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Theil, Inequality Indices and Decomposition

Frank Cowell

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Frank Cowell *

London School of Economics and Political Science

Abstract

Theil's approach to the measurement of inequality is set in the context of subsequent developments over recent decades. It is shown that Theil's initial insight leads naturally to a very general class of decomposable inequality measures. It is thus closely related to a number of other commonly used families of inequality measures

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* Correspondence to: STICERD, London School of Economics and Political Science, Houghton Street, London WC2A 2AE, UK

1 Introduction

Henri Theil's book on information theory (Theil 1967) provided a landmark in the development of the analysis of inequality measurement. The significance of the landmark was, perhaps, not fully realised for some time, although his influence is now recognised in standard references on the analysis of income distribution. Theil's insight provided both a method for thinking about the meaning of inequality and an introduction to an important set of functional forms for modelling and analysing inequality. Theil's structure laid the basis for much of the work that is done on decomposition by population subgroups. The purpose of this paper is to set Theil's approach in the context of the literature that has since developed and to demonstrate that its contribution is more far-reaching than is commonly supposed.

We first introduce a framework for analysis (section 2) and consider Theil's approach to inequality in section 3. Section 4 introduces a general class of inequality indices foreshadowed by Theil's work and section 5 its properties. Section 6 concludes.

2 Analytical framework

2.1 Notation and terminology

Begin with some tools for the description of income distribution. The real number x denotes an individual's income: assume that issues concerning the definition of the income concept and the specification of the income receiver have been settled. Then we may speak unambiguously of an income distribution. Represent the space of all valid univariate distribution functions by $\mathfrak{F}$; income is distributed according to $F \in \mathfrak{F}$ where F has support $\mathfrak{X}$, an interval on the real line $\mathbb{R}$: for any $x \in \mathfrak{X}$, the number $F(x)$ represents the proportion of the population with incomes less than or equal to x.

Standard tools used in distributional analysis can be represented as functionals defined on $\mathfrak{F}$. The mean μ is a functional $\mathfrak{F} \mapsto \mathbb{R}$ given by $\mu(F) := \int x dF(x)$. An *inequality measure* is a functional $I : \mathfrak{F} \mapsto \mathbb{R}$ which is given meaning by axioms that incorporating criteria derived from ethics, intuition or mathematical convenience.

2.2 Properties of inequality measures

Now consider a brief list of some of the standard characteristics of inequality measures.

Definition 1 Principle of transfers. *$I(G) > I(F)$ if distribution G can be obtained from F by a* mean-preserving spread *.*

In order to characterise a number of alternative structural properties of the functional I consider a strictly monotonic continuous function $\tau : \mathbb{R} \mapsto \mathbb{R}$ and let $\mathfrak{X}^{(\tau)} := \{\tau(x) : x \in \mathfrak{X}\} \cap \mathfrak{X}$. A structural property of inequality measures then follows by determining a class of admissible transformations $\mathfrak{T}$.[1] Every $\tau \in \mathfrak{T}$ will have an inverse τ^{-1} and so, for any $F \in \mathfrak{F}$, we may define the τ-transformed distribution $F^{(\tau)} \in \mathfrak{F}$ such that

$$\forall x \in \mathfrak{X}^{(\tau)} : F^{(\tau)}(x) = F\left(\tau^{-1}(x)\right). \tag{1}$$

$F^{(\tau)}$ is the associated distribution function for the transformed variable $\tau(x)$. A general statement of the structural property is

Definition 2 $\mathfrak{T}$-Independence. *For all $\tau \in \mathfrak{T}$: $I\left(F^{(\tau)}\right) = I(F)$.*

Clearly, not all classes of transformations make economic sense. However two important special cases are those of *scale independence*, where $\mathfrak{T}$ consists of just proportional transformations of income by a strictly positive constant, and *translation independence* where $\mathfrak{T}$ consists of just transformations of income by adding a constant of any sign.

