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On longitudinal analysis of poverty conceptualised as a fuzzy state*

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Abstract

When poverty is viewed as a matter of degree, i.e. as a fuzzy measure, two additional aspects are introduced into the analysis compared with the conventional poor/non-poor dichotomous approach: (i) the choice of membership functions i.e. quantitative specification of individuals' or households' degrees of poverty and deprivation; and (ii) the choice of rules for the manipulation of the resulting fuzzy sets, rules defining their complements, intersections, union and averaging. Specifically, for longitudinal analysis of poverty using the fuzzy set approach, we need joint membership functions covering more than one time period, which have to be constructed on the basis of the series of cross-sectional membership functions over those time periods. In this paper we propose a general rule for the construction of fuzzy set intersections, that is, rules for the construction of longitudinal poverty measures from a sequence of cross-sectional measures. On the basis of the results obtained, various fuzzy poverty measures over time can be constructed as consistent generalisations of the corresponding conventional (dichotomous) measures. Examples are rates of any-time, persistent and continuous poverty, distribution of persons and poverty spells according to duration, rates of exit and re-entry into the state of poverty, etc.
The proposed rule has been developed in a logical, step-by-step, manner, satisfying the required marginal constraints. This is important since there are reasons to believe that, hitherto, the rules of fuzzy set operations in the context of multi-dimensional and longitudinal poverty analysis have not been well or widely understood.
In an annex to this paper, we also present some numerical illustrations using survey data from the Italian European Community Household Panel, 1994-2001, with breakdown by Macro-region in Italy. The main objective, however, is to provide quantitative comparison between the conventional and fuzzy approaches. Noteworthy from a methodological point is the difference in the performance of the approaches concerning persistence of poverty. Movements in and out of poverty may be somewhat over-estimated (and hence the persistent or continuous poverty rates under-estimated) with the conventional approach, perhaps because it gives too much weight even to small movements across the poverty line.

Keywords: fuzzy sets, longitudinal and multi-dimensional poverty measures, comparative analysis.
JEL Classification: I31, C42, I32

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1 Introduction

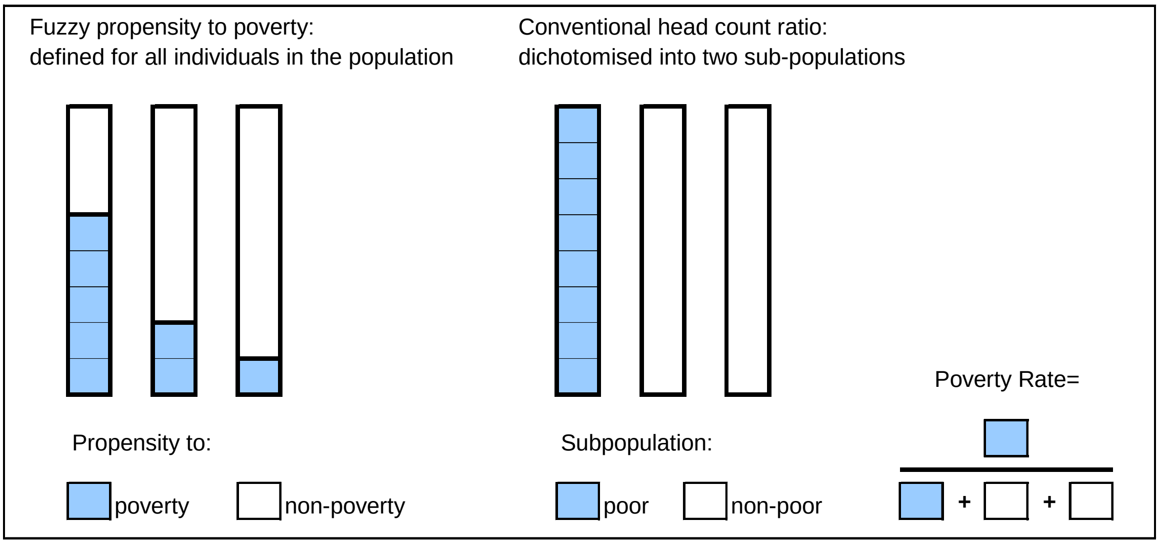
Conventional analyses of poverty often have two main limitations: (i) they are unidimensional, considering only income poverty, and (ii) they need to dichotomise the population into the *poor* and the *non-poor* by means of a *poverty line*. While multidimensionality is being increasingly incorporated into poverty research, little attention has been devoted to the effect of the rigid poor/non-poor dichotomy on the results.

When poverty is viewed as a matter of degree (as distinct from the conventional poor/non-poor dichotomy), that is as a *fuzzy* state, two additional aspects are introduced into the analysis. These concern the choice of (i) membership functions i.e. quantitative specification of individuals' or households' *degrees of poverty*; (ii) and of rules for the manipulation of the resulting fuzzy sets, rules defining their complements, intersections, union and averaging.

We address the second of the above questions in this paper. Specifically, for longitudinal analysis of poverty using the fuzzy set approach, we need longitudinal *joint membership functions* covering more than one time period, which have to be constructed on the basis of the series of *cross-sectional membership functions* over those time periods.

It is useful to begin by a brief clarification of the concept of treating poverty (or more generally, various forms of deprivation) as a matter of degree replacing the conventional classification of the population into a simple dichotomy. Figure 1 illustrates the basic idea. In principle all individuals in a population are subject to poverty, but to varying degrees. We say that each individual has a certain *propensity* to be poor, the population covering the whole range [0,1]. The conventional approach is a special case of this, with the population dichotomised as {0,1}: those with income below a certain threshold are deemed to be poor (i.e. are all assigned a constant propensity=1); others with income at or above that threshold are deemed to be non-poor (i.e. are all assigned a constant propensity=0).

Figure 1. The basic idea of poverty or deprivation as a matter of degree: comparison with the conventional poor/non-poor dichotomy



As to fuzzy sets, the basic idea is as follows. Given a set X of elements $x \in X$, any fuzzy subset A of X is defined as:

$$A = \{x, \mu_A(x)\}$$

where $\mu_A(x)$: X→[0,1] is called the *membership function* (*m.f.*) in the fuzzy subset A. The value $\mu_A(x)$ indicates the degree of membership of x in A. Thus $\mu_A(x) = 0$ means that x does not belong at all to

A, whereas $\mu_A(x) = 1$ means that *x* belongs to A completely. When on the other hand 0< $\mu_A(x)$ <1, then *x* partially belongs to A and its degree of membership of A increases in proportion to the proximity of $\mu_A(x)$ to 1.

For an early application of the ideas of fuzzy sets to the longitudinal study of poverty, see Cheli (1995); some further development of those ideas can be found in Betti, Cheli and Cambini (2004).

In this paper, we take a fresh look at the methodology. The basic rules concerning fuzzy set operations, as relevant for the longitudinal analysis of poverty, are clarified in Section 2.

In the rest of the paper, we develop a general rule for the construction of fuzzy set intersections, that is for the construction of a longitudinal 'joint membership functions' of individuals' propensities to poverty, from a sequence of cross-sectional propensities to poverty. This rule is meant to be applicable to any sequence of "poor" and "non-poor" sets, and satisfies all the relevant constraints. On the basis of the results obtained, diverse fuzzy poverty measures over time can be constructed as consistent generalisations of the corresponding conventional (dichotomous) measures.

We begin in Section 3 with the development and interpretation of what we have termed the 'Composite' set operator for correctly obtaining the intersection of a pair of fuzzy sets, taking into account whether the two sets in the pair are of the same type (e.g., poor/poor) or are of different types (e.g., poor/non-poor). The Composite operator which replaces the widely used 'Standard' operator, was first proposed in Betti and Verma (2004). In Section 4 rules are developed for constructing longitudinal fuzzy measures of persistent poverty. In order to obtain consistent generalisation of these rules, we examine in detail in Section 5 the joint membership functions covering all possible patterns over three years, and present in Section 6 generalisation of the rules for constructing intersections of any sequence of fuzzy membership functions. A number of the longitudinal measures in section 4-6 were first formulated, but without detailed proof, in Verma and Betti (2002).

The fundamental objective of this paper is to clarify and formulate consistent rules for the construction of joint membership functions of the type described above. The aim is to clarify the application of general principles of fuzzy set operations to the specific task of longitudinal analysis of poverty and deprivation.

2 Basic rules concerning fuzzy set operations

We begin from the situation that, for a series of cross-sections (1,… T), each person's propensity to be in poverty (i.e. the person's membership function of the set "poor") is given as:

$$\mu_t \ , \ (t = 1, \ldots, T) \ , \ \mu_t \in [0,1] \tag{1}$$

As necessary, we can also define the complements of the above at each time, i.e. the membership function (m.f.) of the set "non-poor" as $\bar{\mu}_t = 1 - \mu_t$.

Here we take that the degree to which a person is not a member of the set "poor", he/she is a member of the set "non-poor"; in other words, the two sets are taken to form *fuzzy partitions* of the 'universal' set (with m.f.=1 for all units in the population).

For analysis of poverty over time, we need *joint membership functions* (j.m.f.'s) covering more than one time period. Simple examples are the intersection $\mu_1 \cap \mu_2$ giving the j.m.f. of the set "poor at the both times 1 and 2", and the union $\mu_1 \cup \mu_2$ giving the j.m.f. of the set "poor at either of the two times". <u>Here we should clarify that logic operations involving fuzzy sets are defined on their membership functions. Moreover, we should use a unique simbol for each operation: i( ), u( ) and c( ).</u>Similarly $\mu_1 \cap \bar{\mu}_2$ gives the propensity of being poor at time 1 but of escaping from poverty by time 2, and $\bar{\mu}_1 \cap \mu_2$ gives the propensity of being non-poor at time 1, but falling into poverty by time 2.

