Understanding the mechanical relationship between inequality and intergenerational mobility

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Abstract
Income inequality and income intergenerational immobility are positively associated across countries. Here we provide an explanation for this association and show it is mechanically driven by the definition of the intergenerational earnings elasticity. This may hint that higher intergenerational mobility will lead to lower income inequality and vice versa. However, it questions the underlying economic significance of the empirical findings depicted in the so-called “Great Gatsby curve”, as it presents the relationship between two measures of the same underlying mathematical property. We also find a similar relationship between the intergenerational earnings elasticity and the rank-rank slope. We conclude that measuring inequality using the Gini coefficient and intergenerational mobility using intergenerational earnings elasticity or rank-rank slope is fundamentally equivalent.

Keywords: Mobility, inequality, econometrics, stochastic processes

JEL Codes: C0, D0, J0

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1 Introduction

The relationship between economic inequality and intergenerational mobility has recently garnered interest within and outside academic circles. In many studies such a relationship is described, addressing different types of inequality and mobility and different measures of both. One of the most common approaches for quantifying income inequality—the Gini coefficient (Gini [1921])—and one of the most common approaches for quantifying intergenerational mobility—intergenerational earnings elasticity (IGE) (Mulligan [1997] Corak [2006])—are known to be positively associated across countries, and the presentation of their association is known as the Great Gatsby curve (GGC) (Krueger [2012] Corak [2012]).

Here we present a theoretical analysis describing the relationship between IGE and the income Gini coefficient and quantitatively describe their mutual effect. We find that the earnings elasticity definition mechanically implies a degree of inequality. As a consequence, we argue that the Great Gatsby curve is a tautological presentation of a mathematical relationship rather than an indication of a deeper economic relationship. Nevertheless, these findings also indicate that indeed IGE and income inequality are dependent rather than only statistically associated.

Economic inequality and intergenerational mobility are related to both ethical aspects of society and to the prosperity of economies and their output. Intuitively, mobility and inequality may be considered as tightly related and perhaps influential on each other. In terms of ethics, they can be considered as measures of “fairness”. However, while these conceptual statements and perceptions are intuitive, mathematical analysis is required in order to settle whether mobility and inequality are indeed tightly related. When such an analysis is done, clear mathematical definitions of inequality and mobility are required.
The canonical measure of mobility, used in most studies, is the intergenerational earnings elasticity. It measures how “sticky” are earnings across generations within the same family or household, and is relatively simple to estimate (Grawe, 2004; Corak, 2012). However, economic mobility can be measured in alternative methods. An alternative measure widely-used is the Rank-Rank slope (RRS) (Dahl and DeLeire, 2008). Children are ranked based on their incomes relative to other children in the same birth cohort. The parents of these children are ranked based on their incomes relative to other parents with children in these birth cohorts and the RRS is defined as the correlation between children’s and parents’ positions in the income distribution. Chetty et al. (2014a) describe the practical advantage of RRS over the IGE and argue that rank-rank slopes “prove to be much more robust [than IGE] across specifications and are thus more suitable for comparisons across areas from a statistical perspective” (p. 1561).

Other measures can be used as well, such as the autocorrelation of the earnings ranking (Fields and Ok, 1999; Liu et al., 2013; Berman, Peters and Adamou, 2017) and the probability for moving up and down between quantiles (Van de Gaer, Schokkaert and Martinez, 2001; Erikson and Goldthorpe, 2002; Bradbury and Katz, 2009; Marrero and Rodríguez, 2013; Chetty et al., 2014b; Berman, Shapira and Ben-Jacob, 2015).

Mobility is also measured in absolute terms—for example, as the fraction of children earning more than their parents (Chetty et al., 2017). The trends in absolute mobility in income in the United States have also been linked to the growing income inequality during the past few decades (Chetty et al., 2017; Katz and Krueger, 2017; Berman, 2017). However, they are known to be fundamentally different from, and even inversely related to relative mobility measures, such as the IGE (Chetty et al., 2017; Berman, 2017).
Others, such as Leman (1980), Jäger and Holm (2007), Hertel and Groh-Samberg (2014), Clark (2014), Braun and Stuhler (2016), Neidhöfer and Stockhausen (2016), measure and study mobility assuming that not only the previous generation affects the incomes of the current generation, but also earlier generations. However, due to scarcity of reliable data, such measures of mobility are rarely used.

The various measures of economic mobility are partly incompatible, specifically since the IGE is not derived from the income ranking of individuals or families, nor directly from the earnings distribution like the other measures. As a result, mobility can be very high hypothetically, with families shuffled between each other across generations, while the earnings distribution itself is very unequal. Changes in the income inequality, derived from the income distribution, can therefore be independent of mobility and of changes in mobility, as demonstrated empirically by Lee and Solon (2009); Hauser (2010); Chetty et al. (2014b). They show that although income inequality had been increasing rapidly since the 1970s in the US, no significant decrease in the chances of a child to reach the top quintile of the income distribution, starting from the bottom quintile, were observed during this period.

Becker and Tomes (1979) have demonstrated a positive relationship between inequality within a generation and immobility across generations, based on a microeconomic model for a family, assuming utility-maximizing behavior. Maoz and Moav (1999) have reached similar conclusions, also relating mobility and equality to growth. They base their results on a model in which individuals maximize their utility, subject to their cost of acquiring education and inherited income. Galor and Tsiddon (1997), however, obtain different results. They show that “in periods of major technological inventions, a decline in the relative importance of initial conditions raises inequality, enhances mobility, and generates a larger concentration of high-ability individuals in technologically advanced sectors, stimulating future technological progress
and growth” (p. 363). Goldberger (1989) partially criticized these approaches, suggesting that “the structural relationships to be found [...] may not rest on utility maximization [...]” (p. 513).