The following restrictive assumption makes discussion of many issues in inequality analysis much simpler and can be justified by appeal to a number of criteria associated with decomposability of inequality comparisons (Shorrocks 1984, Yoshida 1977).

Definition 3 Additive separability. *There exist functions $\phi : \mathfrak{X} \mapsto \mathbb{R}$ and $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ such that*

$$I(F) = \psi\left(\mu(F), \int \phi(x)dF(x)\right) \tag{2}$$

Given additive separability, most other standard properties of inequality measures can be characterised in terms of the income-evaluation function ϕ and the cardinalisation function ψ.

However, this is just a list of properties that may or may not be satisfied by some arbitrarily specified index. In order to make progress let us briefly consider the alternative ways in which the concept of inequality has been motivated in the economics literature.

[1] See Ebert (1996) for a detailed discussion of this concept.

3 The basis for inequality measurement

3.1 Standard approaches

It is useful to distinguish between the method by which a concept of inequality is derived and the intellectual basis on which the approach is founded. The principal intellectual bases used for founding an approach to inequality can be roughly summarised as follows:

- "Fundamentalist" approaches including persuasive ad hoc criteria such as the Gini coefficient and those based on some philosophical principle of inequality such as Temkin's "complaints" (Temkin 1993). Typically such approaches focus on a concept that quantifies the distance between individual income pairs or between each income and some reference income.

- Approaches derived from an extension of welfare criteria (Atkinson 1970, Sen 1973). These build on standard techniques such as distributional dominance criteria and usually involve interpreting inequality as "welfare-waste."

- Approaches based on an analogy with the analysis of choice under uncertainty (Harsanyi 1953, 1955; Rothschild and Stiglitz 1973). This leads to methods that produce inequality indices on $\mathfrak{F}$ that are very similar to measures of risk defined on the space of probability distributions.

3.2 The Theil approach

Theil added a further intellectual basis of his own. He focused on inequality as a by-product of the information content of the structure of the income distribution. The information-theoretic idea incorporates the following main components (Kullback 1959):

1. A set of possible events each with a given probability of its occurrence.

2. An information function h for evaluating events according to their associated probabilities, similar in spirit to the income-evaluation function ("social utility"?) in welfarist approaches to inequality.

3. The entropy concept is the expected information in the distribution.

The specification of h uses three axioms:

Axiom 1 Zero-valuation of certainty*:* $h(1) = 0$*.*

Axiom 2 Diminishing-valuation of probability*:* $p > p' \Rightarrow h(p) < h(p')$*.*

Axiom 3 Additivity of independent events*:* $h(pp') = h(p) + h(p')$

The first two of these appear to be reasonable: if an event were considered to be a certainty ($p = 1$) the information that it had occurred would be valueless; the greater the assumed probability of the event the lower the value of the information that it had occurred. It is then easy to establish:

Lemma 1 *Given Axioms 1-3 the information function is* $h(p) = -\log(p)$*.*

In contrast to the risk-analogy approach mentioned above Theil's application of this to income distribution replaced the concept of event-probabilities by income shares, introduced an income-evaluation function that played the counterpart of the information function h and specified a comparison distribution, usually taken to be perfect equality. The focus on income shares imposes a requirement of homotheticity – a special case of $\mathfrak{T}$-independence – on the inequality measure and the use of the expected value induces additive separability.

Given an appropriate normalisation using the standard population principle (Dalton 1920) this approach then found expression in the following inequality index

$$I_{\text{Theil}}(F) := \int \frac{x}{\mu(F)} \log\left(\frac{x}{\mu(F)}\right) dF(x) \tag{3}$$

and also the following (which has since become more widely known as the *mean logarithmic deviation*):

$$I_{\text{MLD}}(F) := -\int \log\left(\frac{x}{\mu(F)}\right) dF(x) \tag{4}$$

The second Theil index or MLD is an example of Theil's application of the concept of *conditional entropy*; conditional entropy in effect introduces alternative versions of the comparison distribution and has been applied to the measurement of distributional change (Cowell 1980a).

