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Distribution of Wealth: Theoretical Microfoundations and Empirical Evidence*

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Abstract

This paper studies the dynamics of wealth distribution in an economy where dynasties with different wealth have constant marginal saving rates, firms' productivities are subject to idiosyncratic shocks and factors' returns are determined in competitive markets. Government imposes taxes on capital and labour incomes and redistributes the collected resources to individuals. The equilibrium distribution of wealth is explicitly calculated and it follows a Paretian law in the top tail. The Pareto exponent depends on the saving rate in a nonmonotonic fashion, on the net return on capital, on the growth rate of population and on the degree of portfolio diversification. On the contrary, the bottom tail of the wealth distribution mostly depends on the characteristics of the labour market. The resulting theoretical predictions find a corroboration in the empirical experiences of Italy and United States in the period 1987-2004.

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Contents

1 Introduction

There have been several attempts, in the economic literature, to explain the statistical regularities of the wealth distribution first showed by Pareto (1897) (see Mandelbrot (1960) and Champernowne and Cowell (1998), Cap. 11, and for a review Atkinson and Harrison (1978), Cap. 3, and Davies and Shorrocks (1999)). However, as remarked in Davies and Shorrocks (1999), "research has shifted away from a concern with the overall distributional characteristics, focusing instead on the causes of individual differences in wealth holdings". One of the main reasons of the loss of interest in this field is the lack of a precise economic interpretation of the stochastic processes proposed as explanation of the observed wealth distribution. In the words of Davies and Shorrocks (1999) "[these] models lack of an explicit behavioural foundation for the parameter values and are perhaps best viewed as reduced forms". This makes these stochastic models useless both to understand the causes of an increase in income/wealth inequality and to provide some guide to public policy. Vaughan (1978) and Laitner (1979) represent an attempt to overcome this critique.

The shape of the wealth distribution has also recently rose considerable interest in the econophysics community.[1] The focus has been mostly put on empirical analyses of extensive data sets and on simple models of exchange which largely abstract from microeconomic considerations (see Chatterjee et al. (2005)). Hence, in spite of their relative success in reproducing empirical results, these papers do not face the fundamental drawback (at least from an economic point of view) to give a sound economic explanations of the stochastic laws of wealth accumulation.

In the present paper we try to fill the gap proposing a microfoundation of the dynamics of wealth distribution based on a model where dynasties with different wealth have constant marginal saving rates, firms' productivities are subject to idiosyncratic shocks and factors'returns are determined in competitive markets.[2] Moreover, we consider a Government, which taxes capital and labour incomes and redistributes the revenues to the individuals.

We explicitly calculate the equilibrium distribution of wealth and we find that the shape of the top tail of the distribution follows a Paretian law, whose exponent, which represents a synthetic index of the degree of inequality of the top tail of distribution, depends on the saving rate, the net return on capital, the growth rate of population and the degree of diversification of portfolio. On the contrary, the bottom tail mostly depends on the characteristics of the labour market.

[1] See Chatterjee et al. (2005) for a wide sample of articles on this topics.

[2] Atkinson and Harrison (1978), p.202 distinguish between individual and distributional models, where the first "derive the relationships governing individual wealth-holding and then aggregate these to obtain the overall distribution", while the second "formulate the problem directly in terms of the size distribution". Our approach belongs to the first type of models, but uses the distribution dynamics tools of the second. It is worth noting that already Brown (1976), p.85, suggests this line of research.

In particular, an increase in the taxation of capital income and/or in the degree of diversification of portfolio increases the Pareto exponent. On the contrary, a decrease in the saving rate and/or in the growth rate of population has ambiguous effects on the Pareto exponent: a negative *direct* effect due to the lower accumulation of wealth is contrasted by a positive *induced* effect due to the increase in the return on investment. We show that when technology is Cobb-Douglas there exists an inverted U-shaped relationship between the Pareto exponent and both the saving rate and the growth rate of population. In general, any factor increasing the net return on investment decreases the Pareto exponent. Therefore a technological change favouring capital leads to a decrease in the Pareto exponent via an increase in the return on investment.

The bottom tail of the equilibrium distribution of wealth is instead crucially affected by the structure of labour market, in particular by the cross-section distribution of wages. If the labour market is completely flexible, so that individual wages immediately respond to idiosyncratic shocks, the support of the equilibrium distribution of wealth includes also negative values; on the contrary if all workers receive the same wage, i.e. bargaining in the labour market is completely centralized, shocks are only transmitted through capital returns and the wealth distribution is bounded away from zero. Therefore less flexibility in the labour market means less wealth inequality in the bottom tail of the wealth distribution.

Finally, we show that there could exist an inverted U-shape between the Pareto exponent and the growth rate of the economy if growth is endogenous, while we do not find any relationship if growth is exogenous.

In the final section we corroborate our theoretical results by an empirical analysis. We study the recent trends of wealth inequality in Italy (1987-2004) and United States (1989-2004); the analysis is respectively based on the Survey of Household Income and Wealth (SHIW) and on the Survey of Consumer Finances (SCF). In both countries the top tail of the wealth distribution follows a Pareto distribution. The estimated Pareto exponent is decreasing for both countries in the considered periods. We argue that our model can help to individuate the factors to the base of this decline by pointing to the decrease in the taxation of capital income in both countries; in addition, for Italy a change in its technology in favour of capital and an increase in its saving rate, given its high initial level, while for U.S. a decrease in its saving rate, given its initial low level. Demographic factor seems not to play a relevant role in both countries. Finally, we argue that the increase in the bottom tail of the wealth distribution in both countries (this is more evident for Italy) can be explained by the increase in the flexibility of labour markets: in both countries we find evidence of a lower power of Trade Unions and of an increase in the cross-section variance of the distribution of labour incomes; in addition in Italy we observe a strong increase in the share of non permanent jobs.

The theoretical model is in the spirit of Vaughan (1978), but he considers only two classes of individuals and factors' returns are not determined in competitive markets. The latter is crucial for many findings of the model, as the nonlinear relationships between the Pareto exponent and the saving rate and the growth rate of population and the effects on the bottom tail of distribution of the different degree of flexibility of labour markets. Shorrocks (1975) proposes a similar approach, but he does not consider a general competitive equilibrium model. Garcia-Peñalosa and Turnovsky (2005) proposes a model similar to ours, but they assume an *AK* technology and aggregate shocks to production. Finally, Levy (2003) provides a general discussion on which properties must have the stochastic process of wealth accumulation in order to have an equilibrium distribution of wealth which satisfies the Pareto law. He finds that every agents must have the same investment talent, i.e. the same probability distribution of the returns on her investments; in our model this happens by the competitive capital market, which ensures that every individual has the same investment opportunities. In this respect we generalize Levy (2003)'s results considering also labour incomes and calculating the analytical expression of the equilibrium distribution of wealth. Cagetti and DeNardi (2005) survey the studies of the dynamics of the wealth distribution based on optimizing agents (see, e.g., Alvarez-Pelaez and Diaz (2005), who however follow a deterministic approach).

With respect to the econophysics literature, well summarized by Chatterjee et al. (2005), our paper addresses the main criticisms raised to this approach, see e.g. Chatterjee et al. (2005), p. 51, by providing a microeconomic foundation of the wealth distribution dynamics. This allows us to relate additive stochastic terms in the wealth dynamics to the labour market and multiplicative terms to the capital market. In addition, we provide an explicit relation between the Pareto exponent and key parameters in the economy.

Finally, as regards as the empirical analysis Atkinson (2003) provides a detailed discussion on how the recent dynamics of the distribution of income in OECD countries can be explained by changes in the net return to capital income and technology; his results largely agree with the results of our theoretical model and with our empirical evidence. Piketty (2006) et al. provide further evidence on the factors affecting the wealth inequality in France.

The paper is organized as follows: Section 2 presents the theoretical model; Section 3 shows the evolution of wealth distribution and characterizes the properties of the equilibrium distribution of wealth. Section 4 discusses the empirical evidence supporting our theoretical results. Section 5 concludes. All proofs are relegated in the appendix.

2 The Model

We model a standard competitive economy in which firms demand capital and labour. We assume all the wealth is owned by dynasties, who inelastically offer capital and labour and decide which amount of their disposable income is saved. Wages and interest rate adjust to respectively clear the labour and capital markets. For the sake of simplicity we consider just one type of capital. Hence human capital can be represented by different labour endowments and/or included in the capital stock (in the latter case it is accumulated at the same rate of physical capital).

From a technical point of view, we follow a standard approach to model a stochastic economy, see, e.g., Chang (1988) and Garcia-Peñalosa and Turnovsky (2005). We derive continuum time stochastic equations for the evolution of the wealth distribution. We do this by first analysing the dynamics over a time interval $[t, t + dt)$ and then we let $dt \rightarrow 0$. Stochastic shocks are modelled by differentials of Wiener processes and we focus on the long-run equilibrium.[3] This approach neglects the highly debated issue of time-scale adjustment of different economic variables and the out-of-equilibrium dynamics (see Hicks (1986)).

2.1 Firms

Consider an economy with F firms. Every firm j has the same technology $q(.)$. Its output over the period $(t + dt)$, $dy_j(t)$, is the joint product of its technology and of a random idiosyncratic component $dA_j(t)$:

$$dy_j(t) = q[k_j(t), l_j(t)]dA_j(t), \tag{1}$$

where $k_j(t)$ and $l_j(t)$ are respectively the capital and the labour of firm j at time t and dA_j is a random shock to production. We assume that at time t firm j knows only the distribution of dA_j (see Section 2.4 for the characteristics of the stochastic components of economy).

The presence of a labour augmenting exogenous technological progress can be taken into account assuming that $l_j(t) = l_j^*(t)\exp(\psi t)$, where ψ is the growth rate of technological progress. All the following analysis remains the same, except for the meaning of the per capita variables, which are to be interpreted in efficient units of labour (see Chang (1988), p. 163).

We make the standard assumption that $q(.)$ is an homogeneous function of degree one (i.e. technology has constant returns to scale), with positive first derivatives and negative second derivatives with respect to both arguments. Since $q(.)$ is an homogeneous function of degree one we have that:

$$q(k, l) = lg(k/l) \text{ with } g'(\lambda) > 0 \text{ and } g''(\lambda) < 0, \tag{2}$$

[3] A different method to derive continuous time stochastic differential equations is that of assuming a discrete stochastic dynamics over very small time intervals ϵ. Then increments over times $dt \gg \epsilon$ are obtained summing dt/ϵ increments of the discrete dynamics. Under suitable conditions for the dependence of the the mean and the variance of stochastic increments, the Central Limit Theorem can be applied to yield a stochastic dynamics over time increments dt with Gaussian noise.

where $\lambda = k/l$ is the capital per worker.[4]

Every firm j maximizes its expected profits over the period $(t + dt)$ $d\pi_j$:

$$\max_{k_j,l_j} E\left[d\pi_j\left(t\right)\right] = q[k_j(t), l_j(t)]E\left[dA_j(t)\right] - k_j dr\left(t\right) - k_j db\left(t\right) - l_j dw\left(t\right), \tag{3}$$

at given dr, dw and db, the market interest, the market wage and the capital depreciation respectively over the period $(t + dt)$. From the first order conditions of Problem (3) we have that:

$$dr + db = \frac{\partial q}{\partial k_j} E\left[dA_j\right] \text{ and } \tag{4}$$

$$dw = \frac{\partial q}{\partial l_j} E\left[dA_j\right]. \tag{5}$$

Since $q\left(.\right)$ is an homogeneous function of degree one we have:

$$q = \frac{\partial q}{\partial k_j} k_j + \frac{\partial q}{\partial l_j} l_j, \tag{6}$$

which with Eqq. (3), (4) and (5) implies that in equilibrium the profits of each firm are zero.

After the realization of shock dA_j firm j gets its output and it rewards its factors according to their marginal productivity (see Eqq. (4) and (5)):[5]

$$dr_j = \frac{\partial q}{\partial k_j} dA_j - db = \frac{\left(dr + db\right) dA_j}{E\left[dA_j\right]} - db; \tag{7}$$

$$dw_j = \frac{\partial q}{\partial l_j} dA_j = \frac{dw\, dA_j}{E\left[dA_j\right]}. \tag{8}$$

2.2 Dynasties

Economy is populated by N dynasties and at time t. We use a subscript i to denote dynasty i. Let l_i and p_i be respectively the stock of labour and the current wealth of dynasty i.[6] Her gross income over the period $(t + dt)$, dy_i, is given by:[7]

$$dy_i\left(t\right) = p_i\left(t\right) \sum_{j=1}^{F} \theta_{i,j}\left(t\right) dr_j\left(t\right) + l_i \sum_{j=1}^{F} \phi_{i,j}\left(t\right) dw_j\left(t\right), \tag{9}$$

[4]The CES production function $q(k,l) = \left[\varepsilon k^\gamma + \left(1-\varepsilon\right) l^\gamma\right]^{1/\gamma}$ with $\varepsilon \in (0,1)$ satisfies these hypotheses. Here $\gamma < 1$ tunes the elasticity of substitution between k and l, which is $1/\left(1-\gamma\right)$. For $\gamma \to 0$ we get the Cobb-Douglas production function $q(k,l) = k^\varepsilon l^{1-\varepsilon}$. The function $g(\lambda)$ is given by $g\left(\lambda\right) = \left[\varepsilon\lambda^\gamma + 1 - \varepsilon\right]^{1/\gamma}$, while in the Cobb-Douglas case $g\left(\lambda\right) = \lambda^\varepsilon$.

[5]Here we assume that workers' incomes are subject to the same risk of entrepreneurs. In the real world it is more likely that wages are fix in the short run and the returns on capital absorbs all risk. In Section 3.2.3 we address this possibility and show the crucial implication for the wealth distribution.

[6]Different amounts of labour could be interpreted as different level of abilities (productivities) among individuals. In fact, the wage rate is defined in terms of an unit of labour service: the labour income for dynasty i is equal to $l_i dw$.

[7]If k represents both physical and human capital then dr is the average return of this composite variable.

where $\theta_{i,j}(t)$ is the fraction of wealth p_i invested in firm j at time t ($\sum_{j=1}^{F} \theta_{i,j} = 1$) and $\phi_{i,j}(t)$ is the fraction of labour that dynasty i works in firm j at time t ($\sum_{j=1}^{F} \phi_{i,j} = 1$). Both coefficients $\phi_{i,j}(t)$ and $\theta_{i,j}(t)$ should be thought of as the resulting allocation arising from market interactions at time t. In particular, for any interest rate and wage, they depend on the individual preferences (e.g. the attitude toward risk) and on the possible transaction costs in labour and capital markets (e.g. the presence of a fix cost of investment).

