Segregation, Informativeness and Lorenz Domination *

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Abstract

It is possible to partially order cities with two ethnic groups according to the Lorenz criterion. Similarly, it is possible to partially order cities according to the informativeness of neighborhoods about the ethnic groups of its inhabitants. We show the equivalence of these two orders for the two-group case.

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1 Introduction

Sociologists and economists have been long interested in the problem of how to measure segregation. While early studies restricted attention to segregation between two groups, i.e., blacks and whites, or men and women, later ones developed measures for multigroup cases. One of the difficulties of measuring segregation is that it is not clear what segregation actually means. Massey and Denton [11] identified five dimensions of segregation, which they call evenness, exposure, concentration, centralization and clustering. Each of these dimensions captures some aspect of the idea of segregation. For instance, evenness refers to the similarity among distributions of members of different groups across locations. The more similar these distributions are, the less is the degree of segregation. Exposure, on the other hand, refers to the degree of contact among members of the different groups. Within the two-group case, concentration refers to the relative amount of space occupied by the minority group, and centralization refers to the tendency of the minority group to be located in the center of an urban area. Finally, clustering refers to the tendency of the areas populated by the minority group to be clustered together.

Not only do the various dimensions of segregation refer to different concepts, but their concrete measurement requires different kinds of data. Indeed, while measures of concentration, centralization and clustering require some sort of geographical data, evenness and exposure require information only on the numbers of members of the different groups located in the existing locations. Furthermore, for the measurement of the evenness dimension of segregation only data on the different relative distributions of individuals across locations is required. Although the number of segregation indices is very large, it is safe to say that most of the segregation literature, both theoretical and empirical, focuses on the dimension of evenness dimension, as does the present

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1 See Reardon and Firebaugh [12] for an enumeration and analysis of various multigroup segregation measures. For the two-group case, Massey and Denton [11] provide a comprehensive survey.
The basic model of segregation measuring consists of a list of locations containing different numbers of members of various groups. Papers that focus on residential racial segregation refer to the locations as neighborhoods, and to the groups as ethnic groups. Papers dealing with occupational gender segregation usually use occupations as locations and classify the groups by gender. We will use the language of racial residential segregation, and refer to the list of neighborhoods as cities.

For the two-group case, the literature on segregation borrowed the device of the Lorenz curve from the income inequality literature and applied it to partially order cities. A segregation curve in the context of segregation is the analogue of the Lorenz curve in the context of income inequality. Indeed, recall that for each fraction $p$, the Lorenz curve depicts the proportion of the total income that is owned by the poorest proportion $p$ of the total population. A segregation curve is essentially a Lorenz curve where one group, say blacks, is treated as a population, and the other group, say whites, is treated as income. With this convention, the lower the proportion of whites that live in a neighborhood, the "poorer" is a black individual living there. Thus, for each fraction $p$, a segregation curve describes, the proportion of the total number of whites that share their neighborhoods with the "poorest" proportion $p$ of blacks; in other words, with the proportion $p$ of blacks who live in the neighborhoods with the lowest proportions of whites. Segregation curves appear in the literature as early as in Duncan and Duncan [4].

For each city with two groups one can build its associated segregation curve. The early literature on segregation took advantage of these curves to partially order cities. Specifically, given two cities, their corresponding segregation curves may or may not cross. If they do not cross then the city whose segregation curve lies below the other ones is considered the more segregated one according to the Lorenz criterion.

Following Frankel and Volij [5], one can borrow, this time from the literature on the value of information, another device in order to partially order cities, even for
the multigroup case. Given a city, the location of a randomly selected individual can be seen as a signal of the ethnic group he belongs to. Learning the neighborhood a randomly chosen individual provides some information about his ethnic group. In that sense, the collection of distributions of the various ethnic groups across locations can be seen as an experiment in the sense of Blackwell [1], one in which locations play the role of signals and ethnic groups play the role of states of nature. We can then borrow Blackwell’s partial order of experiments according to their informativeness and apply it to partially order cities. A city whose locations are more informative than another city’s locations will be considered more segregated than the latter.

In this paper we show that, restricted to cities with only two groups, the partial order derived from the segregation curves coincides with the partial order derived from the informativeness of the city’s associated experiment. In that sense, not only is the latter partial order applicable to the multi-group case, but it is also a generalization of the standard order based on segregation curves.

In order to prove this equivalence, we use the axiomatic method. Specifically, we first show that both the partial order induced by the cities’ Lorenz curves and the one induced by the informativeness of the cities’ experiments satisfy four basic axioms. We then show that any partial order that satisfies these four axioms must be an extension of each of the above mentioned orders. Thus these two orders are extensions of each other, which means that they are one and the same order.

The fact that any partial order that satisfies the four axioms must be consistent with the partial order derived from the segregation curves was stated without proof by James and Taeuber [10]. A proof of this result for the case where all locations contain the same number of members of one group (e.g., all occupations contain the same number of women), was proved by Hutchens [7]. Our proof generalizes this result for the case where locations may have any number of members of any group, including zero. The fact that any partial order that satisfies the four axioms must be consistent with the partial order associated with the informativeness of the cities’ experiment
was noted by Frankel and Volij [5] for the special case where all cities have the same ethnic distribution. We prove this result for the case of all cities, independently of their ethnic distribution.

2 Notation

Let $G = \{1, \ldots, |G|\}$ be set of ethnic groups. This set will remain fixed for the whole analysis. A *neighborhood* $n$ is characterized by its racial composition, which is a vector $(T^g_n)_{g \in G}$ of non-negative numbers, at least one of which is positive. The number $T^g_n$ is the number of residents of $n$ that belong to ethnic group $g$. A *city* is a finite collection of neighborhoods such that, for each ethnic group $g$, at least one neighborhood has a positive number of residents of that group. Formally, a city is a pair $\langle N, ((T^g_n)_{n \in N})_{n \in N} \rangle$ such that $N$ is the set of neighborhoods, for each ethnic group $g \in G$, $\sum_{n \in N} T^g_n > 0$, and for each $n \in N$, $\sum_{g \in G} T^g_n > 0$.

Given a city $X = \langle N, ((T^g_n)_{n \in N})_{n \in N} \rangle$, we denote by $T^g(X)$ the total number of residents of group $g$: $T^g(X) = \sum_{n \in N} T^g_n$. When it is clear to which city we are referring, we will write simply $T^g$. We will denote by $t^g_n$ the proportion of individuals of ethnic group $g$ that reside in neighborhood $n$. Formally, $t^g_n = T^g_n / T^g$. Similarly, $p^g_n = T^g_n / \sum_{g \in G} T^g_n$ is the proportion of residents of $n$ that belong to ethnic group $g$. The ethnic distribution of a neighborhood $n$ is given by $(p^g_n)_{g \in G} = (T^g_n)_{g \in G} / \sum_{g \in G} T^g_n$, and the ethnic distribution of a city $X$ is given by $(T^g)_{g \in G} / \sum_{g \in G} T^g$.

3 The Blackwell partial order

Given a set of states of nature $\Omega = \{1, \ldots, K\}$, an experiment provides information about the realized state. Specifically, when the realized state is $k$, the experiment issues a signal with a distribution that depends on $k$. Experiments on $\Omega$ can be described by Markov matrices, whose rows represent the possible states of nature, and
whose columns represent the possible signals. The entry $m_{ks}$ is the probability that the signal $s$ is sent when the realized state is $k$. Conversely, every Markov matrix with $K$ rows can be interpreted as an experiment for $\Omega$. Blackwell [1] (Sherman [13], Stein [14]) partially ordered experiments according to their informativeness, and showed that this partial order has a convenient description in terms of the corresponding matrices.