3.3 Decomposition

The measures founded on the different intellectual bases discussed in subsections 3.1 and 3.2 contrast sharply in their implications for inequality.

The meaning of decomposability can be explained as follows. Suppose that individuals are characterised by a pair (x, a) of income and attributes; the attributes a may be nothing more than a simple indicator of identity. Let the attribute space be A and let Π be a *partition* of A:

$$\Pi := \left\{ A_1, A_2, ..., A_J : \bigcup_{j=1}^{J} A_j = A;\ A_j \cap A_i = \varnothing \text{ if } i \neq j \right\} \tag{5}$$

Let the distribution of x within subgroup j (i.e. where $a \in A_j$) be denoted by $F^{(j)}$ and let the proportion of the population and the mean in each subgroup be defined by

$$\pi_j = \int_{a \in A_j} dF\,(x, a)$$

and

$$\mu_j = \frac{1}{\pi_j} \int_{a \in A_j} x\; dF\,(x, a)$$

Then the minimum requirement for population decomposability is that of subgroup consistency – i.e. the property that if inequality increases in a population subgroup then, other things being equal, inequality increases overall:

Definition 4 *The inequality index satisfies* subgroup consistency *if there is a function* Φ *such that*

$$I(F) = \Phi\,(I_1, I_2, ...I_J;\ \boldsymbol{\pi}, \boldsymbol{\mu}) \tag{6}$$

where

$$I_j := I\left(F^{(j)}\right)$$

$$\boldsymbol{\pi} := (\pi_{1,}\pi_2, ..., \pi_J)$$

$$\boldsymbol{\mu} := \left(\mu_{1,}\mu_2, ..., \mu_J\right)$$

and where Φ *is strictly increasing in each of its first* J *arguments.*

Note that we only need the slightly cumbersome bivariate notation in order to explain the meaning of decomposability. Where there is no ambiguity we shall continue to write F with a single argument, income x.

Now consider each of the types of inequality measure in subsection 3.1 in terms of decomposability. The first group of these measures (the fundamentalist approaches) typically results in measures that are not strictly decomposable by population subgroups: for example it is possible to find cases where the Gini coefficient in a subgroup increases and the Gini coefficient overall falls, violating subgroup consistency. The second group can be made to be decomposable by a suitable choice of welfare axioms. The third group appears to be naturally decomposable because they are based on a standard approach to choice under uncertainty that employs the independence assumption. But clearly this conclusion rests on rather special assumptions: it would not apply if one used a rank-dependent utility criterion for make choices under risk.

However, the Theil indices based on the entropy concept are naturally decomposable by population subgroup. This property does not depend on the additivity of independent events in the information function (Axiom 3) but because of the aggregation of entropy from the individual information components which induces additive separability. This ease of decomposition of his indices was exploited by Theil in a number of empirical applications (Theil 1979a, 1979b, 1989).

3.4 A generalisation

In their original derivation, the Theil measures in section 3.2 use an axiom (#3 in the abbreviated list above) which does not make much sense in the context of distributional shares. It has become common practice to define

$$I_{\mathrm{GE}}^{\alpha}(F) := \frac{1}{\alpha^2 - \alpha} \int \left[\left[\frac{x}{\mu(F)} \right]^{\alpha} - 1 \right] dF(x) \tag{7}$$

where $\alpha \in (-\infty, +\infty)$ is a parameter that captures the distributional sensitivity: if α is large and positive the index is sensitive to changes in the distribution that affect the upper tail; if α is negative the index is sensitive to changes in the distribution that affect the lower tail. The indices (3) and (4) are special cases of (7) corresponding to the values $\alpha = 1, 0$ respectively. Measures ordinally equivalent to those in the class with typical member (7) include a number of pragmatic indices such as the variance and measures of industrial concentration (Gehrig 1988, Hart 1971, Herfindahl 1950).

The principal attractions of the class (7) lie not only in the generalisation of Theil's insights but also in the fact that the class embodies some of the key distributional assumptions discussed in section 2.2.

