Fuzzy Poverty Measurement and Crisp Dominance

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Abstract: In this paper we make several contributions to fuzzy poverty measurement. Specifically, we identify a sensible source for the fuzziness in poverty which leads to a natural way to estimate poverty membership function; we provide an axiomatic characterization for an important class of fuzzy poverty measures; and we derive a set of crisp dominance conditions for fuzzy partial poverty orderings.

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1. Introduction

In 2013, an individual living alone in the continental US is regarded as a poor if his annual income is less than \$11,490. This threshold is referred to as the official US poverty line, which has been in existence since 1965 when it was developed by the economist Mollie Orshansky at the US Social Security Administration. Over the years, the dichotomous nature of the poverty line (i.e., dividing the population into the poor and the non-poor groups) coupled with the official way of poverty calculation has generated much criticism and debate. Sen (1976) pointed out that the official headcount ratio - the percentage of people below the poverty line - does not distinguish between a poor person who is close to the poverty line and a poor person who is at the bottom of income distribution. In this official approach, the drawing of a poverty line is of paramount importance in painting the picture of poverty for a society. Sen (1976) argued that a poverty measure should be sensitive not only to the "incidence of poverty" - whether a person is a poor or not - but also to the level of a poor income and to the distribution of income among the poor. Such a "distribution-sensitive" poverty measure gives less weight to a poor income next to the poverty line than to a poor income further below the poverty line.

Sen's 1976 seminal work focused on the aggregation issue of poverty measurement - aggregating all poor's poverty values into a single value - the issue of poverty identification or "who is poor?" was not particularly addressed. While the use of a single poverty line seems to solve the problem of poverty identification, the issue actually is much more complicated and the study of it over the last two decades has given rise to three new branches of poverty measurement. One issue is whether a single dimension such as income or consumption is adequate in determining an individual's poverty status. It has been argued that an individual's welfare is essentially multidimensional and hence an individual's poverty status must also be evaluated from a multidimensional perspective. Such a consideration has led to the startup of the multidimensional poverty measurement literature (Tsui, 2002; Bourguignon and Chakravarty, 2003; Duclos et al., 2006; Alkire and Foster, 2011).

A second issue is: If income (or any other variable) is deemed to be a proper dimension upon which the poverty status is determined, what income level should be used as the poverty line? Obviously one may argue that \$11,490 is an arbitrary number and some other number would be equally justified as a poverty line. It follows that the poverty comparison between a pair of distributions should be checked for robustness over a range of poverty lines rather than using a single poverty line. This consideration of uncertainty in the poverty line has led to the birth of the literature of partial poverty orderings (Atkinson, 1987; Foster and Shorrocks, 1988; and Zheng, 1999). Finally, the dichotomous nature of a poverty line has recently been questioned. That is, whether we can unequivocally classify a person as a poor or a non-poor. While there are many cases where we can quite confidently determine a person's poverty status (such as a street beggar - a poor; and a millionaire - a non-poor), we also often run into situations where poverty-status classification is not so clear-cut (reasons to be examined later). If we allow an individual's poverty status to have some level of ambiguity and incorporate it into poverty measurement, we are led to the new literature of fuzzy poverty measurement.

This paper focuses on fuzzy poverty measurement and, for simplicity, we consider unidimensional measurement. Specifically, we investigate issues on poverty fuzzification (i.e., allowing poverty to be a fuzzy concept), characterize a class of decomposable (additively separable) fuzzy poverty measures and derive a set of exact dominance conditions for fuzzy poverty orderings. The central stipulation in fuzzy poverty measurement is that the notion of being poor is a vague one - it is impossible for any society to draw a clear demarcation line to separate the poor from the non-poor. Just like many linguistic descriptions such as "tall" and "pretty," there is no sharp borderline for being poor. Instead, the poverty status of an individual is indicated by a real number between zero (clearly a non-poor) and one (clearly a poor). The varying degree of poverty status is referred to as a poverty membership function. The early literature on fuzzy poverty measurement concentrated on deriving poverty membership functions. For example, Cerioli and Zani (1990) introduced, for a given dimension such as income, a straight-line membership function that links between one and zero and decreases linearly with income. Chakravarty (2006) generalizes Cerioli and Zani's membership function to allow it to change non-linearly. While Cerioli and Zani's approach requires a specification of two income levels such that an individual becomes "definitely poor" or "definitely not poor," Cheli and Lemmi (1995) presented a "totally" fuzzy and relative approach in which the degree of poverty membership depends on an individual's relative rank in the distribution.

The specification of a fuzzy membership function is content-dependent, i.e., it depends on the source of the vagueness/fuzziness. In everyday life there are many different sources of vagueness. Hisdal (1986b) documented about a dozen sources of fuzziness but enlisted three as the main sources. Interpreted within the content of poverty measurement, the three sources of fuzziness are: the fuzziness due to the imprecise estimation of income although an exact poverty line can be established; the fuzziness due to the fact that poverty is essentially a multidimensional concept but it is determined by using income alone; and the fuzziness due to the fact that different people may have different ideas about the income level below which an individual is poor. The first source of fuzziness has most commonly been used to justify fuzzy set theoretic approach to poverty measurement. For example, Chakravarty (2006) stated that "it is often impossible to acquire sufficiently detailed information on income and consumption of different basic needs and hence the poverty status of a person is not always clear cut" (p. 51). Shorrocks and Subramanian (1994) also invoked a similar justification but noting that "it does not conform with the strict definition of a fuzzy set."

The shape of a membership function has been a topic of immense interest in the fuzzy-set theory literature. The "aesthetically pleasing" (Beliakov, 1996) S-shape curve has been found to be a common property for many proposed membership functions (Dombi, 1990). Shorrocks and Subramanian (1994) also believed that the poverty membership function "is likely to be " an inverted-S-shape function - "continuous and non-increasing in income" and its slope "first increases and then declines." For the three sources of fuzziness described above, Beliakov (1996) proved that every source gives rise to an averaging process and, consequently, yields a S-shape membership function.

In this paper, we register with the third source of fuzziness as the source of fuzziness in the poverty line. That is, people (voters or social evaluators) have different perceptions about what constitutes poverty. We believe this source of fuzziness is a more sensible explanation for why poverty is a fuzzy predicate than the other two sources; the first source is more about lacking information in computing an accurate poverty line (Qizibash, 2006) while the second source is more about justifying the multidimensional approach to poverty measurement. The third source also allows us to introduce a meaningful "density function" for a poverty membership function - which is akin to the density function for a cumulative distribution function. With this type of poverty membership density function, we are able to extend uniquely any crisp decomposable (additively separable) poverty measure to a fuzzy decomposable poverty measure that Shorrocks and Subramanian (1994) characterized. Our extension does not depend on the hard-to-justify "m-linearity condition" that Shorrocks and Subramanian relied upon; it also clearly avoids the common confusion between "fuzzy measures of poverty" and "measures of the depth of poverty" (Qizibash, 2006). For such a decomposable fuzzy poverty measure, the ranking of income distributions obviously depends on the poverty membership function used. Although the fuzzy set theory literature is filled with methods for membership function construction, no effort has been made to derive a poverty membership function from empirical data. While the approach we take allows us to construct empirical membership functions, in this paper, we do not attempt to specify a poverty membership function beyond the S-shape requirement. What we are interested in achieving in this paper is to derive dominance conditions such that one distribution has no more poverty than another distribution for all possible membership functions with the S-shape. The advantage of this exercise is clear: if a dominance relationship is reported then there is no need to estimate membership functions. One set of dominance conditions we derive turn out to be an interesting device of bifurcated Lorenz dominance.

The rest of the paper is organized as follows. The next section characterizes poverty as a fuzzy concept and specifies the general properties for a poverty membership function. It also discusses the sources of fuzziness that have been considered. Accepting the third source of fuzziness and the corresponding membership functions, Section 3 axiomatically characterizes additively separable fuzzy poverty measures. Section 4 derives a set of dominance conditions for all possible poverty membership functions and/or for all poverty measures. Section 5 concludes.

2. Fuzzy Poverty Set and Poverty Membership Functions

Consider a continuous income distribution represented by a cumulative distribution function F: $\Re_+ \rightarrow [0,1]$. A crisp poverty set of F contains the individuals who are unambiguously identified as poor. This is achieved by employing a singular poverty line z and anyone whose income, x, is less than z is a poor. Denote the poverty set by $A =: \{x | x \in \Re_+ \text{ and } x < z\}$, then the poverty status of an individual can be expressed using the following two-value poverty characteristic function (1 is a poor and 0 is a nonpoor):

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in \bar{A} \end{cases} .$$

The practice of making a sharp distinction between the poor and the nonpoor has been criticized as "difficult" (Shorrocks and Subramanian, 1994) and "nonrealistic" (Cerioli and Zani, 1990). Sen (1981, p. 13) also pointed out that "while the concept of a nutritional requirement is a rather loose one, there is no reason to suppose that the concept of poverty is clear cut and sharp ... a certain amount of vagueness is implicit in both the concepts." Over the last two decades, many poverty measurement researchers have argued that the distinction between the two groups is vague and the notion of being poor is a fuzzy one. They have suggested that the tools developed in the fuzzy set theory should be employed to measure poverty.