Mulligan (1997) presented several theoretical models arguing that “the intergenerational transmission of economic status affects the evolution of inequality over time [...]” (p. 1), but put emphasis on the role parental priorities have on intergenerational mobility, rather than standard utility-maximizing behavior. Corak (2006, 2013) expanded this approach providing a cross-country comparison of intergenerational earnings mobility, demonstrating quantitatively how parental economic status is related to the labor market success of children in adulthood. Corak’s dataset was then used to present the Great Gatsby curve (Krueger, 2012; Corak, 2012) (GGC), depicting the positive relationship found between immobility and inequality. Similar findings were recently described by Neidhöfer (2016), Urrutia and Tavares (2016) and Brahim and McLeod (2016) focusing on Latin-American economies. Clark (2014) provided a wide historical perspective to the processes controlling social and economic mobility in different countries and repeated the same view on its relationship with inequality.

Solon (2004) presented an important theoretical justification, supporting the empirical analysis of Gravel (2004); Krueger (2012); Corak (2013), using a similar approach to that of Becker and Tomes (1979), in which intergenerational earnings elasticity and cross-sectional income inequality are found to be “greater in the presence of stronger heritability, more productive human capital investment, higher returns to human capital, and less progressive public investment in human capital” (p. 44). This conclusion is based on the importance of the return to schooling, associated with lower intergenerational earnings mobility (Solon, 2004; Corak, 2013; Landersø and Heckman, 2017; Neidhöfer, 2016; Rauh, 2016).
In this paper we test whether such a relationship, between intergenerational mobility and economic inequality, found in data, simply stems from the mathematical definition of the intergenerational earnings elasticity. We find that the definition of intergenerational earnings elasticity mechanically drives the creation of inequality within a generation. We also validate our theoretical observation, which echoes the claims made by Goldberger (1989), using the data presented by Corak (2012). We conclude by suggesting, following these findings, that inequality and intergenerational earnings elasticity indeed affect each other, and reducing (increasing) one may reduce (increase) the other. Using the rank-rank slope as a measure of intergenerational mobility, we find a similar result—once again, the lack of mobility mechanically creates some inequality. For this reason, we argue that the Gini coefficient, IGE and RRS all measure the same underlying property of the income distribution, rather than distinct properties of the economy. Hence, the Great Gatsby curve is, to a large extent, a tautological presentation of a mathematical relationship rather than an indication of a deeper economic relationship.

Our findings do not invalidate the views regarding the existence of a deep relationship between intergenerational mobility and income inequality. However, we argue that in order to provide significant empirical evidence for such a relationship, one would need to use intergenerational mobility and inequality measures which can not be mechanically deduced from one another.

1.1 Outline of the Paper

The paper is structured as follows. In Section 2 the background theory and definitions are given and the Great Gatsby curve presented. In Section 3 we use the definition of the intergenerational earnings elasticity to show how it induces inequality within
2 The Great Gatsby Curve

The Gini coefficient is one of the most commonly used measures of economic inequality (Gini [1921]; Cowell [2011]). It is derived directly from the income or wealth distributions and can be defined as half of the relative mean absolute difference (Sen [1973]). Assuming $Y_i$ is the income of the $i$th individual (or household, or family) in a population of size $N$, then the Gini coefficient $G$ is given by:

$$G = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} |Y_i - Y_j|}{2N \sum_{j=1}^{N} Y_j}$$

(2.1)

It follows that the Gini coefficient is invariant under multiplication by a constant and under permutations, implying that if any number of individuals are swapped in terms of their income, for example, it will not affect the income Gini coefficient.

The intergenerational earnings elasticity, described above, is derived from a regression-to-the-mean model. It is defined as the least squares estimate of the coefficient $\beta$ in the following equation (Corak [2006]):

$$\ln Y_{i,t} = \alpha + \beta \ln Y_{i,t-1} + \epsilon_{i,t}.$$  

(2.2)

$Y$ represents the permanent earnings of an individual or a family indexed by $i$. Equation (2.2) is a model for the propagation of $Y_i$ across two generations, indexed $t$ and $t-1$. The parameter $\alpha$ "captures the trend in average incomes across generations, due, for example, to changes in productivity, international trade, technology, or labor
market institutions” (Corak, 2013, p. 81). \( \epsilon \) incorporates additional effects on the \( t \) generation, which are not correlated with the \( t - 1 \) generation and are not consistent, as otherwise would be considered as a part of the trend \( \alpha \). Equation (2.2) implies that the higher \( \beta \) is, or the higher the IGE is, the more it is possible to deduce on the child’s earnings from the parent’s earnings and vice versa.

Regardless of the mathematical independence of the income Gini coefficient and mobility, if measured by quantifying swapped individual incomes, empirical evidence suggests that higher inequality is indeed associated with lower mobility, when measured in terms of IGE. The IGE and the income Gini coefficient were measured cross-sectionally across countries and a plot of these data, referred to as the Great Gatsby curve (Krueger, 2012; Corak, 2012) (GGC), illustrates their association. The GGC is presented in Fig. 1.

### 3 Mobility-Implied Inequality

Based on the above discussion, let us consider a population consisting of \( N \) families (or households), indexed by \( i = 1, \ldots, N \), and each assigned an initial income \( Y_{i,1} \). We denote by \( Y_{i,t} \) the income of the \( i \)-th family at generation \( t \), for \( t = 1, 2, \ldots \). We follow Eq. (2.2), and assume it dictates the dynamics of \( Y_{i,t} \), i.e. knowing \( Y_{i,t} \) allows calculating \( Y_{i,t+1} \) subject to the error term.

Denoting \( X_{i,t} = \ln Y_{i,t} \) and assuming that for a given generation \( t \) the error term, \( \epsilon_{i,t} \), is a normal random variable with variance \( \sigma^2 \) and zero expectation (Corak, 2006), we can rewrite Eq. (2.2), so \( X_{i,t} \) obey a simple autoregressive (AR(1)) model (Hodge, 1966; Clark, 2014; Solon, 2015):

\[
X_{i,t} = \alpha + \beta X_{i,t-1} + \sigma \omega_{i,t},
\]  

(3.1)
Figure 1: The Great Gatsby curve. The intergenerational earnings elasticity measured as the elasticity between paternal earnings and a son’s adult earnings, using data on a cohort of children born during the early to mid 1960s and measuring their adult outcomes in the mid to late 1990s in various countries. The income inequality is measured as the Gini coefficient taking values between 0 and 1. The dashed black curve is the linear least-squares best-fit for these data ($R^2 = 0.6$). This figure and its description are reproduced from the original curve appeared in (Corak, 2012) and based on the data in (Corak, 2006) and (Corak, 2013).
where $\omega_{i,t}$ are random variates of a normal distribution with zero expectation and unity variance.

