects as shown in figure 11. First, there is the rotation of the phase curves clockwise, (0.5, 0.5) being a pivot point. The group 2 line tends to a horizontal position at 0.5 when $n_1 \to 0$. Second, the trajectories originated in zone L, where the level of contribution of the group with less proportion of the total population is higher, tend to the asymmetric equilibrium with more curved forms than in the contrary case. The trajectories with origin in M, where the level of contribution of the group with more individuals is higher, tend to equilibrium following a linear path. Third, trajectories that initially tended to equilibrium (1,1) and which started with a greater proportion of contributors from the more populated rich group now tend to the asymmetric equilibrium where the last group is the contributor one. And trajectories originated with a bigger proportion of contributors from the less populated rich group now tend to equilibrium (1,1). A subsidy focused on any of the two groups, moves the respective curve to the origin, making only stables the asymmetric equilibriums and (1,1). This suggests that, when selecting groups over which we wish to influence with public policies to stimulate their contribution minimizing costs, it is convenient to start with the smallest groups. This will allow two things. First, to reduce or eventually eliminate the trajectories crossing the phase line toward the asymmetric equilibrium where only the most populated group contributes, and take them to (1,1). Second, increase the trajectories that initially pointed towards equilibrium (0,0) and that, crossing or not the phase line, tend to asymmetric equilibrium (1,0).

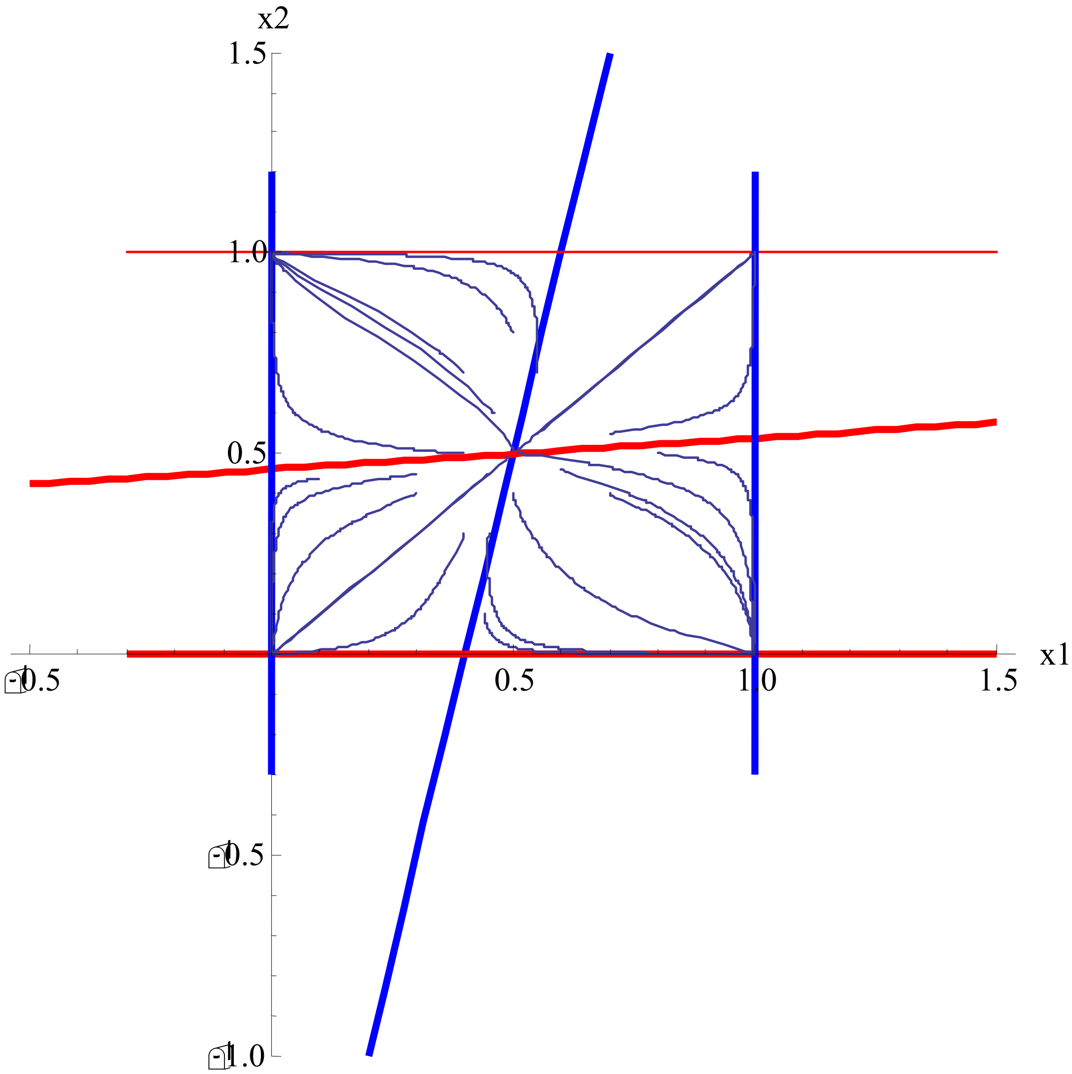

Figure 11. The reduction of n_1 to 0.25.

5.4 A reduction in the public good price: p

Always starting from the Base Diagram, a reduction of the public good price, equal for both rich groups, has the effect of moving the phase line to the left, similar to the combined effect of the increase in the viscosity with the reduction of status effect (figure 8). In Figure 12, the fall of the price from 1 to 0.50 moves point E to the origin, but with a wider opening of the lines, which increases the number of trajectories tending to equilibrium (1,1). This same effect is achieved with a 100% increase in public good utility, taking the value of b from 3 to 6. If the parameter that rewards the behavior considered good increases, or the parameter that punishes that behavior is reduced, the set of points from which the system tends to the desired cooperation is increased.

The fall in the public good price reduces the cost of obtaining social approval. From equation (38) appears that the incentive to deviate from an equilibrium (1,1) is reduced when lowering the price of the public good, which is in fact the deviation pay off. And from the equation (39) it increases the incentive to deviate from an equilibrium (0,0) since the price of the public good is the loss due to deviate. Rege shows the same effect with a government subsidy that reduces the price of the public good. In the Rege model the subsidy has to be bigger than $p-k\lambda$ to move point x' to the origin and transform the equilibrium (0,0), making it unstable. In our model the condition is the same, since it comes from (39). If we call $\hat{p}$ the final price, the condition for the point (0,0) to be unstable is:

$$\hat{p} < \frac{b_1 k}{1+k}$$

As Rege says, the existence of the viscosity is essential for the reduction in price to have the observed effect. If $k=0$ it would be necessary a negative price to achieve the objective. The viscosity makes it possible that the people that play a strategy, for example contributing to the public good, meet with each other even when they represent a very small fraction of the total population. Rege considers that the reduction of the price is made through a government subsidy. In our model this reduction can be made, indirectly by the people through an increase in public good utility. The important discovery of Rege refers to maintening the equilibrium (1,1), when, due to the reduction of the price, the system has moved to a point within zone T of the Base Diagram (figure 5), from where even without the subsidy the same equilibrium can be achieved. This means that the subsidy as well as the reduction of price indirectly obtained constitutes a circumstantial tool, which can be reverted without modification of the final result.