Many linguistic variables such as "tall" and "bald" are considered vague predicates. A vague or fuzzy predicate such as "tall", according to Keefe and Smith (1996) and Qizibash (2006), must possess three characteristics: it involves "borderline" situations where one cannot decide unequivocally "tall" and "not tall"; one cannot establish a sharp boundary between "tall" and "not tall"; and the presence of a Sorites paradox - making a "tall" person a millimeter shorter keeps the person "tall" but repeating the same step many times eventually will make the person "not tall" or "short." Qizibash (2006) argues that the notion of poverty meets all three criteria and hence poverty is indeed a fuzzy predicate: there are cases where a person cannot be obviously classified as a "poor" or a "nonpoor"; there is no clear-cut poverty line to separate the poor from the nonpoor; and any poor person can be made nonpoor by repeatedly increasing his income by a penny but each penny alone does not change his poverty status.

Before proceeding to poverty-fuzziness characterizations, it is important to discern the conceptual difference between fuzziness in poverty and uncertainty in the poverty line. In general, fuzziness may be characterized as inherent imprecision while uncertainty is due to lack of information. The theory of uncertainty in poverty line presupposes the existence of a true and exact poverty line. But the inability to pinpoint the exact value requires us to compute poverty values for a range of poverty lines - each of these lines could equally be the true poverty line. When comparing poverty levels between a pair of income distributions for a range of poverty lines, obviously the direction of poverty ranking can be reversed at different poverty lines.

Requiring a consistent ranking across all poverty lines in the range has given risen to the study of partial poverty orderings. In contrast, the theory of fuzziness in poverty does not assume the existence of a true poverty line. Rather it considers the poverty status not as a two-valued logic (i.e., “to be” or “not to be” in poverty) but as a degree of membership, i.e., it allows a person to be poor to a degree. It is also useful to point out that although conceptually these two notions are distinct, the dominance conditions derived (as we will see later) turn out to be closely connected.

The fuzziness in poverty is characterized through a poverty membership function. To do so, we are required to specify two income levels z_l and z_u ($z_l < z_u$) so that any individual with an income below z_l is definitely counted as a poor while any individual with an income above z_u will definitely be counted as a nonpoor. An immediate criticism of this exercise is: If one cannot specify precisely a value for the poverty line, how could one be expected to specify two exact values for the upper and lower bounds of the fuzzy poverty region? This argument is the very reason behind Cheli and Lemmi's “total fuzzy approach” and is the justification for considering “higher-order vagueness” (Sainsbury, 1991) - there is a vagueness about the degree of membership. A simple response to this criticism, besides the standard responses from the fuzzy-set theories (Qizibash, 2006, p. 14), is that failing to specify two such boundaries, we are unable to proclaim that a street beggar is a definitely poor and a person owning a Ferrari is definitely a non-poor! Certainly the exact specification of z_l and z_u has a flavor of arbitrariness but the source of fuzziness we subscribe to will provide a way to determine their values.

Definition 2.1. The membership of income x is in poverty is represented by a differentiable function $m(x)$ such that

$$m(z_l) = 1, \; m(z_u) = 0 \text{ and } 0 \leq m(x) \leq 1 \text{ for all } z_l < x < z_u \tag{2.3}$$

for some finite values z_l and z_u with $z_l < z_u$.

Clearly, for $z_l < x < z_u$, $m(x)$ must also be non-increasing in x to reflect the fact that an increase in income makes the individual less likely to be a poor. This amounts to assuming $m'(x) \leq 0$. Another commonly assumed property is that $m(x)$ is of an inverted-S shape (the solid curve in Figure 1). That is, there exists a point ξ between z_l and z_u such that

$$\begin{aligned} m''(x) &\leq 0 \text{ for } z_l < x < \xi \text{ and } m''(x) \geq 0 \text{ for } \xi < x < z_u; \\ \text{and } m'(z_l) &= m'(z_u) = 0. \end{aligned} \tag{2.4}$$

Considering all possible membership functions possessing these properties, we can form two sets of membership functions as follows:

$$\mathfrak{M}_1 = \{m | m'(x) \leq 0 \text{ for } z_l < x < z_u\}$$

and

$$\mathfrak{M}_2 = \{m | m(x) \text{ is in } \mathfrak{M}_1 \text{ and satisfies (2.4)}\}.$$

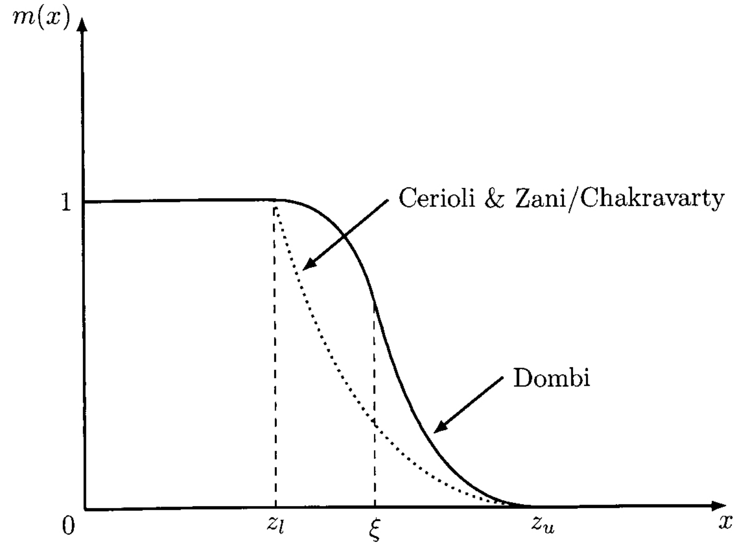

Figure 1: Poverty Membership Functions

In Section 4, we will derive crisp poverty dominance conditions for all membership functions in $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

An example of poverty membership function is

$$m_1(x) = \begin{cases} 1 & \text{if } x \leq z_l \\ \left(\frac{z_u - x}{z_u - z_l}\right)^{\theta} & \text{if } z_l < x < z_u \\ 0 & \text{if } x \geq z_u \end{cases}$$

which Cerioli and Zani (1990) introduced ($\theta = 1$) and Chakravarty (2006) generalized ($\theta > 1$). Another example is

$$m_2(x) = \begin{cases} 1 & \text{if } x \leq z_l \\ \frac{(z_u - x)^2}{(x - z_l)^2 + (z_u - x)^2} & \text{if } z_l < x < z_u \\ 0 & \text{if } x \geq z_u \end{cases}$$

which is a member of the class that Dombi constructed (1990, Theorem 2). Both membership functions are non-increasing in x (and strictly decreasing in x over (z_l, z_u)) and hence are members of $\mathfrak{M}_1$. But $m_1(x)$ is not of an inverted-S shape and thus does not belong to $\mathfrak{M}_2$. It is easy to check that $m_2(x)$ is of an inverted-S shape and is a member of $\mathfrak{M}_2$. Both membership functions are plotted below in Figure 1.

What is the meaning of the m function? The cases of $m = 1$ and 0 are straightforward, but $m = 0.8$? In fact, the difficulty in interpreting a number such as 0.8 is one of the main criticisms about the degree of membership and about the fuzzy set theory in general. The first two sources of fuzziness documented in the Introduction do not seem to provide a sensible interpretation. The third source of fuzziness - different people may perceive poverty differently - does lead to a meaningful interpretation. To state the interpretation, we need to introduce a "density function" - akin to that for a cumulative distribution function - for a poverty membership function.

Definition 2.2. For a poverty membership function $m(x)$, its density function is

$$\rho(x) = |m'(x)| \text{ for all } z_l < x < z_u \tag{2.5}$$

and $\rho(x) = 0$ for any $x \leq z_l$ and for any $x \geq z_u$.

Since poverty is a social concern, it is reasonable to let the society to decide what constitutes poverty. Suppose every member of the society (say a voter) is asked "what do you think a reasonable poverty line x should be?" Now tally up the answers by the value of x. Let $\rho(x)$ denote the proportion of voters in the society who have elected x as the "reasonable poverty line." It follows immediately that $1 - m(x)$ - as a cumulation of $\rho(x)$ - is *the proportion of voters in the society believing the poverty line is at or less than* x; $m(x)$ *is the proportion of voters believing the poverty line is at least* x. These interpretations flow naturally from the definition of the third source of fuzziness; no similar interpretations can be established for the first and second sources of fuzziness. The approach also leads to a natural determination of the values of z_l and z_u: z_l is set at the minimum value of x that voters have elected and z_u is the maximum value of x that voters have chosen. Given the one-to-one correspondence between $m(x)$ and $\rho(x)$, in the rest of the paper, we will use both terms interchangeably.