In this section, we will make use of Blackwell’s informativeness order on experiments in order to define a (segregation) partial order on cities. The idea is to consider a city as an experiment where neighborhoods play the role of signals and ethnic groups the role of states of nature, and say that city $X$ is more segregated than city $Y$ if the neighborhoods of $X$ are more informative about the ethnic group of its residents than the neighborhoods of $Y$.

Let $X = \langle N, ((T_{ng})_{g\in G})_{n\in N} \rangle$ be a city. Let $\phi : \{1, 2, \ldots, |N|\} \rightarrow N$ be an ordering of the neighborhoods. The experiment matrix of $X$ with respect to $\phi$ is the $|G| \times |N|$ matrix

$$M(X, \phi) = (m_{ij})$$

where $m_{ij} = t_{\phi(j)}^i$ is the proportion of individuals of group $i$ that reside in neighborhood $\phi(j)$. Note that $M(X, \phi)$ is a Markov matrix. It represents an experiment in the sense of Blackwell. If we interpret the set of ethnic groups as the set of states of nature and the set of neighborhoods as the set of signals, $m_{ij}$ is the probability that the state is $i$ conditional on having received signal $\phi(j)$. It is the probability that a randomly chosen individual belongs to ethnic group $i$ given that he resides in neighborhood $\phi(j)$.

Let $\mathcal{M}$ be the set of Markov matrices with $|G|$ rows. These matrices can be partially ordered according to their informativeness (Blackwell 1953). Given two matrices $A_{|G| \times |N_A|}, B_{|G| \times |N_B|} \in \mathcal{M}$, we say that $A$ is at least as informative as $B$ if
there is an \( |N_A| \times |N_B| \) Markov matrix \( M \) such that\
\[
B = A \cdot M.
\]

If \( A \) is at least as informative as \( B \), it will remain so even after we permute each of the matrices columns in any arbitrary way. Indeed, let \( P_B \) be a \( |N_B| \times |N_B| \) permutation matrix and let \( P_A \) be a \( |N_A| \times |N_A| \) permutation matrix. If\
\[
B = A \cdot M
\]
then\
\[
B \cdot P_B = A \cdot P_A \cdot P'_A \cdot M \cdot P_B
\]

Since \( P'_A \cdot M \cdot P_B \) is a Markov matrix, we conclude that if \( A \) is at least as informative as \( B \) then \( A \cdot P_A \) is at least as informative as \( B \cdot P_B \).

We now define a partial order on cities based on the informativeness of their respective experiment matrices.

**Definition 1** Let \( X = \langle N_X, ((T^n_g)_{g \in G})_{n \in N_X} \rangle \) and \( Y = \langle N_Y, ((T^n_g)_{g \in G})_{n \in N_Y} \rangle \) be two cities. We say that \( X \) is at least as segregated as \( Y \) according to Blackwell’s criterion, denoted \( X \succ_I Y \), if \( M(X, \phi) \) is at least as informative as \( M(Y, \psi) \) for some orderings \( \phi : \{1, 2, \ldots, |N_X|\} \rightarrow N_X \) and \( \psi : \{1, 2, \ldots, |N_Y|\} \rightarrow N_Y \) of the neighborhoods of \( X \) and \( Y \), respectively.

Note that segregation according to Blackwell’s criterion is well-defined since the informativeness relation on \( M \), is invariant to permutations of columns. Since for most of the analysis the particular ordering of neighborhoods \( \phi \) that is chosen is not important as long as it remains fixed, in what follows we will use, with some abuse of notation, \( M(X) \) instead \( M(X, \phi) \).
3.1 Properties of the Blackwell partial order

Let $\mathcal{C}$ be the set of all cities. A segregation order is a partial order on $\mathcal{C}$. For any $X$ and $Y \in \mathcal{C}$, $X \succ Y$ means that $X$ is as least as segregated as $Y$ according to $\succ$.

Blackwell’s relation $\succ_{I}$ that we have defined above is an example of a segregation order. We will now inquire into the properties that this particular segregation order satisfies.

We say that two cities, $X = \langle N_X, ((T^g_n)_{g \in G})_{n \in N_X} \rangle$ and $Y = \langle N_Y, ((T^g_n)_{g \in G})_{n \in N_Y} \rangle$, are equivalent if there is a one to one mapping $\varphi : N_X \rightarrow N_Y$ such that for all $n \in N_X$, $(T^g_n)_{g \in G} = (T^{\varphi(n)}_g)_{g \in G}$.

Equivalent cities differ only in the names of their neighborhoods. It is clear that two equivalent cities have the same experiment matrices, up to permutation of columns. Therefore, Blackwell’s order satisfies the following axiom.

**Anonymity (ANON)** A segregation order $\succ$ satisfies anonymity if for any two equivalent cities $X$ and $Y$ we have $X \sim Y$.

Consider the city $X = \langle N, ((T^g_n)_{g \in G})_{n \in N} \rangle$ and the city $Y = \langle N, ((\alpha g T^g_n)_{g \in G})_{n \in N} \rangle$ that is obtained by multiplying the number of group $g$ individuals by $\alpha g > 0$, for $g \in G$. Since both cities have the same proportions $t^g_n$, we have that $M(X, \phi) = M(Y, \phi)$ for any ordering $\phi$ of $N$. Therefore, Blackwell’s order satisfies the following axiom.

**Composition Invariance (CI)** Let $X = \langle N, ((T^g_n)_{g \in G})_{n \in N} \rangle$ be a city and let $Y = \langle N, ((\alpha g T^g_n)_{g \in G})_{n \in N} \rangle$ be the city that is obtained by multiplying the number of agents of a group $g$, for $g \in G$, by the same nonzero factor $\alpha g > 0$, in all neighborhoods. A segregation order $\succ$ satisfies composition invariance if for any such cities we have $Y \sim X$.

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$^2$Given $\succ$, the associated relations $\succ$ and $\sim$ are defined as usual. $X \succ Y \iff X \succ Y$ and not $Y \succ X$, and $X \sim Y \iff X \succ Y$ and $Y \succ X$. 

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Composition invariance requires that only the relative distribution of members of a given ethnic group across neighborhoods affects segregation. In particular, the city’s ethnic distribution does not affect segregation.

Let \( X = \langle N, ((T^g_n)_{g \in G})_{n \in N} \rangle \) be a city and consider the city \( Y \) that is obtained from \( X \) by splitting a particular neighborhood \( (T^g_n)_{g \in G} \) into 2 neighborhoods, \( n_1 \) and \( n_2 \) with the same ethnic distribution, namely, \((T^g_n)_{g \in G} = (\alpha T^g_n)_{g \in G} \) and \((T^g_n)_{g \in G} = ((1 - \alpha) T^g_n)_{g \in G} \) for some \( \alpha \in (0, 1) \). Then,
\[
M(Y) = M(X) \cdot I(n, \alpha)
\]
(1)
where \( I(n, \alpha) \) is the (Markov) matrix that is obtained from a \(|N| \times |N|\) identity matrix by splitting the column that corresponds to neighborhood \( n \) into two columns, according to the proportions \( \alpha \) and \( 1 - \alpha \). Furthermore,
\[
M(X) = M(Y) \cdot I(n)
\]
(2)
where \( I(n) \) is the (Markov) matrix that is obtained from a \(|N + 1| \times |N + 1|\) identity matrix by merging the two columns that correspond to \( n_1 \) and \( n_2 \) into one. Equations (1) and (2) imply that \( M(X) \) and \( M(Y) \) are equally informative, and therefore \( X \sim Y \). Consequently, Blackwell’s order satisfies the following axiom.