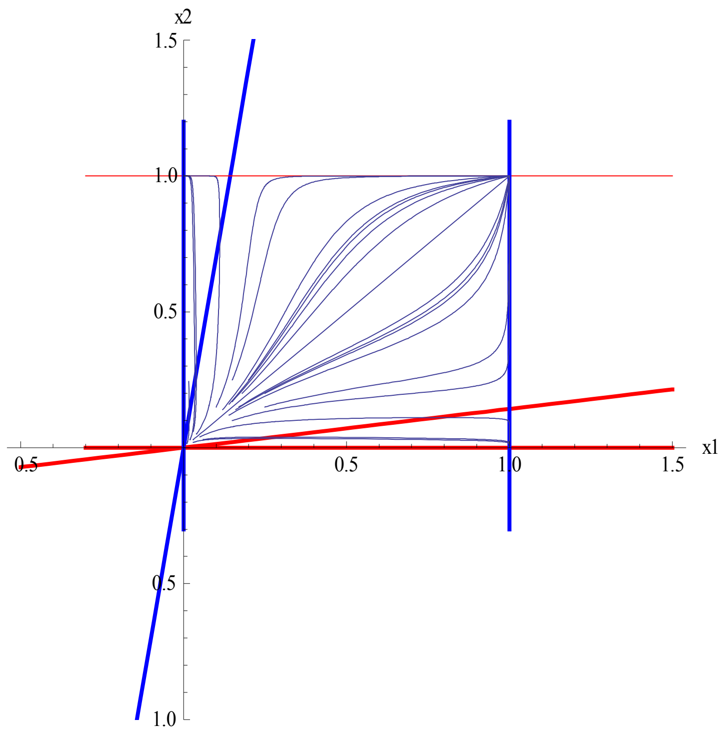

Figure 12. Phase diagram for the unstable equilibrium (0,0) for $\hat{p} < \frac{b_1 k}{1+k}$.

In figure 12 we see some initial asymmetric points, which determine trajectories that cross the phase lines and go towards the asymmetric equilibriums. An additional reduction of 10% in the price, or an equivalent increase of the utility could modify those trajectories sending them towards the equilibrium (1,1).

5.5 Introducing the warm-glow

We could follow Andreoni (2006) showing warm-glow through an additional argument in the utility function. In this way, the function will be widened as follows (see equation (10)):

$$U_i(c_i, q_i, \overline{g}) = u_i(c_i) + f_i(q_i) + y_i(\overline{g}) + s_i(g_i)$$

The fourth argument represents the utility that the person receives from the very fact of giving. But looking at the expression that we have built to capture social approval, it appears the possibility of considering warm-glow as a limit case of social interaction in relation to social approval. The expression of social approval in equation (16) was the following:

$$q_i = \lambda_1 (\mathrm{g_i} - \sigma \overline{\mathrm{g}}) \frac{1}{m} \sum_{h=1}^{m} E_i(g_h)$$

Now, if warm-glow has to capture utility for the very fact of giving, the individuals will not take into account the status effect, so $\sigma = 0$. The expression is reduced to:

$$q_i = \lambda_1 \mathrm{g_i} \frac{1}{m} \sum_{h=1}^{m} E_i(g_h)$$

Social approval, without status effect, is a multiple of the contribution to public good. It would be a contradiction to assume that a person perceives utility by social approval with, for example, $\sigma = 1$, and at the same time also perceives utility for his own contribution, which implies $\sigma = 0$. This observation has made us to discard the fourth argument and consider that the warm-glow effect correspond to the limit case mentioned before. Look at figure 6, the one that shows a widened T zone. Andreoni (2006), concludes that although the warm glow can solve the problem of the efficient provision, since it takes us back to a private good economy, it does not provide an explanation on the reasons of giving. In our vision of the problem the explanation of giving is found in social interaction and in the generation of social approval due to the contribution made. And this explanation contains the limit situation of the warm-glow, where the fact of giving is not related to the search of status, but with the provision of a new good created by social interactions, the relational good described by Uhlaner (1989)[18].

5.6 Phase diagram for the GLOBAL system

[18] Sacco and others (2006), p. 704. Although the search for status brings us to the definition of positional goods (to achieve a better position in relation to another), which can be seen as opposed to a relational good (getting closer to other's position), in this paper both goods also have complementary aspects: "good position may serve to gain desired relations and certain relations may serve to gain a higher position". As the value of σ falls, the importance of the positional good is reduced in favor of the relational good. Uhlaner (1989) defines the relational good as a local public good which only can be produced and consumed through the joint action of a group of individuals, whose identity is relevant.

In section 4.12 we derive the system of diferential equations with global controllers. Figure 13 shows the Base Diagram. The phase lines are overlapped, establishing a full segment between points (1,0) y (0,1) of unstable equilibriums. The change in the control group completely modifies the diagram. Consider a point close to the asymmetric equilibriums, where almost all the members of one of the rich groups contribute, while the opposite situation is found in the other group. With group controllers, a non contributing member will not receive social disapproval, so he would prefer to free-ride and benefit from the public good provided by the other group. Now, with global controllers this behavior receives a greater punishment, generating an incentive to contribute, and moving jointly towards equilibrium (1,1). But we also have the opposite effect, if the initial points determine that the social approval to the contributors does not compensate the cost, the trajectories take both groups to equilibrium (0,0).

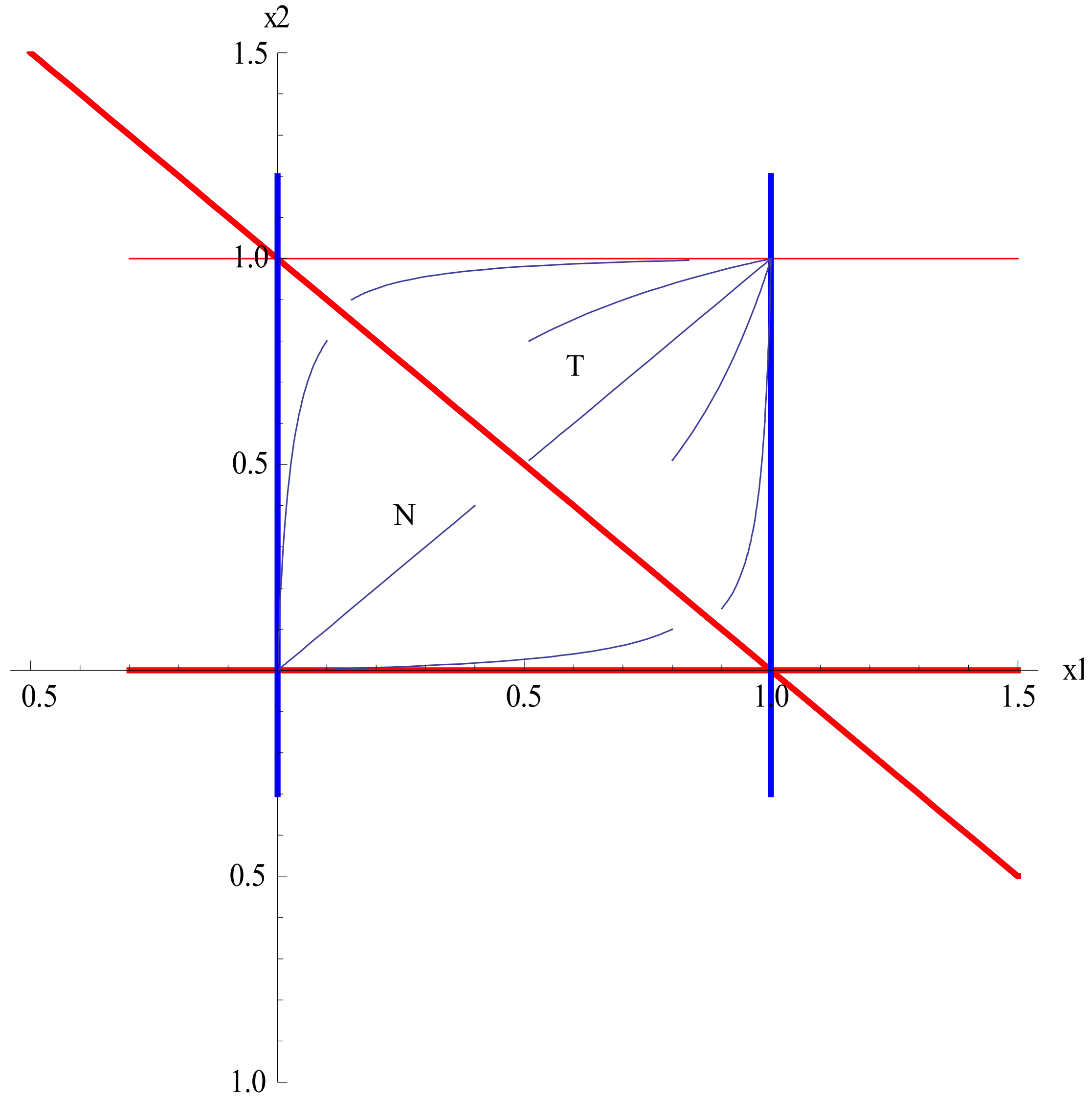

Figure 13. Phase diagram with global controllers (Base Diagram)

Changes in the parameters produce similar effects to those observed previously in the Base Diagram with group controllers, although the results are stronger when facing a 50% reduction in the price of the public good. The phase lines moves to the origin and only remains stable the equilibrium (1,1), see figure 18.