Theorem 1 *A continuous inequality measure $I : \mathfrak{F} \mapsto \mathbb{R}$ satisfies the principle of transfers, scale invariance, and decomposability if and only if it is ordinally equivalent to (7) for some α.*[2]

However it is useful to consider the class (7), and with it the Theil indices, as members of a more general and flexible class. To do this we move

[2] See Bourguignon (1979), Cowell (1980b), Shorrocks (1980, 1984).

away from Theil's original focus on income shares, but retain the use of $\mathfrak{T}$-independence and additive separability.

4 A class of inequality measures

4.1 Intermediate measures

Consider now the "centrist" concept of inequality introduced by Kolm (1969, 1976a, 1976b). This concept has re-emerged under the label "intermediate inequality" (Bossert 1988, 1990). As the names suggest, centrist concepts have been shown to be related in limiting cases to measures described as "leftist" or "rightist" in Kolm's terminology; intermediate inequality measures in their limiting forms are related to "relative" and "absolute" measures. However a general treatment of these types of measures runs into a number of difficulties:

- In some cases the inequality measures are well-defined only with domain restrictions. The nature of these restrictions is familiar from the well-known relative inequality measures which are defined only for positive incomes.
- In the literature results on the limiting cases are available for only a subset of the potentially interesting ordinal inequality indices.

In what follows we consider a general structure that allows one to address these difficulties, that will be found to subsume many of the standard families of decomposable inequality measures, and that shows the inter-relationships between these families and Theil's fundamental contribution.

4.2 Definitions

We consider first a convenient cardinalisation of the principal type of decomposable inequality index:

Definition 5 *For any* $\alpha \in (-\infty, 1)$ *and any finite* $k \in \mathbb{R}_+$ *an* intermediate decomposable inequality measure *is a function* $I_{\text{int}}^{\alpha,k} : \mathfrak{F} \mapsto \mathbb{R}$ *such that*

$$I_{\text{int}}^{\alpha,k}(F) = \frac{1}{\alpha^2 - \alpha} \left[\int \left[\frac{x + k}{\mu(F) + k} \right]^{\alpha} dF(x) - 1 \right] \tag{8}$$

Intermediate measures have usually appeared in other cardinalisations, for example

$$[1+k]\left[1-\left[1+\left[\alpha^2-\alpha\right]I_{\text{int}}^{\alpha,k}(F)\right]^{1/\alpha}\right]$$

(Bossert and Pfingsten 1990, Eichhorn 1988).[3] From (8) we may characterise a class of measures that are of particular interest.

Definition 6 *The* intermediate decomposable class *is the set of functions*

$$\Im := \left\{I_{\text{int}}^{\alpha,k} : \alpha \in (-\infty, 1), 0 < k < \infty\right\} \tag{9}$$

where $I_{\text{int}}^{\alpha,k}$ *is given by definition 5.*

The set $\Im$ can be generalised in a number of ways. Obviously one could relax the domain restrictions upon the sensitivity parameter α and the location parameter k. But more useful insights can be obtained if the possibility of a functional dependence of α upon k is introduced. Let $\mathfrak{T}' \subset \mathfrak{T}$ be the subset of affine transformations and consider $\alpha \in \mathfrak{T}'$ such that

$$\alpha(k) := \gamma + \beta k \tag{10}$$

where $\gamma \in \mathbb{R}, \beta \in \mathbb{R}_+$. Distributional sensitivity depends upon the location parameter k. Then the class $\Im$ in (9) is equivalent to a subset of the following related class of functions