Just as the mean of individual values such as μ_t of the m.f.'s can be seen as the (cross-sectional) poverty rate at time t, the mean of a j.m.f., of for instance $\mu_1 \cap \mu_2$ gives the rate of persistent poverty over the two years.

Fuzzy set operations are a generalisation of the corresponding 'crisp' set operations in the sense that the former reduce to (exactly reproduce) the latter when the fuzzy membership functions, being in the whole range [0,1], are reduced to a series of {0,1} dichotomies:

$$\mu_t \ , \ (t = 1, \ldots, T) \ , \ \mu_t \in \{0,1\} \tag{2}$$

It would be better to find another symbol instead of $\mu_t^{(0)}$. In conventional analysis, using ordinary 'crisp set' formulation, each unit (person) is classified categorically as being "poor" or "non-poor" depending on whether or not the person's income falls below the fixed poverty line defined from income distribution at the time concerned. That is, at each time the individual is determined as belonging wholly to only one of the complementing sets, $\mu_t^{(0)}$ or $\overline{\mu}_t^{(0)}$, and not at all to the other.

The rules for constructing the j.m.f.'s covering more than one period are straightforward in the conventional case. They simply reflect counts of individuals in different states. For instance, joint membership function, $\mu_1^{(0)} \cap \mu_2^{(0)} = \mu_1^{(0)} \cdot \mu_2^{(0)}$, is simply the product of the two cross-sectional m.f.'s, each coded $\{0,1\}$. The corresponding rate is simply the proportion of individuals coded 1 in *both* $\mu_1^{(0)}$ and $\mu_2^{(0)}$. Similarly $\mu_1^{(0)} \cup \mu_2^{(0)} = 1$ if *either* of the two m.f.'s equals 1, and 0 only if both equal 0.

While fuzzy set operations are a generalisation of the corresponding 'crisp' set operations, *there is more than one way in which the fuzzy set operations can be formulated*, each representing an equally valid generalisation of the corresponding crisp set operation. The choice among alternative formulations has to be made primarily on substantive grounds: some options are more appropriate (meaningful, useful, illuminating, convenient) than others, depending on the context and objectives of the application. While the rules of fuzzy set operations cannot be discussed fully in this paper, we need to clarify their application *specifically for the study of poverty and deprivation*.

There are four types of fuzzy set operations on membership functions which are relevant to our application to longitudinal poverty analysis:

- fuzzy intersection,
- fuzzy union,
- fuzzy complement,
- aggregation (or averaging) over fuzzy sets.

Fuzzy intersection and union

Table 1 shows three commonly-used groups of rules – termed Standard, Algebraic and Bounded (Klir and Yuan, 1995) - specifying fuzzy intersection and union. Such rules are 'permissible' in the sense that they satisfy certain essential requirements such as reducing to the crisp set operations with dichotomous variables, satisfying the required boundary conditions, being monotonic and commutative, etc.

In this section we consider these operations applied to only a pair of fuzzy sets, and use simple and general notation as follows. Quantities a and b refer to the membership function of a given individual on the pair of fuzzy sets. These may refer to two time periods, or equally to two different dimensions of deprivation at the same time: the formal rules are identical in the two situations. Similarly, a and b may represent similar states (e.g. poor, poor), or dissimilar states (poor, non-poor), or even states differing to various degrees.

Table 1. Some basic forms of fuzzy operations

Type of operation	Intersection $i(a,b) = a \cap b$	Union $u(a,b) = a \cup b$
Standard	min(a,b)=i_{max}	max(a,b)=u_{min}
Algebraic	a*b	a+b-a*b
Bounded	max(0,a+b-1)	min(1,a+b)

For our application, a most important observation is that the Standard fuzzy operations provide the *largest* (the most loose or the weakest) intersection among all the permissible forms; all other forms give a smaller, or at least no larger, value for the intersection. By contrast, the Standard form gives the *smallest* (the most tight or the strongest) union among all the permitted forms. It is for this reason that these operations have been labelled as i_{max} and u_{min} in Table 1.

Secondly, it should also be noted that, among all the permissible forms, the Standard fuzzy intersection and union are the only ones which satisfy the intuitively and substantively desirable condition of 'idempotency', namely: *i(a,a) = a* and *u(a,a)= a*.

Fuzzy complement

For the complement as well, different forms are permissible. Acceptable forms need to be "involutive":

$$C(c(a)) = c(\bar{a}) = a.$$

The most commonly used and the simplest complement is its Standard form $c(a) = \bar{a} = (1-a)$, whichis also intuitively the most meaningful. We will apply this form throughout.

It must be underlined that the permissible forms of the two operations, intersection and union, go in pairs: to be consistent, it is necessary to select the two from the same row of Table 1, so as to satisfy the De Morgan laws of set operations: $(A \cup B)' = A' \cup B'$; $(A \cup B)' = A' \cap B'$, which in the fuzzy case can be written as:

$$c[i(a,b)] = u[c(a), c(b)]; \qquad c[u(a,b)] = i[c(a), c(b)].$$

Consistency also requires an appropriate form of the complement. Any of the three intersection-union pairs in Table 1 is consistent with the Standard form of the complement, c(a)=1−a.

Fuzzy aggregation and averaging

Aggregation of membership functions over different sets is related to the concept of *fuzzy partitions*. The above definition of the complement is a simple example of a fuzzy partition: the m.f.'s in a pair of complementing sets add up to 1: $a + \bar{a} = 1$. The two sets are taken as fuzzy partitions of the 'universal set' X with m.f.=1 for every unit in the population.

More generally, if for each unit in the population, its m.f. μ in a certain set is decomposed into components μ_j such that $\mu = \Sigma_j \mu_j$, then the μ_j values constitute m.f.'s corresponding to fuzzy partitions of the original set.

This concept of fuzzy partitions is relevant in the specification of marginal constraints which the fuzzy set operations must satisfy, as well be discussed in Section 3 below.

A graphical representation

To elucidate these fuzzy set operational forms, which are central to our methodology, we have developed a graphical representation as shown in Figure 2. The "universal set" X is represented by a rectangle of unit length, and within it are placed the individual's membership functions ($0 \leq a \leq 1$, $0 \leq b \leq 1$) on the two subsets. Different placements correspond to different types of fuzzy set operations.

In the Standard form, appropriate for *similar* sets, the two memberships (a,b) are placed on the same base, so that the smaller (say b) lies completely within the larger (say a). Consequently, their intersection is maximised, so as to equal the smaller of the subsets. By the same token, their union is minimised, so as to equal the larger of the two subsets. The union is represented in the lower part of Figure 2; it shows separately the amount (=a-b in this case) by which it exceeds the intersection of the sets concerned.

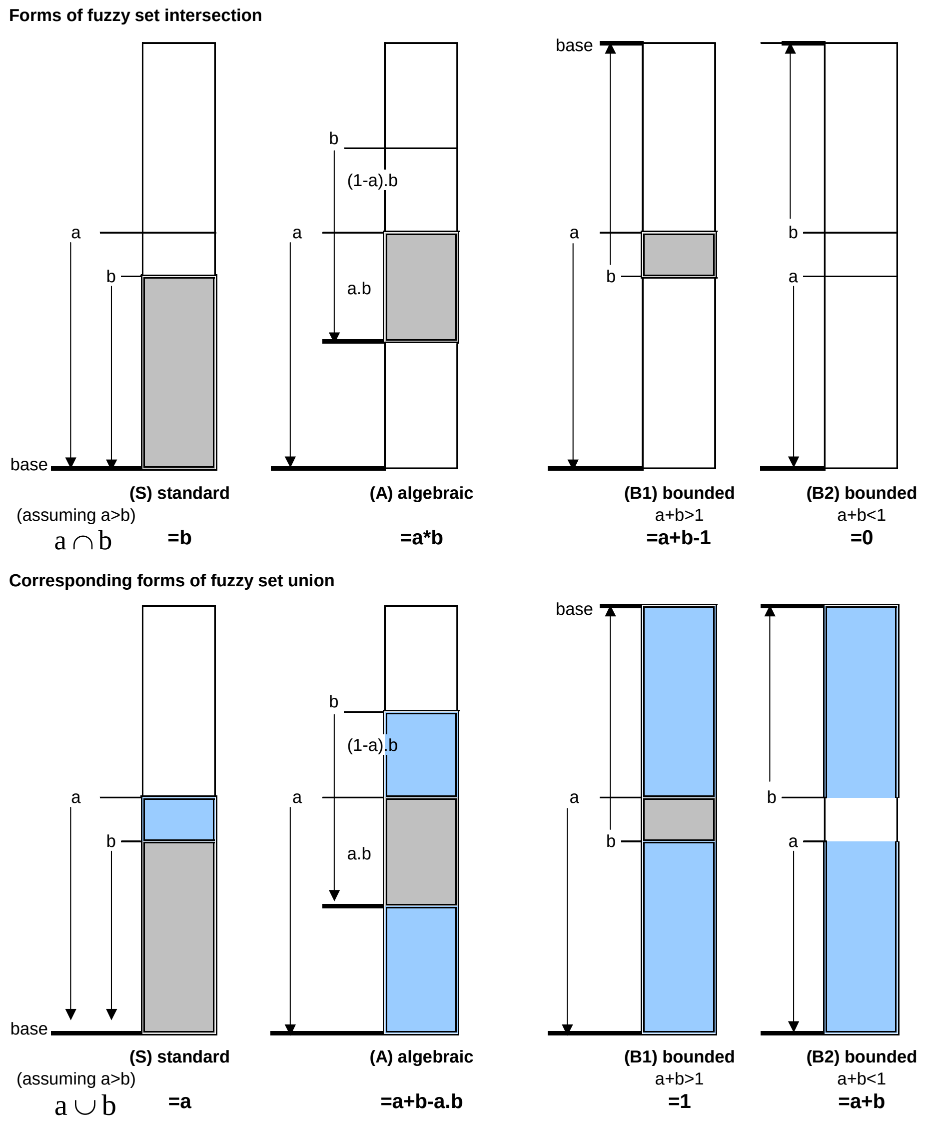
By placing one set higher than the other within X, the overlap (intersection) is generally reduced, and the underlay (union) increased.