**Organizational Equivalence (OE)** Let \( X \in \mathcal{C} \) be a city and let \( (T^g_n)_{g \in G} \) be one of its neighborhoods. Let \( Y \) be the city that results from dividing \( (T^g_n)_{g \in G} \) into 2 neighborhoods, \( (T^g_{n_1})_{g \in G} \) and \( (T^g_{n_2})_{g \in G} \) with the same ethnic distribution. Namely, \( (T^g_{n_1})_{g \in G} = (\alpha T^g_n)_{g \in G} \) and \( (T^g_{n_2})_{g \in G} = ((1 - \alpha) T^g_n)_{g \in G} \) for some \( \alpha \in (0, 1) \).

A segregation order \( \succeq \) satisfies organizational equivalence if for any such cities we have \( Y \sim X \).

Let \( X = \langle N, ((T^g_n)_{g \in G})_{n \in N} \rangle \) be a city and consider the city \( Y \) that is obtained from \( X \) by splitting a particular neighborhood \( (T^g_n)_{g \in G} \) into 2 neighborhoods, \( n_1 \) and \( n_2 \), but with different ethnic distributions. Then
\[
M(X) = M(Y) \cdot I(n)
\]
where, as before, \( I(n) \) is the \(|N + 1| \times |N| \) (Markov) matrix that is obtained from a \(|N + 1| \times |N + 1| \) identity matrix by merging the two columns that correspond to \( n_1 \) and \( n_2 \) into one. Therefore, \( Y \succ_I X \).

On the other hand, as the following lemma states, there is no \(|N| \times |N + 1| \) Markov matrix \( M \) such that \( M(Y) = M(X) \cdot M \). Hence \( Y \succ_I X \).

**Lemma 1** Let \( X \) be an \( n \times m \) Markov matrix and let \( Y \) be an \( n \times (m + 1) \) Markov matrix that is obtained from \( X \) by splitting one of \( X \)'s columns into two, not in a proportional way. Then, there is no Markov matrix \( M \) such that \( Y = X \cdot M \).

**Proof.** See appendix. \( \square \)

Therefore, Blackwell’s order satisfies the following axiom.

**Neighborhood Division Property (NDP)** Let \( X \in \mathcal{C} \) be a city and let \( (T^g_{n})_{g \in G} \) be a neighborhood of \( X \). Let \( Y \) be the city that results from dividing \( (T^g_{n})_{g \in G} \) into 2 neighborhoods, \( (T^g_{n_1})_{g \in G} \) and \( (T^g_{n_2})_{g \in G} \) with different ethnic distributions. Namely, \( (T^g_{n_1})_{g \in G} \neq (\alpha T^g_{n})_{g \in G} \) for any \( \alpha \in [0,1] \). A segregation order \( \succ \) satisfies the neighborhood division property if for any such cities we have \( Y \succ X \).

We summarize the above observations in the following Proposition.

**Proposition 1** The Blackwell segregation order \( \succ_I \) satisfies ANON, CI, OE, and NDP.

We can now state our first result.

**Proposition 2** Let \( \succ \) be a segregation order on \( \mathcal{C} \). It satisfies ANON, CI, OE and NDP if and only if for all two cities \( X, Y \in \mathcal{C} \),

\[
Y \succ_I X \implies Y \succ X \tag{3}
\]

\[
Y \sim_I X \implies Y \sim X \tag{4}
\]
Proposition 2 states that all segregation orders that satisfies ANON, CI, OE and NDP are consistent with Blackwell’s order. Whenever Blackwell’s order ranks two cities, any segregation order that satisfies the above four axioms must rank them in the same way.

Proof. Let \( \succ \) be a segregation order that satisfies (3) and (4). Let \( X \) and \( Y \) be two equivalent cities. Then, by Proposition 1, \( X \sim_I Y \). By (4), \( X \sim Y \). Therefore, \( \succ \) satisfies ANON.

Let \( X = \langle N, ((T_n^g)_{g \in G})_{n \in N} \rangle \) be a city and let \( Y = \langle N, ((\alpha_g T_n^g)_{g \in G})_{n \in N} \rangle \) be the city that is obtained by multiplying the number of agents of a group \( g \) by the same nonzero factor \( \alpha_g > 0 \), for \( g \in G \) in all neighborhoods. Then, by Proposition 1, \( Y \sim_I X \). By (4), \( Y \sim X \). Therefore, \( \succ \) satisfies CI.

Let \( X \in \mathcal{C} \) be a city and let \( (T_n^g)_{g \in G} \) be a neighborhood of \( X \). Let \( Y \) be the city that results from dividing \((T_n^g)_{g \in G}\) into 2 neighborhoods, \((T_{n_1}^g)_{g \in G}\) and \((T_{n_2}^g)_{g \in G}\) with the same ethnic distribution. Then, by Proposition 1, \( Y \sim_I X \). By (4), \( Y \sim X \). Therefore, \( \succ \) satisfies OE.

Let \( X \in \mathcal{C} \) be a city and let \( (T_n^g)_{g \in G} \) be a neighborhood of \( X \). Let \( Y \) be the city that results from dividing \((T_n^g)_{g \in G}\) into 2 neighborhoods, \((T_{n_1}^g)_{g \in G}\) and \((T_{n_2}^g)_{g \in G}\) with different ethnic distributions. Then, by Proposition 1, \( Y \succ_I X \). By (3), \( Y \succ X \). Therefore, \( \succ \) satisfies NDP.

Let now \( \succ \) be a segregation order that satisfies ANON, CI, OE and NDP. Let \( X = \langle N_X, ((T_n^g)_{g \in G})_{n \in N_Y} \rangle \) and \( Y = \langle N_Y, ((T_n^g)_{g \in G})_{n \in N_X} \rangle \) be two cities such that \( Y \succ_I X \). We need to show that (3) and (4) hold. Since \( Y \succ_I X \), there is a Markov matrix \( M = \left( (m_{ij})_{i=1}^{N_Y} \right)_{j=1}^{N_X} \) such that

\[
M(X) = M(Y) \cdot M.
\]

But \( M \) can be written as a product of two matrices

\[
M = \beta \cdot \gamma
\]
where

\[
\beta = \begin{pmatrix}
m_{11} & \cdots & m_{1|N_X|} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & m_{21} & \cdots & m_{2|N_X|} & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & m_{|N_Y|1} & \cdots & m_{|N_Y||N_X|}
\end{pmatrix}
\]

and

\[
\gamma = \begin{pmatrix}
I_{(|N_X| \times |N_X|)} \\
I_{(|N_X| \times |N_X|)} \\
\vdots \\
I_{(|N_X| \times |N_X|)}
\end{pmatrix}
\]

Therefore,

\[M(X) = M(Y) \cdot \beta \cdot \gamma \tag{5}\]

Note that \(M(Y) \cdot \beta\) is the matrix that is obtained from \(M(Y)\) by splitting its \(i\)th column, \(i = 1, \ldots, |N_Y|\), into \(|N_X|\) columns, in the proportions \(m_{ij}, j = 1, \ldots, |N_X|\).