Definition 7 *The* extended intermediate decomposable class *is the set of functions.*

$$\bar{\Im} := \left\{I_{\text{ext}}^{\alpha,k}(F) : \alpha \in \mathfrak{T}', k \in \mathbb{R}\right\} \tag{11}$$

where

$$I_{\text{ext}}^{\alpha,k}(F) := \theta(k)\int\left[\left[\frac{x+k}{\mu(F)+k}\right]^{\alpha(k)} - 1\right]dF(x) \tag{12}$$

and $\theta(k)$ *is a normalisation term given by*

$$\theta(k) := \frac{1+k^2}{\alpha(k)^2 - \alpha(k)}\ , \tag{13}$$

[3] See for example Bossert and Pfingsten (1990) page 129 where the definition (in the present notation) is given as $[1+k]\left[1-\int\left[\frac{x+k}{\mu(F)+k}\right]^{\alpha}dF(x)\right]^{1/\alpha}$. Kolm's standard formulation (Kolm 1976a, page 435) is found by multiplying this by a factor $\frac{\mu(F)+k}{1+k}$; Kolm has suggested a number of other cardinalisations (Kolm 1996, page 17) .

Note that (12) adopts two special forms for the cases $\alpha(k) = 0, 1$. If $\alpha(k) \to 0$ applying L'Hôpital's rule shows that the limiting form (12) is

$$\left[1 + k^2\right] \int \left[\log\left(\frac{\mu(F) + k}{x + k}\right)\right] dF(x)$$

Likewise if $\alpha(k) = 1$ (12) becomes

$$\left[1 + k^2\right] \int \left[\left[\frac{x + k}{\mu(F) + k}\right] \log\left(\frac{x + k}{\mu(F) + k}\right)\right] dF(x)$$

The class of $\bar{\Im}$ will be the primary focus of the rest of the paper.

5 Properties of the class

The class of extended intermediate decomposable measures possesses several interesting properties and contains a number of important special cases.

First, it has the property that it is $\mathfrak{T}'$-independent.

Second, the class is decomposable. Decomposition by population subgroups of any member of $\bar{\Im}$ can be expressed in a simple way. Again take a partition Π consisting of a set of mutually exclusive subgroups of the population indexed by $j = 1, 2, .., J$ as in (5), so that inequality in subgroup j is

$$I_{\text{ext}}^{\alpha,k}\left(F^{(j)}\right) := \theta\left(k\right) \int \left[\left[\frac{x + k}{\mu(F^{(j)}) + k}\right]^{\alpha(k)} - 1\right] dF^{(j)}(x) \tag{14}$$

The inequality in the whole population can be broken down as follows:

$$I_{\text{ext}}^{\alpha,k}\left(F\right) = \sum_{j=1}^{J} w_j I_{\text{ext}}^{\alpha,k}\left(F^{(j)}\right) + I_{\text{ext}}^{\alpha,k}\left(F_\Pi\right) \tag{15}$$

where w_j is the weight to be put on inequality in subgroup j:

$$w_j := \left[\frac{\mu(F^{(j)}) + k}{\mu(F) + k}\right]^{\alpha(k)} \tag{16}$$

and F_Π represents the distribution derived concentrating all the population in subgroup j at the subgroup mean $\mu(F^{(j)})$ so that between-group inequality is given by

$$I_{\text{ext}}^{\alpha,k}\left(F_\Pi\right) = \theta\left(k\right) \sum_{j=1}^{J} \left[\left[\frac{\mu(F^{(j)}) + k}{\mu(F) + k}\right]^{\alpha(k)} - 1\right]. \tag{17}$$

The decomposition relation (15) is clearly easily implementable empirically for any given value of the parameter pair (α, k).

Third, notice that the measure (12) can be written in the form (2) thus

$$\theta(k)\left[\int \frac{\phi(x)}{\phi(\mu(F))} dF(x) - 1\right] \tag{18}$$

where the income-evaluation function ϕ is given by

$$\phi(x) = \frac{1}{\alpha(k)}\left[x + k\right]^{\alpha(k)} , \tag{19}$$

and $\alpha(k), \theta(k)$ are as defined in (10) and (13): the income-evaluation function interpretation is useful in examining the behaviour of the class of inequality measures in limiting cases of the location parameter k. The important special cases of $I_{\text{ext}}^{\alpha,k}(F)$ correspond to commonly-used families of inequality measures:

- The generalised entropy indices are given by $\left\{I_{\text{ext}}^{\alpha,0}\right\}$ (Cowell 1977)
- The Theil indices (Theil 1967) are a subset of these given by the cases $\alpha(0) = 1$ and $\alpha(0) = 0$ (see equations 3 and 4 respectively).
- The Atkinson indices (Atkinson 1970) are ordinally equivalent to a subset of $\left\{I_{\text{ext}}^{\alpha,0}\right\}$:

$$1 - \left[1 + \frac{1}{\theta(0)} I_{\text{ext}}^{\alpha,0}\right]^{1/\alpha(0)}, \alpha(0) < 1.$$

There are other measures that can be shown to belong to this class for certain values of the location parameter k. However, here we encounter a problem of domain for the income-evaluation function ϕ. This problem routinely arises except for the special case where $\alpha(k)$ is an even positive integer;[4] otherwise one has to be sure that the argument of the power function used in (19) is never negative. Because of this it is convenient to discuss two important subcases.

[4]This condition is very restrictive. Indices with values of $\alpha(k) \geq 4$ are likely to be impractical and may also be regarded as ethically unattractive, in that they are very sensitive to income transfers amongst the rich and the super-rich.

5.1 Restricted domain: x bounded below

We first consider the case that corresponds to many standard treatments of the problem of inequality measurement: $-k \leq \inf(\mathfrak{X})$. This restriction enables us to consider what happens as the location parameter goes to (positive) infinity.

Theorem 2 *As $k \to \infty$ the extended intermediate inequality class (11) becomes the class of Kolm indices*

$$\left\{ I_{\mathrm{K}}^{\beta}(F) : \beta \in \mathbb{R}_{+} \right\}$$

where:

$$I_{\mathrm{K}}^{\beta}(F) := \frac{1}{\beta} \left[\int e^{\beta[x-\mu(F)]} dF(x) - 1 \right] \tag{20}$$

Proof. To examine the limiting form of (12) note that the parameter restriction ensures that, for finite $x \in \mathfrak{X}$ and k sufficiently large, we have $\frac{x}{k} \in (-1, 1)$. So, consider the function

$$\chi(x, y, \alpha, k) := \log \left(\frac{\phi(x)}{\phi(y)} \right) = \alpha(k) \left[\log \left(1 + \frac{x}{k} \right) - \log \left(1 + \frac{y}{k} \right) \right]. \tag{21}$$

Using the standard expansion

$$\log (1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - ... \tag{22}$$

and (10) we find

$$\chi(x, y, \alpha, k) = \left[\beta + \frac{\gamma}{k} \right] \left[x - y - \frac{x^2}{2k} + \frac{y^2}{2k} + \frac{x^3}{2k^2} - \frac{y^3}{2k^2} - ... \right]. \tag{23}$$

For finite γ, β, x, y we have:

$$\lim_{k\to\infty} \chi(x, y, \alpha, k) = \beta [x - y] \tag{24}$$

and

$$\lim_{k\to\infty} \theta(k) = \lim_{k\to\infty} \frac{1 + \frac{1}{k^2}}{\left[\beta + \frac{\gamma}{k}\right]^2 - \frac{1}{k}\left[\beta + \frac{\gamma}{k}\right]} = \frac{1}{\beta^2}. \tag{25}$$

So we obtain

$$\lim_{k\to\infty} I_{\mathrm{ext}}^{\alpha,k}(F) = \frac{1}{\beta^2} \int \left[\exp\left(\beta\left[x - \mu(F)\right]\right) - 1 \right] dF(x). \tag{26}$$

■

This family of Kolm indices form the translation-invariant counterparts of the family (7) (Eichhorn and Gehrig 1982, Toyoda 1980).[5]

[5] See Foster and Shneyerov (1999).