In the Algebraic form, membership (b) is placed symmetrically over memberships (a) and (non-a), i.e. each of the two receiving a proportionate share of (b), respectively a*b and (1-a)*b. Hence a*b is the overlap (intersection), while the underlay (union) is [a+(1-a)*b]=[a+b-a*b].

In the Bounded form, appropriate for *dissimilar* sets, the two sets are placed at the opposite ends of X, thus further reducing their intersection to (a+b-1) (which is non-zero only if a+b>1); and increasing their union to (a+b), or to 1 if a+b>1.

We believe that such a graphical representation which is our original proposal (or: which was originally proposed by Betti and Verma) greatly helps in clarifying the meaning of different forms of the fuzzy set operations.

Figure 2. Graphical representation of the fuzzy set operations



3. The Composite fuzzy set operator

In order to clarify the application of general principles of fuzzy set operations to the specific task of longitudinal analysis of poverty, it is instructive to describe in some details the case involving only two time periods. The concept of the Composite fuzzy set operator was first developed in Betti and Verma (2004).

Let a and b be the 'cross-sectional' m.f.'s of the sets "poor" at times 1 and 2, respectively, for a given individual unit. We can also define their complements, the m.f.'s of the corresponding "non-poor" sets, as $\bar{a} = 1 - a$, $\bar{b} = 1 - b$. In longitudinal analysis, four intersection sets can be formed from these, with their joint membership functions under different set operations defined in Table 2.

Table 2. Application of different types of fuzzy intersections over two times periods

Fuzzy intersection	**Standard operator**	**Algebraic operator**	**Bounded operator**	**Betti-Verma Composite operator**		
				Type	Intersection	Union
$a \cap b$	min(a,b)	a*b	max(0,a+b-1)	Standard	min(a,b)	max(a,b)
$\bar{a} \cap \bar{b}$	min(1-a,1-b)= 1-max(a,b)	(1-a)*(1-b)	max(0,1-a-b)	Standard	1- max(a,b)	1- min(a,b)
$a \cap \bar{b}$	min(a,1-b)	a*(1-b)	max(0,a-b)	Bounded	max(0,a-b)	1-max(0,b-a)
$\bar{a} \cap b$	min(1-a,b)	(1-a)*b	max(0,b-a)	Bounded	max(0,b-a)	1-max(0,a-b)

Note: The same rules as above in fact also apply unchanged for the construction of the intersection of two different dimensions of deprivation at a given time.

Table 3. Marginal constraints for fuzzy intersection over two time periods

Time 2 / **Time 1**	**Poor (1)**	**Non-poor (0)**	**Total**
Poor (1)	$a \cap b$	$a \cap \bar{b}$	a
Non-poor (0)	$\bar{a} \cap b$	$\bar{a} \cap \bar{b}$	$\bar{a}$
Total	b	$\bar{b}$	1

The results of these operations need to satisfy certain marginal constraints. These arise from the following considerations. In conventional analysis, to which the fuzzy analysis must reduce with dichotomous m.f.'s, the four intersection (or longitudinal) sets shown in Table 3 correspond to four exhaustive and non-overlapping classes, as do the original cross-sectional sets $(a, \bar{a})$ and $(b, \bar{b})$. Obviously, they must satisfy the marginal constraints shown in Table 3. This is true both at the micro-level (where, in the conventional analysis, only one of the internal cells in the table equals 1 and all other cells equal 0), and hence also at the aggregate level where quantities averaged in cells of the table represent various proportions or rates. These marginal constraints apply with fuzzy sets as well, since it is precisely these proportions or rates which we wish to estimate and compare with conventional analysis.

Cells of Table 3 therefore constitute fuzzy partitions of the universal set with m.f.=1 for all units in the population.

As noted, the Standard operators provide the *largest* intersection and the *smallest* union among all the permitted forms. *It is this factor which makes it inappropriate to use the Standard set operations uniformly throughout in our application to poverty analysis.* If the Standard operation were applied to all the four intersections, $(a,b),(\bar{a},\bar{b}),(a,\bar{b}),(\bar{a},b)$ in Table 3, the sum of membership functions of an individual can be verified to equal $1+2.(s_1 - \max(0,\delta))$, where s_1=min(a,b), and δ=(a+b-1), i.e., to equal (1+2.s_1) for $\delta \leq 0$, and 1+2.(1-s_2) for $\delta > 0$[3], where s_2=max(a,b).

[3] For details, see Betti and Verma (2004).

Hence, because of the 'expansive' nature of the standard intersection, the sum of the resulting membership functions for the four subsets exceeds 1. However, this is in conflict with substantive requirements of our situation, as noted above in the comparison with the conventional approach.

Similarly, it can be verified that uniform application of the Bounded intersection operation would give a sum of the membership functions which generally falls short of 1.

Now it can be easily seen that the Algebraic form "A", applied to all the four intersections, is the *only one* which meets this condition. But despite this numerical consistency, we *do not* regards the Algebraic form to give results which, for our particular application, would be generally acceptable on intuitive or substantive grounds. In fact, if we take the liberty of viewing the fuzzy propensities as probabilities, then the algebraic product rule i(a,b)→joint probability (a,b)=a*b implies zero correlation between the two forms of deprivation, which is clearly at variance with the high positive correlation we expect in the real situation for *similar* states. The rule therefore seems to provide an unrealistically *low* estimate for the resulting membership function for the intersection of two similar states. The Standard rule, giving higher overlaps (intersections) are more realistic for (a,b) representing *similar* states.

By contrast, in relation to *dissimilar* states $(\bar{a}, b)$ and $(a, \bar{b})$ (lack of an overlap between deprivations in two dimensions), the Algebraic rule tend to give *unrealistically high* estimates for the resulting membership function for the intersection, and hence this is true also the Standard rule. The reasoning similar to the above applies: in real situations, we expect large negative correlations (hence reduced intersections) between *dissimilar* states in the two dimensions of deprivation. In fact, it can be easily seen by considering some particular numerical values for $(\bar{a}, b)$ or $(a, \bar{b})$ that Bounded rule, for instance, gives more realistic results for dissimilar states.

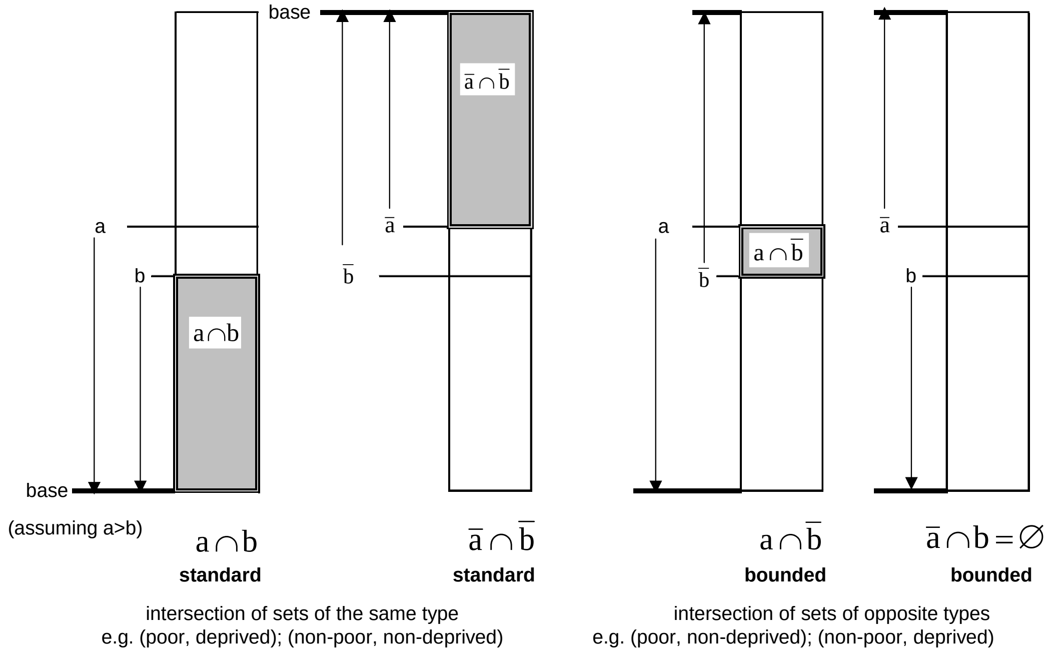
Rule 1: The Composite fuzzy set operation

Given the preceding considerations, the specification of the fuzzy intersection i(a,b) that appears to be the most reasonable for our particular application and that satisfies the above mentioned marginal constraints is of a 'composite' type as follows:

- For sets representing *similar states* - such as the presence (or absence) of poverty at both times - the Standard operation (which provides a larger intersection than the Algebraic operation) is used.
- For sets representing *dissimilar states* - such as the presence of poverty at one time but its absence at the other time - we use the Bounded operation (which provides a smaller intersection than the Algebraic operation).

By applying this Composite intersection, the elements of Table 3 are specified as shown in the three right-hand columns of Table 2. Figure 3 illustrates the Composite operation graphically.

Figure 3. The Composite fuzzy set operations



Note that the satisfaction of the marginal controls implies that the intersection operators are distributive: $a \cap b + a \cap \bar{b} = a \cap (b + \bar{b}) = a$. This is obviously true of the Algebraic operators. It is also true of the Betti-Verma Composite operators since $a \cap b + a \cap \bar{b} = \min(a, b) + \max(0, a - b) = a$.