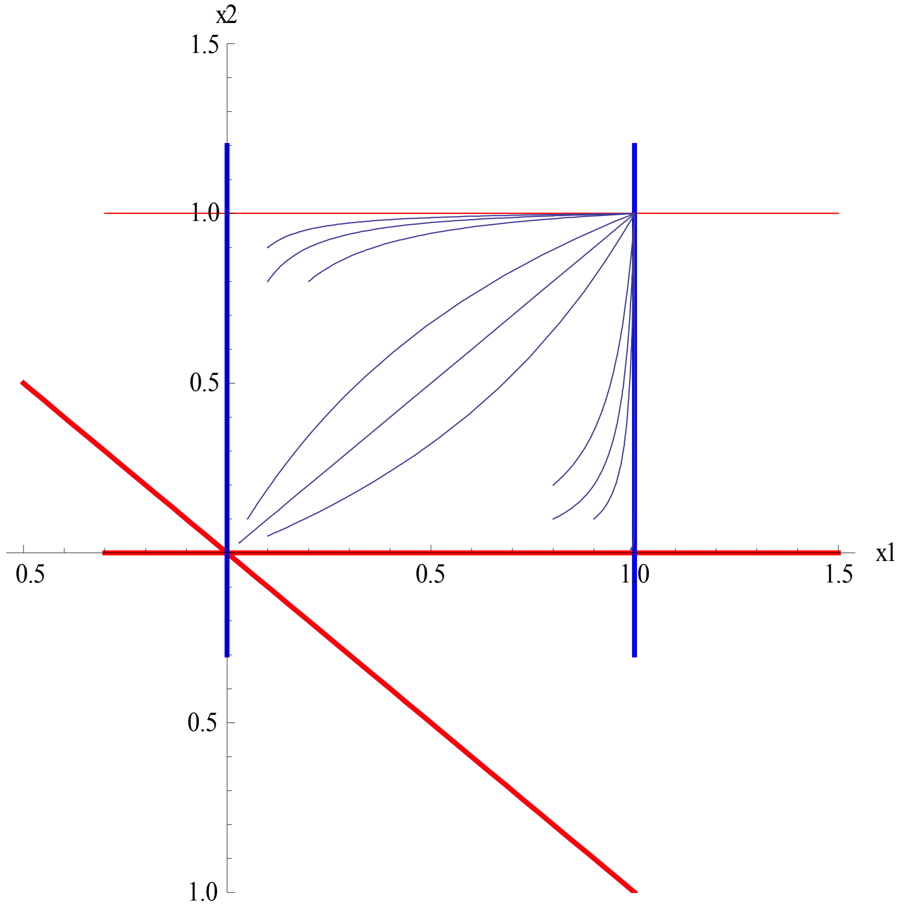

Figure 14. Phase Diagram with global controllers and 50% reduction in public good price

6. Government provision of public good

6.1 An increase in the public good price.

We suppose that the government considers that in equilibrium (1,1) the public good provision is not enough, so imposing a lump sum tax $p\hat{g}$ over each individual, contributes with an additional quantity $\hat{g}$ per person of the society. The cost for each person that whishes to voluntarily contribute increases, because he must add the tax to the price he has to pay for contributing. The new price will be $\hat{p} = p_i(1+\hat{g})$. So there is a level g^* that for all $\hat{g} > g^*$ the stability condition (38) reverts. In this case, the phase lines move to the right leaving point E outside the square. People observe that they can increase their utility not contributing and consuming the private good, since the cost of social approval has increased. In figure 15 we assume that the government decides that the contribution has to be increased by 50%. So, we assume that the public good price is now 1.5. All trajectories converge towards equilibrium (0,0). Notice that the crowding out of the private contribution is stronger than in the traditional model of public good (dollar for dollar), since for each dollar contributed by the government the private sector withdraws two. When the public good price is increased to 1.5, the quantity provided should be the sum of one unit provided by the private sector and 0.5 by the government. This means that if the person reduces his voluntary contribution to zero he is withdrawing 2 unities for each unit provided by the government. The "crowding out" with social approval not only is total but also is accelerated regarding the BBV model[19].

Two additional aspects must be stressed. First, in relation to the long term effects, the symmetry with the subsidy case. Once the government has crowded out the private sector from the equilibrium (1,1) and taken it to (0,0), the fact of going backwards, ceasing the intervention, does not have any effect if in the situation of the Base Diagram the people remain in zone N where they evolutively converge towards equilibrium (0,0). Second, although the government establishes a policy of taxes and public good contributions at a higher level with respect to the contributions made voluntarily by the people, the greater utility in the public good may not compensate the loss of utility due to social approval and the society could be taken then to a Pareto inferior equilibrium. In section 7 we will discuss the welfare aspect of our model.

[19] Bergstrom et al. (1986)