Theorem 3 *As $k \to \infty$ and $\beta \to 0$ (12) converges to the variance*

Proof. An expansion of (26) gives

$$\int \left[\frac{1}{2!}[x - \mu(F)]^2 + \frac{1}{3!}\beta[x - \mu(F)]^3 + \frac{1}{4!}\beta^2[x - \mu(F)]^4 + ... \right] dF(x)$$

As $\beta \to 0$ this becomes the variance. ■

5.2 Restricted domain: x bounded above

A number of papers in the mainstream literature make the assumption that there is a finite maximum income.[6] If we adopt this assumption then it makes sense to consider parameter values such that $-k \geq \sup(\mathfrak{X})$. However, it is immediate that the new parameter restriction again ensures that, for finite $x \in \mathfrak{X}$ and $(-k)$ sufficiently large, we have $\frac{x}{k} \in (-1, 1)$. Therefore the same argument can be applied as in equations (21) to (26) above: again the evaluation function converges to that of the Kolm class of leftist inequality measures.

The behaviour of the evaluation function ϕ as the location parameter changes is illustrated in Figure 1: the limiting form is the heavy line in the middle of the figure. As $k \to +\infty$ the evaluation functions of the $\bar{\Im}$ class approach this from the bottom right; as $k \to -\infty$ the evaluation functions approach it from the top left. Figure 2 shows the relationship of overall inequality to the parameter k when income is distributed uniformly on the unit interval: note that the limiting case (where the inequality measure is ordinally equivalent to the "leftist" Kolm index) is given by the point $1/k = 0$.

5.3 Interpretations

The reformulation (12) is equivalent to (8) in that, given any arbitrary values of the location parameter k and the exponent in the evaluation function (19), one can always find values of γ, β such that $\alpha(k) = \gamma + \beta k$. Clearly there is a redundancy in parameters (for finite positive k one can always arbitrarily fix either γ or β), but that does not matter because the important special cases drop out naturally as we let k go to 0 (Generalised Entropy) or to ∞ (Kolm). Of course the normalisation constant $\theta(k)$ could be specified in some other way for convenience, but this does not matter either.

The general formulation allows one to set up a correspondence between the Generalised Entropy class of measures, including the Theil indices and the

[6] See for example Atkinson (1970).