Also note that \(M(Y) \cdot \beta \cdot \gamma\) is obtained from \(M(Y) \cdot \beta\) by merging the \(i\)th (mod\((|N_X|)|\)) columns together. Therefore, (5) says that \(M(X)\) is obtained from \(M(Y)\) by a successively splitting columns proportionally and then merging columns. Alternatively, the matrix \(M(Y)\) is obtained from \(M(X)\) by splitting its columns, not necessarily in a proportional way, and then merging some proportional columns.

By OE and NDP,

\[Y \succ X\]

For the same reason, if \(X \succ Y\) we would have \(X \succ X\). Consequently, if \(Y \sim X\) then \(Y \sim X\). And if \(Y \succ X\), then the matrix \(M(Y)\) is obtained from \(M(X)\) by splitting its columns, not all in a proportional way, and then merging some proportional columns. Because if the split of columns were all proportional, we would have

\[M(Y) = M(X) \cdot \beta' \cdot \gamma'\]
for some splitting matrix $\beta'$ and merging matrix $\gamma'$, which would imply that $X \succ_1 Y$, contradicting $Y \succ_1 X$. Therefore, $Y$ is obtained from $X$ by splitting some neighborhoods into smaller neighborhoods with different ethnic distributions and then merging some neighborhoods with the same ethnic distributions. By NDP and OE, $Y \succ X$.

4 Two groups: The Lorenz partial order

There is another partial order defined on the class of cities with only two groups. It is known as the Lorenz partial order and it is based on what is known as segregation curves. See Duncan and Duncan [4], James and Taeuber ([15], [10]), Hutchens [7].

Let $G = \{1, 2\}$ be a set of two ethnic groups. The set $G$ of ethnic groups will remain fixed throughout the analysis, and we denote by $C_2$ the set of cities with these two groups. For ease of exposition, we’ll refer to members of ethnic groups 1 and 2 as blacks and whites, respectively. Let $X = \langle N, (B_n, W_n)_{n \in N} \rangle \in C_2$ be a city. For each neighborhood $n \in N$, denote by $p_n$ the proportion of whites in neighborhood $n$. That is, $p_n = W_n / (B_n + W_n)$. We will now build a segregation curve associated with the city $X$. Segregation curves, as experiment matrices in the $n$-group case analyzed in Section 3, will allow us to define a partial order on the set of two group cities. Segregation curves, analogously to experiment matrices, are objects that do not depend on the ethnic distribution of the cities. That is, city $X = \langle N, (B_n, W_n)_{n \in N} \rangle$ and city $\tilde{X} = \langle N, (B_n / B, W_n / W)_{n \in N} \rangle \equiv \langle N, (b_n, w_n)_{n \in N} \rangle$, which is obtained from $X$ by normalizing the groups’ populations so that each group is of size one, will have the same segregation curve. In order to build the segregation curve, let $\phi : \{1, 2, \ldots, |N|\} \to N$ be an ordering of the neighborhoods such that $i \leq j \Rightarrow p_{\phi(i)} \leq p_{\phi(j)}$. Namely, $\phi$ orders neighborhoods in a non-decreasing way according to their proportion whites. Note that

$$p_n \leq p_m \iff w_n / (b_n + w_n) \leq w_m / (b_m + w_m).$$

(6)
That is, ordering the neighborhoods in $\mathcal{N}$ in non-decreasing order of the proportion of whites in $X$ or in its normalized version $\widehat{X}$ results in the same order. Let $\beta_0 = \omega_0 = 0$, and for $i = 1, 2, \ldots, |\mathcal{N}|$, let $\beta_i = \beta_{i-1} + b_{\phi(i)}$ and $\omega_i = \omega_{i-1} + w_{\phi(i)}$. That is, $\beta_i$ is the proportion of blacks that reside in the $i$ neighborhoods with the lowest proportions of whites. Similarly, $\omega_i$ is the proportion of whites that reside in these same neighborhoods. The Lorenz segregation curve of $X$ is the graph that is obtained by plotting the points $(\beta_i, \omega_i)_{i=0}^{\mathcal{N}}$ and connecting the dots. Formally, it is the union of the line segments $\text{seg}[(\beta_{i-1}, \omega_{i-1}), (\beta_i, \omega_i)]$, $i = 1, 2, \ldots, |\mathcal{N}|$, where for any two points $x, y \in \mathbb{R}^2$, $\text{seg}[x, y] = \{(1 - \alpha)x + \alpha y : \alpha \in [0, 1]\}$. Note that the line segment that connects the points $(\beta_{i-1}, \omega_{i-1})$ and $(\beta_i, \omega_i)$ has a slope of $w_{\phi(i)}/b_{\phi(i)}$. Therefore, given (6), this slope is non-decreasing in $i$. Furthermore, the Lorenz curve is invariant to the choice of ordering $\phi$, as long as it satisfies $i \leq j \Rightarrow p_{\phi(i)} \leq p_{\phi(j)}$.

We can use the Lorenz curves to define a segregation order.

Definition 2 Let $X$ and $Y$ be two cities. We say that $Y$ is at least as segregated as $X$ according to the Lorenz criterion, denoted $Y \succeq_L X$, if the Lorenz curve of $Y$ is nowhere above the Lorenz curve of $X$.

4.1 Properties of the Lorenz partial order

We will now check which of the axioms defined in Section 3.1 are satisfied by the Lorenz order. Recall that two cities, $X = \langle N_X, ((T^g_n)_{g \in G})_{n \in N_X} \rangle$ and $Y = \langle N_Y, ((T^g_n)_{g \in G})_{n \in N_Y} \rangle$, are equivalent if there is a one to one mapping $\varphi : N_X \to N_Y$ such that for all $n \in N_X$, $(T^g_n)_{g \in G} = (T_{\varphi(n)}^g)_{g \in G}$.

It is clear that two equivalent cities have the same Lorenz curve. Therefore, the Lorenz segregation order satisfies Anonymity.

Consider now the city $X = \langle N, (B_n, W_n)_{n \in N} \rangle$ and the city $Y = \langle N, (\alpha_1 B_n, \alpha_2 W_n)_{n \in N} \rangle$ that is obtained by multiplying the number of $X$’s members of blacks and whites by $\alpha_1 > 0$ and $\alpha_2 > 0$, respectively. Since neighborhood $n$ in both cities contains the
same proportions $b_n$ of the total number of blacks and the same proportions $w_n$ of the total number of whites, $X$ and $Y$ have the same Lorenz curve. Therefore, the Lorenz order satisfies Composition Invariance.

Let now $X = \langle N, (B_n, W_n)_{n \in N} \rangle$ be a city and consider the city $Y$ that is obtained by splitting a particular neighborhood $n$ into 2 neighborhoods, $n_1$ and $n_2$ with the same ethnic distribution. Namely, $(B_{n_1}, W_{n_1}) = \alpha(B_n, W_n)$ and $(B_{n_2}, W_{n_2}) = (1 - \alpha)(B_n, W_n)$ for some $\alpha \in (0, 1)$. Then we have $b_n = b_{n_1} + b_{n_2}$, $w_n = w_{n_1} + w_{n_2}$, and $p_{n_1} = p_{n_2} = p_n$. Therefore, taking into account (6) both $X$ and $Y$ have the same Lorenz curve. As a result, the Lorenz order satisfies Organizational Equivalence.

In order to motivate the next axiom, consider a city $X = \langle N, (B_n, W_n)_{n \in N} \rangle$. Let $i, j \in N$ be two neighborhoods such that $0 < p_i < 1$ and $p_i \leq p_j$. That is, neighborhood $i$ contains both blacks and whites, but has proportionally less whites than neighborhood $j$.