Any other type of operators, including the Standard and Bounded ones, are *not* distributive in this sense.

Table 4. Longitudinal measures of interest over two time periods for individual i

	Measure	Membership function	Description
1	Never in poverty	$\bar{a}_i \cap \bar{b}_i = 1 - \max(a_i, b_i)$	Poverty at *neither* of the two years
2	Persistent in poverty	$a_i \cap b_i = \min(a_i, b_i)$	Poverty at *both* of the years
3	Exiting from poverty	$a_i \cap \bar{b}_i = \max(0, a_i - b_i)$	Poverty at time 1, but non-poverty at time 2
4	Entering into poverty	$\bar{a}_i \cap b_i = \max(0, b_i - a_i)$	Non-poverty at time 1, but poverty at time 2
5	Ever in poverty	$a_i \cup b_i = \max(a_i, b_i)$	Poverty at *at least one* of the two years

Note that the propensity to be ever in poverty (i.e. in at least one of the two years) equals max(a, b), which can be viewed as any of the three entirely equivalent forms:

- As the complement of cell "0-0" (non-poor/non-poor) in Table 4, or
- As the sum of the membership functions in the other three cells, or
- As the union of (a_i, b_i).

For measures 1, 2 and 5 in Table 4, we are normally interested in population rates. For instance, with w_i as individual sample weights:

Persistent poverty rate = $\Sigma_i w_i.(a_i \cap b_i)/\Sigma_i w_i = \Sigma_i w_i.\min(a_i, b_i)/\Sigma_i w_i$.

For measures 3 and 4, the appropriate denominator is the "at risk" population:

Exit rate = $\Sigma_i w_i.(a_i \cap \bar{b}_i)/\Sigma_i w_i.a_i = \Sigma_i w_i.\max(0, a_i - b_i)/\Sigma_i w_i.a_i$.
Entry rate = $\Sigma_i w_i.(\bar{a}_i \cap b_i)/\Sigma_i w_i.\bar{a}_i = \Sigma_i w_i.\max(0, b_i - a_i)/\Sigma_i w_i.(1-a_i)$.

Briefly, the analysis variables and the cross-sectional fuzzy membership functions have been constructed as follows.

Conventional poverty measures

For the computation presented, each unit (person) is classified as poor or non-poor in relation to a poverty line defined as 60% of the national median. (These computations are performed separately for each wave or each pair of consecutive waves of the survey, and then the results are averaged over those.)

Fuzzy poverty measures

For each survey wave, the membership function (μ_i) for the set "poor" has been specified for each individual i (indexed according to increasing size of equivalised income y_i) following the "Integrated Fuzzy and Relative" (IFR) approach developed in Betti, Cheli, Lemmi and Verma (2005) which combines the two approaches of Cheli and Lemmi (1995) and Betti and Verma (1999)[4]. The details of this approach will not be described in this paper, except to note that the fuzzy membership function for (or the propensity to) income poverty is formulated so as to take into account both the *share* of individuals less poor than the person concerned and the *share* of the total equivalised income received by all individuals less poor than the person concerned:

$$\mu_i = (1-F_i)^{\alpha-1}.[1-L(F_i)] = \left(\frac{\sum_\gamma w_\gamma \mid y_\gamma > y_i}{\sum_\gamma w_\gamma \mid y_\gamma > y_1}\right)^{\alpha-1} . \left(\frac{\sum_\gamma w_\gamma y_\gamma \mid y_\gamma > y_i}{\sum_\gamma w_\gamma y_\gamma \mid y_\gamma > y_1}\right), \tag{3}$$

where F_i is the distribution function of the income distribution, $L(F_i)$ is the corresponding Lorenz ordinate, and parameter α is chosen so that the mean of the μ_i equals to conventional head count ratio H.

The measure as defined above is expressible in terms of the generalised Gini measures (the standard Gini coefficient corresponds to G_α with α=1), which weights the distance (F-L(F)) between the line of perfect equality and the Lorenz curve by a function of the individual's position in the income distribution, giving more weight to its poorer end. It is defined (in the continuous case) as:

$$G_\alpha = \alpha.(\alpha+1).\int_0^1 \left[(1-F)^{(\alpha-1)}.(F-L(F))\right] dF \text{, giving}$$

$$\bar{\mu} = \frac{\alpha + G_\alpha}{\alpha.(\alpha+1)} = H.$$

Increasing the value of exponent α implies giving more weight to the poorer end of the income distribution. In the illustrations here, we have determined this parameter empirically by matching $\bar{\mu}$ to H averaged over the 8 ECHP waves for Italy as a whole. This gives α=4.81, which happens to be close to $1/H$ with H=0.19.

[4] Although we are convinced that an adequate analysis of poverty should be carried out according to a multidimensional approach that include a variety of living conditions indicators of the non-monetary type, for the sake of this research it will be enough to confine our attention to the analysis of income poverty alone.

Applications

In an annex to this paper we provide numerical illustration of the various types of longitudinal measures which can be constructed using the above formulations. The annex tables shows some measures of substantive interest for Italy and her Micro-regions. We compare fuzzy measures against the corresponding conventional measures.

The data used for all illustrations are from Italian European Community Household Panel Survey 1994-2001. In relation to this section, we consider pairs of consecutive waves. Often, for economy in presentation results averaged over 7 such pairs (formed from 8 annual waves of the survey) only will be shown.

4 Persistence of poverty

Analysis of the persistence of poverty over time requires the specification of j.m.f.'s of the type

$$I_T = \mu_1 \cap \mu_2 .... \cap \mu_T \quad \text{and}$$
$$U_T = \mu_1 \cup \mu_2 .... \cup \mu_T \ ,$$

where the first expression is the intersection of a series of T cross-sectional m.f.'s for any individual unit, and the second expression is their union.

Rule 2

Since all sets $\mu_1,.....,\mu_T$ are of the <u>same type</u> (all being propensities to "poverty" rather than to "non-poverty"), the Standard operations apply:

$$I_T = \min(\mu_1, \mu_2, ..., \mu_t, ...\mu_T) \tag{4}$$
$$U_T = \max(\mu_1, \mu_2, ..., \mu_t, ...\mu_T). \tag{5}$$

Clearly these expressions are commutative and associative.

I_T represents the individual's propensity to be poor at all T periods.

U_T is the propensity to be poor at *at least one* of the T periods; the propensity to be non-poor over all T periods is its complement $\overline{U}_T = 1 - U_T$. The same result is obtained by considering intersection of non-poor sets:

$$\bar{I}_T = \min(\overline{\mu}_1, \overline{\mu}_2, ..., \overline{\mu}_t, ...\overline{\mu}_T) = 1 - \max(\mu_1, \mu_2, ..., \mu_T) = 1 - U_T.$$

The propensity of experiencing poverty over any specific sequence of t out of T years is given by the minimum value of cross-sectional propensities μ over those particular years, representing the intersection between the t <u>similar</u> states.

In line with the principle of maximising the intersection when the Standard operation is used, we take the <u>maximum</u> among the values of this intersection over all possible (T!/t!(T-t)!) sequences of t out of T years as the m.f. for the set "poor for at least t out of T years".[5]

Figure 4 illustrates these concepts. The same results can be expressed more easily by considering the ordered sequence

$$(\mu_1, \mu_2, ...\mu_t, ...\mu_T) \Rightarrow (\mu_{[1]} \geq \mu_{[2]} \geq .... \geq \mu_{[t]} \geq ..\mu_{[T]}).$$

The figure on the right shows the same information arranged according to [t]. Clearly:

[5] A more precise proof is given in Section 6 based on 'Rule 5' defined there

Any time poverty:

membership function of the set "poor for at least one year" = $\mu_{[1]}$.

Continuous poverty:

membership function of the set "poor for all the T years" = $\mu_{[T]}$.

These are particular cases of the propensity to be poor for at least t out of T years = $\mu_{[t]}$, the t^{th} largest value.

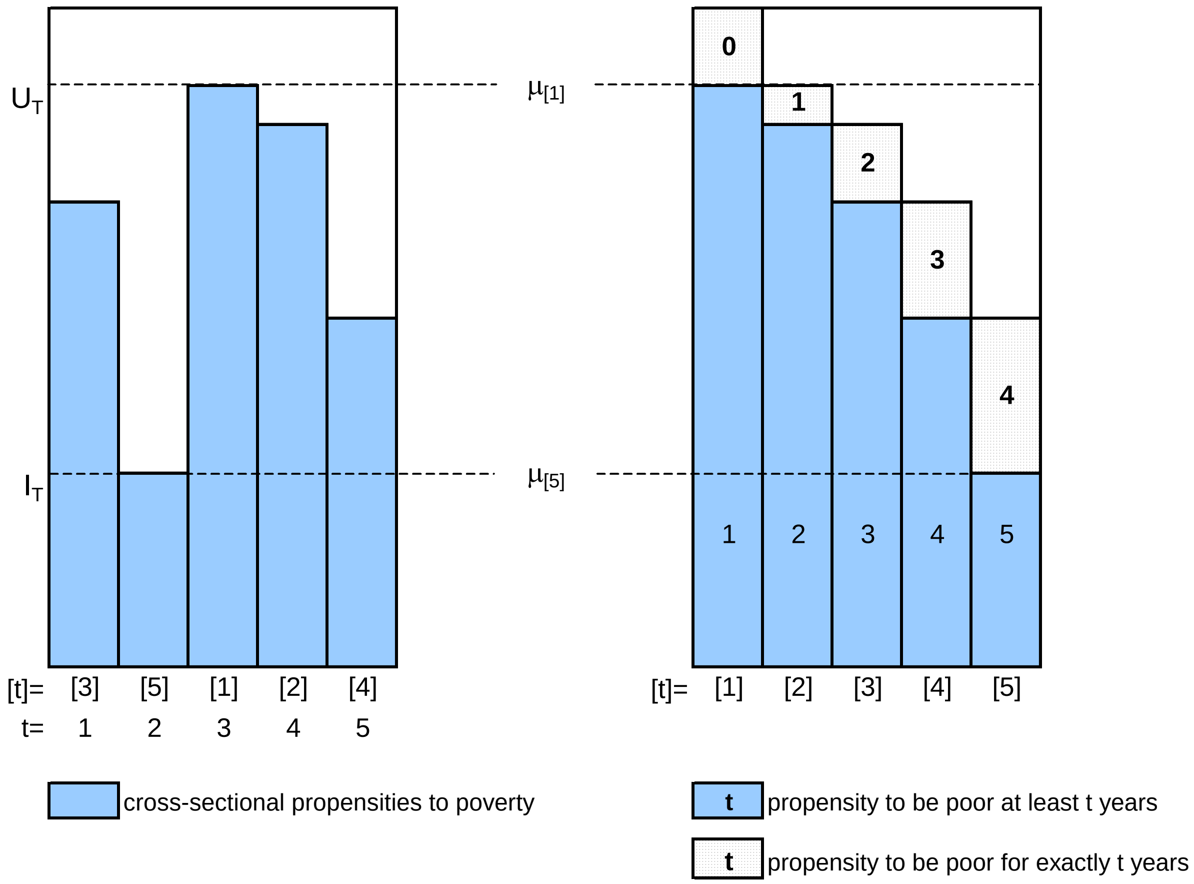
Persistent poverty.