3. Fuzzy Poverty Measures

For a crisp poverty set, issues in poverty measurement have been thoroughly researched and a large literature has been established following Sen's 1976 seminal work. For a given poverty line $z \in \Re_{++}$, a poverty measure $P(F; z)$ indicates the poverty level associated with distribution F. There are two types of poverty measures in the literature: additively separable (also called decomposable) measures and rank-dependent measures. An additively separable poverty measure is

$$P(F; z) = \int_0^z p(x, z) dF(x) \tag{3.1}$$

where $p(x, z)$ is the individual deprivation function with $p(x, z) > 0$ for $x < z$ and $p(x, z) = 0$ for $x \geq z$. We further require that $p(x, z)$ be twice differentiable with respect to x for $x < z$ with $p_x = \frac{\partial p(x,z)}{\partial x} \leq 0$ and $p_{xx} = \frac{\partial^2 p(x,z)}{\partial x^2} \geq 0$.[1] Examples of

[1] These two conditions ensure that a poverty measure satisfies the monotonicity axiom (poverty is reduced if the income of a poor individual is increased) and the transfer axiom (poverty is reduced by a progressive transfer of income among the poor). For surveys on poverty axioms, poverty measures and poverty orderings, see Zheng (1997, 2000), Lambert (2001) and, more recently, Chakravarty (2009).

additively separable poverty measures are the Watts measure with $p(x,z) = \ln z - \ln x$ for $x > 0$, the Clark et al. second measure with $p(x,z) = \frac{1}{\epsilon}[1 - (x/z)^\epsilon]$ and $0 < \epsilon < 1$, the Foster et al. measure with $p(x,z) = (1 - x/z)^\alpha$ and $\alpha \geq 2$, and the Kolm-type constant-distribution-sensitivity (CDS) measure with $p(x,z) = e^{\sigma(z-x)} - 1$ and $\sigma > 0$.

A rank-dependent poverty measure is of the form

$$P(F;z) = \int_0^z q(F,x;z)dF(x) \tag{3.2}$$

for some continuous function q. Examples of rank-dependent poverty measures are the Sen measure with $q(F,x;z) = 2[1 - F(x)/r_z](1 - x/z)$ where r_z is the proportion of the population below the poverty line (the official headcount ratio), the generalized Thon measure with $q(F,x;z) = \frac{1}{c-1}[c - 2F(x)](1 - x/z)$ and $c \geq 2$, and the Clark et al. first measure with $q(F,x;z) = r_z^{\alpha-1}(1 - x/z)^\alpha$ and $\alpha \geq 1$.

For a fuzzy poverty set, the level of poverty is determined not by any single poverty line but instead by a poverty membership function m. To distinguish between these two categories of poverty measures, we follow the literature and label the poverty measures for a crisp poverty set as *crisp poverty measures* and the measures for a fuzzy poverty set *fuzzy poverty measures*. We further denote a fuzzy poverty measure $\Pi(F;m)$. Since a characterization function $\chi_A(x)$ is a degenerated form of poverty membership function, a crisp poverty measure can be written as $\Pi(F;\chi_A)$.

Conceptually, each crisp poverty measure such as the Sen measure can be generalized to a fuzzy poverty measure, and each fuzzy poverty measure should collapse to a crisp poverty measure if the membership function $m(x)$ degenerates to $\chi_A(x)$. The extension from a crisp poverty measure to a fuzzy poverty measure, however, as Shorrocks and Subramanian (1994) demonstrated, is not straightforward and is not even unique. With the exception of Shorrocks and Subramanian (1994), all the generalizations (e.g., Cerioli and Zani (1990), Cheli and Lemmi (1995) and Chakravarty (2006)) in the literature have focused on the fuzzification of the headcount ratio and the fuzzy headcount ratio is simply the sum of poverty memberships:

$$\Pi(F;m) = \int_0^\infty m(x)dF(x).$$

For a general crisp poverty measure such as (3.1) with a poverty deprivation function $p(x,z)$, since it can be written as

$$\Pi(F;\chi_A) = \int_0^\infty p(x,z)\chi_A(x)dF(x)$$

a seemingly natural fuzzy poverty measure could be obtained by replacing $\chi_A(x)$ with $m(x)$. But such a generalization may fail to serve as a reasonable fuzzy poverty measure as Shorrocks and Subramanian (1994) pointed out.

In an important contribution to fuzzy poverty measurement, Shorrocks and Subramanian (1994) proved that for every crisp poverty measure $P(F;z)$, under certain conditions, there is a unique generalization to fuzzy poverty measure

$$\Pi(F;m)=\int_0^\infty P(F;z)d[1-m(z)]. \tag{3.3}$$

A critical assumption employed in the derivation is the m-linearity condition which require $\Pi(F;m)$ to be linear in m, i.e.,

$$\Pi[F;\alpha m_1+(1-\alpha)m_2]=\alpha\Pi(F;m_1)+(1-\alpha)\Pi(F;m_2)$$

for all membership functions m_1 and m_2. But "there is no compelling reason for imposing this requirement" as Shorrocks and Subramanian (1994) recognized and, thus, their characterization of the fuzzy poverty measures (3.3) carries an ad hoc flavor.

In this section, we provide an alternative yet more elementary and intuitive characterization to (3.3) when $P(F;z)$ is additively separable or decomposable. This characterization is based on the source of fuzziness that we subscribe to: voters have different perceptions regarding what constitutes being in poverty. The building block of the characterization is an individual fuzzy deprivation ($\pi(x;m)$ below) and the connector of these deprivations is an equal voting right axiom.

3.1. Decomposable fuzzy poverty measures: individual deprivation

Since we view poverty as a social concern, any individual's poverty status and his level of poverty should be judged and decided by the public or the voters. Suppose a group of voters believe that the "true" poverty line should be z, then an individual with income x will be judged by this group to have the poverty level of $p(x;z)$. The size of this voting group is $\rho(z)$. Considering all groups of voters, we have a distribution of poverty levels represented by $\{p(x;z),\rho(z)\}$ for all $z\in[z_l,z_u]$. For ease of presentation and derivation, we assume that there are k voter groups and denote the fuzzy poverty level voted by the public for the individual with income x as $\pi(x;m)$. This amounts to assuming that there are k discrete poverty lines - a necessary deviation from our continuous poverty-space setting. Denote the k voted poverty lines as $z_1<z_2<\cdots<z_k$, also $p_i=p(x,z_i)$ and $\rho_i=\rho(z_i)$ for $i=1,2,...,k$. The relationship between $\pi(x;m)$ and $\{p(x;z),\rho(z)\}$ can be represented by

$$\pi(x;m)=f(p_{1,}p_2,...,p_k;\rho_{1,}\rho_2,...,\rho_k) \tag{3.4}$$

for some differentiable function f.

What should this relationship be? A natural choice seems to be a simple weighted average of all possible poverty levels that voters have chosen with the weight being the voter density function $\rho(z)$. In what follows, we demonstrate that this choice is justified by the following reasonable and intuitive "equal voting-right" axiom.

The individual-poverty voting axiom. Suppose a fraction ϵ of ρ_j is shifted to ρ_i and denote the new poverty membership function as m', then the resulting change in the poverty level of the individual with income x,

$$\Delta\pi = \pi(x; m') - \pi(x; m),$$

does not depend on any $\rho_i, i = 1, 2, ..., k$.

The essence of the axiom is that all voters carry the same right or weight in determining the poverty level of an individual, it does not matter which group and how large is the group the voter may be in. When a voter changes his vote of poverty line from z_j to z_i with $z_i < z_j$, the poverty membership function changes from m to m' and the individual poverty level π should decrease. The amount of decrease $\Delta\pi$, however, should not depend on ρ_i or ρ_j; it can only depend on $p_i, i = 1, 2, ..., k$. Otherwise, if $\Delta\pi$ depends on the size of ρ_i or ρ_j, then it means that the voter's vote carries different weights in different groups - it depends upon which coalition he chooses to join; in a sense he is no longer an independent voter.

To state and prove the implication of this axiom for $\pi(x; m)$, we also need the following normalization axiom on individual fuzzy poverty deprivation.

The individual normalization axiom. If for some i, $\rho_i = 1$ (thus $\rho_j = 0$ for all $j \neq i$), then $\pi(x; m) = p_i$.

The idea behind this normalization axiom is pretty straightforward: if the entire society agrees upon a single poverty line z_i, then the individual's poverty level π is simply $p_i = p(x; z_i)$.

Proposition 3.1. A differentiable individual fuzzy poverty deprivation function $\pi(x; m)$ satisfies the individual-poverty voting axiom and the individual normalization axiom if and only if it is of the form

$$\pi(x; m) = \sum_{i=1}^{k} p_i \rho_i. \tag{3.5}$$

Proof. First consider a shift of ϵ proportion of voters from ρ_k to ρ_{k-1}, $\epsilon < \rho_k$. That is, some voters change their votes from z_k to z_{k-1}. Denote

$$\tilde{f}(\rho_{1,}..., \rho_2, ..., \rho_k) = f(p_{1,}p_2, ..., p_k; \rho_{1,}..., \rho_2, ..., \rho_k), \tag{3.6}$$

then the change in the individual poverty value of π is

$$\Delta\pi_k = \tilde{f}(\rho_{1,}..., \rho_{k-1} + \epsilon, \rho_k - \epsilon) - \tilde{f}(\rho_{1,}\rho_2, ..., \rho_k).$$

The individual-poverty voting axiom states that $\Delta\pi_k$ can only be a function of ϵ. That is,

$$\tilde{f}(\rho_{1,}..., \rho_{k-1} + \epsilon, \rho_k - \epsilon) - \tilde{f}(\rho_{1,}\rho_2, ..., \rho_k) = h(\epsilon) \tag{3.7}$$

for some differentiable function h.