The disposable income is the result of taxation and redistribution. We assume that capital and labour income are taxed at a flat rate τ_k and τ_l respectively. The resources collected from taxes are redistributed to individuals by lump-sum transfers. Therefore the dynasty i's disposable income over the period $(t + dt)$, dy_i^D, is given by:

$$dy_i^D = (1-\tau_k)\sum_{j=1}^{F} p_i\theta_{i,j}dr_j + (1-\tau_l)\sum_{j=1}^{F} l_i\phi_{i,j}dw_j + \frac{\tau_k}{N}\sum_{j=1}^{F} k_j dr_j + \frac{\tau_l}{N}\sum_{j=1}^{F} l_j dw_j \tag{10}$$

We assume that dynasty i takes her decision according to the following Keynesian consumption function:[8]

$$dc_i = d\bar{c} + c\,dy_i^D + dc^p p_i, \tag{11}$$

where c is the marginal propensity to consume with respect to disposable income, $d\bar{c}$ captures the presence of a minimum consumption $d\bar{c} \geq 0$, which is independent of the level of income and dc^p is the marginal propensity to consume with respect to wealth.[9]

Consumption function (11) appears to be a good compromise between the necessity to consider all the possible relevant variables affecting the individuals consumption decision and the more rigorous method to derive the consumption function from the intertemporal optimizing problem of individual i.[10] In fact, the lack of a closed solution to the latter, excluding a very limited number of cases, would prevent us to find an analytical expression of the the equilibrium distribution of wealth. Moreover, Chang (1988) shows that Eq. (11) with $d\bar{c} = dc^p = 0$ can represent the consumption function of an intertemporal optimizing

[8] The consumption function in the original Keynesian version does not include a term for wealth.

[9] In the empirical section we discuss how $d\bar{c}$ could depend on the average wealth of economy. This is not surprising considering that the minimum consumption should reflect the minimum standard of living, which is strongly relate to the level of development of a country.

[10] See Cagetti and DeNardi (2005) for a survey on the models analysing the wealth distribution with intertemporal optimizing individuals. The key point of this approach is to endogenize the saving rate. A general results is that the actual wealth distributions are not matched under the assumption of homogeneous individual preferences and this is because, under such approach, wealthy individuals tend to save less relatively their income. This is in stark contrast with our assumption of constant marginal saving rate and with the empirical evidence we will show in Section 4. Some authors circumvent this drawback assuming that heterogeneity in the individual saving behaviour of individuals is due to different degrees of risk aversion (i.e. there exist "entrepreneurial" and "worker" individuals) and adding the assumption of altruistic behaviour. From another point of view Eq. (11) can be thought of as the leading order expansion of a generic consumption function $dc(dy_i^D, p_i)$ which neglects higher order differentials.

agent with infinite lifetime when utility function is CES and technology is Cobb-Douglas; in such case Chang (1988), p. 163, shows that s is equal to the intertemporal elasticity of substitution.[11] Finally, in Section 4 we show that consumption function (11) finds a strong empirical corroboration in the Italian data.

Given Eq. (11), dynasty i accumulates her wealth according to:

$$dp_i = sdy_i^D - d\bar{c} - dn_i p_i, \tag{12}$$

where the first term sdy_i^D, with $s \equiv 1 - c$, reflects the relationship between savings and disposable income (s is the marginal saving rate), while the last term, dn_i, arises from the term dc^p, but it may also include demographic effects and any other effect which can directly affect the dynasty i's wealth. We shall assume that dn_i has a stochastic component and for simplicity in the following we will refer to dn_i as a *demographic* component.

2.3 Equilibrium

In the equilibrium of capital market we have that:

$$P = \sum_{i=1}^{N} p_i = \sum_{j=1}^{F} k_j = K, \tag{13}$$

while in the equilibrium of labour market:

$$\sum_{i=1}^{N} l_i = \sum_{j=1}^{F} l_j = L. \tag{14}$$

From Eqq. (7) and (8) we have:

$$E\left[dr_j\right] = dr = \ \forall j \text{ and} \tag{15}$$

$$E\left[dw_j\right] = dw \, \forall j. \tag{16}$$

From Eqq. (4), (5), (15) and (16) we have that:

$$\frac{k_j}{l_j} = \bar{k} = \frac{K}{L} = \frac{P}{L} = \lambda \, \forall j, \tag{17}$$

[11] A limit of our theoretical analysis is to ignore the possible effect of changes in fiscal policy on saving rate. However, ex-ante such effect is ambiguous. In the empirical section we show that in response to a decline in the tax rates both in Italy and in U.S. the saving rate increased in Italy and decreased in U.S.. In Italy such increase could be partially explained by precautionary saving, due to the increase in the expected volatility of future incomes (see Jappelli and Pistaferri (2000b)) and to the change in social security in 1990s (see Attanasio and Brugiavini (2003)). In this regard, to our purposes it is preferable to take saving rates as a parameter to be estimated, being the latter observable, rather than to consider unobservable variables, like the expectations of future incomes, and/or change in welfare policy, which are not easily measurable.

that is every firm j utilizes the same production technique; $\bar{k}$ is the firms' endowment of capital per unit of labour. For convenience we also define the per capita wealth $\bar{p} = \Sigma_{i=1}^{N} p_i/N = P/N$ and the per capita labour endowment $\bar{l} = \Sigma_{i=1}^{N} l_i/N = L/N$. In equilibrium firm j rewards its factors at the following rates:

$$dr_j = dA_j g'(\lambda) - db \text{ and} \tag{18}$$

$$dw_j = dA_j \left[g(\lambda) - \lambda g'(\lambda)\right]. \tag{19}$$

2.4 The Continuum Time Limit

Let us now make the dependence of differentials on the time infinitesimal dt explicit. In particular, we take:

$$dr = \rho dt; \tag{20}$$

$$db = \beta dt; \tag{21}$$

$$dw = \omega dt; \tag{22}$$

$$dA = E\left[dA_j\right] = adt; \tag{23}$$

$$dn = E[dn_i] = \nu dt \text{ and} \tag{24}$$

$$d\bar{c} = \chi dt, \tag{25}$$

where ρ is the interest rate, β the depreciation rate, ω the wage rate, a a scale parameter, ν the growth rate of population plus the marginal effect of wealth on consumption and χ the minimum consumption.

For productivity and demographic shocks we define:

$$d\zeta_j = \frac{dA_j - E\left[dA_j\right]}{a} \text{ and} \tag{26}$$

$$d\xi_i = dn_i - E\left[dn_i\right], \tag{27}$$

so that $E\left[d\zeta_j\right] = E\left[d\xi_i\right] = 0$. We assume that these two sources of randomness, $d\zeta_j$ and $d\xi_i$, are Wiener increments with the following properties:

$$E\left[d\zeta_j d\zeta_{j'}\right] = \Delta dt \delta_{j,j'} \delta\left(t - t'\right) \text{ and} \tag{28}$$

$$E\left[d\xi_i d\xi_{i'}\right] = \Gamma dt \delta_{i,i'} \delta\left(t - t'\right), \tag{29}$$

where Δ and Γ are respectively the variances of productivity and demographic shocks, $\delta_{j,j'} = 1$ if $j = j\prime$ and $\delta_{j,j'} = 0$ otherwise, whereas $\delta\left(t - t'\right)$ is Dirac's δ distribution (random shocks are temporally independent and uncorrelated).

Finally, from Eqq. (7)-(8), (20)-(23) and (26)-(27) we have:

$$\begin{aligned} dr_j &= \rho dt + (\rho + \beta) d\zeta_j \text{ and} \\ dw_j &= \omega dt + \omega d\zeta_j. \end{aligned} \tag{30}$$

The allocation of capital and labour among firms should be such to satisfy the equilibrium conditions (13)-(17). For example, the case in which each dynasty i works just in a single firm $j^*(i)$ (i.e. $\phi_{i,j^*(i)} = 1$ and $\phi_{i,j} = 0$ for $j \neq j^*(i)$) *may not* be compatible with market equilibrium because this may not ensure that every firm has the optimal ratio of capital and labour λ (see Eq. (17)). On the other hand the allocation where each dynasty invests an equal share of her wealth (i.e. $\theta_{i,j} = 1/F$) and contributes the same amount of labour to each firm ($\phi_{i,j} = 1/F$) is always compatible with the market equilibrium.

Consistency with market equilibria imply that the coefficients $\theta_{i,j}$ and $\phi_{i,j}$ carry some dependence on the dynamical variables p_i. Still, market equilibrium is not sufficient to determine unambiguously the value of the coefficients $\theta_{i,j}$ and $\phi_{i,j}$. In what follows, we will largely neglect the dependence of allocations on dynamical variables and treat $\theta_{i,j}$ and $\phi_{i,j}$ as parameters of the economy specifying the degree of concentration of capital and labour investment. Actually, we find it convenient to introduce the variables

$$\Theta_{i,i'} = \sum_{j=1}^{F} \theta_{i,j}\theta_{i',j}, \quad \Omega_{i,i'} = \sum_{j=1}^{F} \theta_{i,j}\phi_{i',j} \text{ and } \Phi_{i,i'} = \sum_{j=1}^{F} \phi_{i,j}\phi_{i',j}. \tag{31}$$

These characterize the structural properties of the economy i.e. the degree of intertwinement of economic interactions. For example $\Theta_{i,i'}$ is a scalar which represents the overlap of investments of dynasty i with those of dynasty i'.

2.5 The Evolution of Wealth Distribution

Proposition 1 shows the dynamics of the dynasty i's wealth.

Proposition 1 *Given the definitions above, the dynasty i's wealth obeys the following stochastic differential equation:*[12]

$$\frac{dp_i}{dt} = s\left[(1-\tau_k)\rho p_i + (1-\tau_l)\omega l_i + \tau_k\rho\bar{p} + \tau_l\omega\bar{l}\right] - \chi - \nu p_i + \eta_i, \tag{32}$$

where η_i is a white noise term with $E[\eta_i(t)] = 0$ and covariance:

$$E[\eta_i(t)\,\eta_{i'}(t')] = \delta(t-t')\,H_{i,i'}[\vec{p}], \tag{33}$$

where

$$\begin{aligned} H_{i,i'}[\vec{p}] = \Delta s^2 \Big\{ & (1-\tau_k)^2(\rho+\beta)^2 p_i p_{i'}\Theta_{i,i'} + (1-\tau_l)^2\omega^2 l_i l_{i'}\Phi_{i,i'} + \\ & + (1-\tau_k)(1-\tau_l)(\rho+\beta)\omega\left[p_i l_{i'}\Omega_{i,i'} + l_i p_{i'}\Omega_{i',i}\right] + \\ & + \frac{\tau_k(\rho+\beta)+\tau_l\omega/\lambda}{N}\left[(1-\tau_k)(\rho+\beta)(p_i\vartheta_i + p_{i'}\vartheta_{i'}) + (1-\tau_l)\omega(l_i\varphi_i + l_{i'}\varphi_{i'})\right] + \\ & + \frac{\left[\tau_k(\rho+\beta)+\tau_l\omega/\lambda\right]^2}{N^2}\sum_{j=1}^{F} k_j^2 \Big\} + \Gamma\delta_{i,i'}p_i^2, \end{aligned}$$

[12]Here we adopt the notation of Langevin equations, see Gardiner (1997).

and

$$\vartheta_i = \sum_{i'=1}^{N} \Theta_{i,i'} p_{i'}, \quad \varphi_i = \sum_{i'=1}^{N} \Omega_{i,i'} p_{i'}. \tag{34}$$

Proof. See Appendix A. ■

Eq. (33) in Proposition 1 shows that the correlation in the shocks hitting two dynasties i and i' arises either because both are investing in the same firms ($\Theta_{i,i'}$), or because one is investing in the firm in which the other is working ($\Omega_{i,i'}$) or because both are working in the same firm ($\Phi_{i,i'}$). Terms ϑ_i and φ_i are respectively the average capital of the firms where dynasty i is investing and working.

3 Infinite Economy

In this section we focus on the limit of infinite economy, where we assume that N and $F \to \infty$. In particular, we focus on the case where $F = fN$, where f is a positive constant. This assumption is not a relevant limit to the analysis because in a real economy N may be of the order of some million and the same applies for the number of firms (or more precisely for the number of possible different types of investment). Given these assumptions, we restrict our attention on the properties of the equilibrium, i.e. we analyse the behaviour of the economy in the limit $t \to \infty$.

Below we first derive the evolution of the aggregate variables and then we focus on two cases: i) the stationary/exogenous growth case and ii) the endogenous growth case.

3.1 Evolution of Aggregate Variables

To derive the dynamics of per capita wealth $\bar{p}$ consider Eq. (32), from which:

$$\frac{1}{N}\frac{d}{dt}\sum_{i=1}^{N} p_i = \frac{d\bar{p}}{dt} = s\left(\rho\bar{p} + \omega\bar{l}\right) - \upsilon\bar{p} - \chi + \bar{\eta},$$

where

$$\bar{\eta}dt = \frac{1}{N}\sum_{i=1}^{N}\left[dp_i - E\left(dp_i\right)\right] = \frac{s(\rho+\beta+\omega/\lambda)}{N}\sum_{j=1}^{F} k_j d\zeta_j + \frac{\nu}{N}\sum_{i=1}^{N} p_i d\xi_i.$$

Proposition 2 shows under which conditions $\lim_{N\to\infty} \bar{\eta}dt = 0$, that is the dynamics of $\bar{p}$ is not stochastic.

Proposition 2 *Assume that there exists a constant $\bar{\theta} > 0$ such that:*

$$\sum_{i=1}^{N} \theta_{i,j} \leq \bar{\theta}\ \forall j, N \tag{35}$$

and that:

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} p_i^2 = 0. \tag{36}$$

Then the per capita wealth $\bar{p}$ follows a deterministic dynamics given by:

$$\frac{d\bar{p}}{dt} = s\left(\rho\bar{p} + \omega\bar{l}\right) - \chi - v\bar{p}. \tag{37}$$

Proof. See Appendix B. ■

Assumption (35) states that the total share of dinasties' investment in each firm is bounded above; this is a equivalent to saying that investment cannot be too unevenly distributed.[13] Notice that taking the sum on j in Assumption (35) yields $\bar{\theta} \geq 1/f = N/F$; this requires that the number of firms growth at least as fast as the number of dynasties. Assumption (36) is a law of large numbers and, in particular, it is an assumption on the shape of the top tail of the distribution of p_i. In fact, if the probability distribution density (*pdf*) of p_i behaves for large p as $f(p) \sim p^{-\alpha-1}$ with $\alpha > 1$, as we will find later, then Assumption (36) holds. If Assumption (36) does not hold then per capita wealth follows a stochastic process and the dynasty i's wealth will fluctuate both for the idiosyncratic shocks and for the fluctuations of the aggregate variables.