A basic property of $AR(1)$ processes, assuming $0 < \beta < 1$, is that in the $t \to \infty$ limit, the distribution of $X_i$ reaches a steady state and follows $\mathcal{N} \left( \frac{\alpha}{1-\beta}, \frac{\sigma^2}{1-\beta^2} \right)$ (Hamilton 1994). Therefore, assuming the convergence is quick\(^1\) the distribution of $Y_i \sim \ln \mathcal{N} \left( \frac{\alpha}{1-\beta}, \frac{\sigma^2}{1-\beta^2} \right)$ is log-normal. According to Crow and Shimizu (1988), the Gini coefficient of $\ln \mathcal{N}(m, s^2)$ is $\text{erf} \left( \frac{1}{2} \right)$. It follows that there exists a steady state Gini coefficient for the income distribution of $Y_i$:

$$G^* = \text{erf} \left( \frac{1}{2} \sqrt{\frac{\sigma^2}{1-\beta^2}} \right),$$

(3.2)

where erf is the Gauss error function—$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

We can see that the underlying process defining $\beta$ using the regression model Eq. (2.2) effectively implies a degree of inequality, when measured in terms of the Gini coefficient. The origin of this inequality is the multiplicative random process dictating the dynamics of incomes, implied by the IGE definition. This process mechanically guarantees, regardless of the initial state at $t = 1$, development of inequality, quantified by the Gini coefficient (see, for a similar effect (Bouchaud and Mézard 2000; Berman, Peters and Adamou 2017)). We note that $G^*$ does not depend on $\alpha$, due to the Gini coefficient invariance to multiplication by constant.

Furthermore, the dependence of the implied Gini coefficient in Eq. (3.2) on $\beta$ demonstrates that as $\beta$ increases (meaning less mobility) $G^*$ increases as well, as long as $\beta < 1$:

$$\frac{\partial G^*}{\partial \beta} = \beta \sqrt{\frac{\sigma^2}{\pi(1-\beta^2)}} e^{-\frac{\sigma^2}{4(1-\beta^2)}} > 0.$$  

(3.3)

\(^1\)we will address this point in 4
Hence, not only that the regression model implies inequality, we also obtain a built-in positive relationship between the IGE and the income Gini coefficient.

If instead of IGE, we would consider the effect of additional generations on the income of the current generation (Jäger and Holm 2007, Hertel and Groh-Samberg 2014, Braun and Stuhler 2016, Neidhöfer and Stockhausen 2016), we could possibly define a set of elasticities, \( \{ \beta_j \} \), which satisfy

\[
\ln Y_{i,t} = \alpha + \sum_{j=1}^{p} (\beta_j \ln Y_{i,t-j}) + \epsilon_{i,t},
\]

where \( p \) is the number of generations taken into account for the effect on the current generation’s incomes.

Equation (3.4) can be rewritten as

\[
X_{i,t} = \alpha + \sum_{j=1}^{p} (\beta_j X_{i,t-j}) + \sigma \omega_{i,t},
\]

similarly to Eqs. (2.2) and (3.1), so that \( X_{i,t} \) now follows an \( AR(p) \) process. Therefore, assuming quick convergence, as we did for the IGE (or the \( p = 1 \) case), the distribution of \( Y_i \) is log-normal. If \( p = 2 \), meaning that we consider the effect of grandfathers’ and fathers’ incomes on sons’ incomes, we obtain mobility-implied inequality:

**Proposition 1** Considering Eq. (3.5) for \( p = 2 \), and assuming \( \beta_1 + \beta_2 < 1 \) and \( \beta_2 - \beta_1 < 1 \), the income distribution converges asymptotically, and the asymptotic income Gini coefficient is

\[
G^* = \text{erf} \left( \frac{1}{2} \sqrt{\frac{\sigma^2}{\beta_2 - \beta_1^2}} \right),
\]

\( G^* \) is an increasing function in both \( \beta_1 \) and \( \beta_2 \). Once again, even if additional gen-
erations are considered, it follows that more mobility directly implies less inequality. This result is illustrated in Fig. 2.

Figure 2: The theoretical relationship between the Gini coefficient and $\beta_1$, in an $AR(2)$ mobility model. The different curves depict an additional dependence on $\beta_2$. The value of $\sigma$ used was 0.4. The results are independent of $\alpha$ (see Eq. (3.6)).

The mobility-implied inequality we obtain is measure-dependent, and for different measures of mobility, the results may differ. Specifically, we find that if mobility is measured by intergenerational elasticity in which the error terms are additive and not multiplicative as in the IGE, we still obtain an mechanical relationship between inequality and immobility, but it is a negative one (see Appendix B).

3.1 Intergenerational earnings elasticity and the rank-rank slope

We have established a mathematical relationship between the IGE, which dictates the dynamics of incomes, and the income Gini coefficient, measuring how unequal
is the income distribution. We wish to test if this relationship is also valid for an alternative definition of intergenerational mobility, the Rank-Rank Slope (RRS) (Dahl and DeLeire, 2008) and follow its definition as described by Chetty et al. (2014a, p. 1561):

“Let $R_i$ denote child $i$’s percentile rank in the income distribution of children and $P_i$ denote parent $i$’s percentile rank in the income distribution of parents. Regressing the child’s rank $R_i$ on his parents’ rank $P_i$ yields a regression coefficient $\rho_{PR} = Corr(P_i, R_i)$, which we call the rank-rank slope. The rank-rank slope $\rho_{PR}$ measures the association between a child’s position in the income distribution and his parents’ position in the distribution.”

Chetty et al. (2014a, p. 1561) further state that “the correlation of log incomes $\rho_{XY}$ [IGE] and the correlation of ranks $\rho_{PR}$ [RRS] are closely related scale-invariant measures of the degree to which child income depends on parent income.”