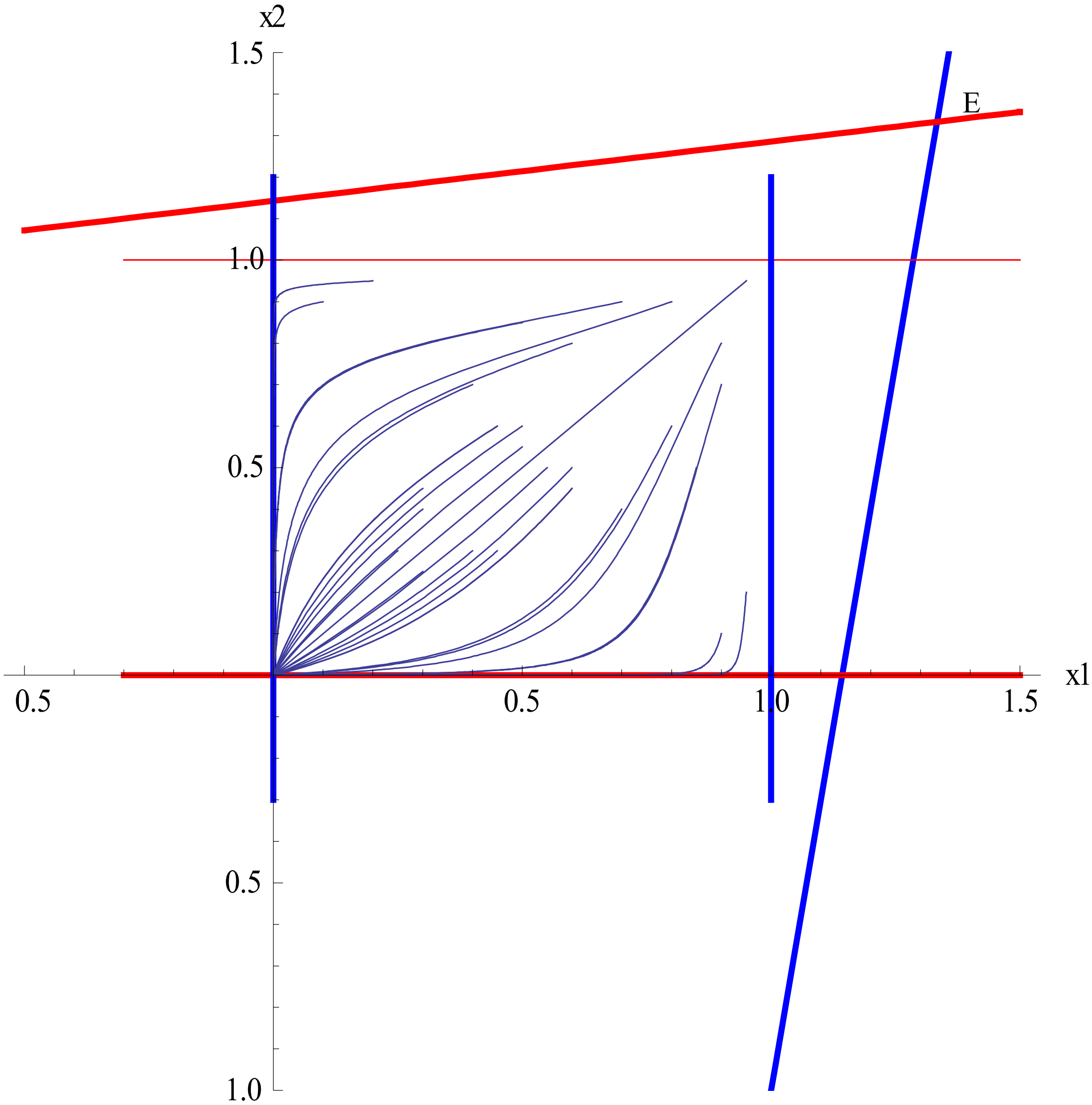

Figure 15. The total crowding out of the private sector by the government as a consequence of an increase of the public good price.

6.2 The model with government intervention

The analysis mentioned before, although it produces a result basically equivalent to the result we are going to see now, is not complete, since it does not take into account the changes produced in the utility function and in the budget constraint of each initial contributor, when introducing the tax. As it is, the model implies that people, when faced to a price increase of the public good, not only reduce their voluntary contribution but also decide not to pay the tax. So, we must modify the model to take into account explicitly the intervention of the government.

Besides, and more important, as Rege says, we must observe the effect produced by government intervention on the subjective rate of social approval, λ. We will consider the stability condition (38). Its reversal can come from the increase of the public good price, already analyzed, and the reduction of the λ rate, if the tax has an indirect effect over λ, reducing its value. This would mean that not only it could crowd out completely the voluntary contribution but also it could erase the social norm of voluntary cooperation, and in consequence the society would become more selfish and individualistic.

In the model without government, the utility derived from the public good for each person in equilibrium (1,1), is constant and equal to b. With the government providing the public good financed through a tax on the contributors, we will see that it will be necessary that the utility conforms to the shape of function (12), with the value of parameter a which assess the concavity.

The changes in the model are the following. First, introduce the government provision of the public good in the utility function, assuming no transaction costs. Function (12) is replaced by:

$$y = b(\overline{g} + tax)^a \tag{50}$$

The final expressions of utility functions are:

$$U_i(c_i, q_i, \overline{g}) = c_i + q_i + b_1(\overline{g} + tax)^{a_1} \tag{51}$$

$$U_j(c_j, q_j, \overline{g}) = c_j + q_j + b_2(\overline{g} + tax)^{a_2} \tag{52}$$

The budget constraints (15) are:

$$Y_i = c_i + p_1 g_i + tax$$

$$Y_j = c_j + p_2 g_j + tax \tag{53}$$

To calculate the exogenous value of λ, without tax, we only need to take into account the excess of utility that each donor would receive if all his group contributes, less the utility loss captured by the price (equations (17)). With the tax, we have to take into account, apart from the price, the utility coming from the public good provided by the government. This provision is independent form the decision of making voluntary contributions or not, then the subjective value of the social approval is reduced. In other words, the net utility that the individuals would reach if there is no free-riding would fall, and in consequence, this last behavior is stimulated. The functions (17) must be replaced, taking into account the function (50), by:

$$\lambda_1 = b_1(1+tax)^{a_1} - b_1(tax)^{a_1} - p_1$$

$$\lambda_2 = b_2(1+tax)^{a_2} - b_2(tax)^{a_2} - p_2 \qquad (54)$$

Now, the subjective rate of social approval does not depend only on the absolute value of each public good unity, measured by the parameter b of the utility function, but also it must be considered the relative value or elasticity of the utility when there are changes in the provision of the public good, measured by the parameter a.

The production functions of social approval (20) and (21) are then expressed as follows:

$$q_i = \left(b_1(1+tax)^{a_1} - b_1(tax)^{a_1} - p_1\right)(\mathrm{g_i} - \sigma\overline{\mathrm{g}})\left(k + (1-k)x_1 - k(1-g_i)\right)$$

$$q_j = \left(b_2(1+tax)^{a_2} - b_2(tax)^{a_2} - p_2\right)(\mathrm{g_j} - \sigma\overline{\mathrm{g}})\left(k + (1-k)x_2 - k(1-g_j)\right) \qquad (55)$$

With these changes, the dynamic of the systems is more complex, due to the influence of the tax and the elasticity a on the ratio λ. With Base Diagram data, figure 16 shows λ in function of a, for $0 \le a \le 1$. This relationship can be linear, concave or convex, starting from a point in the vertical axis corresponding to the negative of the price, p, for $a = 0$, growing up to value $b - p$ for $a = 1$, depending on the tax value. The function will be linear, $\lambda = -p + ba$, for a tax of 0,562, concave for lesser values, and convex for superior values. It can also be studied the function λ in tax terms. This function is decreasing and convex for $0 < a < 1$, being positive for tax values between zero and one ($0 < tax < 1$), if $a > 0.42$, as is shown in figure 17.

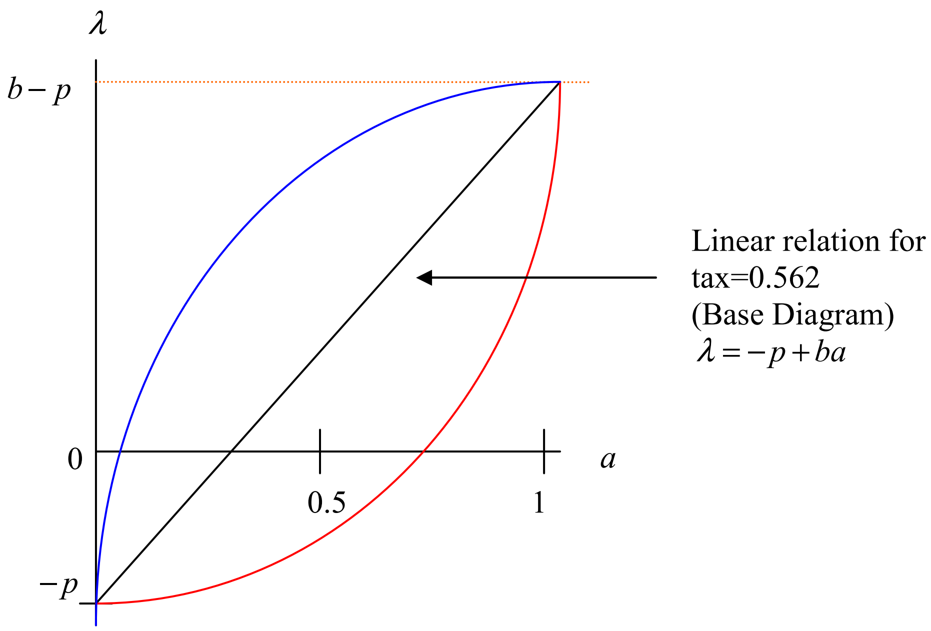

Figure 16. The λ function respect to a.