Substituting for ρ and ω in Eq. (37) from Eqq. (15)-(16) and (20)-(23) we get:

$$\frac{d\bar{p}}{dt} = sag\left(\frac{\bar{p}}{\bar{l}}\right)\bar{l} - \chi - (s\beta + v)\,\bar{p}, \tag{38}$$

which is the well-known equation of the Solow growth model augmented with the minimum consumption χ (note that $ag\left(\bar{p}/\bar{l}\right)\bar{l} = aq\left(\bar{k}, 1\right)\bar{l}$ is the per capita output adjusted for the effective supply of labour of each individual, i.e. it would be per-capita output if $L = N$). Stiglitz (1969) shows that this model can generate many different dynamics according to the parameters' value and to the shape of the production function. In particular, we can have an economy where per capita wealth is i) converging to a positive value/growing at an exogenous rate, ii) growing at an endogenous rate or, finally, iii) converging toward zero; the long-run equilibrium could also depend on the initial level of the per capita wealth. To our purpose case iii) (economy with zero capital) is trivial, so that we analysis in details only cases i) and ii).

3.2 Stationary/Exogenous Growth Case

Heuristically the condition to have an equilibrium with constant per capita wealth is that the growth rate of per capita wealth becomes negative for large value of $\bar{p}$. Moreover, depending

[13] As extreme example consider the case where all dynasties invest in the same firm $j = 1$ all their capital, i.e. $\theta_{i,1} = 1$ and $\theta_{i,j} = 0$ for $j > 1\ \forall i$. Then Condition (35) is violated for $j = 1$.

on the value of the production function in zero, we can have zero, one or two equilibria (even if only one is stable). Let us remark that if there exists an exogenous technological progress, $\bar{p}$ is per capita wealth measured in efficient units; therefore in equilibrium the effective per capita wealth will grow at the exogenous growth rate of technological progress ψ.

Proposition 3 states the conditions for the existence of an equilibrium with a constant and positive per capita wealth.

Proposition 3 *Assume that dynamics of the per capita wealth obeys Eq.(38) and*

$$\lim_{\bar{p}\to\infty} g'\left(\bar{p}/\bar{l}\right) < \frac{s\beta+\nu}{sa}. \tag{39}$$

Then if

$$g\left(0\right) > \frac{\chi}{sa\bar{l}} \tag{40}$$

an equilibrium with constant and positive per capita wealth exists. Otherwise if:

$$g\left(0\right) < \frac{\chi}{sa\bar{l}} \tag{41}$$

and if

$$\exists \bar{p}_1 < \bar{p}_2 \text{ such that } sag\left(\bar{p}_h/\bar{l}\right)\bar{l} = \chi + \left(s\beta+v\right)\bar{p}_h \text{ for } h=1,2, \tag{42}$$

then $\bar{p}_2$ and $\bar{p}_1$ are respectively a local stable and unstable equilibria. Economy converges towards an equilibrium with a per capita wealth equal to $\bar{p}_2$ if and only if $\bar{p}\left(0\right) > \bar{p}_1$, while if $\bar{p}\left(0\right) < \bar{p}_1$ economy converges towards an equilibrium with zero per capita wealth.
When the conditions to have in equilibrium a positive per capita wealth $\bar{p}^*$ are satisfied, then $\bar{p}^*$ solves:

$$sag\left(\bar{p}^*/\bar{l}\right)\bar{l} = \chi + \left(s\beta+v\right)\bar{p}^*, \tag{43}$$

while the interest rate and the wage rate are respectively equal to:

$$\rho^* = ag'\left(\bar{p}^*/\bar{l}\right) - \beta \text{ and} \tag{44}$$

$$\omega^* = a\left[g\left(\bar{p}^*/\bar{l}\right) - \left(\bar{p}^*/\bar{l}\right)g'\left(\bar{p}^*/\bar{l}\right)\right] \tag{45}$$

Proof. See Appendix C.[14] ■

Figures 1 and 2 provide the intuition of results in Proposition 3.

[14] In the case of CES technology the first of the two conditions corresponds to:

$$\varepsilon^{1/\gamma} < \frac{s\beta+v}{sa} \text{ for } \gamma \in \left(0,1\right),$$

while is always satisfied for $\gamma \leq 0$. Condition (40) is satisfied for:

$$\left(1-\varepsilon\right)^{1/\gamma} > \frac{\chi}{sa\bar{l}} \text{ for } \gamma \in \left(0,1\right),$$

while is never satisfied for $\gamma \leq 0$. In the Cobb-Douglas case, i.e. $\gamma = 0$, Conditions (39) and (41) are always satisfied, but $\bar{p}_1 = 0$.

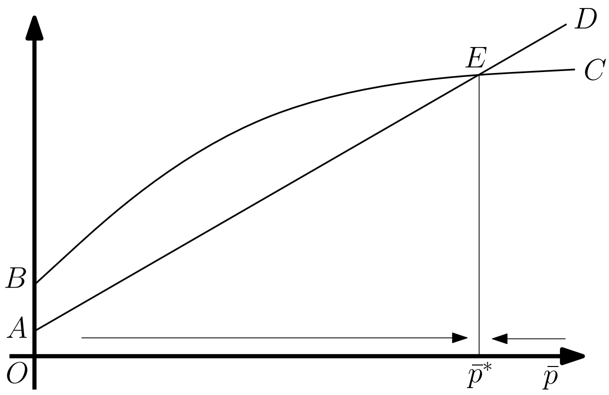

Figure 1: economy with a global stable equilibrium

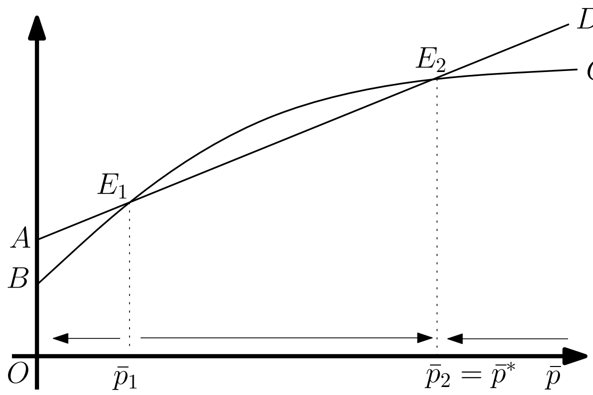

Figure 2: economy with a local stable equilibrium

Figure 1 shows the case of single global stable equilibrium, whereas Figure 2 refers to the case of two equilibria, only one of which is stable (the highest). In the latter case in order to have an equilibrium with a positive per capita wealth it is necessary that the initial value of per capita wealth is higher than $\bar{p}_1$.

Proposition 3 shows that $\bar{p}^*$ positively depends on s and a and negatively on ν and β.[15] The interest rate ρ^* negatively depends on $\bar{p}^*$ and β; note that the effect of a on ρ^* is ambiguous because it has a positive direct effect, but also a negative induced effect: in fact an increase in a increase $\bar{p}^*$, and therefore tends to decrease ρ^*. Finally, in equilibrium the wage rate ω^* depends positively on $\bar{p}^*$.

3.2.1 The Equilibrium Distribution of Wealth

In the following we characterize the equilibrium distribution of wealth for every dynasty i, when economy converges toward a constant and positive per capita wealth $\bar{p}^*$ and we show that Assumption (36) is satisfied in equilibrium.

Proposition 4 *Assume that the infinite dynasty economy converges towards a positive and constant per capita wealth $\bar{p}^*$ and that $s\left[(1-\tau_l)\,\omega^* l_i + \tau_k \rho^* \bar{p} + \tau_l \omega^* \bar{l}\right] > \chi$. Then the equilibrium distribution of p_i is given by:*

$$f_i\left(p_i\right) = \left[\frac{\mathcal{N}}{\left(a_0 + a_1 p_i + a_2 p_i^2\right)^{1+z_1/a_2}}\right] e^{4\left[\frac{z_0 + z_1 a_1/(2a_2)}{\sqrt{4a_0 a_2 - a_1^2}}\right] \arctan\left(\frac{a_1 + 2a_2 p_i}{\sqrt{4a_0 a_2 - a_1^2}}\right)}, \tag{46}$$

[15]This is straightforward given Eq. (43) and the properties of $g(.)$: $g'(.) > 0$ and $g''(.) < 0$.

where

$$
\begin{aligned}
z_0 &= s\left[(1-\tau_l)\,\omega^* l_i + \tau_k \rho^* \bar{p} + \tau_l \omega^* \bar{l}\right] - \chi; \\
z_1 &= \nu - s\,(1-\tau_k)\,\rho^*; \\
a_0 &= \Delta s^2\,(1-\tau_l)^2\,\omega^{*2} l_i^2 \Phi_{i,i}; \\
a_1 &= 2\Delta s^2 (1-\tau_k)(1-\tau_l)\,(\rho^*+\beta)\,\omega^* l_i \Omega_{i,i} \text{ and} \\
a_2 &= \Delta s^2\,(1-\tau_k)^2\,(\rho^*+\beta)^2\,\Theta_{i,i} + \Gamma
\end{aligned}
$$

and $\mathcal{N}$ is a constant defined by the condition $\int_{-\infty}^{\infty} f(p_i)\,dp_i = 1$.

Proof. See Appendix D. ■

Condition $s\left[(1-\tau_l)\,\omega^* l_i + \tau_k \rho^* \bar{p} + \tau_l \omega^* \bar{l}\right] > \chi$ ensures that dynasty with zero wealth have an *expected* positive saving, i.e. she can escape from the zero wealth trap. This also ensures that her average wealth will be positive (see Eq. (75)). Therefore this is not a source of substantial limitation of our analysis.

For large p_i $f(p_i)$ follows a Pareto distribution whose exponent is given by:

$$
\alpha = 1 + 2z_1/a_2 = 1 + 2\frac{\nu - s\,(1-\tau_k)\,\rho^*}{\Delta s^2 (1-\tau_k)^2 (\rho^*+\beta)^2 \Theta_{i,i} + \Gamma}. \tag{47}
$$

We stress that z_1, $a_2 > 0$ (see Condition (39) and Eq. (44)) and hence $\alpha > 1$: this ensures that Assumption (36) is indeed satisfied.

It is worth noticing that while the expected wealth of dynasty i does not depend on individual characteristics of dynasty i, i.e. $E[p_i] = \bar{p}^*$, the distribution does. In particular, $f(p_i)$ depends on $\Theta_{i,i}$, which is an index of the diversification of the dynasty i's portfolio; in particular, $\Theta_{i,i} = 1$ means no diversification and $\Theta_{i,i} = 0$ maximal diversification.[16]

3.2.2 The Determinants of Pareto Exponent

Remark 5 shows the relationships between the top tail of the wealth distribution, i.e. the Pareto exponent, and the main variables and parameters of the model.

Remark 5 *The size of the top tail of wealth distribution measure by (the inverse of) α is an increasing function of Δ, $\Theta_{i,i}$, Γ and β, and a decreasing function of τ_k. Changes in ν, a and s have ambiguous effects on the top tail of wealth distribution.*

Remark 5 provides some insights on which forces affect the top tail of the wealth distribution. First of all we note that labour income does not play any role, on the contrary of the interest rate ρ^*. An increase of the latter tends to increase the size of the top tail by increasing

[16] Castaldi and Milakovic (2006) suggest that also the frequency of the changes in the composition of the wealthiest portfolios can affect the Pareto exponent.

the return to accumulate wealth (i.e. $\partial\alpha/\partial\rho^* < 0$). In this respect the ambiguous relationships between α and s (ν) are due to two competing effects generated by an increase in s (ν): a *direct* effect which tends to decrease (increase) α and an *induced* effect due to the decrease (increase) in the interest rate ρ^*, in turn caused by an increase (decrease) in the equilibrium per capita wealth $\bar{p}^*$. Ex ante it is not possible to determine which effect prevails without specifying the technology. These results highlight the importance to endogenize the returns to factors in order to study the effect of changes in the saving rates (population growth rate) on inequality. Finally, the ambiguous relationship between α and a is due the ambiguous effect of a on ρ^*. In the following we analyse the case of Cobb-Douglas technology.

Cobb-Douglas technology Assume that technology is Cobb-Douglas , i.e. $g = \lambda^\varepsilon$, with $\varepsilon \in (0,1)$ and $\chi = 0$; then in equilibrium:[17]

$$\bar{p}^* = \bar{l}\left(\frac{sa}{s\beta + v}\right)^{1/(1-\varepsilon)}; \tag{48}$$

$$\rho^* = \frac{\varepsilon\left(s\beta + v\right)}{s} - \beta; \tag{49}$$

$$\omega^* = (1-\varepsilon)\, a^{1/(1-\varepsilon)}\left(\frac{s}{s\beta + \nu}\right)^{\varepsilon/(1-\varepsilon)} \text{ and} \tag{50}$$

$$\alpha = 1 + 2\left\{\frac{\nu\left[1-(1-\tau_k)\,\varepsilon\right] + (1-\tau_k)\,(1-\varepsilon)\, s\beta}{\Delta(1-\tau_k)^2\varepsilon^2(s\beta+\nu)^2\Theta_{i,i} + \Gamma}\right\}. \tag{51}$$

First, note that a does not affect the equilibrium interest rate ρ^* and therefore it has no effect on α. As expected, α is negatively related to ε, which measures both the elasticity of product to capital and the share of capital income on total product, via an increase of ρ^*.[18]

[17] We are in the case of two equilibria of Proposition 3, but an equilibrium is trivial, i.e. $\bar{p}_1 = 0$.

[18] Assuming CES technology and $\chi = 0$ we have that:

$$\bar{p}^* = \bar{l}\left\{\frac{1-\varepsilon}{\left[(s\beta+\nu)/sa\right]^\gamma - \varepsilon}\right\}^{1/\gamma} \text{ and}$$
$$\rho^* = \varepsilon a^\gamma\left(\frac{s\beta+\nu}{s}\right)^{1-\gamma} - \beta,$$

from which:

$$\frac{\partial\rho^*}{\partial\gamma} > 0 \Leftrightarrow s\,(a-\beta) > v.$$

Given Condition (39), we have that:

$$\frac{\partial\rho^*}{\partial\gamma} \gtreqless 0 \text{ for } \gamma \in (0,1) \text{ and}$$
$$\frac{\partial\rho^*}{\partial\gamma} < 0 \text{ for } \gamma < 0.$$

Therefore the effect on α of a change in γ is ambigous for $\gamma \in (0,1)$, while it is negative for $\gamma < 0$.

The relationships between α and s and ν are more complex; in particular Remarks 6 and 7 show that these relationships can be nonmonotonic.

Remark 6 *If* $\Gamma' > \nu^2 D$ *then* $\partial\alpha/\partial s > (<)\, 0$ *for* $s < (>)\, \hat{s}$, *otherwise* α *is always increasing in* s, *where* $\hat{s} = \nu \left[-(1+D) + \sqrt{(1-D)^2 + 4\Gamma'/\nu^2} \right] / (2\beta)$, $\Gamma' = \Gamma / \left[\Delta (1-\tau_k)^2 \varepsilon^2 \Theta_{i,i} \right]$ *and* $D = \nu^2 \left[2 - (1-\tau_k)(1+\varepsilon) \right] / \left[(1-\tau_k)(1-\varepsilon) \right]$.