Not only that the correlation of log-incomes and the correlation of ranks are closely related empirically, it can be shown that they are strongly dependent theoretically. The same way the dynamics implied by IGE induce income inequality, they also induce RRS. Specifically, we obtain that the induced RRS is roughly 4.5% lower than the IGE in the long run:

**Proposition 2** Considering Eq. (3.1), assuming $\beta < 1$, the RRS, $\rho_{PR}$, converges approximately to $\frac{3\beta}{\pi}$ in the limit $t \to \infty$.

We also note that the asymptotic RRS is independent of $\sigma$, the variance of the error term in Eq. (3.1) and it is obtained so fast (see also Sec. 4) that we should expect the measured RRS to be very close to the measured IGE. This is illustrated
in Fig. 3. We use the value of the IGE to simulate the income dynamics of \( N = 10^7 \) families and numerically calculate the resulting RRS. Convergence is fast—reached within 1–2 generations.

![Graph showing the relationship between IGE and RRS.](image)

Figure 3: The relationship between IGE and RRS. For each value of \( \beta \) we use the dynamics implied by Eq. (2.2) and numerically simulate the income dynamics of a population of \( N = 10^7 \) families. We obtain that within 2–3 generations the RRS converges to its asymptotic value, which is very similar to the IGE (about 2.5% lower). The small discrepancy observed at higher \( \beta \) between the \( 3\beta/\pi \) and the RRS results is due to higher order effects observed (see approximation in Appendix A.2). The numerical calculations were done for \( \sigma = 0.4 \) (blue) and \( \sigma = 0.8 \) (red) and their results were similar, as expected.

Chetty et al. (2014a) report an IGE estimate of 0.344 and a RRS estimate of 0.341. Based on the data reported by Boserup, Kopczuk and Kreiner (2013); Corak and Heisz (1999), they also estimate the RRS in Denmark as 0.18 and in Canada as 0.174, while Corak (2013) estimate their IGEs as 0.15 and 0.19, respectively. We note, however, that the periods and cohorts used for estimation in Corak (2013) are different from those used in Boserup, Kopczuk and Kreiner (2013); Corak and Heisz (1999). Nevertheless, these empirical observations are consistent with the theor-
ical finding, as the empirical error in measuring both RRS and IGE is larger than the theoretical 4.5% discrepancy between them [Corak, 2006; Chetty et al., 2014a]. These observations imply that the relationship found between the IGE and the Gini coefficient is also expected between RRS and the Gini coefficient. We conclude that measuring inequality using the Gini coefficient and intergenerational mobility by the IGE or the RRS is fundamentally equivalent.

3.2 Modification: Constant error term

So far we assumed that the error term $\epsilon_{i,t}$ is different for every generation $t$ and for each family $i$. One might argue that in the same family the error should reflect inherited skill and ability, which may increase the family income relative to the overall population or decrease it. An important modification of the model, which considers this argument, is such that the error term in Eq. (3.1) is fixed in time. We consider

$$X_{i,t} = \alpha + \beta X_{i,t-1} + \sigma \omega_i,$$

assuming that $\omega_i$ are normal iid random variables with zero expectation and unity variance.

Following Eq. (3.7) it is possible show that if $\beta < 1$, the distribution of $X$ again converges to a normal distribution:

**Proposition 3** Assuming $\beta < 1$ and a fixed error term $\omega_i$, the distribution of $X$ converges to a normal distribution, and $\mathcal{N}\left(\frac{\alpha}{1-\beta}, \left(\frac{\sigma}{1-\beta}\right)^2\right)$.

Consequently, $Y_i$ follow a log-normal distribution—$\ln \mathcal{N}\left(\frac{\alpha}{1-\beta}, \left(\frac{\sigma}{1-\beta}\right)^2\right)$ and the steady state Gini coefficient for the income distribution is [Crow and Shimizu, 1988]:
\[ \tilde{G}^* = \text{erf} \left( \frac{\sigma}{2(1-\beta)} \right). \] (3.8)

Similarly to the previous configuration, while \( \beta < 1 \), \( \tilde{G}^* \) increases with \( \beta \):

\[ \frac{\partial \tilde{G}^*}{\partial \beta} = \frac{e^{-\frac{\sigma^2}{4(1-\beta)^2} \sigma}}{\sqrt{\pi} (1-\beta)^2} > 0 \] (3.9)

The theoretical dependencies of the asymptotic Gini coefficient on the IGE according to Eq. (3.2) and Eq. (3.9) are illustrated in Fig. 4. It depicts how the implied income inequality increases with immobility.

Figure 4: The theoretical relationship between the Gini coefficient and the intergenerational earnings elasticity. The different curves depict the dependence of the IGE (\( \beta \)) on \( G^* \) (dashed) and \( \tilde{G}^* \) (solid) for different values of \( \sigma \) (0.2 (blue), 0.5 (red) and 1 (purple)).

The analysis of both configurations demonstrates that the mobility-implied in-
equality is a general result, as long as mobility is defined as the IGE or RRS. This is driven by the assumption that the income dynamics are affected by multiplicative random processes, which induce inequality (Bevan and Stiglitz, 1980; Stiglitz, 2015; Bouchaud and Mézard, 2000; Berman, Peters and Adamou, 2017). We note again that hypothetically, when alternative definitions of mobility are considered, and do not incorporate multiplicative noise implicitly or explicitly, mobility can very high even when the income distribution is very unequal.

The relationships found between $G^*$, $\beta$ and $\sigma$, in both configurations, also indicate the existence of degeneracy in the system. The same value of $G^*$ or $\tilde{G}^*$ may be obtained for different values of $\sigma$ and $\beta$. The results depicted in Fig. 4, for example, demonstrate that $\tilde{G}^*(\beta = 0.48, \sigma = 0.5) = \tilde{G}^*(\beta = 0.79, \sigma = 0.2) = 0.5$. Therefore, at least theoretically, a difference between the income inequality of two populations can be explained not only by a difference of their IGE, but also by different $\sigma$. This implies that when the IGE and the Gini coefficient of different economies are compared in practice, substantial differences in $\epsilon_i$ between these economies, should be also taken into consideration. Nevertheless, in any case, we find a positive relationship between $G$ and $\beta$.