We may define persistent poverty as the propensity to be poor over at least a majority of the T years, i.e. over at least t years, with

t=int(T/2)+1, the smallest integer strictly larger than (T/2).

For instance, for a T= 4 or 5 year period, 'persistent' would refer to poverty for at least 3 years; for T =6 or 7, it would refer to poverty for at least 4 years, etc. The required propensity to persistent poverty is the $[\text{int}(T/2)+1]^{th}$ largest value in the sequence $(\mu_1,\ldots\ldots\mu_T)$.

Figure 4. Time spent in poverty according to its duration



The propensity to be poor for exactly t years turns out to be:

$\mu_{[t]} - \mu_{[t+1]}$, t = 0 to T, with $\mu_{[0]}$ defined as 1 and $\mu_{[T+1]}$ as 0.

These propensities are displayed as lightly shaded areas in Figure 4.

The above are all generalisation of the corresponding concepts in the conventional analysis, and are reduced to the latter with dichotomous {0,1} membership functions.

With the conventional poor/non-poor dichotomy, any individual spends some specified numbers of years between 0 and T in the state of poverty during the interval T. With poverty treated as a matter of degree, any particular individual is seen as contributing to the *whole distribution*, from 0 to T, of the number of years spent in poverty.

Over an interval of T years the proportion of the time spent in poverty by the i-th individual is:

$$t_i = \sum_{t=1}^{T} t \cdot \left(\mu_{[t]} - \mu_{[t+1]}\right) / T = \sum_{t=1}^{T} \mu_t \Big/ T,$$ <u>However this concept does not convince me…</u>

i.e. simply the mean over the T periods of an individual's cross-sectional propensities to poverty, and the distribution of the population according to the number of years (from 0 to T) spent in poverty is estimated as

$$p_t = \sum_i w_i \left(\mu_{[t],i} - \mu_{[t+1],i}\right) / \sum_i w_i$$

where i refers to an individual unit and w_i to its sample weight.

Marginal constraints

It is desirable that operational rule (equation 4) defining the intersection of the sets ($\mu_1, \ldots \ldots \mu_T$) is consistent with marginal constraints in the following sense.

The above sequence is just one of all possible sequences of length T, in which any element t can take one of two values, μ_t and its complement $\overline{\mu}_t = (1-\mu_t)$. There are 2^T such sequences. In the conventional analysis, these represent 2^T exhaustive and non-overlapping classes, with each individual unit belonging to one and only one of these, i.e. having some particular pattern (k) of poverty and non-poverty over the T years. Population totals or proportions over any grouping of these patterns are clearly additive. The same consistency must also hold under fuzzy conceptualisation.

(1) Overall constraint

The most important marginal constraint is that, with fuzzy conceptualisation as well, the individual's propensities over all possible patterns k sum up to 1:

$$\sum_{k=1}^{2^T} I_T^{(k)} = 1, \tag{6}$$

where index k refers to a particular pattern. In other words, each joint membership functions $\left(I_T^{(k)}\right)$ refers to one of the 2^T fuzzy partitions of the universal set.

(2) Row marginals

Next in importance are marginal constraints of the form

$$I_t + I_{t-1} \cap \overline{\mu}_t = I_{t-1}, \ t = 1 \text{ to } T \tag{7}$$

where I_t is the intersection of the <u>first</u> t term in the time sequence ($\mu_1, \ldots \mu_t, \ldots \mu_T$), i.e.

$$I_t = \left(\mu_1 \cap \mu_2 \ldots\ldots\ldots \cap \mu_t\right) = \min\left(\mu_1 \ldots\ldots\ldots \mu_t\right). \tag{8}$$

Note that for t=2, this reduces to the marginal constraint for 2 periods (Table 3),

$$\mu_1 \cap \mu_2 + \mu_1 \cap \bar{\mu}_2 = \mu_1,$$

Observing that, by Rule 2, $I_t = I_{t-1} \cap \mu_t = \min(I_{t-1}, \mu_t)$,

equation 7 requires that the second term on its left is defined using the Bounded intersection (in exactly the same way as for two periods described earlier) as follows:

$$I_{t-1} \cap \bar{\mu}_t = \max[0, \min(\mu_1, \mu_2, \ldots\ldots, \mu_{t-1}) - \mu_t] = \max[0, I_{t-1} - \mu_t]. \qquad (9)$$

We may also write an expression similar to equation 7 for the complementing sets:

$$\bar{I}_t + \bar{I}_{t-1} \cap \mu_t = \bar{I}_{t-1},$$

where $\bar{I}_t$ is the intersection of first t terms in the sequence $(\bar{\mu}_1, \bar{\mu}_2, \ldots\ldots\ldots, \bar{\mu}_T)$.

(3) All other marginals

The marginal constraint described so far correspond to control totals for rows in a table like Table 3.

There are also additional constraints to be considered, such as of the type:

$$(\mu_1 \cap \mu_2 \cap \mu_3) + (\mu_1 \cap \bar{\mu}_2 \cap \mu_3) = (\mu_1 \cap \mu_3),$$

in which aggregation is taken over complementing elements of a pair which occurs before the last element in the time sequence (over pair $\mu_2, \bar{\mu}_2$ in the above example). Such constraints correspond, for instance, to column totals in Table 3.

Constraints of this type may perhaps be considered less important in practice than constraints of the type (2) and (1) described above. At the same time they are more difficult to define. We will return to these additional constraints in the next section; these are not covered (though also not necessarily violated) by Rule 3 below defining j.m.f. for a sequence of a particular type, namely $(\mu_1, \mu_2, \ldots\ldots\ldots \mu_{T-1}, \bar{\mu}_T)$ or its complementing type $(\bar{\mu}_1, \bar{\mu}_2, \ldots\ldots\ldots \bar{\mu}_{T-1}, \mu_T)$.

Rule 3

Equation 9 means that the intersection of a sequence of t sets, given that the first (t-1) sets are of the same type (eg, each term representing the individual's propensity to poverty) and the last term is of the opposite type (representing propensity to non-poverty) is to be determined as follows.

(i) First the intersection of the first (t-1) terms of the same type is taken using the Standard operation.

(ii) And then the intersection of the result of (i) is taken with the last set t of the opposite type, using the Bounded operation.

Consider all possible intersections which can be formed from a general sequence over years 1 to t. There are 2^t such operations.

Since at each year we have two complementary sets, these can be seen as 2^{t-1} pairs of sets: a pair being formed by the two longitudinal sets in which the first (t-1) elements are identical, and the last elements are complements of each other. Let $I_t^{(k)}$ and $I_t'^{(k)}$ be the complementing sets in pair k, with k=1 to 2^{t-1}. Equation 9 ensures that the system satisfies 2^{t-1} marginal constraints of the form:

$$I_{t-1}^{(k)} = I_t^{(k)} + I_t'^{(k)}.$$

This rule is a generalisation of the Composite rule defining the intersection of pairs of cross-sectional sets (Rule 1, Section 3) and of Rule 2 in this section defining the intersection of a sequence (μ_1,

μ_2,……) of sets of the same type. Note that unlike the earlier rules, this generalisation is generally NOT commutative or associative.[6]

Hence condition in equation 7 may be interpreted as implying marginal constraints as follows. Given the joint membership function I_{t-1} of similar states up to time (t-1), the j.m.f.'s of the two complementing sets formed from it in the following period are its fuzzy partitions.

Repeating this procedure from t=T back to t=1 implies that the overall constraint (equation 6) is automatically satisfied by the procedure.

Applications

Using a balanced panel over 8 waves of the Italian ECHP, the above procedures have been applied to estimate the following measures.[7]

1. Rate of continuous poverty, i.e. poverty over all of the 8 years covered in the panel.
2. Rate of persistent poverty, i.e. poverty over at least 5 of the 8 years.
3. Rate of any-time poverty, i.e. experience of poverty over 1 or more of the 8 years.
4. Mean proportion of the time spent in poverty.
5. Distribution of the population according the number of years, 0 to 8, spent in poverty.

Numerical results, also comparing these measures using the conventional (poor/non-poor) approach with those using the fuzzy approach, are presented and discussed in an annex to this paper.