Differentiate (3.7) with respect to ϵ,

$$\tilde{f}_{k-1}(\rho_{1,}..., \rho_{k-1}+\epsilon, \rho_k-\epsilon) - \tilde{f}_k(\rho_{1,}...\rho_{k-1}+\epsilon, \rho_k-\epsilon) = h'(\epsilon).$$

Let $\epsilon \to 0$, we have

$$\tilde{f}_{k-1}(\rho_{1,}..., \rho_{k-1}, \rho_k) - \tilde{f}_k(\rho_{1,}...\rho_{k-1}, \rho_k) = h'(0)$$

or

$$\hat{f}_{k-1}(\rho_{k-1}, \rho_k) - \tilde{f}_k(\rho_{k-1}, \rho_k) = h'(0) \equiv c_k \tag{3.8}$$

where $\hat{f}(\rho_{k-1}, \rho_k) = \tilde{f}(\rho_{1,}..., \rho_{k-1}, \rho_k)$.

Using the method of Lagrange (e.g., Dennemeyer, 1968, p.17) to the quasi-linear partial differentiation equation (3.8), we solve the following subsidiary equations

$$\frac{d\rho_{k-1}}{1} = \frac{d\rho_k}{-1} = \frac{du}{c_k}$$

which yield the following intergrals

$$\rho_{k-1} + \rho_k = v \text{ and } c_k\rho_k + u = w$$

for some constants v and w. The general solution to (3.8) is

$$s(v, w) = s(\rho_{k-1}+\rho_k, c_k\rho_k+u) = 0 \tag{3.9}$$

for some arbitrary function s. Solving (3.9) for u, we obtain

$$\hat{f} = u = r(\rho_{k-1}+\rho_k) - c_k\rho_k \tag{3.10}$$

for some differentiable function r. Restoring $\tilde{f}$, we have

$$\tilde{f}(\rho_{1,}..., \rho_k) = \tilde{r}(\rho_{1,}..., \rho_{k-1}+\rho_k) - c_k\rho_k \tag{3.10a}$$

for some differentiable function $\tilde{r}$.

Now consider a further shift of $\epsilon(< \rho_{k-1})$ proportion of voters from ρ_{k-1} to ρ_{k-2}. Similarly we have

$$\hat{f}_{k-2}(\rho_{k-2}, \rho_{k-1}+\rho_k) - \hat{f}_{k-1}(\rho_{k-2}, \rho_{k-1}+\rho_k) = c_{k-1} \tag{3.8a}$$

where $\hat{f}$ is similarly defined as above and c_{k-1} is a constant. The general solution to (3.8a) is

$$\tilde{f}(\rho_{1,}..., \rho_k) = \tilde{r}(\rho_{1,}..., \rho_{k-2}+\rho_{k-1}+\rho_k) - c_{k-1}\rho_{k-1} - c_k\rho_k \tag{3.10b}$$

for some differentiable function $\tilde{r}$ and some constants c_{k-1} and c_k. Repeating this process of voter shifting from ρ_{k-2} to ρ_{k-3}, from ρ_{k-3} to ρ_{k-4}, ..., and finally from ρ_2 to ρ_1, we eventually arrive at

$$\begin{aligned} \tilde{f}(\rho_{1,}..., \rho_k) &= \tilde{r}(\rho_1 + \rho_2 + \cdots + \rho_{k-1} + \rho_k) - c_2\rho_2 - \cdots - c_{k-1}\rho_{k-1} - c_k\rho_k \\ &= \tilde{r}(1) - c_2\rho_2 - \cdots - c_{k-1}\rho_{k-1} - c_k\rho_k \qquad (3.10c) \\ &= \tilde{r}(1)(\rho_1 + \rho_2 + \cdots + \rho_{k-1} + \rho_k) - c_2\rho_2 - \cdots - c_{k-1}\rho_{k-1} - c_k\rho_k \\ &= \tilde{r}(1)\rho_1 + (1 - c_2)\rho_2 + \cdots + (1 - c_k)\rho_k. \end{aligned}$$

Here we have used the fact that $\rho_1 + \rho_2 + \cdots + \rho_{k-1} + \rho_k = 1$. Let $\tilde{c}_1 = \tilde{r}(1)$ and $\tilde{c}_i = (1 - c_i)$ for $i = 2, ..., k$ and restoring f as defined in (3.6), we have

$$f(p_{1,}p_2, ..., p_k; \rho_{1,}..., \rho_2, ..., \rho_k) = \sum_{i=1}^{k} \tilde{c}_i(p_{1,}p_2, ..., p_k)\rho_i$$

for some continuous functions $\tilde{c}_i(p_{1,}p_2, ..., p_k), i = 1, 2, ..., k$. Finally, for each i, let $\rho_i = 1$ and thus $\rho_j = 0$ for $j \neq i$, the normalization axiom entails

$$\tilde{c}_i(p_{1,}p_2, ..., p_k) = p_i, i = 1, 2, ..., k$$

which leads to (3.5). This proves the necessity of the proposition. The sufficiency is obvious. □

3.2. Decomposable fuzzy poverty measures: population aggregated poverty

To aggregate individual poverty levels across the population, let's also consider a discrete population F with n income groups represented by $\{(x_j, \theta_j)\}$ where $x_j, j = 1, 2, ..., n$, is the income level of group j and θ_j is the population share of group j. We assume $x_1 < x_2 < \cdots < x_n$ and denote an individual's fuzzy poverty deprivation in group j as $\pi_j \equiv \pi(x_j; m)$. The aggregate fuzzy poverty measure of F and individual fuzzy poverty deprivations π_is are related via

$$\Pi(F; m) = g(\pi_1, \pi_2, ..., \pi_n; \theta_1, \theta_2, ..., \theta_n) \qquad (3.11)$$

for a differentiable and increasing function g and we need to determine the functional form of g.

Once again, we view the aggregation process as a voting process - each individual "votes by foot" on the aggregate poverty by choosing the income group he wants to be part of. Similar to the determination of individual fuzzy poverty deprivation, we consider the following two axioms on the aggregation process.

The population-poverty voting axiom. Suppose a fraction ϵ of θ_j is shifted to θ_l and denote the new distribution function as F', then the resulting change in the aggregate level of poverty of the population,

$$\Delta\Pi = \Pi(F'; m) - \Pi(F; m),$$

does not depend on any $\theta_j, j = 1, 2, ..., n$.

The population normalization axiom. If for some j, $\theta_j = 1$ (thus $\theta_l = 0$ for all $l \neq j$), then $\Pi(F; m) = \pi_j$.

The population-poverty voting axiom states that each individual's contribution to the aggregate poverty does not depend on the size of the group the individual falls into; there is no gain from strategic coalition-formation. The population normalization axiom claims that if all individuals have the same income (so they are in the same income group) then the aggregate poverty level is the same as the poverty level of any individual in the group. Note that in this case even though people have the same level of income but they still possess different perceptions about what constitutes poverty and hence there is still a fuzzy poverty membership function.

We are now in the position to state the main result of this section.

Proposition 3.2. A differentiable fuzzy poverty measure $\Pi(F; m)$ defined in (3.11) satisfies the population-poverty voting axiom and the population normalization axiom if and if it is of the form

$$\Pi(F; m) = \frac{1}{n} \sum_{j=1}^{n} \pi(x_j; m)\theta_j. \tag{3.12}$$

Proof. The proof is a replication of the proof for Proposition 3.1 with p_i being replaced by π_j and ρ_i being replaced by θ_j. □

Translated back into the continuous space, the individual fuzzy poverty deprivation is

$$\pi(x; m) = \int_0^\infty p(x, z)\rho(z)dz$$

and the aggregate fuzzy poverty measure is

$$\begin{aligned} \Pi(F; m) &= \int_0^\infty \pi(x; m)dF(x) = \int_0^\infty \int_0^\infty p(x, z)\rho(z)dzdF(x) \\ &= \int_0^\infty \left[\int_0^\infty p(x, z)dF(x)\right]\rho(z)dz = \int_0^\infty P(F; z)\rho(z)dz \qquad (3.13) \end{aligned}$$

which is the same as that derived by Shorrocks and Subramanian (1994). Thus, Proposition 3.2 provides a simple and intuitive characterization to Shorrocks and Subramanian's fuzzification proposal for additively separable poverty measures.

As for rank-dependent fuzzy poverty measures, since

$$\begin{aligned} \Pi(F; m) &= \int_0^\infty \left[\int_0^\infty q(F, x; z)dF(x)\right]\rho(z)dz \\ &= \int_0^\infty \left[\int_0^\infty q(F, x; z)\rho(z)dz\right]dF(x), \qquad (3.14) \end{aligned}$$

we can provide a similar interpretation for $\Pi(F;m)$. Denote $\tilde{\pi}(x;m) = \int_0^\infty q(F,x;z)\rho(z)dz$, then $\tilde{\pi}(x;m)$ can also be interpreted as an individual fuzzy poverty deprivation. A result similar to Proposition 3.2 can be established to show that $\Pi(F;m)$ is a weighted average of $\tilde{\pi}(x;m)$. But a similar characterization for $\tilde{\pi}(x;m)$, hence Proposition 3.1, cannot be established since the $q(F,x;z)$ function in $\tilde{\pi}(x;m)$ may rely on a complex combination between the rank of income and the level of income and, consequently, the individual-poverty voting axiom is no longer valid. It would certainly be useful and interesting to provide an axiomatic characterization for $\tilde{\pi}(x;m)$.