4 Empirical Evidence

The analysis above demonstrated that theoretically, the positive association between income inequality and IGE is mechanically driven by the IGE definition. We will now show that the empirical evidence for this relationship is consistent with the theoretical findings. We use the expressions found in Eq. (3.2) and Eq. (3.9), testing them with respect to data. Using these equations for such a comparison, we explicitly make the following assumptions:
1. The error term $\epsilon$ is normally distributed — this is supported by the data, based on the shape of the error term of the linear regressions done for estimating $\beta$ (Grawe, 2004; Corak, 2006; 2016; Neidhöfer and Stockhausen, 2016). We have also shown that this assumption leads to a log-normal income distribution in the long run, consistent with a simple Gibrat’s law (Gibrat, 1931) (also see Salem and Mount, 1974; Benabou, 2000). A different choice of distribution for the error terms will be inconsistent with this empirical observation.

2. $\sigma$ is fixed for all families and within generations — the error term of the linear regressions used for estimating $\beta$ is slightly different for the various economies analyzed. However, the value of $\sigma$ is relatively similar, whereas the values of $\beta$ and the Gini coefficient of the different countries significantly differ (Grawe, 2004; Corak, 2006; Neidhöfer and Stockhausen, 2016). Following Eq. (3.8) it is also possible to calculate $\sigma$ for each country reported by Corak (2012), according to its value of $\beta$ and Gini coefficient. These values of $\sigma$ lie within a narrow band with an average of 0.39 and a standard deviation of 0.07.

3. $\alpha = 0$ — we know that $\alpha$ does not affect the Gini coefficient since it is invariant under multiplication by a constant factor.

4. The Gini coefficient reaches its asymptotic value very quickly so that the presented data reflects $G^*$ — using a Monte-Carlo simulation we were able to estimate the number of generations taken to reach an steady state $G^*$ and $\tilde{G}^*$ starting from a perfectly equal economy (meaning that every family has exactly the same income, taken as 1). For the parameters considered, meaning $\sigma \in [0.2, 0.6]$ and $\beta \in [0.1, 0.7]$, convergence (defined as being as close as 0.01% to $G^*$) was reached within 2–5 generations. From which we can assume that in practice, when we don’t start from a perfectly equal economy, and the changes
between generations are much less extreme, steady state is reached within 1–2 generations (see also Appendix C).

We can now rewrite Eq. (3.2) and Eq. (3.9) as

\[
\beta = \sqrt{1 - \frac{\sigma^2}{(2 \text{erf}^{-1}(G^*))^2}}
\]

\[
\tilde{\beta} = 1 - \frac{\sigma}{2 \text{erf}^{-1}(\tilde{G}^*)},
\]

where \(\tilde{\beta}\) denotes the value of \(\beta\) corresponding to \(\tilde{G}^*\).

Following the above assumptions and using Eq. (4.1), we calculate the dependence of \(\beta\) and \(\tilde{\beta}\) on \(G^*\) and \(\tilde{G}^*\), respectively. We use fixed \(\sigma\) values in the interval [0.3, 0.5]. The results are presented in Fig. 5 together with the cross-country data provided by Corak (2012). Corak (2006) reports confidence intervals of approximately ±15% on the estimations of \(\beta\), which we follow in order to provide a range of plausible values for the theoretical calculations.

These results indicate that the theoretical relationship between \(\tilde{\beta}\) and \(\tilde{G}^*\) well agrees with the data. For \(\sigma = 0.4\) we obtain a better description of the measured data than the linear fit in Fig. 1 \((R^2 = 0.7\) compared to \(R^2 = 0.6\)), providing a mechanical explanation to the empirical relationship found (the best fit to the data is at \(\sigma = 0.378\)—\(R^2 = 0.73\)). We conclude, therefore, that the modified model, in which the random noise is fixed in time for each family, provides a better description of the dynamics implied by the regression model Eq. (2.2), when compared to the original version of the model.
Figure 5: Comparison between theoretical calculations and cross-country data. The thick colored curves present $\beta$ vs. $G^*$ (top) and $\hat{\beta}$ vs. $\hat{G}^*$ (bottom), for different values of $\theta$—$\theta = 0.3$ (green), $\theta = 0.4$ (red) and $\theta = 0.5$ (magenta). Translucent envelopes indicate $\pm 15\%$ standard error in $\beta$. The blue circles are the same as in Fig. 1 for the measured IGE and Gini coefficient in different countries. The dashed black curve is the linear least-squares best-fit for these data ($R^2 = 0.7$).
5 Discussion and Conclusions

We derived a theoretical relationship between intergenerational earnings elasticity and income inequality and showed that inequality is inevitably implied by immobility due to the effect of a multiplicative noise term. Our derivation assumed that the regression model according to which the earnings elasticity is defined suggests certain intergenerational income dynamics. Following these dynamics the intergenerational earnings elasticity and the income Gini coefficient are strongly dependent. As a consequence, we may argue that the Great Gatsby curve is to a large extent a presentation of a mathematical relationship rather than an indication of a deeper economic relationship. However, these findings indicate that indeed the two variables are dependent rather than only statistically associated. We found that if the rank-rank slope is used as the measure of intergenerational mobility, rather than the intergenerational earnings elasticity, a similar relationship is found.

However, we find that if mobility is measured by intergenerational elasticity, in which the error terms are additive and not multiplicative as in the IGE, the relationship between inequality and immobility is negative.

We also found that the resulting mathematical relationship agrees with the available cross-country inequality and mobility data. We note, however, that this good agreement with data is subject to several simplifying assumptions regarding the statistical properties of the error term in the regression model (see Sec. 4). These assumptions are essential for enabling the calculation and can be justified, but require further investigation in the future.