Let $\varepsilon \in (0, W_i]$, and let $Y$ be the city that is obtained from $X$ by moving $\varepsilon$ whites from neighborhood $i$ to neighborhood $j$. That is $Y = \langle N, (B'_n, W'_n)_{n \in N} \rangle$ in which $(B'_i, W'_i) = (B_i, W_i - \varepsilon)$, $(B'_j, W'_j) = (B_j, W_j + \varepsilon)$, and $(B'_n, W'_n) = (B_n, W_n)$ for all $n \neq i, j$. Then we have that

$$w_n/b_n = w'_n/b'_n \text{ for all } n \neq i, j$$

$$w'_i/b'_i < w_i/b_i \leq w_j/b_j < w'_j/b'_j.$$ 

It can be checked that the Lorenz curve of $X$ lies nowhere below the Lorenz curve of $Y$ while it is not true that the Lorenz curve of $Y$ lies nowhere below the Lorenz curve of $X$. Therefore, the Lorenz order satisfies the following axiom.

**The Whites' Transfer Principle (WT)** For any city $X = \langle N, (B_n, W_n)_{n \in N} \rangle$, let $i, j \in N$ be two neighborhoods such that $B_i W_i > 0$ and

$$\frac{W_i}{B_i + W_i} \leq \frac{W_j}{B_j + W_j}$$

Let $\varepsilon \in (0, W_i]$, and let $Y$ be the city that is obtained from $X$ by moving $\varepsilon$
whites from neighborhood \( i \) to neighborhood \( j \). A segregation order \( \succeq \) on \( C_2 \) satisfies the whites’ transfer principle if for any such cities we have \( Y \succ X \).

Analogously, the Lorenz order satisfies the following axiom.

**The Blacks’ Transfer Principle (BT)** For any city \( X = (N, (B_n, W_n)_{n \in N}) \), let \( i, j \in N \) be two neighborhoods such that \( B_i W_i > 0 \) and

\[
\frac{B_i}{B_i + W_i} \leq \frac{B_j}{B_j + W_j}
\]

Let \( \varepsilon \in (0, B_i] \), and let \( Y \) be the city that is obtained from \( X \) by moving \( \varepsilon \) blacks from neighborhood \( i \) to neighborhood \( j \). A segregation order \( \succeq \) on \( C_2 \) satisfies the blacks’ transfer principle if for any such cities we have \( Y \succ X \).

We can summarize the two axioms in the following.

**The Transfer Principle (T)** A segregation order \( \succeq \) on \( C_2 \) satisfies the transfer principle if it satisfies both WT and BT.

Note the the transfer principle is the only axiom that introduces some symmetry between the ethnic groups. This symmetry, however, is very weak. Many non-symmetric segregation orders satisfy the transfer principle.

We summarize the above observations in the following Proposition.

**Proposition 3** Lorenz’s segregation order \( \succeq_L \) satisfies ANON, CI, OE, and T.

Frankel and Volij (2011) show that OE and T imply NDP. The next claim shows that, assuming OE, T and NDP are in fact equivalent.

**Claim 1** Let \( \succ \) be a segregation order on \( C_2 \) that satisfies OE. Then, it satisfies T if and only if it satisfies NDP.
Proof. Let \( \succ \) be a segregation order that satisfies OE and T. We will show that satisfies NDP as well. Let \( X \) be a city and let \( n \) be a neighborhood of \( X \). Let \( Y \) be the city that results from dividing \( n \) into two neighborhoods, \( n_1 \) and \( n_2 \). Assume that \( n_1 \) and \( n_2 \) don’t have the same ethnic distributions. Further assume without loss of generality that the proportion of blacks is higher in \( n_1 \) than in \( n_2 \): \( B_{n_1}W_{n_2} > B_{n_2}W_{n_1} \).

Neighborhood \( n \) in city \( X \) can be written \( (B_n,W_n) = (B_{n_1} + B_{n_2},W_{n_1} + W_{n_2}) \). Let \( \alpha = \frac{B_{n_1} + W_{n_1}}{B_n + W_n} \) and let \( X' \) be the city that results from \( X \) by splitting neighborhood \( n \) into the following two neighborhoods: \( n'_1 = \alpha (B_n,W_n) \) and \( n'_2 = (1 - \alpha) (B_n,W_n) \). By organizational equivalence, \( X \sim X' \). Since \( B_{n_1}W_{n_2} > B_{n_2}W_{n_1} \) we have

\[
B_{n_1} > \alpha B_n.
\]

Transfer \( B_{n_1} - \alpha B_n > 0 \) blacks from from \( n'_2 \) to \( n'_1 \). (Since \( B_{n_1} - \alpha B_n < (1 - \alpha)B_n \), this can be done.) Further transfer the same amount of white from \( n'_1 \) to \( n'_2 \). The city that results is \( Y \). By the transfer principle, this operation strictly raises segregation: \( Y \succ X' \sim X \), so by transitivity, \( Y \succ X \).

We will now show that NDP implies WT. The proof that it also implies BT is analogous and is left to the reader. Let now \( X = \langle N, (B_n, W_n)_{n \in N} \rangle \), and let \( i, j \in N \) be two neighborhoods such that \( B_iW_i > 0 \) and

\[
B_iW_j \geq B_jW_i
\]

Let \( \varepsilon \in (0, W_i] \), and let \( Y \) be the city that is obtained from \( X \) by moving \( \varepsilon \) whites from neighborhood \( i \) to neighborhood \( j \). That is \( Y = \langle N, (B'_n, W'_n)_{n \in N} \rangle \) in which \( (B'_i, W'_i) = (B_i, W_i - \varepsilon) \), \( (B'_j, W'_j) = (B_j, W_j + \varepsilon) \), and \( (B'_n, W'_n) = (B_n, W_n) \) for all \( n \neq i, j \). We need to show that \( Y \succ X \). If \( B_j = 0 \), then \( Y \) is the result of splitting neighborhood \( (B_i, W_i) \) into \( (B_i, W_i - \varepsilon) \) and \( (0, \varepsilon) \) and then merging \( (0, \varepsilon) \) with \( (0, W_j) \). By NDP, the splitting operation increases segregation, and by OE, the merging of two neighborhoods with the same proportion of whites leaves segregation unchanged. Therefore, \( Y \succ X \).
If $B_j > 0$, define the following values:

$$\alpha = \frac{W_j + \varepsilon}{B_j}$$
$$\beta = \frac{W_i - \varepsilon}{B_i}$$
$$\gamma = \frac{\varepsilon}{\alpha - \beta} = \frac{\varepsilon B_i B_j}{B_i (W_j + \varepsilon) - B_j (W_i - \varepsilon)}$$

Since $W_j B_i \geq W_i B_j$, $\gamma, \beta, \alpha > 0$. Split $(B_i, W_i)$ into $(B_i - \gamma, W_i - \alpha \gamma)$ and $(\gamma, \alpha \gamma)$. Similarly, split $(B_j, W_j)$ into $(B_j - \gamma, W_j - \beta \gamma)$ and $(\gamma, \beta \gamma)$. (This can be done because $\gamma < \min\{B_i, B_j\}$.) Indeed,

$$\gamma = \frac{\varepsilon B_i B_j}{B_i (W_j + \varepsilon) - B_j (W_i - \varepsilon)} < \frac{B_i B_j}{B_i + B_j}$$