Proof. The proof directly follows from the derivative of α expressed in Eq. (51) with respect to s. ■

This nonmonotonic relationship is due to the alternate prevalence between the direct and the induced effect, whose strengths depend on the level of per capita wealth, that is on the level of saving rate. In particular, the induce effect prevails for low levels of s, while the direct effect prevails for high values of s. Figure 3 reports a numerical example.[19]

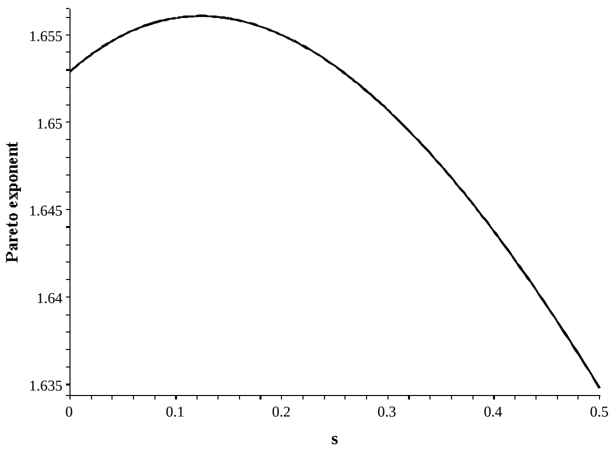

Figure 3: the nonmonotonic relationship between the Pareto exponent and saving rate

Therefore a decrease in s can lead to a decline in the Pareto exponent if s is sufficiently low (under 0.13 in Figure 3). Otherwise an increase in s determines a decrease in α. Remark 7 states that a similar nonmonotonic relationship holds also for ν.

Remark 7 *If* $\Gamma' > s^2\beta^2$ *then* $\partial\alpha/\partial\nu > (<)\, 0$ *for* $\nu < (>)\, \hat{\nu}$, *otherwise* α *is always increasing in* ν, *where* $\hat{\nu} = s\beta \left[-(1+D) + \sqrt{(1-D)^2 + 4\Gamma'} \right] /2$, $\Gamma' = \Gamma / \left[\Delta (1-\tau_k)^2 \varepsilon^2 \Theta_{i,i} \right]$ *and* $D = \left[(1-\tau_k)(2-\varepsilon) - 1 \right] / \left[1 - (1-\tau_k)\varepsilon \right]$.

Proof. The proof directly follows from the derivative of α expressed in Eq. (51) with respect to ν. ■

[19] The parameters assume the following values: $\nu = 0.01$, $\beta = 0.01$, $\chi = 0$, $\varepsilon = 0.35$, $a = 0.5$, $\bar{l} = 1$, $\tau_k = 0.20$, $\tau_l = 0.3$, $\Gamma = 0.015$, $\Delta = 9000$, $\Theta = 0.1$, $\Phi = 1$, $\Omega = 1$.

In this case the direct effect prevails for low level of ν, while the induced effect prevails for high values. Figure 4 reports a numerical example.[20]

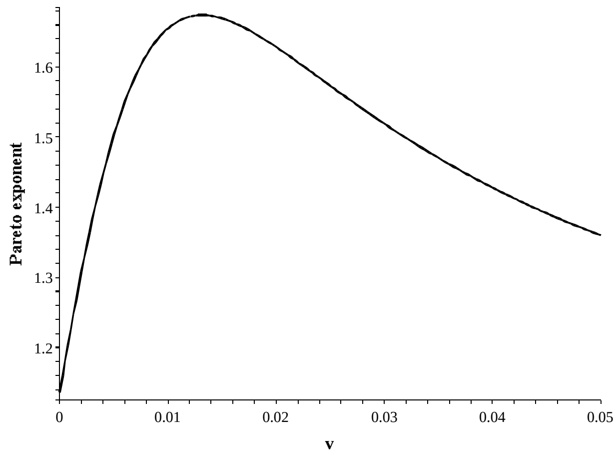

Figure 4: the nonmonotonic relationship between the Pareto exponent and growth rate of population

Countries with a low and declining growth rate of population (lower than 0.01, e.g. Italy) should show a decline in α as well as countries with high and increasing growth rate of population. This result could explained the empirical evidence in Laitner (2001).

3.2.3 Staggered Wages

We have just seen that labour market does not affect the top tail of the wealth distribution; however it is crucial for the shape of bottom tail, i.e. for the poorest dynasties. We assumed that wages are perfectly flexible, but in the real labour markets wages are generally fixed in the short period and productivity shocks are absorbed by the returns on capital (see Garcia-Peñalosa and Turnovsky (2005) for a similar point).[21] To investigate the implications of this fact on the wealth distribution we assume that all wages in the economy are set to the expected level of productivity, that is:

$$dw_j = \frac{\partial q}{\partial l_j} E\left[dA_j\right] = dw \; \forall j. \tag{52}$$

This means that the cross-section variance of labour incomes is zero: this framework can represent an economy in which Trade Unions have strong market power, such that the bargaining on labour market is completely centralized. It is worth remarking that wages follow

[20] The parameters assume the following values: $s = 0.2$, $\beta = 0.01$, $\chi = 0$, $\varepsilon = 0.35$, $a = 0.5$, $\bar{l} = 1$, $\tau_k = 0.20$, $\tau_l = 0.3$, $\Gamma = 0.015$, $\Delta = 9000$, $\Theta = 0.1$, $\Phi = 1$, $\Omega = 1$.

[21] In general, wages flexibility depends on many factors such as the presence in the labour markets of strong Trade Unions and on the possibility to stipulate short-term labour contracts between workers and firms.

the marginal productivity of labour and therefore there does not exist unemployment. The actual profits are defined by:

$$d\pi_j(t) = q[k_j(t), l_j(t)]dA_j(t) - dr_j k_j - dbk_j - dwl_j$$

and since in equilibrium $d\pi_j(t) = 0$,[22] we have that:

$$dr_j = q[k_j(t), l_j(t)]dA_j(t)/k_j - db - dwl_j/k_j. \tag{53}$$

All the results in Proposition 1 are unchanged and therefore p_i follows again Eq. (32), with the noise term η_i which satisfies $E[\eta_i(t)] = 0$ and Eq. (33), but

$$\lim_{N\to\infty} H_{i,i'}[\vec{p}] = \left[\Delta s^2(1-\tau_k)^2(\rho+\beta)^2\Theta_{i,i'} + \Gamma\delta_{i,i'}\right] p_i p_{i'}. \tag{54}$$

Eq. (54) reflects the fact that now labour market is not a source of shocks for the dynamics of dynasty i's wealth .

Under Assumption (36), again $\bar{p}$ has a deterministic dynamics given by Eq. (37), so that the results in Proposition 3 also hold in the case of staggered wages. Therefore the per capita wealth, wages and interest rate are not affected by the assumption of staggered wages. However, the equilibrium distribution of wealth changes.

Proposition 8 *Assume that economy converges towards a positive and constant per capita wealth* $\bar{p}^*$ *and that* $s\left[(1-\tau_l)\,\omega^* l_i + \tau_k \rho^* \bar{p} + \tau_l \omega^* \bar{l}\right] > \chi$. *Let* $f^{SW}(p_i)$ *be the equilibrium distribution of* p_i *when* $N \to \infty$. *Then:*

$$f^{SW}(p_i) = \frac{\mathcal{N}^{SW}}{a_2 p_i^{2(1+z_1/a_2)}} e^{-\left(\frac{2z_0}{a_2 p_i}\right)}, \tag{55}$$

where $\mathcal{N}^{SW}$ *is a constant defined by the condition* $\int_{-\infty}^{\infty} f^{SW}(p_i)\,dp_i = 1$ *and* z_0, z_1 *and* a_2 *are the same as in Proposition 4.*

Proof. See Appendix E. ■

For large p_i the equilibrium distribution $f^{SW}(p_i)$ follows a Pareto distribution whose exponent is equal to $\alpha^{SW} = 1 + 2z_1/a_2$; this is the same of the model with perfectly flexible wages (see Eq. (47)). The distribution is instead markedly different for small values of p_i. The intuition is that wages' behaviour is crucial for the low wealth dynasties, and, in particular, a lower volatility of wages decreases the size of the bottom tail of the wealth distribution because poor dynasties have an income largely dependent of wages (the cross-section volatility of wages is zero when wages are staggered). To support this intuition note that $\lim_{p_i\to 0+} f^{SW}(p_i) = 0$, while $\lim_{p_i\to 0+} f(p_i) > 0$. Figure 5 shows a numerical example of two wealth distributions when technology is Cobb-Douglas.[23]

[22] In the framework profits are zero because the returns on capital is residual with respect to the wages, therefore the owners of capital takes all net product not distributed to the workers.

[23] The parameters assume the following values: $\nu = 0.01$, $s = 0.20$, $\beta = 0.01$, $\chi = 0$, $\varepsilon = 0.35$, $a = 0.5$, $\bar{l} = 1$, $\tau_k = 0.20$, $\tau_l = 0.3$, $\Gamma = 0.015$, $\Delta = 30$, $\Theta = 0.1$, $\Phi = 1$, $\Omega = 1$.

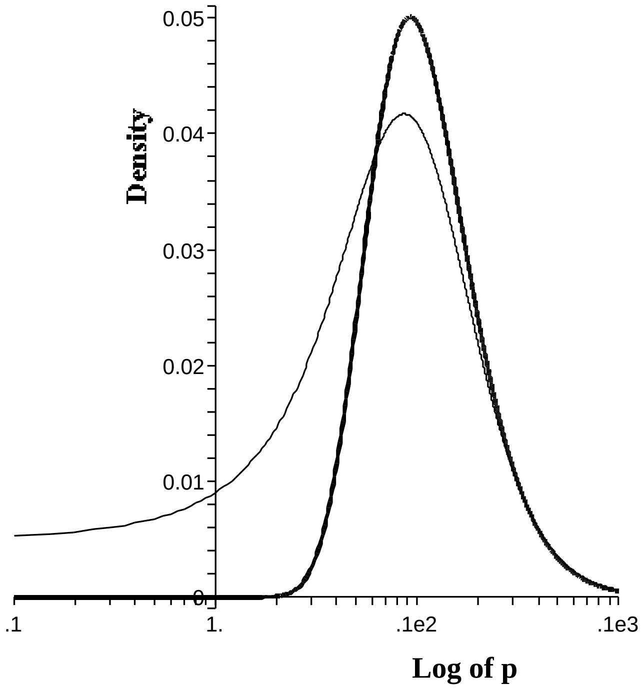

Figure 5: comparison between the wealth distributions with perfectly flexible (thin line) and staggered (thick line) wages

The thin line represents the density of wealth distribution $f(p_i)$ for the case of perfectly flexible wages, while the thick line represents the density $f^{SW}(p_i)$ for the case of staggered wages. Figure 5 confirms that, when wages are staggered, the bottom tail of the wealth distribution has a lower size and there is no dynasties with negative wealth; finally, as expected, the top tail is not affect from this change in the labour market.

3.3 Endogenous Growth Case

Figures 6 and 7 report two cases where the growth of per capita wealth in equilibrium is due to the ongoing accumulation of wealth (and not by the possible exogenous technological progress).

The worth of this case derives from: i) the major focus on the capital return, which decides the shape *also* of the bottom tail of the wealth distribution and ii) the relationship between the wealth inequality and the endogenous growth rate of economy.

Proposition 9 states the necessary and sufficient conditions under which per capita wealth

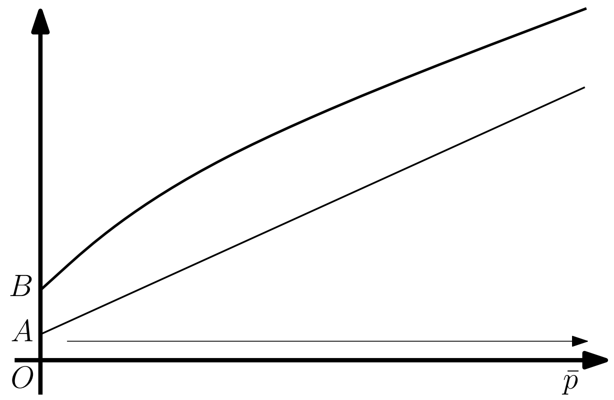

Figure 6: Expanding economy

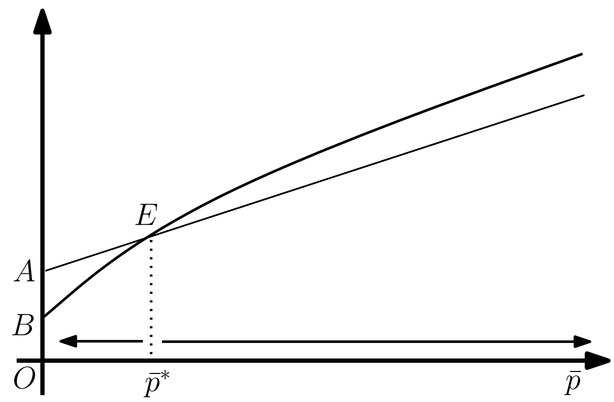

Figure 7: Expanding economy with a unstable equilibrium

grows at a positive growth rate in equilibrium, i.e. there is endogenous growth.[24]

Proposition 9 *Assume that:*

$$\lim_{\bar{p}\to\infty} g'\left(\bar{p}/\bar{l}\right) > \frac{s\beta+v}{sa}; \tag{56}$$

then if:

$$g(0) > \frac{\chi}{sa\bar{l}} \tag{57}$$

in equilibrium per capita wealth will be growing at the following rate:

$$\psi^{EG} = \lim_{\bar{p}\to\infty} s[ag'\left(\bar{p}/\bar{l}\right) - \beta] - \nu, \tag{58}$$

indipendent of initial per capita wealth. Differently, if:

$$g(0) < \frac{\chi}{sa\bar{l}}, \tag{59}$$

then in equilibrium per capita wealth will be growing at constant rate ψ^{EG} if and only if the initial per capita wealth is sufficient high. Assume that in equilibrium per capita wealth growth at rate ψ^{EG}; then the factor returns are given by:

$$\rho \to \rho^* = \lim_{\bar{p}\to\infty} ag'\left(\bar{p}/\bar{l}\right) - \beta \tag{60}$$

$$\omega \to \omega^* = \lim_{\bar{p}\to\infty} a\left[g\left(\bar{p}/\bar{l}\right) - \left(\bar{p}/\bar{l}\right)g'\left(\bar{p}/\bar{l}\right)\right]. \tag{61}$$

[24] Assuming CES technology Condition (56) corresponds to:

$$\varepsilon^{1/\gamma} > \frac{s\beta+v}{sa} \text{ for } \gamma\in(0,1),$$

while it is never satisfied for $\gamma \leq 0$. Condition (57) is satisfied for:

$$(1-\varepsilon)^{1/\gamma} > \frac{\chi}{sa\bar{l}} \text{ for } \gamma\in(0,1)$$

and it always satisfied for $\gamma \leq 0$. For the Cobb-Douglas technology Condition (56) is never satisfied.