Our findings also suggest that the definition of the intergenerational earnings elasticity as a measure for mobility is questionable and should be considered cautiously. Inequality measures which are directly derived from the income distribution,
such as the Gini coefficient, are mathematically independent of mobility definitions considering permutations between individual incomes, transfers or ranking changes which have no effect on the income distribution. Chetty et al. (2014b) reported that while income inequality increased in the US during the past several decades, mobility, measured by the probability of moving from a low income quintile to a high income quintile between childhood and adulthood, has not changed significantly. This questions once again whether a causal positive relationship between inequality and immobility indeed exists. These arguments also fuel the practical question regarding how should mobility be measured and how should its effect on the economy as a whole be estimated. However, our findings support theoretical explanations that find a causal relationship between mobility and inequality, such as those given by Becker and Tomes (1979); Mulligan (1997); Maoz and Moav (1999); Solon (2004); Landersø and Heckman (2017). The single most important possible explanation for this relationship, given by these different models, is that in countries characterized by high immobility, it is harder for poor individuals to acquire education and new skills, which perpetuates the income gap.

The analysis presented here also provides a mathematical mechanism for the relationship between three variables—\( G^* \), \( \beta \) and \( \sigma \). Even though this mechanism is driven by the definition of \( \beta \), it is still possible to argue that the reduction of \( \beta \) or \( \sigma \) might reduce inequality and vice versa. Therefore, policy aiming to reduce inequality will likely increase intergenerational mobility as well and vice versa. Since mobility is generally perceived as important for prosperity, this supports the argument that inequality may be harmful to growth rates indirectly by its effect on mobility (Galor and Zeira, 1993; Aghion and Bolton, 1997; Maoz and Moav, 1999).
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A Proofs

A.1 Proof of Proposition 1

We start from Eq. (3.5):
\[
X_{i,t} = \alpha + \sum_{j=1}^{2} (\beta_j X_{i,t-j}) + \sigma \omega_{i,t},
\]  
(A.1)
where \(\alpha\) is common income growth trend between consecutive generations, and \(\sigma\) is the standard deviation of the error term. \(\beta_1\) and \(\beta_2\) are the intergenerational elasticities—\(\beta_1\) (\(\beta_2\)) measures the effect of the fathers’ (grandfathers’) generation on the sons’ generation.

Convergence is obtained if \(\beta_1 + \beta_2 < 1\) and \(\beta_2 - \beta_1 < 1\). Since fathers’ effect is larger than the grandfathers’ effect, \(\beta_1 > \beta_2\). For \(p = 1\), most of the \(\beta\) values obtained were less than 0.5, and \(\beta_1\) is expected to be ever lower. Therefore, both criteria are most likely to be met and the distribution of \(X_i\) reaches a steady state and follows (Hamilton, 1994):
\[
N \left( \frac{\alpha}{1 - \beta}, \frac{\sigma^2}{(1 + \beta_2^2) \left[ (1 - \beta_2)^2 - \beta_1^2 \right]} \right).
\]  
(A.2)

Therefore, assuming the convergence is quick, as was illustrated for IGE, the distribution of \(Y_i\) is log-normal. According to Crow and Shimizu (1988), the Gini coefficient of \(\ln N(m, s^2)\) is \(\text{erf}\left(\frac{z}{2}\right)\). It follows that there exists a steady state Gini coefficient for the income distribution of \(Y_i\):
\[
G^* = \text{erf} \left( \frac{1}{2} \sqrt{\frac{\sigma^2}{(1 + \beta_2^2) \left[ (1 - \beta_2)^2 - \beta_1^2 \right]}} \right).
\]  
(A.3)

\]
A.2 Proof of Proposition 2

We start from Eq. (3.1)

\[ X_{i,t} = \alpha + \beta X_{i,t-1} + \sigma \omega_{i,t}, \quad (A.4) \]

where \( \omega_{i,t} \) are random variates of a normal distribution with zero expectation and unity variance.

We assume \( \beta < 1 \), and look at the converged distribution which follows \( \mathcal{N} \left( \frac{\alpha}{1-\beta}, \frac{\sigma^2}{1-\beta} \right) \) (see Sec. 3). The percentile rank of an individual \( i \), with log-income \( X_i \), would therefore be

\[ P_i = \frac{1}{2} \left( 1 + \text{erf} \left( \sqrt{\frac{1-\beta^2}{2\sigma^2}} \left( X_i - \frac{\alpha}{1-\beta} \right) \right) \right). \quad (A.5) \]

Assuming a converged distribution, the percentile rank of the next generation \( R_i \) would be

\[ R_i = \frac{1}{2} \left( 1 + \text{erf} \left( \sqrt{\frac{1-\beta^2}{2\sigma^2}} \left( \alpha + \beta X_i + \sigma \omega_i - \frac{\alpha}{1-\beta} \right) \right) \right), \quad (A.6) \]

for the same \( X_i \) as in \( P_i \), and where \( \omega_i \) are random variates of a normal distribution with zero expectation and unity variance and \( \text{erf} \) is the error function.

We wish to calculate \( \rho_{PR} = \text{Corr} (P_i, R_i) \) in this case:

\[ \rho_{PR} = \frac{E[P_i R_i] - E[P_i] E[R_i]}{\sqrt{(E[P_i^2] - E[P_i]^2)(E[R_i^2] - E[R_i]^2)}} \quad (A.7) \]

By definition \( E[P_i] = E[R_i] = 1/2 \) and similarly \( E[P_i^2] = E[R_i^2] = 1/3 \) and therefore \( \rho_{PR} = 12 E[P_i R_i] - 3 \). We will explicitly calculate \( E[P_i R_i] \) using Eq. (A.5) and Eq. (A.6)
\[ E[P_i R_i] = \frac{1}{4} + \frac{1}{4} E \left[ \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( X_i - \frac{\alpha}{1 - \beta} \right) \right) \right] + \]
\[ \frac{1}{4} E \left[ \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( \alpha + \beta X_i + \sigma \omega_i - \frac{\alpha}{1 - \beta} \right) \right) \right] + \]
\[ \frac{1}{4} E \left[ \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( X_i - \frac{\alpha}{1 - \beta} \right) \right) \times \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( (\alpha + \beta X_i + \sigma \omega_i) - \frac{\alpha}{1 - \beta} \right) \right) \right]. \]