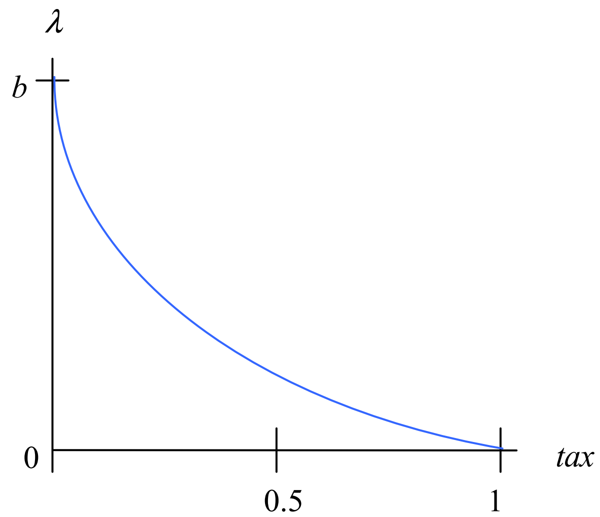

Figure 17. The λ function with respect to tax, for $a = 0.42$ and Basic Diagram.

These relations confirm our previous result on a strong crowding out of voluntary contributions, since for a giving value of a, the rate of reduction of λ is inceasing with respect to tax.

Using the tax model, figure 18 shows a movement of the phase lines, where point E is again outside the square, with Base Diagram parameters value, $a_1 = a_2 = 2/3$ and a

0.5 tax value. The crowding-out is total and the trajectories have the same behavior as in figure 15.

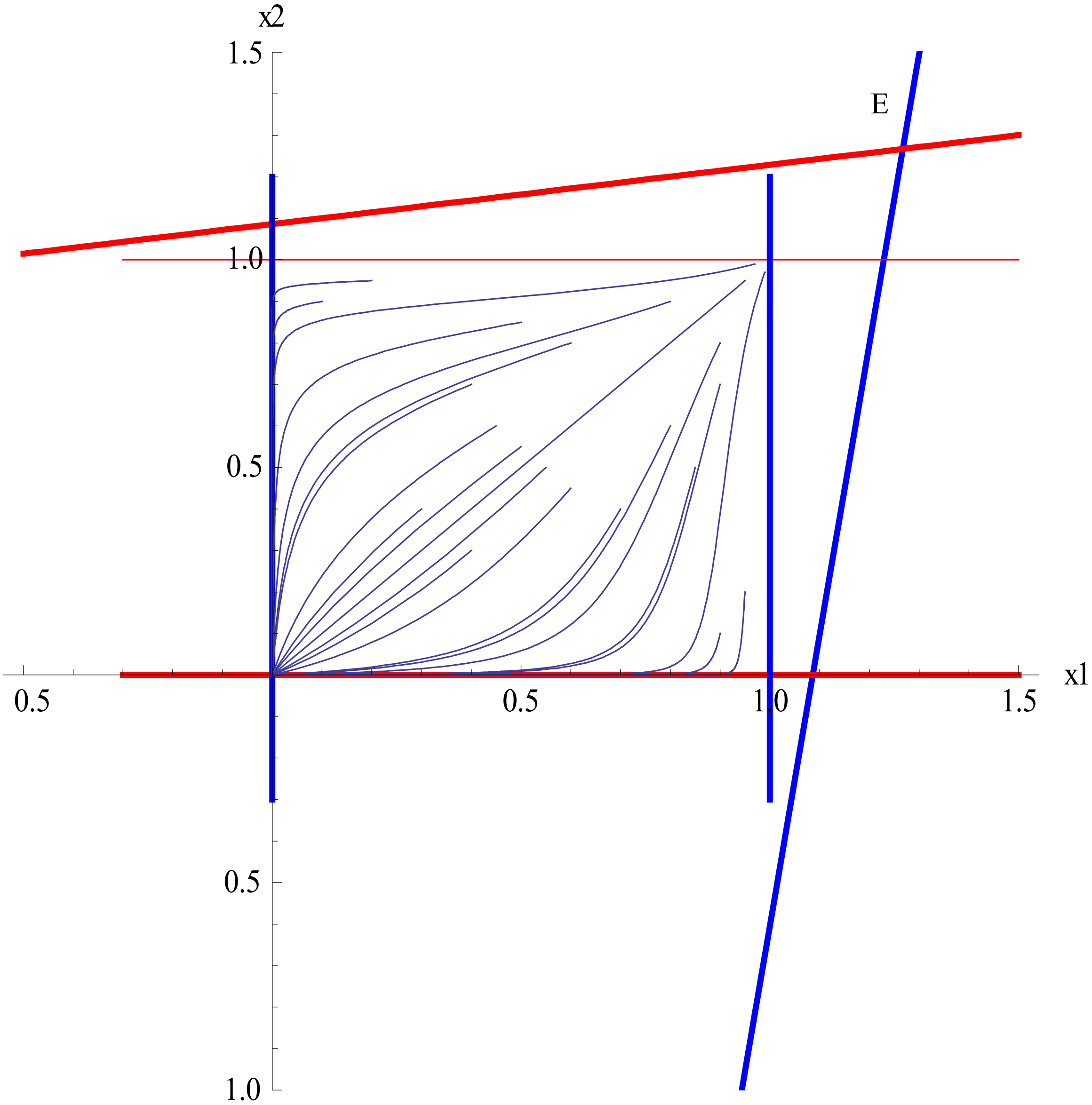

Figure 18. The complete crowding out of private contributions for 0.5 tax, with $a_1 = a_2 = 2/3$.

If the government establishes increasing taxes of the same absolute value, from, for example, 0.1 up to 0.5, the function $\lambda(tax)$ demands that the initial movements of the phase lines be relatively stronger than the final ones. It is another way of showing that the crowding out effect is stronger than the one seen in the BBV model.

7. The evolutive equilibriums and welfare

The social approval production functions, $q_i = \lambda_1 (g_i - \sigma \overline{g}) \frac{1}{m} \sum_{h=1}^{m} E_i(g_h)$, are interdependent through the average contribution $\overline{g}$. This characteristic produces three externalities. The status parameter, σ, produces a negative externality. The increase of the contribution of all persons except i, reduces the social approval that this last receives. A second externality can be observed if we consider, as Hollander does, that the approval ratio, λ, also depends on the average contribution. In this case, see equation (9), the sign of the externality effect can not be clearly established. Finally, since the average contribution is also the supply of the public good, there is a third positive externality from the production to the joint consumption of the public good (the classic externality in the definition of a public good). The addition of these externalities will establish that the equilibrium will not be efficient in terms of Pareto, although it is probable that the effects could be partially compensated.

7.1 Externalities compensation: The lottery model of Morgan

Morgan (2000), from a non profit organization point of view (NPO), compares two financial sources: the voluntary contributions and the sell of a lottery. The idea is to find a mechanism that does not use government imposition but instead incentives people to provide the efficient quantity of the public good. Groves and Ledyard (1977) as well as Walker (1981) developed mechanisms using the fiscal system and Nash type interactions to achieve efficient allocations for the public good. This allows Morgan to say that "purely as a theoretical exercise, the decentralized public goods provision problem has largely been solved"[20]. But, Chen and Tang (1998) [21], when describing different results of laboratory experiments with those mechanisms, observe that they

[20] Morgan (2000), p. 762.
[21] Morgan (2000), p. 762.

have not been very satisfactory and conclude that "behavior is better described by various learning models than by equilibrium predictions".