Proof. See Appendix F. ■

It is worth noting that in equilibrium this economy has a behaviour similar to an AK model (Barro and Sala-i-Martin (1999)), i.e. a model where marginal and average product of capital are constant (or, more precisely, are bounded below). Moreover, the equilibrium interest rate ρ^*, taking β as given, is determined only by technology. Finally, ω^* is finite.

The growth rate ψ^{EG} can be written as a function of interest rate, i.e.:

$$\psi^{EG} = s\rho^* - \nu, \tag{62}$$

which shows that growth rate ψ^{EG} is positively related to the level of saving rate s and negative related to ν. Interestingly ψ^{EG} is independent of the flat tax rate on capital τ_k and on the spread of individual portfolios $\Theta_{i,i}$.[25] Moreover, ψ^{EG} positively depends on the return on capital ρ^*. Therefore all changes in the technology which increases the capital return cause an increase in ψ^{EG} as well. Assuming *CES* technology, i.e. $g = [1 - \varepsilon + \varepsilon\lambda^\gamma]^{1/\gamma}$, we have that:

$$\psi^{EG} = sa\varepsilon^{1/\gamma} - \nu, \tag{63}$$

from which it is clear that ψ^{EG} positively depend on ε and γ.

Proposition 10 shows the dynamics of the dynasty i's wealth in the case of endogenous growth.

Proposition 10 *Let u_i be the relative per capita wealth of dynasty i, i.e. $u_i = p_i/\bar{p}$. The dynasty i's wealth obeys the following stochastic differential equation:*

$$\lim_{\bar{p}\to\infty} \frac{du_i}{dt} = s\rho^*\tau_k(1 - u_i) + \tilde{\eta}_i, \tag{64}$$

where $\tilde{\eta}_i = \eta_i/\bar{p}$ is a white noise term with $E\left[\tilde{\eta}_i\left(t\right)\right] = 0$ and covariance:

$$E\left[\tilde{\eta}_i\left(t\right)\tilde{\eta}_{i'}\left(t'\right)\right] = \delta\left(t - t'\right)H_{i,i'}\left[\vec{u}\right], \tag{65}$$

where:

$$\lim_{\bar{p}\to\infty}\lim_{N\to\infty} H_{i,i'}[\vec{u}] = \left[\Delta s^2(1-\tau_k)^2(\rho^* + \beta)^2\Theta_{i,i'} + \Gamma\delta_{i,i'}\right]u_i u_{i'}.$$

Proof. See Appendix G. ■

In Proposition 10 the ongoing growth of wealth suggested to express the dynamics of per capita wealth of dynasty i in term of the average per capita wealth. Notice that in the limit $\bar{p} \to \infty$ wages do not play any role in the dynamics of relative per capita wealth (see Eq. (64)). This is essentially due to the fact that $\omega^*/\bar{p}$ goes to zero in the equilibrium.

Proposition 11 states the equilibrium distribution of the relative per capita wealth u_i.

[25] The independence of ψ^{EG} from τ_k is due to the assumption of constant saving rate. In the endogenous growth theory where the saving rate s is optimally chosen, s positively depends on the net return on capital $(1-\tau_k)\,\rho^*$. Hence s decreases with τ_k.

Proposition 11 *Assume that per capital wealth is growing at the rate ψ^{EG}; then the equilibrium distribution $f^{EG}(u_i)$ of $u_i = p_i/\bar{p}$ is given by:*

$$f^{EG}(u_i) = \frac{\mathcal{N}^{EG}}{\left[\Delta s^2(1-\tau_k)^2(\rho^* + \beta)^2\Theta_{i,i} + \Gamma\right] u_i^{\alpha^{EG}+1}} e^{-(\alpha^{EG}-1)/u_i}, \tag{66}$$

where $\mathcal{N}^{EG}$ is a constant defined by the condition $\int_{-\infty}^{\infty} f^{EG}(p_i)\, dp_i = 1$ and

$$\alpha^{EG} = 1 + 2\frac{s\rho^*\tau_k}{\Delta s^2(1-\tau_k)^2(\rho^* + \beta)^2\Theta_{i,i} + \Gamma} \tag{67}$$

is the Pareto exponent.

Proof. The proof follows the same steps reported in Appendix E taking $\mu(u_i) = s\rho^*\tau_k(1-u_i)$ and $\sigma^2(u_i) = \left[\Delta s^2(1-\tau_k)^2(\rho^* + \beta)^2\Theta_{i,i} + \Gamma\right] u_i^2$. ■

The Pareto exponent α^{EG} is always greater than 1 for $\tau_k, s > 0$ and therefore Assumption (36) is satisfied. Notice that the limit $\tau_k \to 0$ of this result does not reproduce the behaviour of the economy with $\tau_k = 0$. It can be shown that in the latter case u_i has a non stationary lognormal distribution. Finally, notice that the Pareto exponent is continuous across the transition from a stationary to an endogenously growing economy

$$\lim_{s\rho^*-\nu \to 0^-} \alpha = \lim_{s\rho^*-\nu \to 0^+} \alpha^{EG},$$

though it has a singular behavior in the first derivative (with respect to ν or s).

3.3.1 Pareto Exponent and the Growth Rate of Economy

As for the case of stationary/exogenous growth case the Pareto exponent α^{EG} negatively depends on τ_k and positively on $\Theta_{i,i}$ and β. However, now it is independent of ν: in fact, in an economy where wealth accumulation is the source of the long-run growth the demographic factor does not affect the equilibrium interest rate (which is determined only by the technology) and the level of per capita wealth. The relationships between α and saving rate s and interest rate ρ^* is again inverted U-shaped, as showed in Remarks 12 and 13.

Remark 12 $\partial\alpha^{EG}/\partial s > (<)\, 0$ *for* $s < (>)\, \hat{s}$, *where* $\hat{s} = \sqrt{\frac{\Gamma}{\Delta\Theta_{i,i}\rho^*\tau_k}} / \left[(1-\tau_k)(\rho^* + \beta)\right]$

Proof. The proof directly follows from the simple derivative of α^{EG} reported in Eq. (67) with respect to s. ■

Remark 13 $\partial\alpha^{EG}/\partial\rho^* > (<)\, 0$ *for* $\rho^* < (>)\, \hat{\rho}^*$, *where* $\hat{\rho}^* = \sqrt{\beta^2 + \Gamma/\left[\Delta s^2\Theta_{i,i}(1-\tau_k)\right]}$

Proof. The proof directly follows from the simple derivative of α^{EG} in reported Eq. (67) with respect to ρ^*. ■

Remarks 12 and 13 can imply a nonmonotonic relationship between α^{EG} and the growth rate ψ^{EG}. In other words the relation between the inequality and growth is not univocal because we could observe both an increase and a decrease in α^{EG} as the economy increases its growth rate ψ^{EG}.

Summing up, the endogenous growth model appears to be the limiting case where labour market does not affect the shape of wealth distribution. In this respect it appears not very realistic, but it gives us some insights on the possible relationship between growth rate and the shape of the top tail of wealth distribution.

4 Empirical Evidence

In this section we contrast the theoretical findings of the previous section with empirical evidence. Households appears the best unit of observation to test our model. We consider two datasets: the Survey of Household Income and Wealth (SHIW), which provides information on saving, income and wealth for a large sample of Italian households and the Survey of Consumer Finances (SCF), which provides, among many other variables, the net wealth of a large sample of U.S. households.[26] The comparison of these two datasets is very complex; therefore we will consider them separately.[27]

The analysis aims to test if our model is able to reproduce the *qualitative* changes in the wealth distribution. Many reasons suggest to keep our analysis at qualitative level: i) our datasets span at most 17 years, which may be a too short period for the convergence to equilibrium of the wealth distribution;[28] ii) our model does not take into account many social/cultural factors affecting the distribution of wealth of a country and iii) many factors can simultaneously push in the same directions (e.g. a contemporaneous decrease in capital taxation and in the saving rate) and their effects are strongly nonlinear, so it may be hard to disantagle the effect of each variable on the wealth distribution.

[26]In general net wealth includes all marketable assets of households. SHIW and SCF are respectively available on the following websites: http://www.bancaditalia.it/ and http://www.federalreserve.gov/pubs/oss/oss2/scfindex.html. We refer to these websites for more details on the two datasets.

[27]A first attempt to provide comparable data on these two datasets is LWS project, see www.lisproject.org/lws.htm.

[28]Notice that Shorrocks (1975) shows that, for the type of stochastic process which describes the wealth accumulation of dynasties, the estimate of the Pareto exponent can have a non monotonous behaviour as the actual distribution converges to the equilibrium one.

4.1 Italy

The SHIW includes data on many economic variables, among which net wealth, savings and disposable income for about 8000 Italian families for the years 1987, 1989, 1991, 1993, 1995, 1998, 2000, 2002 and 2004.[29] Brandolini et al. (2004) and Jappelli and Pistaferri (2000) present a detailed analysis of the SHIW and we refer to them for more details. By the estimate of transition matrix between the different waves of the net wealth, where states are defined by the quintiles of distribution, we calculate the *asymptotic half life*, i.e. the speed of convergence of actual distribution to the equilibrium distribution.[30] It ranges from 2.04 in 1991-1993 to 3.66 in 1993-1995, and, on average, is equal to 2.55; this means that on average 10.2 (i.e. $2.55 \times 2\times$number of lags) years are necessary to have the complete effect on the wealth distribution of an exogenous shock (e.g. a change in the fiscal policy). Therefore the period of observation appears to be sufficiently wide.[31]

4.1.1 The Saving Function

The average saving rates of Italian households in the period 1987-2004 are reported in Table 1.

Year	1987	1989	1991	1993	1995	1998	2000	2002	2004
Average saving rate	0.216	0.275	0.293	0.277	0.228	0.285	0.275	0.273	0.251

Table 1: Italian average saving rates. Source: SHIW

The average saving rate appears remarkable constant over time with exception of 1987 and 1995. In order to verify that Eq. (11) can represent the effective consumption function of Italian families we estimate the following equation:[32]

$$S_i = -\chi + s y_i^D - c^p p_i,$$

for the eight different years, where all variable are expressed in nominal liras. Table 2 reports the result of the estimates.

[29] In the SHIW the codes of the net wealth, the disposable income, the labour income, the entrepreneurial income, savings and the households' weights are respectively W, Y2, YL, YM, S2 and PESOFIT.

[30] All the statistical analysis is performed by R and all codes and datasets are available on Davide Fiaschi's website (http://www-dse.ec.unipi.it/fiaschi/).

[31] The asymptotic half life between 1993 and 1995 calculated on the estimated transition matrix reported in Jappelli and Pistaferri (2000) is equal to 3.62, but there the states are defined by the quartiles of the wealth distribution.

[32] We test the possibility of a nonlinear relation between saving, disposable income and net wealth. While we can reject this hypothesis for disposable income, net wealth appears to have a significant nonlinear relationship with saving (which changes over time). We ignore this fact because, as it is clear from the estimates reported in Table 2, the effect of net wealth on saving is negligible.

Est.\Year	1987	1989	1991	1993	1995	1998	2000	2002	2004
$\hat{\chi}$	6.62e+3	7.24e+3	9.00e+3	1.22e+4	1.46e+4	1.50e+4	1.74e+4	1.50e+4	2.47e+4
$\hat{s}$	0.4825	0.5052	0.5502	0.5802	0.608	0.6018	0.6363	0.5769	0.7389
$\hat{c}^p$	0.0080	0.00465	0.00359	0.00025	0.00674	0.00338	0.0042	0.00417	0.00842

Table 2: estimate of saving function ($\hat{\chi}$ is expressed in current million liras). Source: SHIW

All the estimated parameters result highly significant (we do not report t-statistics for simplicity of exposition). The estimate of $\hat{\chi}$ is increasing over time; this is likely due to the positive inflation rate and to the increase in the per capita wealth.[33] The estimate of $\hat{s}$ varies from 0.4825 to 0.7389, but overall $\hat{s}$ results increasing in the period. The estimates of $\hat{c}^p$ is more volatility, but the impact of net wealth on saving results negligible (see Paiella (2004)). Moreover, the net wealth and the net disposable income result highly correlated (from 0.54 to 0.65) and this could introduce a bias in the estimates. Table 3 reports the estimates when c^p is neglected.

Est.\Year	1987	1989	1991	1993	1995	1998	2000	2002	2004
$\hat{\chi}$	5.89e+3	7.35e+3	7.59e+3	1.21e+4	1.31e+4	1.32e+4	1.45e+4	1.43e+4	2.31e+4
$\hat{s}$	0.4331	0.4937	0.4984	0.5809	0.5338	0.5585	0.5604	0.5381	0.6535

Table 3: estimate of saving function without wealth ($\hat{\chi}$ is expressed in current liras). Source: SHIW

Again all the estimated parameters result highly significant and overall $\hat{s}$ is increasing. Table 3 shows how year 1993 is a turning point in the estimate of s; after such year $\hat{s}$ becomes stable around 0.55-0.54. The big increase in $\hat{s}$ recorded in 1993 should be due to the severe crisis of Italian economy in 1992-1993, followed to the devaluation of lira and to the tight fiscal policy. The increase in the economic uncertainty pushed households to increase their savings (in particular their precautionary savings). In 2004 there was another big increase in the marginal saving rate, but we cannot know if such increase is temporary.

Summing up Eq. (11) appears to be adequate to represents the consumption function of Italian households and overall the saving rate appears to be increasing in the period.

4.1.2 Changes in the Top Tail of the Wealth Distribution

The estimates in Tables 2 and 3 show that between 1987 and 2004 there was a substantial increase in the marginal saving rate (see Tables 2 and 3). In the same period there was

[33] As previously argued, the minimum consumption is to be related to a minimum standard of living. Then an increase in the per capita wealth increases also χ. This intuition is confirmed by the fact that the ratio $\hat{\chi}/\bar{p}$ is nearly constant in the period.

a remarkable decrease in the taxation of capital income. In particular, in the recent years all capital gains are subject to a flat tax rate τ_k equal to 12.5% whereas taxation on personal income[34] which was strongly progressive until the mid of 1990s, has decreased starting since 1995 (e.g. the highest marginal tax rate on income decreased from 51% to 45.5%).

Another relevant phenomenon in the period was the decrease in the share of labour income on the aggregate product (in our model given by $\omega^*/g\left(\bar{p}^*/\bar{l}\right)$), from 0.46 in 1990 to 0.416 in 2004 (see Ministero dell'Economia e delle Finanze (2005)); the possible correction for the self-employed workers does not change the scenario but only the magnitude of the decrease (from 66.9 in 1990 to 59.0 in 2004, see also Jones (2003)). Assuming Cobb-Douglas technology (see Eqq. (48)-(48)) $\omega^*/g\left(\bar{p}^*/\bar{l}\right) = 1 - \varepsilon$, which would imply that ε has increased and, as a consequence, the gross return on capital has increased as well.