Therefore,

$$\gamma (B_i + B_j) < B_i B_j$$

or,

$$B_j (\gamma - B_i) + \gamma B_i < 0$$

which implies that $\gamma < B_i$. A similar argument shows that $\gamma < B_j$.) By NDP the resulting city is more segregated. Now merge $(B_i - \gamma, W_i - \alpha \gamma)$ with $(\gamma, \beta \gamma)$ and also merge $(B_j - \gamma, W_j - \beta \gamma)$ with $(\gamma, \alpha \gamma)$. Since

$$\frac{W_i - \alpha \gamma}{B_i - \gamma} = \beta$$
$$\frac{W_j - \beta \gamma}{B_j - \gamma} = \alpha$$

by OE this merger does not affect segregation. The resulting pair of neighborhoods is

$$(B_i, W_i - \alpha \gamma + \beta \gamma) \text{ and } (B_j, W_j - \beta \gamma + \alpha \gamma)$$

which happen to be $(B_i, W_i - \varepsilon)$ and $(B_j, W_j + \varepsilon)$. ■

**Corollary 1** The Lorenz order is consistent with Blackwell’s order.
Proof. By Claim 1, the Lorenz order satisfies ANON, CI, OE and NDP. By Proposition 2, it is consistent with Blackwell’s ordering.

It turns out that Blackwell’s order is consistent with the Lorenz order as well. This follows from the following Proposition.

Proposition 4 Let $\succ$ be an order on $C_2$. It satisfies ANON, CI, OE and T if and only if for all two cities $X,Y \in C_2$,

$$Y \succ_L X \implies Y \succ X \quad (7)$$

$$Y \sim_L X \implies Y \sim X \quad (8)$$

Proof. Let $\succ$ be a segregation order that satisfies (7) and (8). Let $X$ be a city and let $Y$ be the city that is obtained from $X$ by means of any of the transformations related to the axioms of ANON, CI, or OE. Then, by Proposition 3, $X \sim_L Y$. By (8), $X \sim Y$. If $Y$ is obtained from $X$ after the transfer related to the transfer principle, then by Proposition 3 $Y \succ_L X$. By (7), $Y \succ X$.

Let now $\succ$ be a segregation order that satisfies ANON, CI, OE and T. Let $X = \langle N, (B_n, W_n)_{n \in N} \rangle$ and $Y = \langle N', (B'_n, W'_n)_{n \in N'} \rangle$ be two cities. We need to show (7) and (8).

Case 1: For each $n \in N$, $B_n > 0$, and for each $n \in N'$, $B'_n > 0$.

Since both $\succ$ and $\succ_L$ satisfy CI, we can assume without loss of generality that $\sum_{n \in N} B_n = \sum_{n \in N'} B'_n = \sum_{n \in N} W_n = \sum_{n \in N'} W'_n = 1$. Since both $\succ$ and $\succ_L$ satisfy OE, we can assume without loss of generality that $|N| = |N'|$. Since both $\succ$ and $\succ_L$ satisfy ANON, we can further assume that $N = N' = \{1, 2, \ldots I\}$ and that the neighborhoods are ordered in non-decreasing order of proportion of whites, i.e., $n < m \implies p_n \leq p_m$ and $p'_n \leq p'_m$. Lastly, since both $\succ$ and $\succ_L$ satisfy OE, we can assume without loss of generality that $B_n = B'_n$ for all $n \in \{1, 2, \ldots I\}$. (If the numbers of blacks and whites in each neighborhood were integers, or at least rationals, we could have assumed further that $B_n = B'_n = k$ for all $n \in \{1, 2, \ldots I\}$ for some integer $k$.)
Therefore, we can denote $X$ by $(b_n, w_n)_{n=1}^I$ and $Y$ by $(b_n, w'_n)_{n=1}^I$. If $Y \sim_L X$ then $w_n = w'_n$ as well for $n = 1, \ldots, I$. Consequently, $X = Y$ and therefore $Y \sim X$.

Assume therefore that $Y \succ_L X$. Then,

$$\sum_{n=1}^I w_n \geq \sum_{n=1}^I w'_n \quad i = 1, \ldots, I \quad (9)$$

$$\sum_{n=1}^I w_n = \sum_{n=1}^I w'_n$$

with strict inequality for some $i \in \{1, 2, \ldots, I - 1\}$.

Let $k$ be the largest index $n$ such that $w_n > w'_n$. Since (9) holds with strict inequality for some $i \in \{1, 2, \ldots, I - 1\}$, such $k$ exists.

We must have that for some index $n > k$, $w_n < w'_n$. For otherwise, by definition of $k$, we should have

$$\sum_{n=k+1}^I w_n = \sum_{n=k+1}^I w'_n$$

and therefore

$$\sum_{n=k}^I w_n > \sum_{n=k}^I w'_n$$

which implies

$$\sum_{n=1}^{k-1} w_n < \sum_{n=1}^{k-1} w'_n$$

contradicting (9). Consequently we can find the smallest index $n$ such that $n > k$ and $w_n < w'_n$. Call that index $j$. Then we have

$$w_k > w'_k$$

$$w_j < w'_j$$

Also, since $k < j$, we have

$$\frac{w_k}{b_k + w_k} \leq \frac{w_j}{b_j + w_j} \quad (10)$$

$$\frac{w_k}{b_k + w'_k} \leq \frac{w_j}{b_j + w'_j}$$
Let $\delta = \min \{ w_k - w_k', w_j - w_j \} > 0$, and let $Y^* = (b_n, w^*_n)_{n=1}^I$ be the city that is obtained from $Y$ by moving $\delta$ whites from neighborhood $j$ to neighborhood $k$. That is,

$$
\begin{align*}
  w^*_k &= w'_k + \delta \\
  w^*_j &= w'_j - \delta \\
  w^*_n &= w'_n \quad n \neq k, j
\end{align*}
$$

First, we want to check that $Y^*$’s neighborhoods are ordered in a non-decreasing proportion of whites. Namely, $n < m \Rightarrow p^*_n \leq p^*_m$. For this it is enough to show that

$$
  p^*_k \leq p^*_{k+1} \tag{11}
$$

and

$$
  p^*_{j-1} \leq p^*_j \tag{12}
$$

We have

$$
\begin{align*}
  p^*_k &= \frac{w^*_k}{b_k + w'_k} \\
   &= \frac{w'_k + \delta}{b_k + w'_k + \delta} \\
   &\leq \frac{w'_k + (w_k - w'_k)}{b_k + w'_k + (w_k - w'_k)} \\
   &= \frac{w_k}{b_k + w_k} \\
   &\leq \frac{w'_{k+1}}{b_{k+1} + w'_{k+1}} \\
   &\leq \frac{w'_{k+1}}{b_{k+1} + w'_{k+1}}
\end{align*}
$$

where the last inequality follows from the fact that, by definition of $k$, $w'_{k+1} \geq w_{k+1}$.

If $k + 1 < j$, then $\frac{w'_{k+1}}{b_{k+1} + w'_{k+1}} = p^*_{k+1}$ and the inequality (11) is shown.
Similarly,

\[
p_j^* = \frac{w_j^*}{b_j + w_j^*} = \frac{w'_j - \delta}{b_j + w'_j - \delta} \geq \frac{w'_j - (w'_j - w_j)}{b_j + w'_j - (w'_j - w_j)} = \frac{w_j}{b_j + w_j} \geq \frac{w_{j-1}}{b_{j-1} + w_{j-1}} \leq \frac{w'_{j-1}}{b_{j-1} + w'_{j-1}}
\]

(14)

where the last inequality follows from the fact that, by definition of \( j \), \( w_{j-1}' \leq w_j \). If \( k < j - 1 \), then \( \frac{w'_{j-1}}{b_{j-1} + w_{j-1}} = p_{j-1}^* \) and the inequality (12) is shown.