5 Joint membership functions covering 3 time-periods

The 8 intersection (or longitudinal) sets for 3 time-periods

In order to generalise the above rules further, it is instructive to consider the longitudinal situation involving 3 time-periods in some details.

The $2^3 = 8$ intersections are shown in Table 5.

Again we use simplified notation, with (a,b,c) as the cross-sectional propensities to poverty at year 1, 2 and 3, respectively, and $(\bar{a}, \bar{b}, \bar{c})$ as their complements.

The table shows only a subset of marginal constraints, each concerning a pair of complementing sets involving c and $\bar{c}$, such as the pair $(a\ b\ c,\ a\ b\ \bar{c})$.

The marginals are simply the 4 intersections determined by applying Rule 1 (the Composite operator) to the first 2 time periods. As already seen in Section 3, these intersections themselves satisfied all the 5 marginal constraints involved in a 2 time period situation shown earlier (Table 3). Note specifically that they sum to 1 for any individual as required by the overall constraint.

---

[6] For instance, expressions like $(\mu_1) \cap (\mu_2 \cap \bar{\mu}_3)$ or $(\mu_2) \cap (\mu_1 \cap \bar{\mu}_3)$ do not necessary equal to $(\mu_1 \cap \mu_2) \cap \bar{\mu}_3$ as defined above. In fact, the intersection between the two components in parenthesis in either of the first two expressions has not been defined so far in this paper, it involves intersection of components which are neither entirely similar nor entirely dissimilar to each other.

[7] A balanced panel means that only individuals present in all the 8 waves in the sample are retained in the analysis.

Table 5. Membership functions for the 8 intersections sets for 3 time-periods

<table>
<tr><td></td><td></td><td>c</td><td>$\bar{c}$</td><td>marginal</td></tr>
<tr><td rowspan="2">a</td><td>b</td><td>(1)
min(a, b, c)</td><td>(2)
max(0, min(a,b)-c)</td><td>(1)+(2)=
min(a, b)</td></tr>
<tr><td>$\bar{b}$</td><td>(3)
max[0, min(a, c)-b]</td><td>(4)
max(0, a-max(b,c))</td><td>(3)+(4)=
max(0, a-b)</td></tr>
<tr><td rowspan="2">$\bar{a}$</td><td>b</td><td>(5)
max(0, min(b, c)-a)</td><td>(6)
max[0, b-max(a, c)]</td><td>(5)+(6)=
max(0, b-a)</td></tr>
<tr><td>$\bar{b}$</td><td>(7)
max(0, c-max(a, b))</td><td>(8)
1-max(a, b, c)</td><td>(7)+(8)=
1-max(a, b)</td></tr>
</table>

Cells (1) and (8) are determined by application of Rule 2.

Cells (2) and (7) are determined by application of Rule 3. In accordance with that rule, the row marginal constraint for each pair ((1)+(2), and (7)+(8)) is automatically satisfied.

For instance, min(a, b, c) + max(0, min(a, b) – c) ≡ min(a, b).

We may also interpret the above as follows. Given the marginal constraint (1)+(2) (determined from Rule 1), and the value in cell (1) (determined by the Rule 2), it is required that their difference, (2), is determined as by Rule 3. The same applies to cell (7).

Rule 4

Cells (4) and (5) involve sequences of the same type: $(a\ \bar{b}\ \bar{c})$ or $(\bar{a}\ b\ c)$.

We propose to construct the j.m.f. over such sequences in the following form:

$$a \cap \bar{b} \cap \bar{c} = (a) \cap (\bar{b} \cap \bar{c}) = \max[0, a - \max(b, c)]$$
$$\bar{a} \cap b \cap c = (\bar{a}) \cap (b \cap c) = \max[0, \min(b, c) - a].$$

That is, we *first* take the intersection of the two adjoining similar states (eg, $b \cap c$) using the Standard operator, and then take the intersection of the result with the state $(\bar{a})$ of a different type using the Bounded operator.

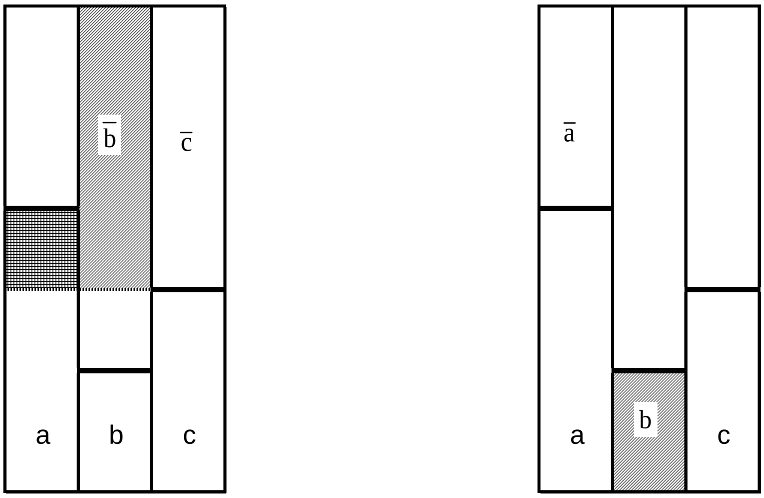
It is useful to formulate this rule in somewhat broader terms so that it is a generalisation of Rule 3 involving any number of time periods.

Consider a sequence of cross-sectional propensities of the following form:

$$(\mu_1, \mu_2, ...\mu_t, \bar{\mu}_{t+1}, \bar{\mu}_{t+2}, ..., \bar{\mu}_T),$$

i.e. the first t terms are all of the same type, and the remaining (T-t) are all of the opposite type. The intersection of these T sets is interpreted as the propensity (the j.m.f.) of being in poverty continuously for the first t years, and then of continuously being non-poor for the remaining (T-t) years. We propose to construct this j.m.f. as follows.

Figure 5. Application of Rule 4 for intersections involving 3 time periods



$\bar{b} \cap \bar{c} = 1 - \max(b, c)$

$a \cap (\bar{b} \cap \bar{c})$
$= \max[0, a - \max(b, c)]$

$b \cap c = \min(b, c)$

$\bar{a} \cap (b \cap c)$
$= \max[0, \min(b, c) - a] = 0$

$$\begin{aligned} f &= \mu_1 \cap \mu_2 \ldots\ldots \cap \mu_t \cap \bar{\mu}_{t+1} \cap \bar{\mu}_{t+2} \ldots\ldots\ldots \cap \bar{\mu}_T \\ &= (\mu_1 \cap \mu_2 \ldots\ldots \cap \mu_t) \cap (\bar{\mu}_{t+1} \cap \bar{\mu}_{t+2} \ldots\ldots\ldots \cap \bar{\mu}_T) \end{aligned}$$

That is:

(i) We first construct the intersection for each of the 2 parts (each made up of consecutive similar states) using the Standard operators.

(ii) And then we construct the intersection of the two resulting sets of different types by using the Bounded operator.

This gives the result:

$$f = \max[0, \min(\mu_1, \mu_2, \ldots.., \mu_t) - \max(\mu_{t+1}, \mu_{t+2}, \ldots\ldots, \mu_T)].$$

The application of this rule gives cells (4) and (5) of Table 5. This rule is illustrated graphically in Figure 5. We consider cells (3) and (6) in the next section.

6 Generalisation to the intersection of any sequence of fuzzy membership functions

Cells (3) and (6) in Table 4 have been obtained from cells (4) and (5) as defined above, and the marginal constraint for the rows concerned.

It can be seen that in order to ensure consistency, these results require the following grouping for the construction of the required intersections in cells (3) and (6):

$a \cap \bar{b} \cap c = (a \cap c) \cap \bar{b}$; and similarly $\bar{a} \cap b \cap \bar{c} = (\bar{a} \cap \bar{c}) \cap b$.

This procedure can be seen as involving a rearrangement of the elements in the sequence into two groups, formed by putting together all elements of the same type, irrespective of their actual temporal location.

Figure 6 shows the operations involved graphically. With correct placement of the m.f.'s of different types at opposite ends of the unit rectangle, it can be seen that the order of elements is immaterial in determining their intersection. Intersection $a \cap \bar{b} \cap c$, for example, is simply the area when all the three elements – a, $\bar{b}$ and c – overlap. Such consistent representation gives a strong intuitive justification for the rule proposed.

On the basis of the above, we formulate the following rule in somewhat broader terms. Consider any sequence of cross-sectional propensities. It can always be expressed in the form:

$$(\ldots, \mu_{t_1}, \ldots)\ (\ldots, \mu_{t_2}, \ldots)$$

where t_1 indicates T_1 elements of the same type in one group, and t_2 indicates T_2 elements of the opposite type in the other group; clearly $T_1 + T_2 = T$.

Rule 5

(i) Sort the elements into 2 groups by type, for instance all T_1 elements of one type followed by all T_2 elements of the other type.

(ii) Construct the intersection for each group involving elements of the same type using the Standard operator.

(iii) Finally, construct the intersection of the two results of the above operation using the Bounded operator.

Rules 5 subsumes all previously observed rules, which are merely special cases of it.

Since the temporal order of cross-sectional propensities is immaterial in the construction of their intersection using this rule, those propensities are entirely inter-changeable in the application of the rule. It follows that Rule 5 satisfies all the required marginal constraints, and not only the ones along rows of a table such as Table 5.

Also with the results being independent of the temporal order, we may view the application of this rule as being "without memory". [8] More precisely perhaps, we may designate it as a procedure "without chronology": the outcome depends on the whole 'history' (i.e., the specified type of cross-sectional sets in the time sequence t=1 to T, and the associated membership functions); but it does not depend on the actual chronology, the temporal sequence, of those cross-sections.

An illustration: Persistent poverty

The following is a more transparent proof of the result obtained in Section 4 above.