The changes in the annual growth rate of population are negligible: it slightly increases from 0.05% in the period 1981-1990 to 0.17% in the period 1991-2000, but its absolute magnitude is very low.[35]

Our model predicts that an increase in saving rate has an ambiguous effect on the Pareto exponent. For high level of the saving rate such effect may be negative (see Remark 6). A change in the fiscal policy in favour of capital income should unambiguously decrease the Pareto exponent. The same effect obtains under a change in technology in favour of the return on capital. The change in the growth rate of population appears to be not significant. Finally, the average growth rate of per capita GDP in 1991-2000 was the half of the one in 1981-1990 (i.e. 1.2% vs 2.4%).[36] According to the endogenous growth model the relationship between growth rate and Pareto exponent can be inverted U-shaped (see Section 3.3.1), therefore the decrease in the growth rate starting from a low initial growth should lead to a decrease of the Pareto exponent. Therefore, overall we expect a decrease in the Pareto exponent.

Figures 8 reports in the y-axis the log of cumulative density of about top 800 Italian households (10% of sample), and in the x-axis the log of normalized net wealth, i.e. the net wealth of households normalized with respect to the average wealth (this is to control for the growth of the average wealth over the period).

Figure 8 highlights how the relationship between the log of cumulative density and the log of normalized net wealth is approximatively linear, which agrees with the theoretical distributions reported in Propositions 4, 8 and 9.[37] Figure 9 reports the estimated of the Pareto exponents of the top tail of distribution (10% of sample) for the eights years, corresponding

[34]Personal income includes income from labour and from profits. In the second part of 1990s a credit tax (called *DIT*: dual income tax) was introduced for the firms reinvesting their profits: this further decreases the effective taxation on profits.

[35]Source: Penn World Table 6.1 (http://pwt.econ.upenn.edu/).

[36]Source: Penn World Table 6.1.

[37]Adjusted R^2 is equal to 0.99 for both years.

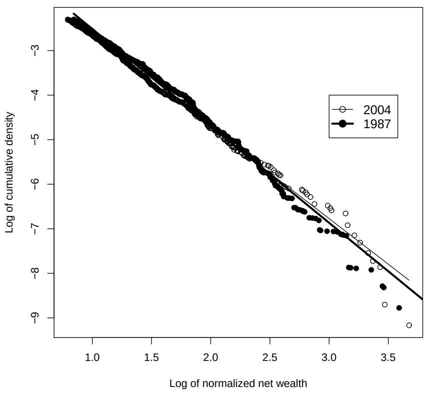

Figure 8: estimate of the top tail (10%) of the Italian wealth distribution.

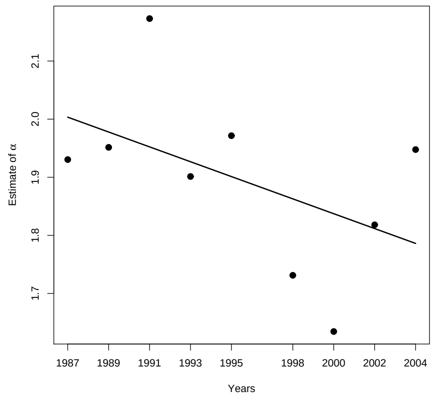

Figure 9: estimate of Pareto exponent by the Hill's Estimator

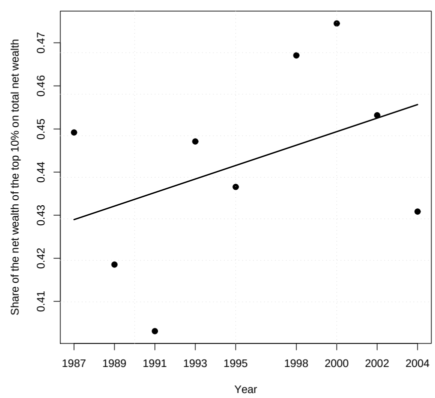

Figure 10: share of the net wealth of top 10% Italian households

to the expression in Eqq. (47) and (67), by the Hill's Estimator.[38]

Figure 9 supports the predictions of our theoretical model.[39] As we said, the presence of nonlinearities and the availability of limited time-series make difficult to disentangle the individual effects of the changes in the saving rates, fiscal policy and technology. Moreover, it is not an easy task to have a plausible estimate of the variance of random components (i.e. Δ and Γ) and of the portfolio diversification of dynasties (i.e. $\Theta_{i,i}$).

To highlight the consequence of this decline in the Pareto exponent in terms of wealth inequality Figure 10 reports the share of the net wealth of top 10% on the total net wealth.

The comparison of Figure 9 and 10 shows the strong correlation between the Pareto exponent and the share of the net wealth of top 10%: a rough linear regression shows that Pareto exponent declines from 2.0 in 1987 to 1.79 in 2004 and the share of top 10% increases

[38]See Embrechts et al. (1997) for more details. We use a more general formula of the estimator of the one reported in Embrechts et al. (1997), which allows for weighted observations. In particular, given the ranked vector of households' wealth $(p_1, p_2, ..., p_N)$, where $p_1 \geq p_2 \geq ... \geq p_N$, and the vector of households' weights $(\lambda_1, \lambda_2, ..., \lambda_N)$, where $\Sigma_{j=1}^{N} \lambda_j = N$, we have that:

$$\hat{\alpha}_z = \frac{\Sigma_{j=1}^{z} \lambda_j \left(\log p_j - \log p_z\right)}{\Sigma_{j=1}^{k} \lambda_j}$$

represents the Hill's Estimator of the Pareto exponent of the distribution of the z wealthy households (in our case z is about 800 in every year). Embrechts et al. (1997), p. 336, show that this estimate is consistent and $\sqrt{\Sigma_{j=1}^{z} \lambda_j} \left(\hat{\alpha}_z - \alpha\right) \longrightarrow N\left(0, \alpha^2\right)$.

[39]The small sample used in the estimates make the standard errors of the estimates very large, so that the differences among the Pareto exponents of different years are only partially statistically significant (this can be also due to the relative small period of observation). Moreover, here we are interested more to the tendency over time than the exact values of the Pareto exponent. Moreover, in 1989, 1991 and 1993 the Hill's plots show that the estimate of Pareto coefficient is not convergent in the top first decile. This could mean that the observed wealth distribution was far from its equilibrium in those years.

from 0.43 in 1987 to 0.46 in 2004.

4.1.3 Change in the Bottom Tail of the Wealth Distribution

The comparison between Proposition 4 and 8 shows how labour market crucially affects the bottom tail of wealth distribution, i.e. in the latter workers are the majority. This is confirmed by data: Italian households with a positive labour income have an average net wealth slightly below the per capita net wealth $\bar{p}$ ($0.96\bar{p}$ in 1987 and $0.92\bar{p}$ in 2004), while the households with positive entrepreneurial income have an average net wealth slightly below $2\bar{p}$ ($1.98\bar{p}$ in 1987 and $1.94\bar{p}$ in 2004).

The Italian labour market appears to be progressively increasing in flexibility: the share of permanent jobs has decreased in favour of the share of non-permanent jobs and the market power of Trade Unions is steadily declining. To have an idea of the magnitude of this phenomenon Table 4 reports the share of non-permanent jobs on the total employees, excluding the self-employed (the share of the latter on the total employees is stable around 0.27-0.28 over the period 1993-2003).

Year	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003
Share	0.062	0.068	0.073	0.073	0.079	0.086	0.095	0.101	0.098	0.097	0.097

Table 4: non-permanent jobs on the total employees, excluding the self-employed. Source: Ministero dell'Economia e delle Finanze (2005).

Moreover, in the same period we observe a strong decrease in the net union membership. The share of memberships, excluding self employed and retired, on the total labour force declines from 36.5% in 1985 to 30.9% in 1997 (see Golden et al. (2004)). Both phenomena should lead to an increase in the cross-section variance of labour incomes. Figure 11 corroborates this intuition.

Figure 11 reports the distribution of the log of normalized (gross) labour incomes of Italian households in 1987 and 2004 (in both periods data are normalized to the average). We observe a clear increase in the size of the bottom tail and, in general, an increase in the variance of the distribution. The latter is confirmed by the increased Gini index of the distribution of labour incomes, which rises from 0.583 in 1987 to 0.615 in 2004 (see Figure 12).[40]

The comparison between Propositions 4 and 8 suggest that an increase in the cross-section variance of the distribution of labour incomes would imply an increase in the size of the bottom tail of the wealth distribution (see Figure 5). Figures 13 and 14 respectively report

[40]See also Brandolini et. al. (2001), who find a wide increase in the earnings dispersion of the Italian households in the early 1990s.

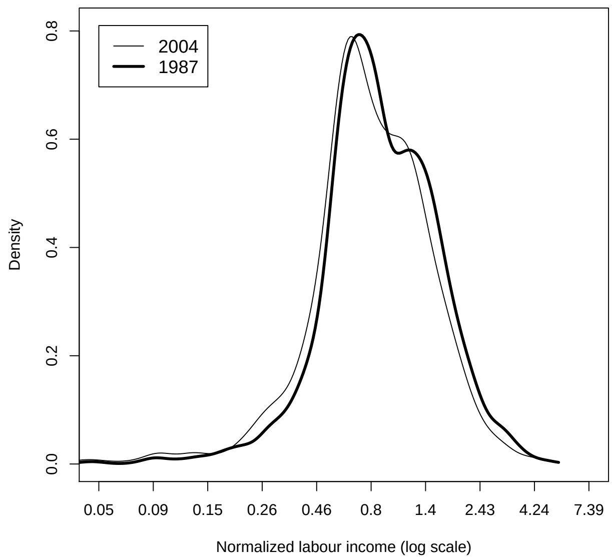

Figure 11: estimate of distribution of Italian labour incomes

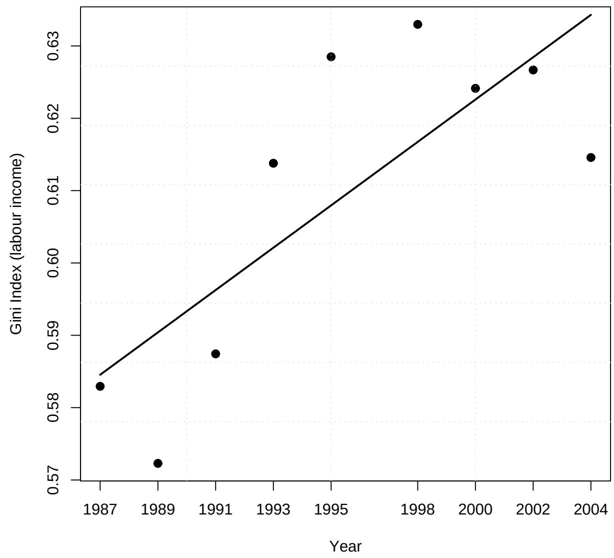

Figure 12: Gini index of Italian labour income distribution

the nonparametric estimate of the wealth distribution in 1987 and 2004 (only for households with positive net wealth) and the share of households with negative net wealth for all available years.[41]

Both figures support the prediction that the size of the bottom tail of the wealth distribution has increased in the period 1987-2004 as a consequence of the increase in the flexibility of Italian labour market.

To conclude, we observe that the changes in the top and bottom tails of the wealth distribution suggest an increase in the inequality of wealth distribution; however, the inspection of Figure 13 shows a strong increase in the density around the mean (1 in Figure 13), which means less inequality in the middle of distribution. In a synthetic index of inequality the latter effect may outweigh the increase in the inequality on both tails. Indeed the Gini index of the wealth distribution shows a decrease from 0.622 in 1987 to 0.604 in 2004 (0.622 in 1987 is to consider very carefully since already in 1989 Gini index is equal to in 0.591).[42]

4.2 U.S.

We take the data on the wealth of U.S. households from the SCF: the following years are available: 1989, 1992, 1995, 1998, 2001 and 2004. The number of U.S. households included in the SCF is increased from 3143 in 1989 to 4442 in 2004. Data on the net wealth and income

[41] For the nonparametric density estimation we used the packages "sm" with the standard settings (in particular the normal optimal smoothing bandwidth), see Bowman and Azzalini (2005).

[42] In particular, Gini index of wealth distribution is equal to 0.62 in 1987, 0.59 in 1989, 0.59 in 1991, 0.63 in 1993, 0.61 in 1995, 0.63 in 1998, 0.63 in 2000, 0.62 in 2002 and 0.60 in 2004.

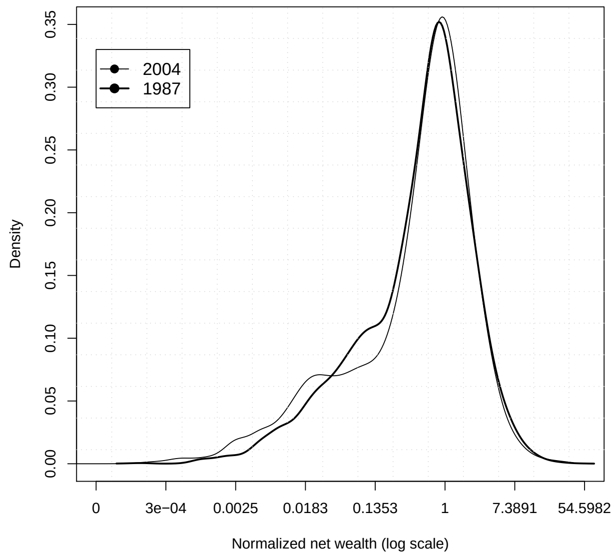

Figure 13: comparison between the distribution of wealth in 1987 and in 2002 (only Italian households with positive net wealth)

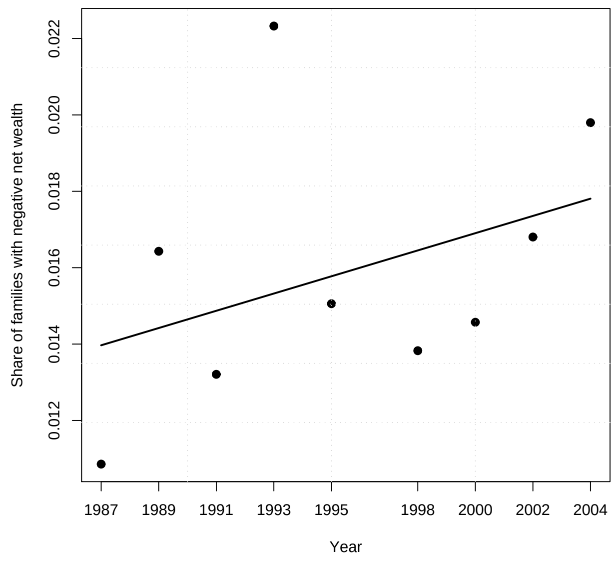

Figure 14: share of Italian households with negative

are easily available on the SCF's website, but unfortunately neither savings nor earnings are available.[43] Also panel information on households in the sample are not included. We refer to Wolff (2004) for more details on the SCF.[44]

We partially try to fill this gap by using data on labour income reported in PSID.[45] PSID also reports data on wealth. However, the wealthy households are strongly underrepresented, so that we prefer the SCF for studying the wealth distribution. Still the panel framework of PSID allows us to have an estimate of the speed of convergence of actual wealth distribution to the equilibrium distribution.[46] The estimated asymptotic half life is equal to 3.46 in 1999-2001 and 3.81 in 2001-2003, i.e. on average it needs 14.5 year to have the com-

[43]The variable "SAVING" in the SCF cannot be used in the estimate of the saving function because its definition does not match the standard definition of saving (i.e. it is not defined as the not-consumed disposable income).