If \( k + 1 = j \), using (13), (14) and (10) we obtain

\[
p_k^* = \frac{w_k^*}{b_k + w_k^*} \leq \frac{w_k}{b_k + w_k} \leq \frac{w_j}{b_j + w_j} \leq \frac{w_j^*}{b_j + w_j^*} = p_j^*
\]

which shows both (11) and (12). That is, \( Y^* \) is ordered in non-decreasing order of proportion of whites, i.e., \( n < m \Rightarrow p_n^* \leq p_m^* \).

We now need to show that \( Y^* \succ_L Y^* \succ_L X \).

We now check that \( Y^* \succ_L X \).

\[
\sum_{n=1}^{i} w_n \geq \sum_{n=1}^{i} w'_n = \sum_{n=1}^{i} w_n^* \quad i = 1, \ldots, k - 1
\]

Since \( w_k = w'_k + (w_k - w'_k) \geq w'_k + \min \{ w_k - w'_k, w'_j - w_j \} = w_k^* \)

\[
\sum_{n=1}^{k} w_n = \sum_{n=1}^{k-1} w_n + w_k \geq \sum_{n=1}^{k-1} w_n^* + w_k^* = \sum_{n=1}^{k} w_n^*
\]

Since \( w'_n = w_n^* \) for \( n = k + 1, \ldots, j - 1 \),

\[
\sum_{n=1}^{i} w_n = \sum_{n=1}^{k} w_n + \sum_{n=k+1}^{i} w_n \geq \sum_{n=1}^{k} w_n^* + \sum_{n=k+1}^{i} w_n^* = \sum_{n=1}^{i} w_n^* \quad i = k + 1, \ldots, j - 1
\]

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Since $w_k' + w_j' = w_k^* + w_j^*$ and $w_n' = w_n^*$ for $n = j + 1, \ldots, I$,

$$\sum_{n=1}^{i} w_n \geq \sum_{n=1}^{i} w_n' = \sum_{n=1}^{i} w_n^* \quad i = j, \ldots, I$$

with equality for $i = I$.

Similarly, we now check that $Y \succ_L Y^*$.

$$\sum_{n=1}^{i} w_n^* = \sum_{n=1}^{i} w_n' \quad i = 1, \ldots, k - 1$$

$$\sum_{n=1}^{i} w_n^* = \sum_{n=1}^{i} w_n' + \delta > \sum_{n=1}^{i} w_n' \quad i = k, \ldots, j - 1$$

$$\sum_{n=1}^{i} w_n^* = \sum_{n=1}^{i} w_n' \quad i = j, \ldots, I$$

Consequently, $Y \succ_L X$.

Since either $w_k = w_k^*$ or $w_j = w_j^*$ (or both), the number of indices $n$ such that $w_n \neq w_n^*$ is lower than the number of indices $n$ such that $w_n \neq w_n'$. Therefore, it is possible to move from $Y$ to $X$ by means of a finite number of transfers of whites from neighborhoods with proportionally more whites to neighborhoods with proportionally less whites. By T, $Y \succ X$

Case 2: There is $n \in N$, and $n' \in N'$ such that $B_n = B_n' = 0$.

Since both $\succ$ and $\succ_L$ satisfy OE, we can assume without loss of generality that $|N| = |N'|$. By OE, we can assume without loss of generality that both in $X$ and in $Y$, there is only one neighborhood with no blacks. Since both $\succ$ and $\succ_L$ satisfy ANON, we can further assume that $N = N' = \{1, 2, \ldots, I\}$ and that the neighborhoods are ordered in non-decreasing order of proportion of whites, i.e., $n < m \Rightarrow p_n \leq p_m$ and $p_n' \leq p_m'$. Since both $\succ$ and $\succ_L$ satisfy CI, we can assume without loss of generality that $\sum_{n \in N} B_n = \sum_{n \in N'} B_n' = \sum_{n \in N} W_n = \sum_{n \in N'} W_n' = 1$. Lastly, since both $\succ$ and $\succ_L$ satisfy OE, we can assume without loss of generality that $B_n = B_n'$ for all $n \in \{1, 2, \ldots, I\}$. The rest of the proof is identical to the one in Case 1.

Case 3: For each $n \in N$, $B_n > 0$, and there is $n' \in N'$ with $B_n' = 0$. Since both $\succ$ and $\succ_L$ satisfy OE, we can assume without loss of generality that in $Y$, there is only
one neighborhood with no blacks. Furthermore, by OE we can assume without loss of generality that $|N| = |N' - 1|$. Since both $\succeq$ and $\succeq_L$ satisfy ANON, we can further assume that $N = \{1, 2, \ldots I\}$, $N' = \{1, 2, \ldots I + 1\}$ and that the neighborhoods are ordered in non-decreasing order of proportion of whites. Since both $\succeq$ and $\succeq_L$ satisfy CI, we can further assume that $N = f_1; 2; \ldots I$ and $N_0 = f_1; 2; \ldots I + 1$ and that the neighborhoods are ordered in non-decreasing order of proportion of whites. Since both $\succeq$ and $\succeq_L$ satisfy CI, we can assume without loss of generality that $P_{n^2 N_B n} = P_{n^2 N_0 B_0 n} = 1$. Lastly, since both $\succeq$ and $\succeq_L$ satisfy OE, we can assume without loss of generality that $B_n = B_0 n$ for all $n \in \{1, 2, \ldots I\}$.

Therefore, we can denote $X$ by $(b_n; w_{nI})_{n=1}^I$ and $Y$ by $(b_n; w'_{nI})_{n=1}^{I+1}$ (where $b_{I+1} = 0$ and $w'_{I+1} > 0$). In this case, $Y \sim_L X$ is impossible. Assume therefore that $Y \succ_L X$. Let $\varepsilon = b_I - w'_{I+1}$ and let $Y = (\overline{b}_n, w'_{nI})_{n=1}^{I+1}$ be the city that is obtained from $Y$ by relocating $\varepsilon$ blacks from neighborhood $I$ to neighborhood $I + 1$. That is, $\overline{b}_n = b_n$ for $n = 1, \ldots I - 1$, $(\overline{b}_I, w'_I) = (b_I - \varepsilon, w'_I)$ and $(\overline{b}_{I+1}, w'_{I+1}) = (\varepsilon, w'_{I+1})$. By T, $Y \succ Y$. Note that neighborhoods $I$ and $I + 1$ have the same proportion of whites. As a result, $Y$’s neighborhoods are ordered in a non-decreasing order of proportion of whites. Furthermore, it can be seen that $Y \succeq_L X$. By construction, $Y$ has no neighborhoods with 0 blacks. Therefore, by Case 1, $Y \succeq X$. By transitivity, $Y \succeq X$. ■

Corollary 2 The Blackwell and the Lorenz orders on $C_2$ are the same.

Proof. By Claim 1, the Lorenz order satisfies Anon, CI, OE and NDP. By Proposition 2, it is consistent with Blackwell’s ordering. By Claim 1, the Blackwell order satisfies Anon, CI, OE and T. By Proposition 4, it is consistent with the Lorenz ordering. ■

5 Appendix

Lemma 2 Let $X$ be an $n \times m$ Markov matrix and Let $Y$ be an $n \times (m + 1)$ Markov matrix that is obtained from $X$ by splitting one of $X$’s columns into two, not in a proportional way. Then, there is no Markov matrix $M$ such that $Y = XM$. 