\tag{A.8}

\[ \frac{1}{4} E \left[ \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( X_i - \frac{\alpha}{1 - \beta} \right) \right) \right] \text{ and } \frac{1}{4} E \left[ \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( (\alpha + \beta X_i + \sigma \omega_i) - \frac{\alpha}{1 - \beta} \right) \right) \right] \]

are 0 by definition and therefore

\[ \rho_{PR} = 12 E[P_i R_i] - 3 = \]
\[ 3E \left[ \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( X_i - \frac{\alpha}{1 - \beta} \right) \right) \times \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( (\alpha + \beta X_i + \sigma \omega_i) - \frac{\alpha}{1 - \beta} \right) \right) \right]. \]

\tag{A.9}

We denote \( E \left[ \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( X_i - \frac{\alpha}{1 - \beta} \right) \right) \times \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( (\alpha + \beta X_i + \sigma \omega_i) - \frac{\alpha}{1 - \beta} \right) \right) \] by \( K \). In order to calculate \( K \), we expand the error functions \( \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( X_i - \frac{\alpha}{1 - \beta} \right) \right) \) and \( \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( (\alpha + \beta X_i + \sigma \omega_i) - \frac{\alpha}{1 - \beta} \right) \right) \) as a second order power series around \( \frac{\alpha}{1 - \beta} \).
\[
\text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( X_i - \frac{\alpha}{1 - \beta} \right) \right) \times \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2\sigma^2}} \left( (\alpha + \beta X_i + \sigma \omega_i) - \frac{\alpha}{1 - \beta} \right) \right) \approx \\
\sqrt{\frac{2(1 - \beta^2)}{\pi \sigma^2}} \text{erf} \left( \sqrt{\frac{1 - \beta^2}{2}} \omega_i \right) \left( X_i - \frac{\alpha}{1 - \beta} \right) + \frac{2\beta (1 - \beta^2)}{\pi \sigma^2} e^{-\frac{1 - \beta^2}{2} \omega_i} \left( X_i - \frac{\alpha}{1 - \beta} \right)^2.
\]

(A.10)

Averaging over \( X_i \sim \mathcal{N} \left( \frac{\alpha}{1 - \beta}, \frac{\sigma^2}{1 - \beta^2} \right) \) and \( \omega_i \sim \mathcal{N} (0, 1) \), we obtain \( K \approx \beta / \pi \), hence

\[
\rho_{PR} = 3K \approx \frac{3\beta}{\pi}. \tag{A.11}
\]

A.3 Proof of Proposition 3

Following Eq. (3.7), and assuming \( \beta < 1 \), we can write for \( t > 0 \):

\[
X_{i,t+1} = (\alpha + \sigma \omega_i) + \beta X_{i,t} = (\alpha + \sigma \omega_i) + \beta \left( (\alpha + \sigma \omega_i) + \beta X_{i,t-1} \right) = \\
(\alpha + \sigma \omega_i) + \beta \left( (\alpha + \sigma \omega_i) + \beta \left( (\alpha + \sigma \omega_i) + \beta X_{i,t-2} \right) \right) = \cdots = \\
(\alpha + \sigma \omega_i) \left[ 1 + \beta + \beta^2 + \cdots + \beta^t X_{i,1} \right]. \tag{A.12}
\]

Without loss of generality we assume \( X_{i,1} = 1 \), and obtain that in the limit \( t \to \infty \), which is equivalent to assuming fast convergence (see Sec. 4), the log-income of family \( i \) is

\[
X_i^* = \frac{\alpha + \sigma \omega_i}{1 - \beta}. \tag{A.13}
\]
Hence $X_i$ follow a normal distribution—$\mathcal{N}\left(\frac{\alpha}{1-\beta}, \left(\frac{\sigma}{1-\beta}\right)^2\right)$. 
B  Modification: Additive dynamics

The IGE definition implicitly assumes multiplicative dynamics of income intergenerationaly (see Eq. (2.2)). Let us now modify Eq. (2.2) so that the dynamics are additive. The motivation for this modification is that it allows us to test the importance of the multiplicative versus additive dynamics assumption. While assuming multiplicative dynamics is a more natural choice when growth processes are in discussion (Gibrat, 1931), additive dynamics are often considered, for simplicity. By considering the following modified equation, we will estimate the effect of this choice on the results:

\[
Y_{i,t} = \alpha + \beta Y_{i,t-1} + \tilde{e}_{i,t}, \tag{B.1}
\]

where the notations are the same as in Eq. (2.2), considering \( \tilde{e}_{i,t} \sim \ln \mathcal{N}(0, \sigma^2) \).

We assume the error terms \( \tilde{e}_{i,t} \) now follow a log-normal distribution and not normal, in order to avoid negative values, and to make sure the steady state distribution of \( Y_{i,t} \) is approximately log-normal and not normal, which is completely atypical of income distributions.

The modified definition of intergenerational immobility as \( \beta \), as implied by Eq. (B.1) is very different from the intergenerational earnings elasticity (see Eq. (2.2)). However, it is intuitively clear that this definition also provides a measure of predictability of the earnings of a younger generation based on the earnings of the older generation, where a small \( \beta \) indicates low predictability, hence high mobility and vice versa.

Similarly to Eq. (2.2), it follows that Eq. (B.1) is \( AR(1) \). Therefore, if \( \beta < 1 \), the distribution of \( Y_i \) reaches a steady state.