The propensity to be poor in exactly t out of T years is the sum of j.m.f.'s over all sequences with t cross-sectional sets of the type "poor" and the remaining (T-t) of the type "non-poor". For any particular sequence of this type, rearrange the sets such that the *first* t terms are of the type "poor". With Rule 5, the j.m.f. for the particular sequence is:

$$f=\max[0, \min(\mu_1, \mu_2, \ldots, \mu_t) - \max(\mu_{t+1}, \mu_{t+2}, \ldots, \mu_T)],$$

which is non-zero only for one sequence in which the first group contains the t largest m.f.'s. With [t] denoting the ordered sequence of decreasing μ values, the required j.m.f. becomes:

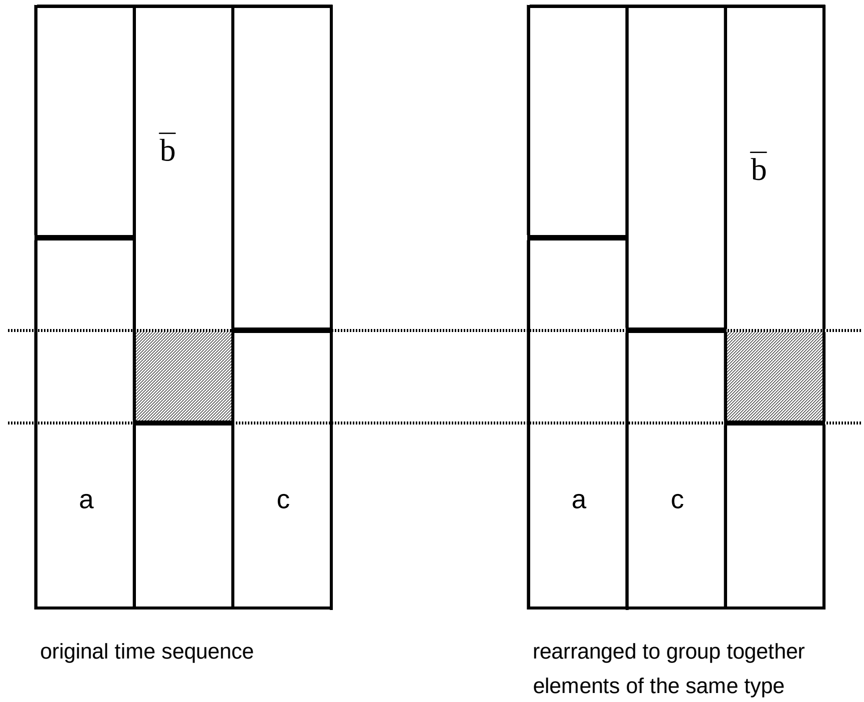
Poor (exactly t out of T years): $\mu_{[t]} - \mu_{[t+1]}$,

and by simple addition:

Poor (at least t out of T years) $\mu_{[t]}$, the t^{th} largest value.

[8] In Betti, Cheli and Cambini (2004), we believe that a particular case of this rule was interpreted in error as being "with first-order memory".

Figure 6. Application of Rule 5 for intersection involving 3 time periods.



$a \cap \bar{b} \cap c = (a \cap c) \cap \bar{b} = \max[0, \min(a, c) - b]$ (Note. In this illustration, $a \cap c = c$.)

Rates of exit and re-entry

Given the state of poverty at time 1, and also at a later time (t-1), what is the proportion exiting from poverty at time t=2, 3, …?

Given the state of poverty at time 1, but of non-poverty at a later time (t-1), what is the proportion which has re-entered poverty at time t=3, 4, ….?

In conventional analysis, the above rates are computed simply from the count of persons in various states:

For exit rate, the numerator is the count of persons poor at times 1 and (t-1), but non-poor at time t; the denominator is the count of all persons who are poor at times 1 and t-1 (and are present in the sample at time t).

For re-entry rate, the numerator is the count of persons poor at time 1, non-poor at time (t-1), but poor again at time t. The denominator is the count of persons who are poor at time 1 and non-poor at time (t-1) (and are present in the sample at time t).

The construction of these measures using fuzzy m.f.'s is also straightforward. With μ_t as a person's propensity to poverty at time t, the person's contribution of these rates is as follows.

Exit rate:

Numerator $(\mu_1 \cap \mu_{t-1}) \cap \bar{\mu}_t = \max[0, \min(\mu_1, \mu_{t-1}) - \mu_t]$

Denominator $(\mu_1 \cap \mu_{t-1}) = \min(\mu_1, \mu_{t-1})$

Re-entry rate:

Numerator $\mu_1 \cap \bar{\mu}_{t-1} \cap \mu_t = (\mu_1 \cap \mu_t) \cap \bar{\mu}_{t-1} = \max[0, \min(\mu_1, \mu_t) - \mu_{t-1}]$

Denominator $\mu_1 \cap \bar{\mu}_{t-1} = \max[0, \mu_1 - \mu_{t-1}]$

Figure 7. Exit and re-entry properties of an individual

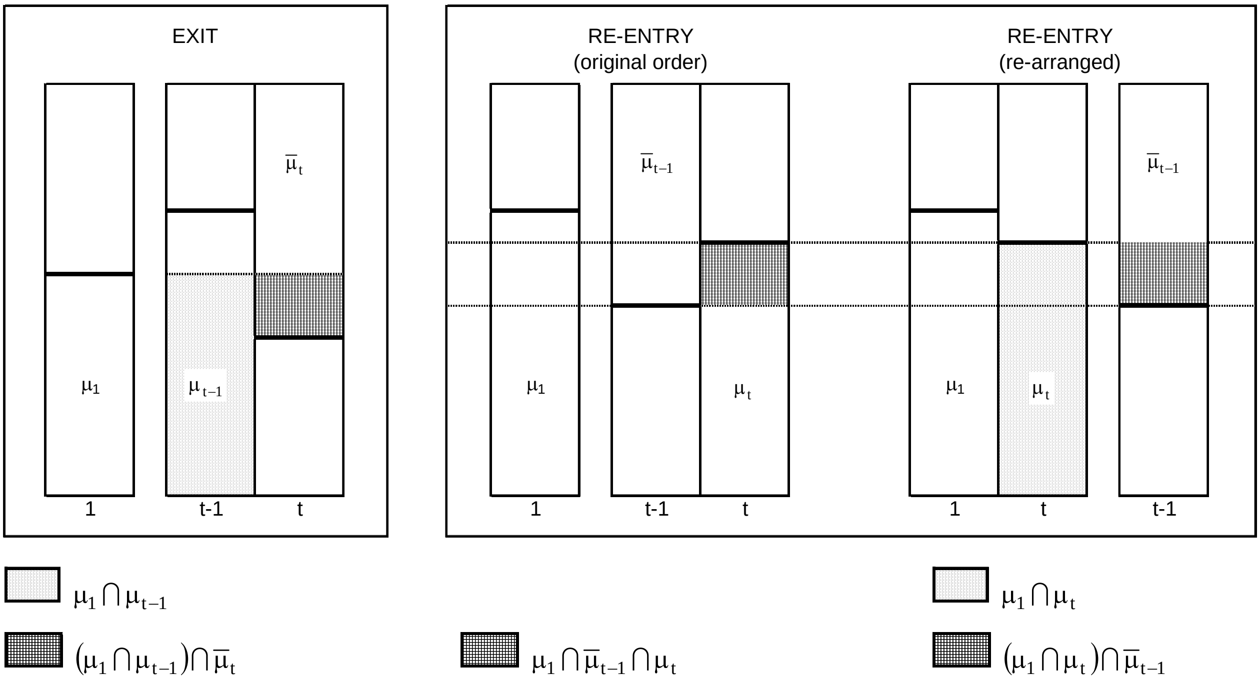


Figure 7 illustrates the procedure for constructing the required intersections, given a person's propensities to poverty at time 1, (t-1) and t.

Note that the result is identical in the two versions of the graphical representation of re-entry propensities. However, rearranging, the time periods as in the second version is illuminating and directly gives the required analytical expressions.

Gross exit rate over a period 2 to T years has been defined as the sum of exit rates experienced by population poor at the starting point and also at the immediately preceding time, even though some of these exits may have been preceded by other exits and re-entries at earlier periods,

Similarly, gross re-entry rate is the sum of yearly re-entries from year 3 to Y.

The difference between the two gross rates gives the net exit rate, aggregated over the period. It approximates net change in the cross-sectional rates between the beginning and the end periods.

Applications

As before we have constructed these measures for Italy and her Macro-regions using Italian ECHP survey.

Numerical results, also comparing these measures using the conventional (poor/non-poor) approach with those using the fuzzy approach, are presented and discussed in an annex to this paper.

Noteworthy from a methodological point is the difference in the performance of the conventional and the fuzzy approaches, especially concerning the estimated incidence of continuous poverty. It appears that movements in and out of poverty tend to be somewhat over-estimated (and hence the persistent or continuous poverty rates under-estimated) with the conventional approach, presumably because it gives too much weight even to small movements across the poverty line.

7. Concluding remarks

When poverty is viewed as a matter of degree in contrast to the conventional poor/non-poor dichotomy, that is, as a fuzzy state, two additional aspects are introduced into the analysis.

(i) The choice of membership functions i.e. quantitative specification of individuals' or households' degrees of poverty and deprivation.

(ii) And the choice of rules for the manipulation of the resulting fuzzy sets, rules defining their complements, intersections, union and averaging. Specifically, for longitudinal analysis of poverty using the fuzzy set approach, we need *joint membership functions* covering more than one time period, which have to be constructed on the basis of the series of cross-sectional *membership functions* over those time periods.

We have aimed to address in this paper the second of the above questions.