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Proof. Let $X$ be an $n \times m$ Markov matrix and let $Y$ be an $n \times (m+1)$ Markov matrix that is obtained from $X$ by splitting one of $X$’s columns into two, not in a proportional way. Assume without loss of generality that $X$’s $m$th column is the one that is split. That is, for all rows $i = 1, \ldots, n$, $y_{im} + y_{i(m+1)} = x_{im}$. And for all rows $i = 1, \ldots, n$, and columns $j = 1, \ldots, m-1$, $y_{ij} = x_{ij}$.

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1(m-1)} & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{n(m-1)} & x_{nm} \end{pmatrix}$$

$$Y = \begin{pmatrix} x_{11} & \cdots & x_{1(m-1)} & y_{1m} & y_{1(m+1)} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{n(m-1)} & y_{nm} & y_{n(m+1)} \end{pmatrix}$$

Since column $m$ is split in a non-proportional way, we can find two rows $i$ and $i'$ such that $x_{im}, x_{i'm} \neq 0$. Without loss of generality assume that $i = 1$ and $i' = 2$.

Let $A = \{(\alpha, (1-\alpha), 0, \ldots, 0) \in \mathbb{R}^n : 0 \leq \alpha \leq 1\}$ be a set of decisions. A decision rule for $X$ is an $m \times n$ matrix $D = (d_{ji})$ where each of its rows belongs to $A$. (A decision rule for $X$ is a function that maps each signal $j = 1, \ldots, m$ to a decision $(d_{j1}, d_{j2}, 0, \ldots, 0) \in A$). Let $\mathcal{D}$ be the set all all decision rules for $X$. Given a decision rule $D \in \mathcal{D}$ for $X$, its associated expected outcome is

$$\text{diag}(XD) = \left(\sum_{j=1}^{m} x_{1j}d_{j1}, \sum_{j=1}^{m} x_{2j}d_{j2}, 0, \ldots, 0\right) \in \mathbb{R}^n$$

We denote by $\mathcal{B}(X, A)$ the set of expected outcomes associated with some decision rule for $X$. That is,

$$\mathcal{B}(X, A) = \{(a_1, a_2, 0, \ldots, 0) \in \mathbb{R}^n : (a_1, a_2, 0, \ldots, 0) = \text{diag}(XD) \text{ for some } D \in \mathcal{D}\}.$$ 

Similarly, A decision rule for $Y$ is an $(m+1) \times n$ matrix $D' = (d'_{ij})$ where each of its rows belongs to $A$. (A decision rule for $Y$ is a function that maps each signal $j = 1, \ldots, m+1$ to a decision $(d_{j1}, d_{j2}, 0, \ldots, 0) \in A$). Let $\mathcal{D}'$ be the set all all decision rules for $Y$. Given a decision rule $D' \in \mathcal{D}'$ for $Y$, its associated expected outcome is

$$\text{diag}(YD') = \left(\sum_{j=1}^{m+1} y_{1j}d'_{j1}, \sum_{j=1}^{m+1} y_{2j}d'_{j2}, 0, \ldots, 0\right) \in \mathbb{R}^n$$

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We denote by $\mathcal{B}(Y, A)$ the set of expected outcomes associated with some decision rule for $Y$. That is,

$$\mathcal{B}(Y, A) = \{(a_1, a_2, 0, \ldots, 0) \in \mathbb{R}^n : (a_1, a_2, 0, \ldots, 0) = \text{diag}(YD') \text{ for some } D' \in \mathcal{D}' \}.$$

We will show that $\mathcal{B}(Y, A) \nsubseteq \mathcal{B}(X, A)$. That is, that $X$ is not at least as informative as $Y$. By Blackwell’s theorem, it will then follow that there is no Markov matrix $M$ such that $Y = XM$.

For any $(a_1, a_2, 0, \ldots, 0) \in \mathbb{R}^n$ let $f(a_1, a_2, 0, \ldots, 0) = x_{2m}a_1 + x_{1m}a_2$.

Let $D^* \in \mathcal{D}$ be a decision rule such that

$$f(\text{diag}(XD^*)) \geq f(\text{diag}(XD)) \text{ for all } D \in \mathcal{D}.$$ 

$D^*$ exists, since we are dealing with a maximization of a continuous function on a compact set. It is enough to show that there exists $D' \in \mathcal{D}'$ such that

$$f(\text{diag}(YD')) > f(\text{diag}(XD^*))$$

because it will then follow that $\text{diag}(YD') \in \mathcal{B}(Y, A)$ and $\text{diag}(YD') \not\in \mathcal{B}(X, A)$.

Suppose without loss of generality that

$$y_{1m}y_{2(m+1)} > y_{2m}y_{1(m+1)}. \quad (15)$$

Let

$$D^* = \begin{pmatrix}
    d_{11} & d_{12} & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    d_{(m-1)1} & d_{(m-1)2} & 0 & 0 \\
    d_{m1} & d_{m2} & 0 & \cdots & 0
\end{pmatrix}$$

Note that $\text{diag}(XD^*) = \left( \sum_{j=1}^{m} x_{1j}d_{j1}, \sum_{j=1}^{m} x_{2j}d_{j2}, 0, \ldots, 0 \right)$. Therefore,

$$f(\text{diag}(XD^*)) = x_{2m} \sum_{j=1}^{m} x_{1j}d_{j1} + x_{1m} \sum_{j=1}^{m} x_{2j}d_{j2}.$$
We will now build the desired $D'$. The first $m - 1$ rows of $D'$ are identical to the first $m - 1$ rows of $D^*$. The last two rows of $D'$ are

\[
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0
\end{pmatrix}
\]

That is, $D' = \begin{pmatrix}
  &  &  &  &  \\
  & & &  \\
  & & &  \\
  & & &  \\
  &  &  &  &  \\
  & & &  \\
  & & &  \\
  & & &  \\
  &  &  &  &  \\ d_{11} & d_{12} & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
 d_{(m-1)1} & d_{(m-1)2} & 0 & 0 \\
  1 & 0 & 0 & \cdots & 0 \\
  0 & 1 & 0 & \cdots & 0
\end{pmatrix}.$

Note that $\text{diag}(YD') = \sum_{j=1}^{m+1} x_jd_{j1} + y_{1m} + \sum_{j=1}^{m-1} x_{2j}d_{j2} + y_2(m+1)$. Therefore, $f(\text{diag}(YD')) = x_{2m}(\sum_{j=1}^{m-1} x_jd_{j1} + y_{1m}) + x_{1m}(\sum_{j=1}^{m} x_{2j}d_{j2} + y_2(m+1))$. Consequently,

\[
f(\text{diag}(YD')) - f(\text{diag}(XD^*)) = x_{2m}y_{1m} + x_{1m}y_2(m+1) - (x_{2m}x_{1m}d_{m1} + x_{1m}x_{2m}d_{m2}) = x_{2m}y_{1m} + x_{1m}y_2(m+1) - x_{2m}x_{1m} - (y_{2m} + y_2(m+1)) y_{1m} + (y_{1m} + y_1(m+1)) y_2(m+1) - (y_{2m} + y_2(m+1))(y_{1m} + y_1(m+1)) = y_{1m}y_2(m+1) - y_{2m}y_1(m+1) > 0
\]

where the inequality follows from (15). \hfill \blacksquare
References