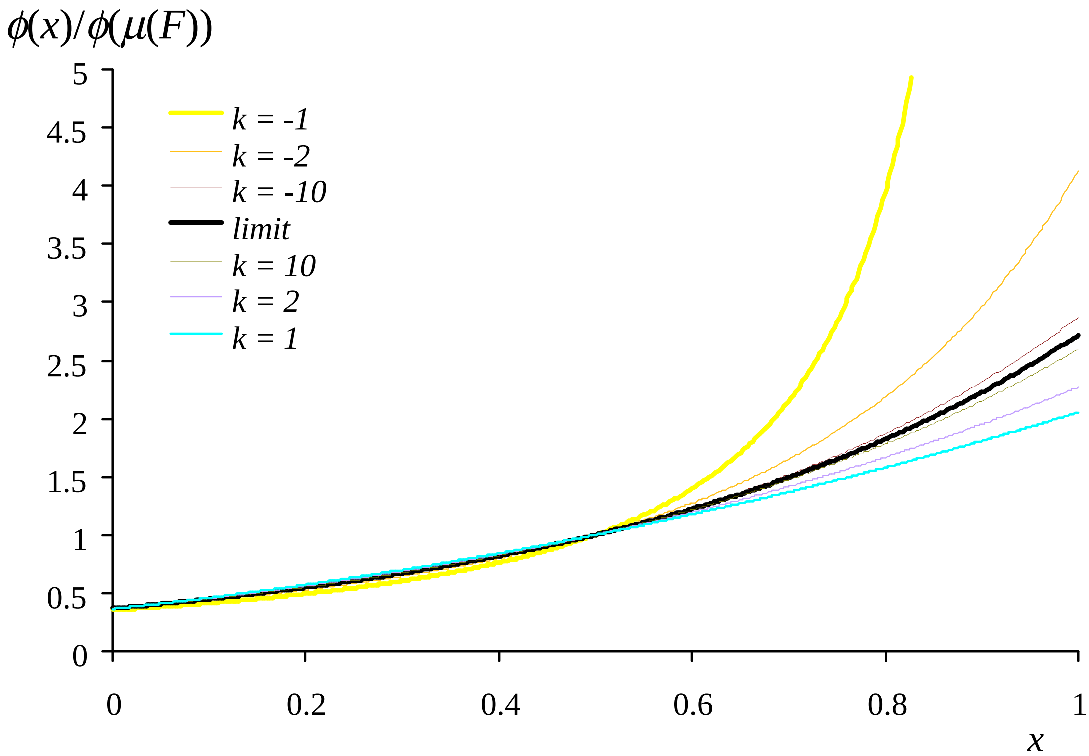

Figure 1: Values of $\phi(x)/\phi(\mu(F))$ as k varies: $\mathfrak{X} = [0,1], \gamma = 0.5, \beta = 2, \mu(F) = 0.5$

Kolm leftist class of measures ($k = \infty$). Consider, for example the subclass that is defined by the restriction $\beta = \gamma$

$$I_{\text{ext}}^{\beta,k}(F) := \theta(k) \int \left[\left[\frac{x+k}{\mu(F)+k} \right]^{\beta[1+k]} - 1 \right] dF(x) \tag{27}$$

Putting $k = 0$ one immediately recovers the Generalised Entropy class with parameter β. However, letting $k \to \infty$ Theorem 2 gives the Kolm index with parameter β.

6 Conclusion

Theil's seminal contribution led to a way of measuring inequality that has much in common with a number of families of indices that have become standard tools in the analysis of income distribution. Indeed, in examining

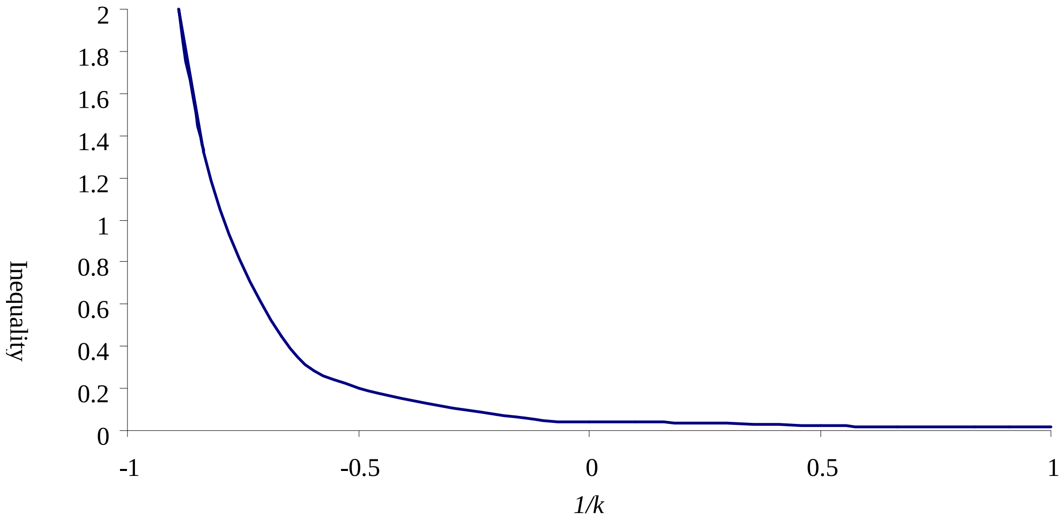

Figure 2: Inequality and k for a rectangular distribution on $[0, 1]$.

some of the widely used families of inequality indices, it is clear that a relatively small number of key properties characterise each family and the sets of characteristic properties bear a notable resemblance to each other. However, Theil's approach has a special advantage in that his basis for measuring inequality naturally leads to a decomposable structure, whereas decomposability has to be imposed as an extra explicit requirement in alternative approaches to inequality. This paper has further shown that many of these standard families of inequality measures are in fact related to the original Theil structure.

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