We have proposed a general rule for the construction of fuzzy set intersections, that is for the construction of a longitudinal poverty measures from a sequence of cross-sectional measures under fuzzy conceptualisation. This general rule is meant to be applicable to any sequence of "poor" and "non-poor" set, and it satisfies all the marginal constraints. On the basis of the results obtained, various fuzzy poverty measures over time can be constructed as consistent generalisations of the corresponding conventional (dichotomous) measures.

The proposed rule has been developed in a logical, step-by-step, manner, satisfying the required marginal constraints. This is important since there are reasons to believe that, hitherto, the rules of fuzzy set operations in the context of multi-dimensional and longitudinal poverty analysis have not been well or widely understood.

Figure 8 illustrates the proposed rules for the construction of fuzzy set intersections.

We begin with two most important and basic rules. The first, termed here 'Rule 1', defines the intersection of a pair of fuzzy sets taking into account whether the two sets in the pair are of the same type (e.g., poor/poor), or of different types (e.g., poor/non-poor). This is the Composite set operator which replaces the Standard operator which has been used mostly (where? In other contexts?).

'Rule 2' deals with a sequence of sets of the same type (e.g., a series of cross-sectional propensities to poverty, or to non-poverty). It provides the basis for the study of persistence of poverty over time using the fuzzy conceptualisation.

These two rules imply as matter of logical consistency what we have called 'Rule 3'; the latter is also a generalisation of and subsumes the preceding rules. All these rules can be viewed as applied successively, term by term starting from the first, in the original time sequence of the constituent sets[9]:

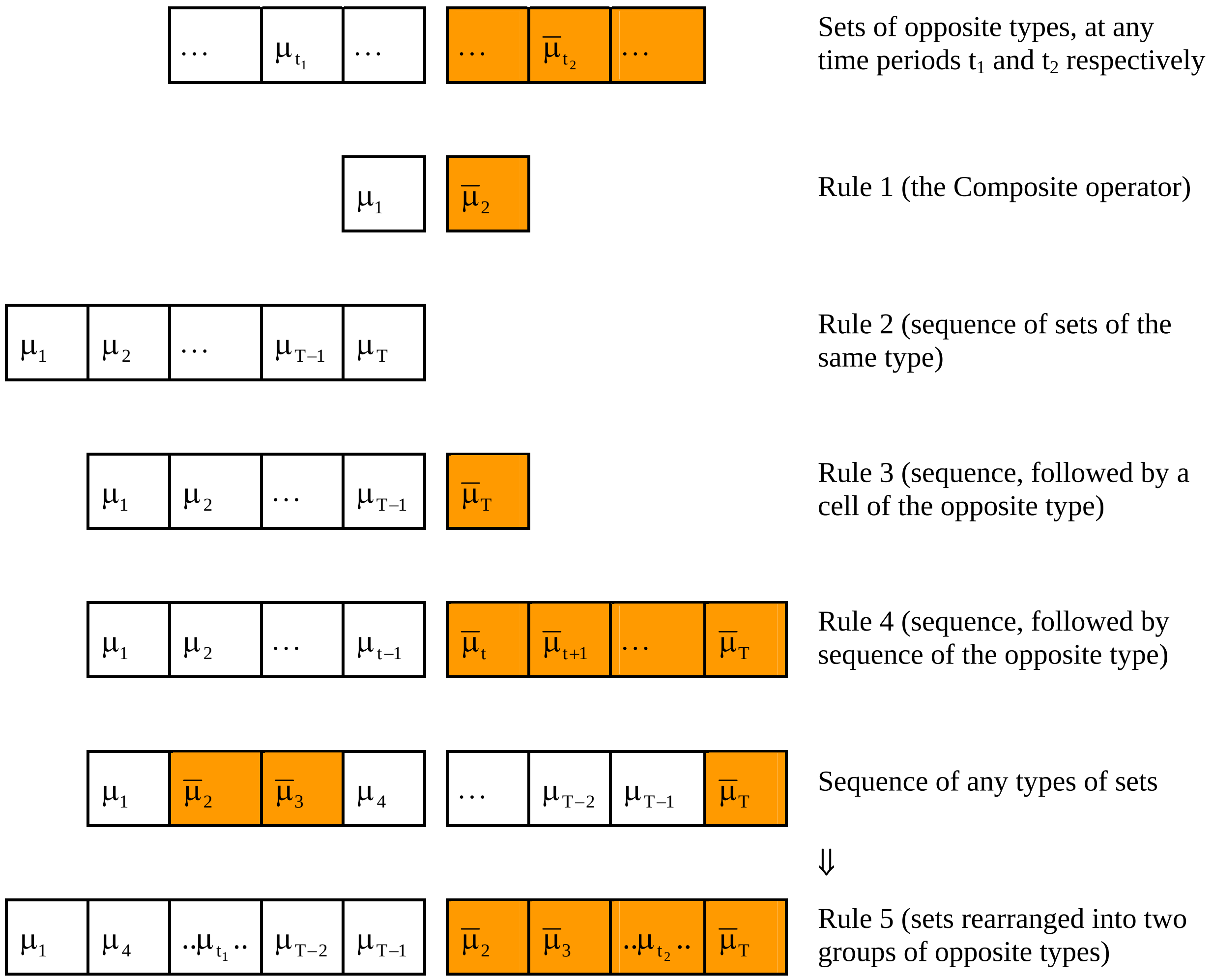
$$\mu_1 \cap \mu_2 \cap \mu_3 \cap .....\cap \mu_{T-1} \cap \bar{\mu}_T = \left[\left\langle ...\left\{\left(\mu_1 \cap \mu_2\right) \cap \mu_3\right\} \cap .....\right\rangle \cap \mu_{T-1}\right] \cap \bar{\mu}_T . \tag{10}$$

'Rule 4' is a generalisation of Rule 3 (and of course also of the preceding rules). However, this may not be the only possible generalisation. It involves dealing *separately* with two sequences, each of a uniform type but different from the other, and putting them together only at the end. Unlike equation 10, however, this rule cannot be viewed as applied successively in the original time sequence of the constituent sets.

The final rule ('Rule 5') is a generalisation of all of the above, and in this sense the only rule needed. In fact, it is *implied* by them as matter of logical consistency.

[9] All intersections up to μ_{T-1} for sets of the same type are constructed using the Standard operator, and the intersection of the result with the last term of the opposite type using the Bounded operator.

Figure 8. Fuzzy set intersection: Rules for the construction of a longitudinal measure (joint membership function) from a sequence of cross-sectional measures (membership functions)



We have discussed elsewhere approaches and procedures for constructing measures (propensities) of income poverty and of combining them with similarly constructed measures of non-monetary deprivation using the fuzzy set approach (Betti, Cheli, Lemmi and Verma, 2005). In fact, the procedures for combining fuzzy measures in multiple dimensions at a given time are identical, in formal terms, to the procedures described here for combining fuzzy cross-sectional measures over multiple time periods, as noted in Section 2.

We plan to present, in a separate paper, numerical results of these procedures applies to measures of multidimensional poverty and deprivation, and to combinations of such measures.

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Annex: Summary of the procedure

Let, for a series of cross-sections (1,…t,…T), each person's propensity to be in poverty (i.e. the person's membership function of the set "poor") be given as:

$$(\mu_1,\ \ \mu_2,.......\mu_t,.....,\mu_T)=[0,1].$$

We also define the complements of the above at each time, i.e. the membership function (m.f.) of the set "non-poor" as $\overline{\mu}_t = 1-\mu_t$.

Let S(1,2,….,T) be a *particular pattern* of T "poor" and "non-poor" sets for which the j.m.f. is required. Let the elements (cross-sectional sets) of this pattern be grouped into two parts:

$$S_1 = (....,t_1,....)\ ,\ \ S_2 = (....,t_2,....),$$

where t_1 indicates any of T_1 elements of the same type (say, "poor") in the first group, and t_2 any of T_2 elements of the group of the opposite type ("non-poor"), with $T_1 + T_2 = T$. Let:

$$m_1 = \min(....,\mu_{t_1},....)\ ;\ \ M_2 = \max(....,\mu_{t_2},....).$$

The required j.m.f. for the particular pattern of interest is given by:

$$\text{JMF} = \max(0, m_1 - M_2). \qquad (11)$$

Different types of longitudinal measures correspond to, or can be simply derived from, different patterns S.

For instance, for the propensity to be poor at time 1, non-poor at time 2, and then re-entering poverty at time 3, we have:

$$S_1 = (1,3),\ S_2 = (2),\ \ \text{JMF} = \max(0, \min(\mu_1,\mu_3) - \mu_2).$$

For "continuously poor", S_1=S, S_2=0, giving:

$$\text{JMF} = \min(\mu_1,\ \ \mu_2,.......\mu_t,.....,\mu_T).$$

We can also express (11) in terms of cross-sectional propensities to be "non-poor". Noting that $\min(\mu) = 1-\max(\overline{\mu})$, an equivalent expression is:

$$\text{JMF} = \max(0, \overline{m}_2 - \overline{M}_1), \qquad (12)$$

where $\overline{M}_1 = \max(....,\overline{\mu}_{t_1},....) = 1-m_1$, $\overline{m}_2 = \min(....,\overline{\mu}_{t_2},....) = 1-M_2$.

For instance, for "never poor"(or "continuously non-poor"), all the elements of the particular sequence S of interest are of the type "non-poor", hence S_2=S, S_1=0, giving:

$$\text{JMF} = \min(\overline{\mu}_1,\ \ \overline{\mu}_2,.......\overline{\mu}_t,.....,\overline{\mu}_T) = 1-\max(\mu_1,\ \ \mu_2,.......\mu_t,.....,\mu_T).$$

The propensity to be "ever poor" is the complement of the above:

$$\text{JMF} = \max(\mu_1,\ \ \mu_2,.......\mu_t,.....,\mu_T).$$