4. Crisp Poverty Dominance

The fuzzy poverty measure characterized in the previous section involves a crisp poverty measure and a poverty membership function. In (crisp) poverty comparisons, it is recognized that the use of any single poverty measure is arbitrary and multiple poverty measures should be consulted. As mentioned in the Introduction, the uncertainty in choosing "proper" poverty measures has led to the development of the literature of poverty-measure partial poverty orderings. Once a dominance relationship is found to hold between a pair of income distributions, all poverty measures satisfying certain conditions (or in a class) will agree on the poverty ranking. For fuzzy poverty measurement, there is also a similar issue of poverty-measure partial poverty ordering. That is, for a given poverty membership function $m(x)$, one may wish to know the conditions under which all fuzzy poverty measures will reach the same conclusion on poverty orderings.

Conceptually, the specification of a poverty membership function should be less arbitrary and more "scientific" than the choice of a poverty measure. Over the last half-century since Zadeh (1965) introduced the notion of fuzzy set, various techniques of quantifying fuzziness have been proposed (see, for example, Chapter 10 of Klir an Yuan (1995)). These techniques have found numerous applications in a wide range of areas such as in the designing and manufacturing of washing machine and rice cooker (see, for example, a recent survey on industrial applications of fuzzy control by Precup and Hellendoorn, 2011). Researchers, however, do not appear to agree on the existence of a precise membership function. Dubois and Prade (1980, p. 2), for example, believed that "precise membership values do not exist by themselves, they are tendency indices that are subjectively assigned by an individual or a group" and "the grades of membership reflect an 'ordering' of the objects in the universe." The entire school of "higher-order vagueness" (e.g., Sainsbury, 1991) also doubts the existence of a precise membership function on the philosophical ground. In fuzzy poverty measurement, the voter-frequency membership approach we follow in this paper may appear to be free of the arbitrariness that Dubois and Prade alluded to, but the fact that voters *subjectively* decide what constitute poverty suggests some degree of non-uniqueness in computing the membership values. This non-uniqueness points to the usefulness in checking the robustness of poverty rankings by different poverty membership functions. This is a new type of partial poverty orderings and it is the main focus of this section.

Before presenting any new dominance conditions, we need to define two types of additively separable poverty measures: absolute poverty measures and relative poverty measures. Later on, we will establish dominance conditions for the two types of measures. An additively separable (crisp) poverty measure defined by (3.1) is absolute if

$$P(F;z) = \int_0^z \tilde{p}(z-x)dF(x) \tag{4.1}$$

and relative if

$$P(F;z) = \int_0^z \tilde{p}(\frac{x}{z})dF(x) \tag{4.2}$$

for some differentiable function $\tilde{p}$. That is, an absolute poverty measure remains unaffected if all incomes and the poverty line are changed by the same dollar amount; a relative poverty measure remains unaffected if all incomes and the poverty line are changed by the same percentage. Of the poverty measure examples given immediately after (3.1), the first three (Watts, Clark et al. and Foster et al.) are relative and the last one (the Kolm-type) is absolute.

For ease of reference in later presentation, we further specify four subclasses of absolute and relative poverty measures:

$$\begin{aligned}
\mathfrak{P}_1^a &= \{P(\cdot;\cdot\}|P \text{ is (4.1) with } \frac{\partial \tilde{p}}{\partial x} \leq 0\} \\
\mathfrak{P}_2^a &= \{P(\cdot;\cdot\}|P \text{ is (4.1) with } \frac{\partial \tilde{p}}{\partial x} \leq 0 \text{ and } \frac{\partial^2 \tilde{p}}{\partial x^2} \geq 0\} \\
\mathfrak{P}_1^r &= \{P(\cdot;\cdot\}|P \text{ is (4.2) with } \frac{\partial \tilde{p}}{\partial x} \leq 0\} \\
\mathfrak{P}_2^r &= \{P(\cdot;\cdot\}|P \text{ is (4.2) with } \frac{\partial \tilde{p}}{\partial x} \leq 0 \text{ and } \frac{\partial^2 \tilde{p}}{\partial x^2} \geq 0\}.
\end{aligned}$$

4.1. Poverty-membership partial ordering for a given (crisp) poverty measure

Recall that we consider two classes of membership functions $\mathfrak{M}_1$ and $\mathfrak{M}_2$ - members in $\mathfrak{M}_1$ are non-increasing in income (i.e., $m'(x) \leq 0$ or $\rho(x) = -m'(x) \geq 0$) and members in $\mathfrak{M}_2$ are non-increasing and are of inverted-S shape (i.e., there exists a point $\xi \in (z_l, z_u)$ such that $m''(x) \leq 0$ for $x \in (z_l, \xi)$ and $m''(x) \geq 0$ for $x \in (\xi, z_u)$; and $m'(z_l) = m'(z_u) = 0$). The following proposition establishes the partial poverty ordering conditions for all membership functions in each of the two classes.

Proposition 4.1. For any crisp poverty measure $P(\cdot;\cdot)$ and any two income distributions F and G that have the same poverty membership function $m(x)$, F has no more poverty then G, i.e., $\Pi(F;m) \leq \Pi(G;m)$, for all membership functions in $\mathfrak{M}_1$ if and only if

$$P(F;z) \leq P(G;z) \text{ for all } z \in [z_l, z_u]; \tag{4.3}$$

and $\Pi(F;m) \leq \Pi(G;m)$ for all membership functions in $\mathfrak{M}_2$ and a given ξ if and only if

$$\begin{aligned} \int_z^{\xi} P(F;t)dt &\leq \int_z^{\xi} P(G;t)dt \text{ for all } z \in [z_l, \xi] \text{ and} \\ \int_{\xi}^{z} P(F;t)dt &\leq \int_{\xi}^{z} P(G;t)dt \text{ for all } z \in [\xi, z_u]. \end{aligned} \tag{4.4}$$

Proof. For a fuzzy poverty measure $\Pi(F;m)$ with crisp poverty measure $P(\cdot;\cdot)$, via (3.13), distribution F has no more poverty than distribution G if and only if

$$\int_{z_l}^{z_u} P(F;z)\rho(z)dz \leq \int_{z_l}^{z_u} P(G;z)\rho(z)dz$$

or

$$\Pi(F;m) - \Pi(G;m) = \int_{z_l}^{z_u} [P(F;z) - P(G;z)]\rho(z)dz \leq 0. \tag{4.5}$$

For a membership function $m(x)$ in $\mathfrak{M}_1$, the only requirement on $\rho(x)$ is $\rho(x) \geq 0$. The sufficiency of condition (4.3) is obvious. The necessity of (4.3) can also be easily verified using the method of contradiction that Atkinson (1987) employed. That is, if (4.3) is false and $P(F;z) > P(G;z)$ over some open interval (a,b), then we can choose

$$\rho(z) = \frac{1}{b-a} \text{ for } a < z < b \text{ and } \rho(z) = 0 \text{ elsewhere.}$$

It follows that

$$\Pi(F;m) - \Pi(G;m) = \frac{1}{b-a}\int_a^b [P(F;z) - P(G;z)]dz > 0$$

which contradicts (4.5).

To prove (4.4), denote $\Delta P(z) \equiv P(F;z) - P(G;z)$, $\Delta\Pi(m) \equiv \Pi(F;m) - \Pi(G;m)$ and further expand $\Delta\Pi(m)$ as follows:

$$\begin{aligned} \Delta\Pi(m) &= \int_{z_l}^{\xi} \Delta P(z)\rho(z)dz + \int_{\xi}^{z_u} \Delta P(z)\rho(z)dz \\ &= \int_{z_l}^{\xi} \rho(z)d\left[\int_{\xi}^{z} \Delta P(t)dt\right] + \int_{\xi}^{z_u} \rho(z)d\left[\int_{z}^{\xi} \Delta P(t)dt\right] \\ &= \left\{\rho(z)d\left[\int_{\xi}^{z} \Delta P(t)dt\right]\Bigg|_{z_l}^{\xi} - \int_{z_l}^{\xi} \rho'(z)\left[\int_{\xi}^{z} \Delta P(t)dt\right]dz\right\} \\ &\quad + \left\{\rho(z)d\left[\int_{z}^{\xi} \Delta P(t)dt\right]\Bigg|_{\xi}^{z_u} - \int_{\xi}^{z_u} \rho'(z)\left[\int_{z}^{\xi} \Delta P(t)dt\right]dz\right\}. \end{aligned}$$

Since $\rho(z_l) = \rho(z_u) = 0$ by the inverted-S-shape assumption for an $m(x)$ in $\mathfrak{M}_2$, we have

$$\rho(z)d\left[\int_\xi^z \Delta P(t)dt\right]\Big|_{z_l}^{\xi} = 0 \text{ and } \rho(z)d\left[\int_z^\xi \Delta P(t)dt\right]\Big|_{\xi}^{z_u} = 0.$$