Morgan's theorem 1 establishes that a raffle or lottery with a fixed price provides a greater quantity of the public good than the voluntary provision. The cause of this difference is found in the negative externality produced by the lottery, which favorable compensates the positive externality of the public good, reducing the free-riding problem. Morgan´s theorem 2 establishes that with a price big enough, a lottery can induce a result as close as we want to the Pareto optimal.

For future research we propose the following question: in our social approval model, the externalities generated can have a similar effect as in Morgan´s, which could imply that the equilibrium (1,1) is as close as we want to the optimum?

7.2 Welfare in the Rege and Hollander models

We have already mentioned in section 3 the Rege result regarding that the utility of a person in particular is greater in the stationary state(1,1), with respect to (0,0), in terms of private consumption, in a quantity equal to λ, although in these equilibriums there is no social approval (since we suppose $\sigma = 1$). The reduction of utility due to applying private consumption to the financing of the public good is over compensated with the utility increase resulting from the joint consumption of the public good. So, Rege concludes that the movement from the non contribution state, to the state where everybody contributes, is an achievement in the Pareto sense, but it can not be established how close from the Pareto optimum (PO) that result is placed.

Hollander goes a step further, introducing an element we use in our model that totally modifies the conception of the problem. It is the status effect, σ. With $\sigma < 1$, in the equilibrium (1,1) people receive utility from the social approval. In the provision of public good corresponding to the Samuelson condition compared with the one in Hollander, it must be taken into account that in this last case we can have a smaller contribution due to the positive externality pointed by Samuelson, but, on the other hand, the difference can be more than compensated for the bigger contribution coming from the negative externality of the social approval. "One might perhaps expect that state or market provides more of the collective good than a system of social exchange (he refers to an ideal market that can comply with the Samuelson condition)”[22]. But surprisingly, the contrary also can be found. A strong enough desire of social approval can make the quantity of social equilibrium to be higher than the corresponding to the PO with state provision. The intuition behind this result is found in the fact that if we have Kantian people that decide to contribute voluntarily the level corresponding to the optimum, they would also have the utility coming from the social approval. Although the individuals do not act in accordance to the Kantian mandate, Hollander proposition 4 states that if the system of social interchange provides a higher level of public good with respect to the one provided by the government to comply with Samuelson condition, in this last case the individuals will have a smaller level of welfare.

Social approval is a subproduct of the contribution to public good, if this is made voluntarily. In itself, it is a good that does not require resources for its production. In consequence, the social interaction allows producing more goods with the same quantity of resources. And even if we could not apply Hollander proposition 4, because we do not know if the provision of the public good in the evolutionary equilibrium is greater or smaller than in Samuelson's, the final utility could be bigger when adding the one coming from social approval.

7.3 Welfare of the evolutive model of voluntary redistribution

Redistribution is a public good with distinct characteristics[23]. First, as in the general case, it is a public good for the ensemble of potential contributors. That is why we consider the utility of the rich groups that decide the amount of their contributions. But also the proper public good is the welfare of a group of people, the poor, which is improved by means of the transfers received. The general welfare is composed by the aggregation of the utilities of the three groups.

In our model, Nash equilibrium corresponding to the classic stage game of the public good is (0,0). The equilibrium (1,1) is another Nash equilibrium, which not only presents a greater utility (we always suppose that $\lambda > 0$), for the two ich groups, than (0,0), but it also includes the utility of the transfers towards the set of poor, so it is an

[22] Hollander (1990), p. 1164.
[23] See Kolm (2005) that deals with this subject as a techincal note.

improvement in Pareto terms. Both asymmetric Nash equilibriums, for the same reasons, are Pareto better than (0,0). We can not say that it is always an improvement in Pareto terms when going from one of the asymmetric equilibriums to (1,1). The set of poor people obviously improves. The rich group not contributing could see his final utility reduced if, the improvement in terms of social approval, is smaller than the loss due to less private consumption. It is the case of the Base Diagram and $\sigma = 1$, where the welfare improvement of the poor group requires a reduction of the welfare corresponding to the rich. This does not happen for the values of the parameters of the Base Diagram and $\sigma = 0.5$, where the equilibrium (1,1) is a Pareto improvement compared to the asymmetric ones.

7.3.1 Utilities for the GROUP system without taxes

We start with the case corresponding to the Base Diagram, with group controllers and without taxes. In Nash equilibrium (0,0) the utility is reduced to the one corresponding to each rich income. The other Nash equilibriums show improvements in Pareto terms. In Table 1 we show the levels of utility for each person of the society. We suppose a utility aggregated function for the group of K poor individuals, considered as a whole, linear with respect to the transfers received from the rich:

$$\sum_{k=1}^{K} U_k(g_i, g_j) = \sum_{i=1}^{I} g_i + \sum_{j=1}^{J} g_j$$

We must calculate the utility for each one of the three equilibriums that show an improvement in the Pareto sense and for each of the rich group. To do so, we use indistinctly one of the utility functions (22) or (23), since we are considering $a_1 = a_2 = 0.5$ and $Y_i = Y_j$. Replacing the values of the parameters corresponding to the Base Diagram we have the following utility function:

$$U_i(c_i, \overline{g}) = Y - g_i + (3-1)(g_i - (1.0)\overline{g})\big(0.2 + (1 - 0.2)x_1 - 0.2(1 - g_i)\big) + 3\overline{g}^{0.5}$$

<u>Equilibrium (1,1):</u> In this Nash equilibrium the increased utility is achieved only from the consumption of the public good. The utility value for any contributor is:

$$U(c, \overline{g}) = Y - 1 + 3.1^{0.5} = Y - 1 + 3 = \underline{Y + 2}$$

The loss of utility due to the lower private consumption is absorbed by the utility of the public good, remaining a positive net result. Notice that as we assume, as Rege, $\sigma = 1$, in the equilibrium there is no social approval. No contributor has any incentive to deviate since the utility corresponding to bigger consumption of the private good (+1), is smaller than the social disapproval coming from the control group formed by all the population except himself (-1.6). The total utility of the deviating individual is:

$$U(c,\overline{g})_{deviate} = Y + (3-1)(\text{-}1.0)\left(1 \text{-} 0.2\right) + 3 = Y + (2)(\text{-}1.0)\left(0.8\right) + 3 = Y \text{-} 1.6 + 3 = \underline{Y + 1.4}$$

In the case of $\sigma = 0.5$, shown in table 2, with social approval production in equilibrium, the total utility increases 50% :

$$U(c,\overline{g}) = Y - 1 + (2)(1 \text{-} 0.5) + 3.1^{0.5} = Y - 1 + 1 + 3 = \underline{Y + 3}$$

Here again, no one has incentives to deviate since the deviation utility corresponding to the greater consumption of the private good (+1), is lower than the social disapproval coming from the control group formed by all the population except himself. (-1.8).

<u>Equilibriums (1,0) or (0,1):</u> In these two asymmetric equilibriums each group in the case of the Base Diagram receives the same utility. Supposing that the group of rich type 1 is formed by contributors and those of type 2 by non contributors, the utilities are:

$$U_i(c_i,\overline{g}) = Y - 1 + (3-1)(1 \text{-} (1.0).0.5)\left(0.2 + (1 \text{ - } 0.2) \text{-} 0.2(1 - g_i)\right) + 3.(0.5)^{0.5}$$

$$= Y - 1 + 1 + 3.(0.7071) = \underline{Y + 2.12}$$

$$U_j(c_j,\overline{g}) = Y + (3-1)(-(1.0)0.5)\left(0.2 \text{-} 0.2\right) + 3(0.5)^{0.5}$$

$$= Y + 3(0.5)^{0.5} = \underline{Y + 2.12}$$

For the group of contributors the utility cost in terms of private good is compensated with the one coming from social approval. For the group of non contributors its utility comes from the public good consumption as free riders. It has to be mentioned, as we saw at the beginning of this section, that the utility of both groups of rich is higher in the asymmetric equilibriums with respect to the (1,1). The contributors win more utility from the social approval in relation to the loss due to the lower level of public good provision. The non contributors win more utility from the private good compared with what they loose by the reduction of the public good. But including the beneficiary group of the transfers, the utility of the society as a whole is higher in the (1,1).