[44]It is worth noting that in our calculations we use the weights reported in SCF's website and this determines the differences with Wolff (2004)'s results (the main difference is for low wealth/income households, which he says to be underrepresented in the sample). These differences are relevant both in magnitude, e.g. Gini index of the wealth distribution is equal to 0.805 in 2001 in our calculations while it is equal to 0.826 in Wolff (2004) and over time, e.g. Gini index is slightly decreasing between 1989 and 2001 in Wolff (2004) (0.832 in 1989 vs 0.826 in 2001) and increasing in our calculations (respectively 0.787 vs 0.805). In this regard Davies and Shorrocks (1999), in suggesting to use SCF for the analysis of U.S. wealth distribution, warn about the controversy on which weights must be used.

[45]We take data directly from website http://psidonline.isr.umich.edu/.

[46]In particular we use the following variables (we report the PSID code): ER417, S517 and S617 (households' wealth in 1999, 2001 and 2003 respectively) and FCWT99, ER20394 and ER24179 (sample weights for 1999, 2001 and 2003)

plete effect of a shock on the wealth distribution. Also in this case the period for which data are available appears to be sufficiently wide.

4.2.1 Change in the Top Tail of the Wealth Distribution

Table 5 highlights that the average tax rate has slightly decreased over the period 1985-2003, but the main benefits are for the top 1% income people (about -6.5%), while top 25% and top 50% income people have a lower decrease (about -2.5%); it is worth remarking that the major changes happened at the end of 1980s.

Year	Total	Top 1%	Top 5%	Top 10%	Top 25%	Top 50%
1985	13.89	30.87	24.07	21.34	17.80	15.59
2003	11.90	24.31	20.74	18.49	15.38	13.35

Table 5: average tax rates on income 1985-2003. Source: IRS (http://www.irs.gov/taxstats/)

Also the tax on capital income decreased. Table 6 reports the average and the marginal federal tax rates on the corporate profits in 1994 and in 2002.[47]

Year	1994	2002
Average tax rate	27.43	25.58
Marginal tax rate	26.82	23.63

Table 6: federal corporate tax rates in 1985 and in 2002. Source: our calculations on IRS data

The decline in taxation of corporate profits is particularly strong in the marginal tax rate and the marginal tax rate is lower than the average tax rate (this is due to the higher tax credit to the biggest corporations). Finally, the Jobs and Growth Tax Relief Reconciliation Act of 2003 provides further concessions for households and a major cut in the tax rates on capital gains and dividends.

Table 7 shows how average saving rates of U.S. households has fallen in the period 1985-2003, starting from an already low initial level.

As regards possible changes in technology in the period 1985-2004 the share of labour income on the aggregate product did not change appreciably and it was about equal to 0.67 (see Jones (2003)).

There is virtually no change in the annual growth rate of population: 0.93% in the period 1981-1990 and 0.97% in the period 1991-2000.[48] Finally, the average growth rate of per capita

[47] Average tax rate is the ratio between the total income after tax credit and income subject to tax, while the marginal tax rate is the same ratio calculated for the corporations in the highest class in terms of assets (over 2.500 millions of dollars).

[48] Source: Penn World Table 6.1.

Year	1985	1990	1995	2000	2001	2002	2003	2004
Average saving rate	9.0%	7.0%	4.6%	2.3%	1.8%	2.4%	2.1%	1.8%

Table 7: average saving rates of the US households 1985-2004. Source: BEA (http://www.bea.doc.gov/bea/dn1.htm)

GDP in 1991-2000 is the same of the one in 1981-1990 (i.e. 2.2% vs 2.3%).[49]

Our theoretical model suggests that the decrease in the tax rate on capital income should lead to a decrease in the Pareto exponent of the distribution of wealth. Also the decrease in the saving rate could induce a decrease in the Pareto exponent given its low initial level (see Remarks 6 and 12). Finally, the growth rate of population and per capita GDP would not affect the Pareto exponent given their substantial stability.

In the calculation of the Pareto exponent we consider top 10% of U.S. households, excluding top 0.5% because the extreme top tail appears to be underrepresented and this could bias the estimate of Pareto exponent.[50] For example to respect the privacy of the richest U.S. households, SCF does not explicitly consider the 400 wealthiest people included in the Forbes list whose total wealth account for 1.5% in 1989 and 2.2% in 2001 of total U.S. wealth (see Kennickell (2003), p. 3). For the wealthiest people we refer to Klass et al. (2006), who show that the distribution of wealth for the people in the Forbes list follows a Pareto distribution, whose Pareto exponent decreased from 1.6 in 1988 to 1.2 in 2003 (see also Castaldi and Milakovic (2006)).

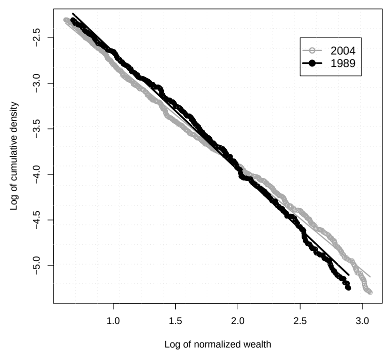

Figure 15: estimate of the top 10% of U.S. wealth distribution

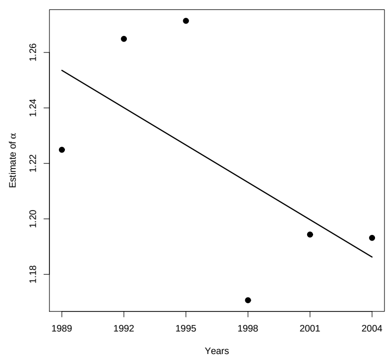

Figure 16: estimate of the Pareto exponent by the Hill's Estimator

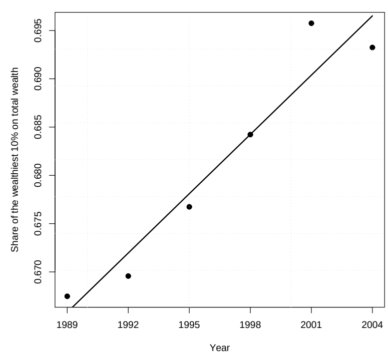

Figure 17: share of the net wealth of top 10% U.S. households

Figure 15 shows that top tail follows a Pareto law in 1989 and 2004. The linear regression in Figure 16 highlights that Pareto exponent decreases from 1.25 in 1989 to 1.19 in 2004; this

[49]Source: Penn World Table 6.1.

[50]Comparable results are obtained by excluding 0.1% of top tail. We choose to exclude 0.5% of top tail because regressions presents an higher adjusted R^2.

agree with Klass et al. (2006)'s results. Figure 17 confirms the inverse relationship between the Pareto exponent and the size of top tail distribution: a decline in the Pareto exponent from 1.25 to 1.19 implies an increase in the share of top 10 from 0.67 to 0.69.

Finally, it is worth noting that Wolff (2004) says that a big increase in the inequality of the U.S. wealth distribution happened between 1983 and 1989. We did not analyse data of 1983 but, given the strict relationship between the Pareto exponent and the wealth inequality, we could argue that the decrease in the progressivity and in the average tax rates during 1980s reported in Tables 5 and 6 could be one of the sources of this increase in the inequality.

4.2.2 Change in the Bottom Tail of the Wealth Distribution

As we discussed above the labour market is the key aspect for the shape of the bottom tail of the wealth distribution according to our model. In the U.S. labour market the share of non-permanent jobs has not changed significantly in the last 15 years, but the net union density sharply declined from 17.2 in 1985 to 13.5 in 2000 (see Golden et al. (2004)). Figure 11 reports the estimate of distribution of the log of normalized (gross) labour income in 1989 and 2001 and Figure 19 the Gini index of labour income distribution for the years 1981, 1989, 1996 and 2002.[51]

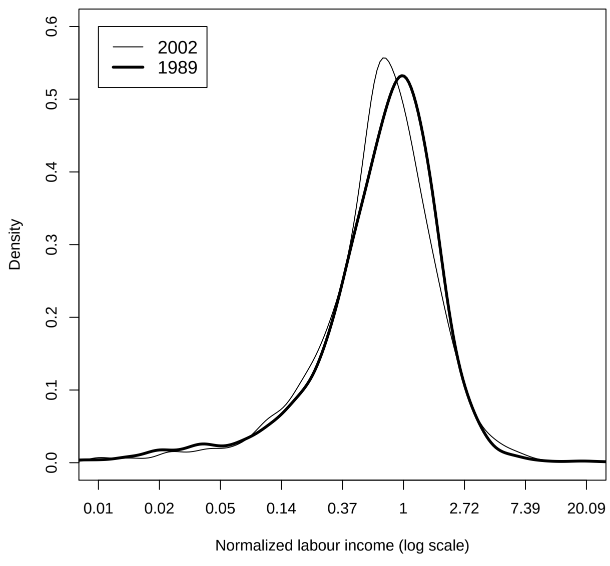

Figure 18: estimate of distribution of U.S. labour income

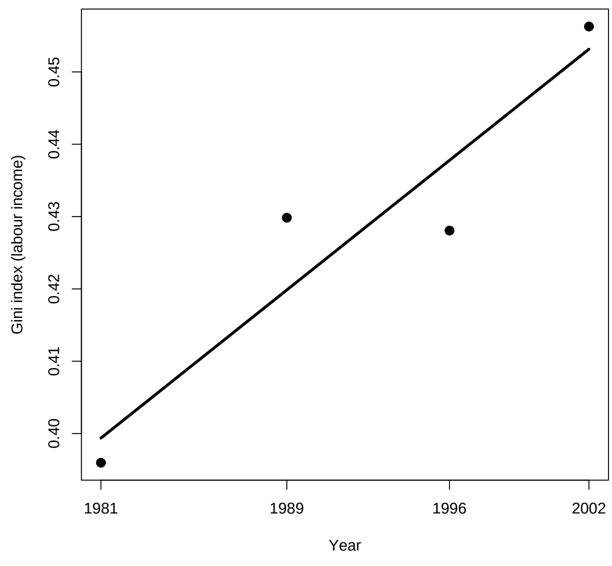

Figure 19: Gini index of U.S. labour income distribution

The peak of the estimated density shows a clear shift below 1 and a consequent increase in the size of the bottom tail. The increase in the cross-section variance of distribution is

[51]In particular in the estimate we consider the labour income of the head of family (in PSID database ER24116, ER12080, V18878 and V8690 are the codes for labour income in 2002, 1996, 1989 and 1981 respectively and ER24179, ER12084, V18945 and V8727 for sample weights for 2002, 1996,1989 and 1981 respectively).

confirmed by the increase in the Gini index of distribution of labour incomes, which rises from 0.43 in 1989 to 0.456 in 2002 (see Figure 19). Therefore we should expect an increase in the bottom tail of the wealth distribution.

Figure 20 reports the estimated U.S. distribution of wealth in 1989 and 2001 (only households with positive wealth), while Figure and 21 the share of U.S. households with negative net wealth.

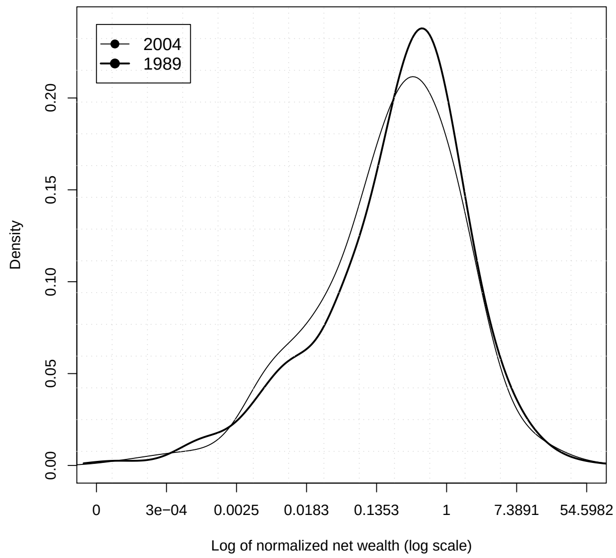

Figure 20: comparison between the U.S. distributions of wealth in 1989 and in 2001 (only households with positive wealth)

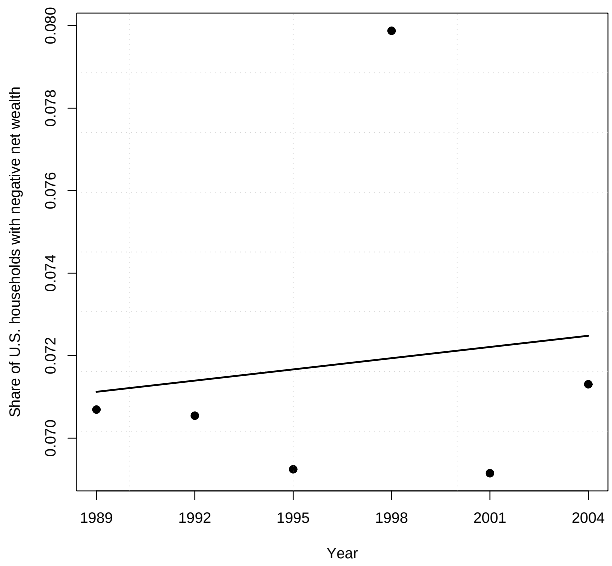

Figure 21: share of U.S. households with negative net wealth

The comparison between the wealth distributions of 1989 and 2004 reported in Figure 20 highlights that the share of U.S. households with a positive net wealth below the average has increased (the two distributions cross in correspondence of about 0.38 in Figure 21. The share of households with negative net wealth appears to be almost constant (see Figure 21) between 1989 and 2004.

Overall, the changes in the top and bottom tails of the wealth distribution suggest that the inequality of wealth distribution has increased. The downward shift in the peak of the distribution also contributes to the increase in inequality. This is confirmed by the Gini index of the wealth distribution which is increases from 0.787 in 1989 to 0.808 in 2004.

5 Conclusions and future research

This paper can be seen as a first step toward a general equilibrium theory of the distribution of wealth. We characterize the equilibrium distribution of wealth in an economy with a large number of firms and dynasties, who interacts through the capital and the labour markets. The top tail of the equilibrium distribution of wealth is well-represented by a Pareto distribution, whose exponent depends on saving rate, net return on capital, growth rate of population, tax on capital income and the degree of portfolio diversification. On the contrary the bottom tail mostly depends on the institutional setting of labour market: a labour market with a centralized bargaining implies a lower wealth inequality. We also highlight how shocks to firms' productivity and dynasties' wealth can affect the wealth distribution, even when the economy is very large. Moreover, we find that the passage from a stationary (or exogenously growing) regime to an endogenously growing one is mirrored by changes in the wealth distribution. First, the labour and the capital sectors decouple in an economy with endogenous growth, with labour having no effect on the stationary wealth distribution. Second, the Pareto exponent has a singularity across the transition. For example, the exponent does not depend on the growth rate of population in the endogenous growth case, whereas it depends on it in a possibly nonmonotonic way in the stationary economy.