Therefore, we have

$$\begin{aligned}\Delta\Pi(m) &= -\int_{z_l}^{\xi} \rho'(z)\left[\int_\xi^z \Delta P(t)dt\right]dz - \int_\xi^{z_u} \rho'(z)\left[\int_z^\xi \Delta P(t)dt\right]dz \\ &= \int_{z_l}^{\xi} \rho'(z)\left[\int_z^\xi \Delta P(t)dt\right]dz + \int_\xi^{z_u} [-\rho'(z)]\left[\int_z^\xi \Delta P(t)dt\right]dz.\end{aligned}$$

Since $\rho'(z) \geq 0$ (or $m''(z) \leq 0$) over (z_u, ξ) and $\rho'(z) \leq 0$ (or $m''(z) \geq 0$) over (ξ, z_u), the sufficiency of (4.4) is straightforward. Following Atkinson's proof of his condition IIA, the necessity of (4.4) could be proved by using a well-known result that a convex (or a concave) function can be uniformly approximated by piecewise linear functions. We can also construct a contradiction directly as follows. Suppose the first part of (4.4) is false and $\int_z^\xi \Delta P(t)dt > 0$ for $z \in (c, d) \subset (z_l, z_u)$, then we can construct a $\rho(z)$ such that[2]

$$\rho'(z) = \gamma > 0 \text{ for } c < z < d \text{ and } \rho'(z) = 0 \text{ for } z \in (z_l, z_u)\backslash(c, d).$$

It follows that $\Delta\Pi(m) = \gamma \int_c^d \left[\int_z^\xi \Delta P(t)dt\right] dz > 0$ - the required contradiction. □

The conditions derived in Proposition 4.1 are valid for any crisp poverty measure. Condition (4.3) can be viewed as a first-order condition while condition (4.4) can be regarded as of second-order nature. Clearly, the first-order condition implies the second-order conditions. The first-order condition requires a dominance of crisp poverty values over the entire poverty border region $[z_l, z_u]$. This result is very intuitive: if any value $\tilde{z}$ within $[z_l, z_u]$ can be a poverty line, then by choosing a membership function such that $\rho(z) > 0$ over $(\tilde{z} - \varepsilon, \tilde{z} + \varepsilon)$ and $\rho(z) = 0$ elsewhere, (4.5) collapses to $P(F; \tilde{z}) \leq P(G; \tilde{z})$ as $\varepsilon \to 0$.

The second-order condition (4.4) is interesting: it is a bifurcated Lorenz-type dominance condition (Figure 2). The device is constructed as follows: from the switch point ξ, poverty values are cumulated toward both ends. Since $P(\cdot; z)$ generally

[2] An example of $\rho(z)$ is

$$\rho(z) = \begin{cases} \gamma c + \theta & \text{if } z_l < z < c \\ \gamma z + \theta & \text{if } c < z < d \\ \gamma d + \theta & \text{if } d < z < z_u \end{cases}$$

where $\gamma > 0$ and $\theta > 0$ satisfy

$$w\gamma + (z_u - z_l)\theta = 1$$

with $w = \frac{1}{2}(c^2 - d^2) - cz_l + dz_u > 0$. It is easy to verify that such a $\rho(z)$ is a "density function" for a poverty membership function. Note that the definition excludes a small neighborhood at z_l and z_u to enable $\rho(z_l) = \rho(z_u) = 0$.

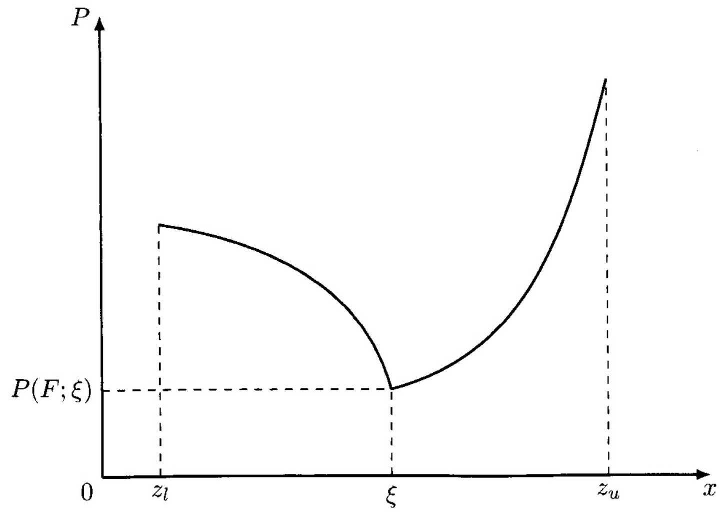

Figure 2: Bifurcated Dominance

increases as z increases, the left panel of the device is a concave curve while the right panel is a convex one. Distribution F has less poverty than distribution G if and only if the device of F lies below that of G.

The switching point ξ is significant in the condition. It can be viewed as the center of the poverty border region (i.e., between z_l and z_u). For example, in 2013, the official poverty line of \$11,490 can be regarded as the center and 25% above the value (i.e., \$14,362.5) is the upper bound z_u, and 25% below the value (i.e., \$8,617.5) is the lower bound z_l. In condition (4.4), if ξ approaches either end of the poverty border region, only one part of the condition survives and it becomes a simple cumulation of the first-order condition (4.3). For example, if $\xi \to z_l$, the second-order condition becomes

$$\int_{z_l}^{z} P(F;t)dt \leq \int_{z_l}^{z} P(G;t)dt \text{ for all } z \in [z_l, z_u].$$

If one is unsure about the location of ξ or one allows ξ to vary over the entire poverty border region, then the second-order condition may collapse to the first-order condition. To see this, in the first condition of (4.4), let $z = \xi - \varepsilon$ for some small $\varepsilon > 0$ and dividing both sides by ϵ, we have

$$\frac{1}{\varepsilon}\int_{\xi-\varepsilon}^{\xi} P(F;t)dt \leq \frac{1}{\varepsilon}\int_{\xi-\varepsilon}^{\xi} P(G;t)dt. \tag{4.6}$$

Let $\varepsilon \to 0$ and taking limits of the both sides, we would obtain $P(F;\xi) \leq P(G;\xi)$ and condition (4.3) ensues as ξ varies over $[z_l, z_u]$. This result is summarized in the following corollary.

Corollary 4.1. The second-order condition (4.4) becomes the first-order condition (4.3) if ξ can take any value in $[z_l, z_u]$.

4.2. Poverty-measure partial ordering for a given membership function

As afore-suggested, it would be useful to extend poverty-measure partial oderings to fuzzy poverty measurement. Here a poverty membership function is given and we wish to establish dominance conditions for one distribution to have unambiguously less poverty than another distribution as judged by all possible fuzzy poverty measures. Similar to the practice in the literature of crisp-poverty-measure partial orderings, we limit our derivations to additively separable (decomposable) fuzzy poverty measures.[3]

For an additively separable (decomposable) fuzzy poverty measure, distribution F has no more poverty than distribution G if and only if

$$\Pi(F;m) = \int_0^\infty \int_0^\infty p(x,z)\rho(z)dzdF(x) \leq \int_0^\infty \int_0^\infty p(x,z)\rho(z)dzdG(x) = \Pi(G;m)$$

which becomes (by specifying the poverty border region)

$$\int_{z_l}^{z_u} \rho(z)\left[\int_0^z p(x,z)dF(x)\right]dz \leq \int_{z_l}^{z_u} \rho(z)\left[\int_0^z p(x,z)dG(x)\right]dz. \quad (4.7)$$

In crisp partial poverty orderings ($z_u \to z_l$), the poverty line is the same in all poverty deprivation computations. The comparisons of poverty values are equivalent to the comparisons of income shortfalls from the (same) poverty line. In fuzzy poverty orderings (4.7), however, poverty values computed from different poverty lines are compared (i.e., $p(x,z)$ is compared with $p(x',z')$ for $x \neq x'$ and $z \neq z'$) and, consequently, a general dominance condition for (4.7) with all possible poverty deprivation functions seems elusive if not impossible to establish. To gain some tractability, in what follows we limit our investigation to the two afore-defined types of poverty measures, namely the absolute classes ($\mathfrak{P}_1^a$ and $\mathfrak{P}_2^a$) and the relative classes ($\mathfrak{P}_1^r$ and $\mathfrak{P}_2^r$). Either type of poverty measures enables the distances between incomes and the poverty lines to be quantified (i.e., $z - x$ or x/z) and compared.[4] Below we first consider absolute poverty measures.

Proposition 4.2. For a given membership density function $\rho(z)$ with $z \in [z_l, z_u]$, condition (4.7) holds for all poverty measures in $\mathfrak{P}_1^a$ if and only if

$$\int_{z_l}^{z_u} \rho(z)F_z(z-y)dz \leq \int_{z_l}^{z_u} \rho(z)G_z(z-y)dz \text{ for all } y \in [0, z_u] \quad (4.8)$$

where $F_z(x)$ is $F(x)$ censored at z: $F_z(x) = F(x)$ for $x < z$ and $F_z(x) = 1$ for $x \geq z$ ($G_z(x)$ is similarly defined); and condition (4.7) holds for all poverty measures in $\mathfrak{P}_2^a$

[3] Note that the dominance conditions derived for crisp decomposable measures often are only sufficient for rank-based poverty measures such as the Sen measure - see Zheng (2000) for a detailed exposition.