Nobody has incentives to deviate. If a contributor does deviate, his utility increases due to the bigger private consumption (+1), but it is reduced by the fall in social approval (-1.8). Total utility of the deviating contributor is calculated as:

$$U_i(c_i,\overline{g})_{deviate} = Y + (2)(\text{-}0.5)\left(0.8\right) + 2.12 = Y - 0.80 + 2.12 = \underline{Y + 1.32}$$

If a non contributor deviates, its utility is worse due to the smaller private consumption (-1), and he can not revert this effect with the generation of social approval (+0.20) since his control group is formed in greater proportion by non contributors which do not give him any approval. His total utility is calculated as:

$$U_j(c_j,\overline{g})_{deviate} = Y - 1 + (3-1)(1-1.0(0.5))\left(0.2\right) + 3(0.5)^{0.5} = Y - 1 + 0.20 + 2.12 =$$

$$\underline{Y + 1.32}$$

In the case of $\sigma = 0.5$, the total utility of the contributor increases 24% compared to the Base Diagram, this difference comes from the increase in the social approval, since as his contribution is higher than the average, it is more valued by his control group:

$$U_i(c_i,\overline{g}) = Y - 1 + (3-1)(1\text{-}(0.5).0.5)\left(0.2 + (1\ \text{-}\ 0.2)\text{-}0.2(1-g_i)\right) + 3.(0.5)^{0.5}$$

$$= Y - 1 + 1.50 + 3.(0.7071) = Y - 1 + 1.50 + 2.12 = \underline{Y + 2.62}$$

As it can be expected, the utility of non contributors is not modified by variations in σ, it only perceives the utility coming from the public good provided by group 1, and in consequence acts as a perfect free-rider:

$$U_j(c_j,\overline{g}) = Y + (3-1)(-(0.5)0.5)\left(0.2\text{-}0.2\right) + 3(0.5)^{0.5}$$

$$= Y + 3(0.5)^{0.5} = \underline{Y + 2.12}$$

Again, nobody has incentives to deviate. If a contributor does deviate, he improves his utility due to bigger private consumption (+1), but he reduces it by the fall of social approval (-1.9). If a non contributor deviates, he worsen his utility due to less private consumption (-1), but cannot revert this effect with the generation of social approval (+0.30) since his control group is formed by a greater proportion of non contributors which do not give him any approval.

Notice that, when there is social approval in the equilibrium, the utility in (1,1) is higher to those corresponding to the asymmetric equilibriums, reversing the result obtained with $\sigma = 1$.

In table 1, the utility of the contributors is only 15% higher in the equilibrium where both groups contribute compared with the equilibrium where only one group contributes. This is due to the counter effect between the utility coming from the public good and the utility coming from the social approval. While in the asymmetric equilibriums the difference of utility for the contributors is proportional bigger (+24%) compared to those that do not contribute. This shows that the free rider effect that benefits the non contributors is over compensated by the social approval of the contributors.

We have to make two remarks regarding the utilities referred to in tables 1 and 2. First, they are based on the assumption that the utilities of the three groups are cardinally measurable and totally comparables. Second, they represent the values corresponding to the evolutive equilibriums which by ESS definition are all Nash equilibriums, so they are not payments corresponding to a stage game between both groups of rich where the strategies are contributing or not.

The Base Diagram considers that both groups of rich are formed by the same number of people. In the case of table 2 the total utility in the equilibrium where everybody contribute is more the 39% than the one corresponding to the asymmetric equilibriums, while, leaving aside the income of each group of rich, the utility generated by the public good and the social approval is 50% higher than the one corresponding to the poor group. This last difference can measure the strength of the social attraction value that the group of poor has over potential donors.

Supposing that the model parameters produce the equilibrium (1,1), is possible to characterize the contribution $g = 1$ as the one corresponding to the necessary level to arrive to the Pareto optimum in the Samuelson condition (SPO). We can also suppose that each group of rich solves a game of the Guttman-Danziger[24] type to assess the optimal contribution and then acts in accordance to the dynamic of the replicator. The application of the Proposition 4 of Hollander would allow us to say that the Nash equilibrium (1,1) not only complies with the Samuelson condition but also generates a relational good, the social approval, which brings the society to a superior PO. In other words, the neoclassical reasoning could be inverted. Instead of using the theory of the public good to demonstrate that the private sector should be completely crowded out, to achieve a SPO, the same theory could show that the government should be completely crowded out so that the private sector can generate the social interaction that produces a

[24] Guttman (1978,1987), Dantziger (1991)

new good, and in this way, the result would be a Community Pareto optimum (CPO) better than the SPO.

Utility / Equilibriums	Rich 1	Rich 2	Poor	Total $I=J$
(0,0)	YI	YJ	0	$2YI$
(1,0)	$(Y+2.12)I$	$(Y+2.12)J$	I	$2YI+5.24I$
(0,1)	$(Y+2.12)I$	$(Y+2.12)J$	J	$2YI+5.24I$
(1,1)	$(Y+2)I$	$(Y+2)J$	$I+J$	$2YI+6I$

Table 1. Utilities in the GROUP system. Basic Diagram. $I=J$

Utility / Equilibriums	Rich 1	Rich 2	Poor	Total $I=J$
(0,0)	YI	YJ	0	$2YI$
(1,0)	$(Y+2.62)I$	$(Y+2.12)J$	I	$2YI+5.74I$
(0,1)	$(Y+2.12)I$	$(Y+2.62)J$	J	$2YI+5.74I$
(1,1)	$(Y+3)I$	$(Y+3)J$	$I+J$	$2YI+8I$

Table 2. Utilities for the GROUP system. Basic Diagram with $\sigma=0.5$, $I=J$.

The calculation of utilities in the GLOBAL system, with global controllers, for $\sigma=1$ and for $\sigma=0.5$, using the utility functions 44 or 45, shows, as expected, that the values of the equilibrium (1,1) are the same that those shown in the previous system. The difference is, as seen in figure 18, that with global controllers the asymmetric equilibriums are unstable.

8 Summary and conclusions.

Self-organization has two properties. The generation of endogenous structural changes in the society and the production of new facts, not foreseen in the initial people plans. Income redistribution could be treated as an endogenous structural change. The production and consumption of a relational good, the social approval, is a new fact not present in the initial people plan. Redistribution as a public good when treated under the optimization theory can not overcome the Samuelson condition. We need to deal with it from a different angle: the evolution theory embedded in a learning process. For the construction of our model we use tools from an optimization model (Hollander 1990) and from an evolutionary one (Rege 2004).

The Myerson concept of viscocity allows us to define a coordination game. The Homans concept of social interaction, traduced into the subjective rate of social approval, leads us to measure the utility for one persone to be part of a society where every one contributes to the public good. The status effect of Hollander, σ, shows the importance of the negative externality that arises from the average contribution in a group.

We model a society formed of three groups, two rich and one poor, in order to allow the consideration of free rider behavior. We derive the dyamic replicator equation for the contributor's proportion for each rich type group. The phase diagram of this system of non linear differential equations shows us four possible stable Nash equilibriums. We want to find the conditions under wich any initial contribution point could start a trajectory going to the equilibrium were everybody contributes.

First, we found that the reduction of σ produces the phase lines to open, tend to be parallels to the axes and move to the origin. The limit case, when that parameter equals zero, correspond to Andreoni´s warm glow. Second, our model continues to be a coordination game for a viscocity value up to 2/3, higher than in Rege where the limit is ½. That limit fraction even increases as the status effect diminishes. For a 0.5 status effect the game is still a coordination one up to 4/5. Third, the effects over the phase diagram of a non equal proportion of rich in each group allow us to suggest the selection of smaller groups to stimulate their contribution with public policies.