**Proposition 4** Considering Eq. (B.1), assuming \( \beta < 1 \) and \( \tilde{e}_{i,t} \sim \ln \mathcal{N}(0, \sigma^2) \), it follows that the distribution of \( Y_i \) reaches a steady state approximately follows a log-
normal distribution with

\[
Y_i \sim \ln \mathcal{N} \left( \frac{\alpha + e^{\frac{\sigma^2}{2}}}{1 - \beta} - \frac{1}{2} \ln \left( 1 + \frac{(e^{\sigma^2} - 1) e^{\sigma^2}}{(\alpha + e^{\frac{\sigma^2}{2}})^2 (1 + \beta)} \right) \right),
\]

(B.2)

Proof:

Since \( \beta < 1 \) guarantees convergence, it follows from the Fenton-Wilkinson approximation that the steady distribution of \( Y_i \) is approximately log-normal (Fenton, 1960) and we denote \( Y_i \sim \ln \mathcal{N} (m, s^2) \). We wish to calculate \( m \) and \( s^2 \) using \( \alpha \), \( \sigma \) and \( \beta \). We can take the expectation value and the variance of each of Eq. (B.1) sides and obtain:

\[
e^{m + \frac{s^2}{2}} = \alpha + \beta e^{m + \frac{s^2}{2}} + e^{\frac{s^2}{2}}
\]

(B.3)

and

\[
(e^{s^2} - 1) e^{2m + s^2} = \beta^2 (e^{s^2} - 1) e^{2m + s^2} + (e^{\sigma^2} - 1) e^{\sigma^2}.
\]

(B.4)

From Eq. (B.3) we get \( e^{m + \frac{s^2}{2}} = \frac{\alpha + e^{\frac{s^2}{2}}}{1 - \beta} \), and therefore

\[
e^{2m + s^2} = \frac{\left( \alpha + e^{\frac{s^2}{2}} \right)^2}{(1 - \beta)^2}.
\]

(B.5)

Substituting Eq. (B.5) in Eq. (B.4) and rearranging the terms we get

\[
(e^{s^2} - 1) = \frac{(e^{\sigma^2} - 1) e^{\sigma^2}}{(\alpha + e^{\frac{\sigma^2}{2}})^2 (1 + \beta)} 1 - \beta
\]

(B.6)
It now follows from Eq. (B.3) and Eq. (B.6) that:

\[ s^2 = \ln \left( 1 + \left( \frac{e^{\sigma^2} - 1}{\alpha + e^{\sigma^2}} \right)^2 \frac{1 - \beta}{1 + \beta} \right), \quad \text{(B.7)} \]

\[ m = \frac{\alpha + e^{\sigma^2}}{1 - \beta} - \frac{1}{2} \ln \left( 1 + \left( \frac{e^{\sigma^2} - 1}{\alpha + e^{\sigma^2}} \right)^2 \frac{1 - \beta}{1 + \beta} \right). \quad \text{(B.8)} \]

Now, since \( Y_i \sim N \left( \frac{\alpha + e^{\sigma^2}}{1 - \beta} - \frac{1}{2} \ln \left( 1 + \left( \frac{e^{\sigma^2} - 1}{\alpha + e^{\sigma^2}} \right)^2 \frac{1 - \beta}{1 + \beta} \right), \ln \left( 1 + \left( \frac{e^{\sigma^2} - 1}{\alpha + e^{\sigma^2}} \right)^2 \frac{1 - \beta}{1 + \beta} \right) \right), \)
we can simply calculate the Gini coefficient of the steady state income distribution (we now denote as \( \hat{G}^* \)), as done previously (Crow and Shimizu 1988). We get:

\[ \hat{G}^* = \text{erf} \left( \frac{1}{2} \sqrt{\ln \left( 1 + \left( \frac{e^{\sigma^2} - 1}{\alpha + e^{\sigma^2}} \right)^2 \frac{1 - \beta}{1 + \beta} \right)} \right). \quad \text{(B.9)} \]

In contrast to the results of the original model and to modification 1, \( \hat{G}^* \) decreases with \( \beta \). \( \hat{G}^* \) is continuous if \( \beta \in [0, 1] \), and for \( \beta = 0 \), we get \( \hat{G}^* > 0 \) and \( \hat{G}^* = 0 \) for \( \beta = 1 \). We also obtain that \( \alpha \) affects the resulting Gini coefficient, since its effect is not the multiplication of the distribution by constant, but the addition of a constant to the distribution. Therefore, as \( \alpha \) is larger, the Gini coefficient, which measures relative inequality, will be smaller. These results are illustrated in Fig. 6.

The analysis of this modification to the model indicates that the multiplicative dynamics implied by the definition of the IGE are essential for obtaining a positive relationship between the income Gini coefficient and the income intergenerational elasticity. It demonstrates that had the mobility not been defined by the IGE (or RRS), the relationship between mobility and inequality might have been the opposite.
Figure 6: The theoretical relationship between the Gini coefficient and the intergenerational earnings elasticity for modification 2 of the model, assuming additive income dynamics. The different curves depict the dependence of $\beta$ on $\hat{G}^*$ for different values of $\sigma$ and $\alpha$.

As is observed in the GGC. Therefore, the conclusions of models describing the relationship between mobility and inequality may rely on whether they consider additive or multiplicative income dynamics, or more generally on the definition of intergenerational mobility.
C Fast Convergence of the Gini Coefficient

In the discussion above, we assumed that in the case of $\beta < 1$, the convergence of the income distribution to its asymptotic shape, and hence the convergence of the income Gini coefficient to its asymptotic value, is very fast. In order to test this assumption we use the dynamics implied by the IGE definition and simulate the incomes of $N = 10^7$ families for different $\beta$ values. We assume that the initial income of each family is the same ($X = 1$), meaning a perfectly equal distribution. As illustrated in Fig. 7, convergence is reached within 1–5 generations. Without assuming an initially perfectly equal distribution, convergence would be reached within 1–2 generations, and we conclude that it is indeed fast enough to treat the Gini coefficient in its asymptotic value. We note, however, that as $\beta$ gets larger, the convergence is slower. Therefore, the fast convergence assumption when $\beta$ is about 0.8 or larger, is less valid—in such a case convergence will take 5–6 generations.
Figure 7: The convergence of the Gini coefficient in the non-modified model (left) and in the fixed error term modification (right). We simulate $N = 10^7$ families for 5 generations, assuming $\sigma$ is 0.4, for $\beta = 0.1$ (blue) and $\beta = 0.4$ (red). The black lines are the theoretical asymptotic Gini coefficients.