[4] One could also consider a more general intermediate poverty measure, $\tilde{p}(\frac{z-x}{z^\eta})$, which contains both relative and absolute measures as special cases ($\eta = 1$ and 0, respectively). For a general treatment of intermediate poverty measures, see Zheng (2007).

if and only if

$$\int_0^y \int_{z_l}^{z_u} \rho(z) F_z(z-t) dz dt \leq \int_0^y \int_{z_l}^{z_u} \rho(z) G_z(z-t) dz dt \text{ for all } y \in [0, z_u]. \tag{4.9}$$

Proof. Denote $y \equiv z - x$ for $x < z$ (and $y \equiv z$ for $x \geq z$) and denote y's cumulative distribution functions $\tilde{F}_z(y)$ and $\tilde{G}_z(y)$, respectively. Then $\tilde{F}_z(y)$ is related to $F(x)$ via

$$\begin{aligned} \tilde{F}_z(y_0) &= P\{y \leq y_0\} = P\{z - x \leq y_0\} = P\{x \geq z - y_0\} \\ &= 1 - F_z(z - y_0) \end{aligned} \tag{4.10}$$

where $F_z(x) = F(x)$ for $x < z$ and $F_z(x) = 1$ for $x \geq z$. We then have

$$\begin{aligned} \Pi(F; m) &= \int_{z_l}^{z_u} \rho(z) \left[\int_0^z \tilde{p}(z - x) dF(x) \right] dz = \int_{z_l}^{z_u} \rho(z) \left[\int_z^0 \tilde{p}(y) dF(z - y) \right] dz \\ &= \int_{z_l}^{z_u} \rho(z) \left[\int_z^0 \tilde{p}(y) d\{1 - \tilde{F}_z(y) \right] dz \text{ (by (4.10))} \\ &= \int_{z_l}^{z_u} \rho(z) \left[\int_0^z \tilde{p}(y) d\tilde{F}_z(y) \right] dz \\ &= \int_{z_l}^{z_u} \int_0^{z_u} \rho(z) \tilde{p}(y) \tilde{f}_z(y) dy dz \end{aligned}$$

where $\tilde{f}_z(y)$ is the density function of $\tilde{F}_z(y)$ with $\tilde{f}_z(y) = 0$ for $y \geq z$. In the last step, we have also replaced the upper bound z with z_u which is the maximum value of z. Reverse the order of integration,

$$\begin{aligned} \Pi(F; m) &= \int_0^{z_u} \int_{z_l}^{z_u} \rho(z) \tilde{p}(y) \tilde{f}_z(y) dz dy \\ &= \int_0^{z_u} \tilde{p}(y) \int_{z_l}^{z_u} \rho(z) \tilde{f}_z(y) dz dy \\ &= \int_0^{z_u} \tilde{p}(y) d \left[\int_{z_l}^{z_u} \rho(z) \tilde{F}_z(y) dz \right] \end{aligned}$$

and (4.7) becomes

$$\int_0^{z_u} \tilde{p}(y) d \left[\int_{z_l}^{z_u} \rho(z) \tilde{F}_z(y) dz \right] \leq \int_0^{z_u} \tilde{p}(y) d \left[\int_{z_l}^{z_u} \rho(z) \tilde{G}_z(y) dz \right]. \tag{4.7a}$$

For all poverty measures in $\mathfrak{P}_1^a$, $\tilde{p}(0) = 0$ and $\tilde{p}'(y) = -\frac{\partial \tilde{p}}{\partial x} > 0$, the necessary and sufficient condition for (4.7a) can be shown (again using the Atkinson method of contradiction) to be

$$\int_{z_l}^{z_u} \rho(z) \tilde{F}_z(y) dz \geq \int_{z_l}^{z_u} \rho(z) \tilde{G}_z(y) dz \text{ for all } y \in [0, z_u]. \tag{4.11}$$

For all poverty measures in $\mathfrak{P}_2^a$, $\tilde{p}(0) = 0$, $\tilde{p}'(y) > 0$ and $\tilde{p}''(y) < 0$, the necessary and sufficient condition for (4.7a) can also be shown to be

$$\int_0^y \int_{z_l}^{z_u} \rho(z)\tilde{F}_z(t)dzdt \geq \int_0^y \int_{z_l}^{z_u} \rho(z)\tilde{G}_z(t)dzdt \text{ for all } y \in [0, z_u]. \quad (4.12)$$

Substituting (4.10) into (4.11), we have

$$\int_{z_l}^{z_u} \rho(z)[1 - F_z(z-y)]dz \geq \int_{z_l}^{z_u} \rho(z)[1 - G_z(z-y)]dz \text{ for all } y \in [0, z_u] \quad (4.11a)$$

or equivalently (because $\int_{z_l}^{z_u} \rho(z)dz = 1$),

$$\int_{z_l}^{z_u} \rho(z)F_z(z-y)dz \leq \int_{z_l}^{z_u} \rho(z)G_z(z-y)dz \text{ for all } y \in [0, z_u]. \quad (4.11b)$$

Condition (4.9) is obtained similarly. □

Conditions (4.8) and (4.9) generalize the partial ordering conditions for crisp poverty measures. It is well known that the first-order dominance condition for all poverty measures (or the relative and the absolute subclasses) with $p_x(x, z) < 0$ and for a single poverty line z is $F(x) \leq G(x)$ over $x \in [0, z]$. Here for poverty orderings with fuzzy poverty measures, we compare the weighted censored *cdf* functions over $[0, z_u]$, $\int_{z_l}^{z_u} \rho(z)F_z(z-y)dz$ and $\int_{z_l}^{z_u} \rho(z)G_z(z-y)dz$. If the poverty border region degenerates to a single point z, i.e., $z_u \to z_l$, condition (4.8) collapses to $F_z(z-y) < G_z(z-y)$ which is equivalent to $F(x) < G(x)$ for all $x \in [0, z]$ - thus (4.8) includes the ordinary first-order condition as a special case.

For the relative classes of poverty measures, we have:

Proposition 4.3. For a given membership density function $\rho(x)$ with $x \in [z_l, z_u]$, condition (4.7) holds for all poverty measures in $\mathfrak{P}_1^r$ if and only if

$$\int_{z_l}^{z_u} \rho(z)F_z(yz)dz \leq \int_{z_l}^{z_u} \rho(z)G_z(yz)dz \text{ for all } y \in [0, 1] \quad (4.8a)$$

where $F_z(x)$ is $F(x)$ censored at z: $F_z(x) = F(x)$ for $x < z$ and $F_z(x) = 1$ for $x \geq z$ ($G_z(x)$ is similarly defined); and condition (4.7) holds for all poverty measures in $\mathfrak{P}_2^r$ if and only if

$$\int_0^y \int_{z_l}^{z_u} \rho(z)F_z(tz)dzdt \leq \int_0^y \int_{z_l}^{z_u} \rho(z)G_z(tz)dzdt \text{ for all } y \in [0, 1]. \quad (4.9a)$$

Proof. Denote $y \equiv \frac{x}{z}$ for $x < z$ (and $y \equiv 1$ for $x \geq z$) and y's cumulative distribution functions $\tilde{F}_z(y)$ and $\tilde{G}_z(y)$, respectively. Then $\tilde{F}_z(y)$ is related to $F(x)$ through

$$\tilde{F}_z(y_0) = P\{y \leq y_0\} = P\{\frac{x}{z} \leq y_0\} = P\{x \leq y_0 z\} = F_z(y_0 z)$$

where again $F_z(x) = F(x)$ for $x < z$ and $F_z(x) = 1$ for $x \geq z$. The rest of the proof is similar to that for the case of absolute poverty measures (but noting that now $\tilde{p}'(y) < 0$ and $\tilde{p}''(y) > 0$). □

It is interesting to note one more difference between crisp poverty orderings and fuzzy poverty orderings: for a given (and same) poverty line, both the relative class and the absolute class of poverty measures yield the same dominance condition; the condition for the class of relative fuzzy poverty measures and the class of absolute poverty measures - a counterexample can be easily constructed - could be quite different.

4.3. Poverty-membership-measure partial ordering

Finally one may naturally be interested in establishing dominance conditions for all possible additively separable fuzzy poverty measures - that is, for all possible crisp decomposable poverty measures and all possible membership functions. Given the results obtained above, we can achieve this by following two routes: considering all membership functions first, then requiring the results to hold for all poverty measures; or considering all poverty measures first, then requiring the results to hold for all membership functions.