Fourth and most important, government intervention. The government provision of the public good has a similar effect than an increase in its price. The crowding out of voluntary contibutions is complete and quicker than in the BB model. The important discovery of Rege refers to keeping up the equilibrium (1, 1), when, due to the reduction of price, the system has moved to a point in the T zone of the Base Diagram (figure 8), from which even without the subsidy the same equilibrium can be achieved. This means that a government subsidy, as well as the reduction of the price indirectly obtained, are circumstantial tools, which can be reverted without modifying the final result.

We find that the tax has to be taken into account in the equation defining the rate of social approval, in the utility functions and in the function of social approval production. When we explicitly introduce the tax in our functions the crowding out is even stronger and depends also on the utility elasticity of the public good function.

Fifth, the observer or controller of each individual behavior can be of two kinds: group or global controller. When we use the last one it reduces the number of possible equilibriums, as it makes the assimmetric ones unstable.

Sixth, the model works under the effect of three externalities which could differ in sign value. Then, the equilibrium would not be Pareto efficient unless they compensate each other. For this reason the equilibrium where everybody contributes can not be characterize as a Pareto optimum. Nevetheless, it is a Pareto improvement compared with the non contribution situation.

Seventh, the social approval is a relational good produced as a by product when the status effect is less than one. The social approval utility adds to the traditional goods (private and public) utility. In this way an evolutionary equilibrium could be a Pareto improvement compared to the one that complies with the Samuelson condition. In other words, the neo classical reasoning could be inverted. Instead of using the theory of the public good to demonstrate that the private sector has to be completely crowded out to achieve a Pareto optimal of the Samuelson type (SPO), the same theory could show that the government has to be totally crowded out to allow the private sector to generate the social interaction that produces a new good and in this way the result will constitute a communitarian Pareto optimal superior to SPO.

Among the possible future lines of investigation we consider studying the relation between the theory of the public choice and the spontaneous redistribution, to explore answers to the following question: if the people behave guided by an utility function that values the consumption of social approval, why no political party chooses frankly the proposal of eliminating the distributive function from the public budget, reducing taxes corresponding to this function and allow the voluntary action? This investigation should follow the lines traced by the work of Mancur Olson on the logic of the collective action, and has to study the different aspects of the theory of the public choice, in particular the theory of the burocracy, by William Niskanen. Another way of studying this subject would be to think if the redistribution function implemented in a coercitive manner by a government can be a spontaneous or endogenous result produced by the society, as it is suggested by Gosta-Esping Andersen (1990).

REFERENCES

- Andreoni, J., 2006. *Philanthopy,* Cap. 18 in: *Handbook of the Economics of Giving, Altruism and Reciprocity*, Vol. 2, Editores: S-C. Kolm and J.M Ythier. North-Holland.
- Becker, G.S., 1974. A Theory of social interactions. *Journal of Political Economy*. 82:1063-93.
- Beito, D.T., Gordon P., and Tabarrok, A. 2002. *The Voluntary City. Choice, Community, and Civil Society*, Ann Arbor: The University of Michigan Press.
- Bergstrom, T. C., Blume L. E., and Varian, H.R. 1986. On the private provision of public goods. *Journal of Public Economics*. 29:25-49.
- Binmore, K., 1994. *Teoría de Juegos*, Madrid: McGraw-Hill.
- Borgers, T. and Sarin, R. 1997. Learning through reinforcement and replicator dynamics. *Journal of Economic Theory*. 77: pp. 1-14
- Bush, R.R. y Mosteller 1955. *Stochastic Models for Learning*. New York, Wiley.
- Chen, Y and Tang, F-F 1998. Learning and incentive-compatible mechanisms for public goods provision. *Journal of Political Economy*, 106: 633-662.
- Coleman, J. 1990. *Foundations of Social Theory*. Cambridge, Mass. Harvard University Press.
- Cross, J.G. 1973. A Stochastic learning model of economic behavior. *The Quarterly Journal of Economics*, Vol. 87, No. 2.: pp. 239-266.
- Danziger, L. and Schnytzer, A. 1991. Implementing the Lindahl voluntary-exchange mechanism. *European Journal of Political Economy*. 7:55-64.

- De la Fuente, A. 2000. *Mathematical Methods and Models forEconomists*, Cambridge: Cambridge University Press.
- Guttman, J.M. 1978. Understanding collective action: matching behavior. *The American Economic Review*, Vol. 68, Nº2: pp. 251-255.
- Guttman, J.M. 1987. A non-Cournot model of voluntary collective action. *Economica*, New Series, Vol. 54, No. 213: 1-19.
- Holländer, H. 1990. A Social exchange approach to voluntary cooperation. *The American Economic Review*, Vol 80, Nº 5: 1157-1167.
- Homans, G.C. 1961. *Social Behavior: Its Elementary Forms*. New York: Harcourt, Brace&World, Inc.
- Hurwicz, L. 1972. On informationally decentralized systems. In C. B. McGuire and R. Radner, eds., Decision and Organization: a Volume in Honor of Jacob Marshak. North-Holland. pp. 297-336.
- Josef Hofbauer, J.,Oechssler, J. and Riedel, F. 2008. Brown–von Neumann–Nash dynamics: the continuous strategy case. In www.sciencedirect.com.
- Morgan, J. 2000. Financing public goods by means of lotteries. *The Review of Economic Studies*, 67: 761-784.
- Myerson, R.B. 2007 Fundamental theory of institutions: a lecture in honor of Leo Hurwicz. See http://home.uchicago.edu/~rmyerson/hurwicz.pdf.
- Myerson, R.B., Pollock, G.B, y Swinkels J.M. (1991). Viscous population equilibria. *Games and Economic Behavior*. 3:101-109.
- Olson, M. 1965. *The Logic of Collective Action. Public Goods and the Theory of Groups*. Cambridge, Mass: Harvard University Press.
- Rege, M. 2004. Social norms and private provision of public goods. *Journal of Public Economic Theory*, 6 (1):65-77.
- Rubio de Urquía, R., 2003. *Estructura fundamental de la explicación de procesos de autoorganización mediante modelos teórico-económicos*", Cap. 1 in Rubio de Urquía, R., Vázquez, F. Y Muñoz Pérez, F., *Procesos de Autoorganización*, Instituto de Investigaciones Económicas y Sociales "Francisco de Vitoria", Madrid: Unión Editorial.

- Sacco, P.L., Vanin, P. y Zamagni, S. 2006. *The Economics of Human Relationships,* Cap. 9, in: *Handbook of the Economics of Living, Altruism and Reciprocity*, Vol. 1, Editores: S-C. Kolm y J.M Ythier. North-Holland.
- Samuelson, P.A., 1954. The Pure theory of public expenditure. *Review of Economics and Statistics*, 387-389.
- Stigler G.J. and Becker G.S. 1977. De Gustibus non est disputandum. The *American Economic Review*, Vol.67, Nº2. pp. 76-90.
- Sugden, R. 1982. On the Economics of Philanthropy. *The Economic Journal*, Vol. 92, Nº 366: 341-350.
- Uhlaner, C.J. 1989. Relational goods and participations: incorporting sociability into a theory of racional action. *Public Choice*, 62, (3): 253-85.
- Vara Crespo, O. and Rodríguez García-Brazales, A. 2003. El Concepto de autoorganización en Carl Menger y en Friedrich A. Hayek Cap. 6 in Rubio de Urquía, R., Vázquez, F. Y Muñoz Pérez, F., *Procesos de Autoorganización*, Instituto de Investigaciones Económicas y Sociales "Francisco de Vitoria", Madrid: Unión Editorial.