For example, if we wish to derive dominance conditions for all membership functions in $\mathfrak{M}_1$ and for all absolute poverty measures in $\mathfrak{P}_1^a$, we could start from Proposition 4.1 and require the results to hold for all absolute poverty measures. This approach amounts to requiring (4.3) to hold for all absolute poverty measures with $\tilde{p}'(x) > 0$ and for all $z \in [z_l, z_u]$, which yields $F(x) \leq G(x)$ for all $x \in [0, z_u]$. We could also start from Proposition 4.2 and require the results to hold for all membership functions in $\mathfrak{M}_1$. This alternative approach amounts to requiring (4.8) to be valid for all possible membership density functions $\rho(z)$. We would have $F(z-y) \leq G(z-y)$ for all $x \in [0, z_u]$ and all $z \in [z_l, z_u]$. Clearly, by setting $x = z - y$, this route also leads to $F(x) \leq G(x)$ for all $x \in [0, z_u]$.

As another example, if we want to establish dominance conditions for all membership functions in $\mathfrak{M}_2$ and all absolute poverty measures in $\mathfrak{P}_1^a$, we could start from (4.8) and require $\rho'(z)$ to change sign from positive to negative at $z = \xi$. Following the proof of the second part of Proposition 4.1, we would be able to establish the following condition

$$\begin{aligned}
\int_s^\xi F_z(z-y)dz &\leq \int_s^\xi G_z(z-y)dz \text{ for all } y \in [0, \xi] \text{ and } s \in [z_l, \xi], \text{ and} \\
\int_\xi^s F_z(z-y)dz &\leq \int_\xi^s G_z(z-y)dz \text{ for all } y \in [0, z_u] \text{ and } s \in [\xi, z_u].
\end{aligned}$$

But the first part of the condition can be simplified to $F_\xi(\xi - y) \leq G_\xi(\xi - y)$ for all $y \in [0, \xi]$ since it implies $F_s(s-y) \leq G_s(s-y)$ for any $s \leq \xi$. The condition $F_\xi(\xi - y) \leq G_\xi(\xi - y)$ further turns out to be a part of the second condition (by letting $s \to \xi$). The second part of the condition, however, cannot be simplified any further. The alternative route - starting from (4.4) and requiring them to hold for all $\tilde{p}(y)$s with $\tilde{p}'(y) \geq 0$ - would also lead to the same conclusion (by applying crisp poverty ordering results to each part of (4.4)). Note that the dominance condition now consists of only the right half of the bifurcated device in Figure 2.

The following two propositions summarize the results of partial poverty orderings for all membership functions in $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and for all poverty measures in the relative and absolute classes, respectively. The additional results beyond what we have demonstrated above can also be easily verified.

Proposition 4.4. For any distributions F and G: (*i*) $\Pi(F;m) \leq \Pi(G;m)$ for all membership functions in $\mathfrak{M}_1$ and all poverty measures in $\mathfrak{P}_1^a$ if and only if

$$F(x) \leq G(x) \text{ for all } x \in [0, z_u]; \tag{4.13}$$

(*ii*) $\Pi(F;m) \leq \Pi(G;m)$ for all membership functions in $\mathfrak{M}_1$ and all poverty measures in $\mathfrak{P}_2^a$ if and only if

$$\int_0^x F(t)dt \leq \int_0^x G(t)dt \text{ for all } x \in [0, z_u]; \tag{4.14}$$

(*iii*) $\Pi(F;m) \leq \Pi(G;m)$ for all membership functions in $\mathfrak{M}_2$ and all poverty measures in $\mathfrak{P}_1^a$ if and only if

$$\int_\xi^s F_z(z-y)dz \leq \int_\xi^s G_z(z-y)dz \text{ for all } y \in [0, z_u] \text{ and } s \in [\xi, z_u]; \tag{4.15}$$

and (*iv*) $\Pi(F;m) \leq \Pi(G;m)$ for all membership functions in $\mathfrak{M}_2$ and all poverty measures in $\mathfrak{P}_2^a$ if and only if

$$\int_0^y \int_\xi^s F_z(z-t)dzdt \leq \int_0^y \int_\xi^s G_z(z-t)dzdt \text{ for all } y \in [0, z_u] \text{ and all } s \in [\xi, z_u]. \tag{4.16}$$

Proposition 4.5. For any distributions F and G: (*i*) $\Pi(F;m) \leq \Pi(G;m)$ for all membership functions in $\mathfrak{M}_1$ and all poverty measures in $\mathfrak{P}_1^r$ if and only if

$$F(x) \leq G(x) \text{ for all } x \in [0, z_u];$$

(*ii*) $\Pi(F;m) \leq \Pi(G;m)$ for all membership functions in $\mathfrak{M}_1$ and all poverty measures in $\mathfrak{P}_2^r$ if and only if

$$\int_0^x F(t)dt \leq \int_0^x G(t)dt \text{ for all } x \in [0, z_u];$$

(*iii*) $\Pi(F;m) \leq \Pi(G;m)$ for all membership functions in $\mathfrak{M}_2$ and all poverty measures in $\mathfrak{P}_1^r$ if and only if

$$\int_{\xi}^{s} F_z(yz)dz \le \int_{\xi}^{s} G_z(yz)dz \text{ for all } y \in [0,1] \text{ and all } s \in [\xi, z_u]; \tag{4.15a}$$

and (iv) $\Pi(F;m) \le \Pi(G;m)$ for all membership functions in $\mathfrak{M}_2$ and all poverty measures in $\mathfrak{P}_2^r$ if and only if

$$\int_{0}^{y}\int_{\xi}^{s} F_z(xz)dzdt \le \int_{0}^{y}\int_{\xi}^{s} G_z(xz)dzdt \text{ for all } y \in [0,1] \text{ and all } s \in [\xi, z_u]. \tag{4.16a}$$

In empirical applications, one may want to combine the conditions for the relative class and the absolute class of poverty measures to strengthen the robustness of poverty orderings.

5. Summary and Conclusion

Judging an individual "to be" or "not to be" in poverty is an important first question in poverty measurement. The almost four-decade long poverty measurement research started from Sen (1976), however, has not paid sufficient attention to this important question. The dichotomous treatment of the poor and the non-poor has largely remained the "rule of the land" in empirical poverty studies. But the concept of poverty, as many have argued, is not a crisp one and there exist "borderline" situations where it is difficult to proclaim an individual to be a definite poor or a definite non-poor. Poverty, like many linguistic variables such as "tall" and "pretty," is a fuzzy predicate and poverty measurement must take this fuzziness into account. The fuzzy set theory, developed over the last-half century, can be fruitfully employed to extend crisp poverty measurement into fuzzy poverty measurement.

At the heart of fuzzy poverty measurement is a poverty membership function - it quantifies the degree of an individual's belonging to the poverty population. The calibration of a membership function, however, depends on the source of fuzziness. In the paper we argued that the most plausible source is that people (or voters) have different perceptions about what constitutes poverty. This difference-in-perception characterization of fuzziness renders a direct estimation of the poverty membership function: the degree of poverty membership for an individual with income x is simply the cumulative percentage of voters who believe the poverty line should be at least x. In the paper, we considered two sets of membership functions: the first set contains all functions decreasing in income while the second set contains all inverted-S shape membership functions (Figure 1). To provide a meaningful interpretation for the membership function, we further defined a "density function" which is the proportion of the voters who believe the poverty line to be at a given income level. With the help of two intuitive equal voting-right axioms, the difference-in-perception approach also provides an axiomatic characterization to the class of decomposable fuzzy poverty measures that Shorrocks and Subramanian (1994) defined. Compared with their characterization, ours is more elementary and intuitive.

The other major contribution of the paper was to provide a set of dominance conditions for fuzzy partial poverty orderings. Since a fuzzy poverty measure involves the choices of a crisp poverty measure and a poverty membership function, there are three types of partial poverty orderings we need to consider: the orderings by all possible membership functions with a given crisp poverty measure, the orderings by all crisp poverty measures with a given membership function, and the orderings by all possible crisp poverty measures *and* all membership functions. The first-order dominance condition we derived for all membership functions is akin to a first-order stochastic dominance applied to vectors of sorted poverty values. But the second-order condition is a bifurcated Lorenz-type dominance with origin at the "switch point" of the membership functions (Figure 2). The dominance conditions for all crisp poverty measures that we established are for two types of decomposable poverty measures, namely the absolute classes and the relative classes. All these conditions can be easily applied to real income data.

We hope that the discussions presented and the results derived in the paper would be useful to poverty researchers. We also hope that the fact that poverty is a fuzzy predicate will be taken more seriously in future poverty measurement research. Moving forward, there are a number of specific issues remain to be topics for further research. First, the empirical estimation of a poverty membership function. Although our approach makes it possible to estimate a membership function from empirical data, no such empirical studies have been done. To implement the estimation we may need to collect special data on people's perception of poverty. Second, the axiomatic characterization we developed in Section 2 is only for decomposable (additively separable) fuzzy poverty measures. It is useful to develop a similar characterization for all rank-based poverty measures such as the Sen measure. Third, the dominance conditions we derived for crisp poverty measures are only for the absolute and relative subclasses. It would certainly be helpful to know whether similar dominance conditions exist for all poverty measures. Finally, in considering different membership functions of the inverted-S shape we have maintained a same "switch point" assumption - the same ξ value for all distributions. This assumption would certainly be viewed as too restrictive as different societies or different populations may have different "switch points." It would also be useful to extend our conditions to the cases with different values of ξ.

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