However, our framework neglects important factors which have been shown to have a relevant impact on the wealth distribution, such as for example the possible optimizing behaviour of agents, the age structure of the population and bequests (see Davies and Shorrocks (1999)). Moreover, our results are relative to the equilibrium distribution of wealth: the analysis of out-of-equilibrium behaviour seems a necessary extension, also to take into account the speed of convergence of the actual distribution to its equilibrium and the possible nonmonotonic behaviour (see Atkinson and Harrison (1978), p. 227). In this respect the Italian data we analysed shows clear deviations from equilibrium, over time scale of few years. We conjecture that the latter are related to possible soaring in the real estate prices, yet another effect not considered in our model.

The lack of space has limited our analysis in many stimulating directions. We did not deepen the relationship between the distribution of wealth and the distribution of income. Heuristically we can argue that in our model the wealth inequality is always higher than the income inequality because generally only a small share of current income derives from wealth (empirically plausible returns on wealth are well below 10%); the other part of income derives from wages and from Government transfers, which are more equally distributed across dynasties.

Two further extensions look promising. The first is related to the cases where labour market and capital market have different speeds of adjustment to equilibrium. It seems realistic to assume that labour market adjusts at a slower pace than the capital sector. In such a situation, productivity shocks would impact mostly the capital sector. In this sense, the case of staggered wages considered here might be thought of as the extreme case where wages evolve over infinitely slower time scales. Naively speaking, we expect that a slow speed of adjustment in labour market decreases the cross-section variance of wages and hence reduces inequalities in the low tail of the wealth distribution. However, full account of these issues would entail dealing with situations where firms are constrained in the choice of the factors of production, with consequent underutilization of factors (i.e. unemployment).

The second interesting extension is the case in which aggregate wealth exhibits a non deterministic behaviour. In the light of our findings, this can arises because of correlations in productivity shocks, which were neglected here, because dynasties concentrate their investments in few firms/assets or because the number of firms/assets is much smaller than the number of dynasties. This extension would draw a theoretical link between the dynamics of the wealth distribution, firm size distribution and the business cycle.

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A Proof of Proposition 1

Given Eqq. (12), (18) and (19) we have

$$dp_i = s\left\{\left[\sum_{j=1}^{F}(1-\tau_k)\,dr_j\theta_{i,j}p_i(t) + (1-\tau_l)\,dw_j\phi_{i,j}l_i\right] + \right.$$
$$\left. + \frac{1}{N}\sum_{i'=1}^{N}\left[\sum_{j=1}^{F}\tau_k dr_j\theta_{i',j}p_{i'}(t) + \tau_l dw_j\phi_{i',j}l_{i'}\right]\right\} - d\bar{c} - dn_i(t)\,p_i. \tag{68}$$

Taking a continuum time limit the dynamics of p_i is described by the Langevin equation (see Gardiner (1997)):

$$\frac{dp_i}{dt} = F_i[\vec{p}] + \eta_i, \tag{69}$$

where $E[\eta_i(t)] = 0$ and the covariance of η_i is given by:

$$E[\eta_i(t)\,\eta_{i'}(t')] = H_{i,i'}[\vec{p}]\,\delta(t-t').$$

In Eq.(69),

$$F_i[\vec{p}] = \lim_{dt\to 0}\frac{E[dp_i]}{dt} \text{ and} \tag{70}$$

$$H_{i,i'}[\vec{p}] = \lim_{dt\to 0}\frac{1}{dt}E\left[(dp_i - E[dp_i])\,(dp_{i'} - E[dp_{i'}])\right]. \tag{71}$$

From Eq. (68) we have that:

$$E[dp_i] = s\left[(1-\tau_k)\,drp_i + (1-\tau_l)\,dwl_i + \tau_k dr\bar{p} + \tau_l dw\bar{l}\right] - d\bar{c} - np_i, \tag{72}$$

which together with Eq. (70) and Eqq. (20)-(25) leads to:

$$F_i[\vec{p}] = s\left[(1-\tau_k)\,\rho p_i + (1-\tau_l)\,\omega l_i + \tau_k\rho\bar{p} + \tau_l\omega\bar{l}\right] - \chi - \nu p_i.$$

In order to compute $H_{i,i'}[\vec{p}]$ note that from Eqq. (68) and (72):

$$dp_i - E[dp_i] = s\sum_{j=1}^{F}\left\{(1-\tau_k)\,(\rho+\beta)\,p_i\theta_{i,j} + (1-\tau_l)\,\omega l_i\phi_{i,j} + \right. \tag{73}$$
$$\left. \frac{1}{N}\left[\tau_k(\rho+\beta) + \tau_l w/\lambda\right]k_j\right\}d\zeta_j - p_i d\xi_i$$

from which:

$$H_{i,i'}[\vec{p}] = \Delta s^2\left\{(1-\tau_k)^2(\rho+\beta)^2 p_i p_{i'}\Theta_{i,i'} + (1-\tau_l)^2\omega^2 l_i l_{i'}\Phi_{i,i'} + \right.$$
$$(1-\tau_k)(1-\tau_l)\,(\rho+\beta)\,\omega\left[p_i l_{i'}\Omega_{i,i'} + l_i p_{i'}\Omega_{i',i}\right] +$$
$$+\frac{\tau_k(\rho+\beta)+\tau_l\omega/\lambda}{N}\left[(1-\tau_k)(\rho+\beta)(p_i\vartheta_i + p_{i'}\vartheta_{i'}) + (1-\tau_l)\omega(l_i\varphi_i + l_{i'}\varphi_{i'})\right] +$$
$$\left. + \frac{\left[\tau_k(\rho+\beta)+\tau_l\omega/\lambda\right]^2}{N^2}\sum_{j=1}^{F}k_j^2\right\} + \Gamma p_i^2\delta_{i,i'},$$

where we used Definitions (21), (28) and (29) and the parameters $\Theta_{i,i'}$, $\Omega_{i,i'}$ and $\Phi_{i,i'}$ are defined in Eq. (31) and ϑ_i, φ_i in Eq. (34).

QED

B Proof of Proposition 2

From Eq. (73) we have that:

$$\bar{\eta}dt = \frac{1}{N}\sum_{i=1}^{N}\left[dp_i - E\left(dp_i\right)\right] =$$

$$= s\sum_{j=1}^{F}\left\{\left(1-\tau_k\right)\left(\rho+\beta\right)\frac{\sum_{i=1}^{N}p_i\theta_{i,j}}{N} + \right.$$

$$\left. + \left(1-\tau_l\right)\omega\frac{\sum_{i=1}^{N}l_i\phi_{i,j}}{N} + \frac{1}{N}\left[\tau_k\left(\rho+\beta\right)+\tau_l\omega/\lambda\right]k_j\right\}d\zeta_j - \frac{\sum_{i=1}^{N}p_i d\xi_i}{N} =$$

$$= \frac{1}{N}\left\{s\left(\rho+\beta+\omega/\lambda\right)\sum_{j=1}^{F}k_j d\zeta_j - \sum_{i=1}^{N}p_i d\xi_i\right\},$$

since $k_j = \sum_{i=1}^{N}p_i\theta_{i,j}$, $l_j = \sum_{i=1}^{N}l_i\phi_{i,j}$ and $k_j = \lambda l_j \ \forall j$. In order to have no stochastic fluctuation in $\bar{p}$ we need $E\left[\bar{\eta}dt\right] = 0$ and $E\left[\left(\bar{\eta}dt\right)^2\right] = 0$. It is straithforward to see that:

$$\lim_{N\to\infty} E\left[\bar{\eta}dt\right] = 0$$

and that:

$$\lim_{N\to\infty} E\left[\left(\bar{\eta}dt\right)^2\right] =$$

$$= \lim_{N\to\infty}\frac{1}{N^2}\left\{s^2\left(\rho+\beta+\omega/\lambda\right)^2\Delta\sum_{j=1}^{F}k_j^2 - \Gamma\sum_{i=1}^{N}p_i^2\right\}dt = 0 \Leftrightarrow \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{F}k_j^2 = 0.$$

Since:

$$\sum_{j=1}^{F}k_j^2 = \sum_{i=1}^{N}p_i p_{i'}\Theta_{i,i'} \leq \sum_{i=1}^{N}p_i^2\Theta_{i,i'} = \sum_{i=1}^{N}p_i^2\sum_{j=1}^{F}\theta_{i,j}\sum_{i'=1}^{N}\theta_{i',j} \leq \bar{\theta}\sum_{i=1}^{N}p_i^2, \tag{74}$$

here in the first inequality we used the Cauchy's inequality (see Hardy et al. (1954)):

$$\sum_k a_k b_k \leq \sqrt{\sum_k a_k^2 \sum_k b_k^2}$$

setting $k = i, i'$ and $a_k = p_i\sqrt{\Theta_{i,i'}}$ and $b_k = p_{i'}\sqrt{\Theta_{i,i'}}$. In the last passage of Eq. (74) we use Assumption (35) and $\sum_{j=1}^{F}\theta_{i,j} = 1$.

QED

C Proof of Proposition 3

From Eq. (38) we see that Condition (39) ensures that for large value of $\bar{p}$ $d\bar{p}/dt < 0$; in fact:

$$\lim_{\bar{p}\to\infty} \frac{d\bar{p}}{dt} < 0 \Leftrightarrow \lim_{\bar{p}\to\infty} \frac{g\left(\bar{p}/\bar{l}\right)\bar{l}}{\bar{p}} = \lim_{\bar{p}\to\infty} g'\left(\bar{p}/\bar{l}\right) < \frac{s\beta+\nu}{sa}.$$

Condition (40) states that in $\bar{p} = 0$ $d\bar{p}/dt > 0$. Since $g\,(.)$ is continuos, always increasing and concave, then there exists only one value of $\bar{p}$, $\bar{p}^*$, such that $d\bar{p}/dt = 0$, i.e.:

$$sag\left(\bar{p}^*/\bar{l}\right)\bar{l} = \chi + (s\beta + v)\,\bar{p}^*.$$

On the contrary Condition (41) states that in $\bar{p} = 0$ $d\bar{p}/dt < 0$. This means that $\bar{p} = 0$ is an equilibrium. Condition (42) states that two other equilibria exist. Without an analytical proof, a simple inspection of Figure 2 shows that the low equilibrium is unstable, while the high equilibrium is locally stable. Eqq. (44) and (45) are directly derived by Eqq. (15) and (16), taking into account Eqq. (18) and (19).

QED

D Proof of Proposition 4

In the infinite dynasty economy, when per capita wealth converges to its equilibrium level, dp_i/dt depends on the wealth of all the other dynasties only through the per capita wealth $\bar{p}^*$. Hence the determination of the marginal distribution of p_i reduces to a single dynasty problem:

$$\frac{dp_i}{dt} = \mu\,(p_i) + \eta_i; \tag{75}$$
$$E\left[\eta_i\,(t)\,\eta_i\,(t')\right] = \sigma^2\,(p_i)\,\delta\,(t - t')$$

where, from Proposition 1, we have:

$$\mu\,(p_i) = z_0 - z_1 p_i; \tag{76}$$
$$\sigma^2\,(p_i) = \lim_{N\to\infty} H_{i,i}\,[\vec{p}] = a_0 + a_1 p_i + a_2 p_i^2, \tag{77}$$

with:

$$\begin{aligned}
z_0 &= s\left[(1-\tau_l)\,\omega^* l_i + \tau_k \rho^* \bar{p} + \tau_l \omega^* \bar{l}\right] - \chi;\\
z_1 &= \nu - s\,(1-\tau_k)\,\rho^*;\\
a_0 &= \Delta s^2\,(1-\tau_l)^2\,\omega^{*2} l_i^2 \Phi_{i,i};\\
a_1 &= 2\Delta s^2 (1-\tau_k)(1-\tau_l)\,(\rho^* + \beta)\,\omega^* l_i \Omega_{i,i};\\
a_2 &= \Delta s^2\,(1-\tau_k)^2\,(\rho^* + \beta)^2\,\Theta_{i,i} + \Gamma.
\end{aligned}$$

Notice that the last two terms in braces in the expression of $H_{i,i}[\vec{p}]$ vanish in the limit $N \to \infty$.

The Fokker-Planck equation corresponding to Eq.(75) is given by (see Gardiner (1997)):

$$\frac{\partial f(p_i)}{\partial t} = -\left[\frac{\partial \mu(p_i)}{\partial p_i} - \frac{1}{2}\frac{\partial^2 \sigma^2(p_i)}{\partial p_i}\right] f(p_i).$$

Since $z_0 > 0$, in equilibrium $\partial f(p_i)/\partial t = 0$, that is:

$$\frac{\partial \mu(p_i)}{\partial p_i} f(p_i) = \frac{1}{2}\frac{\partial^2 \sigma^2(p_i)}{\partial p_i^2} f(p_i)$$

Take $\varphi = \sigma^2(p_i) f(p_i)$, then:

$$2\frac{\mu(p_i)}{\sigma^2(p_i)} = \frac{\partial \varphi}{\partial p_i},$$

that is:

$$\varphi = Be^{2\int dp_i \mu(p_i)/\sigma^2(p_i)},$$

where B is a constant; finally:

$$f(p_i) = \left(\frac{B}{\sigma^2(p_i)}\right) e^{2\int dp_i \mu(p_i)/\sigma^2(p_i)}. \tag{78}$$

The integral in Eq. (78) is given by:

$$\int dp_i \mu(p_i)/\sigma^2(p_i) = -\left(\frac{z_1}{2a_2}\right) \ln \sigma_{p_i}^2 + 2\left(\frac{z_0 + \frac{z_1 a_1}{2a_2}}{\sqrt{4a_0 a_2 - a_1^2}}\right) \arctan\left(\frac{a_1 + 2a_2 p_i}{\sqrt{4a_0 a_2 - a_1^2}}\right),$$

from which:

$$f(p_i) = \left[\frac{\mathcal{N}}{\left[a_0 + a_1 p_i + a_2 p_i^2\right]^{1+z_1/a_2}}\right] e^{4\left(\frac{z_0 + z_1 a_1/(2a_2)}{\sqrt{4a_0 a_2 - a_1^2}}\right) \arctan\left(\frac{a_1 + 2a_2 p_i}{\sqrt{4a_0 a_2 - a_1^2}}\right)},$$

where $\mathcal{N}$ is such that $\int_{-\infty}^{\infty} f(p_i)\, dp_i = 1$. Finally we notice that the distribution $f(p_i)$ is well-defined if $\sqrt{4a_0 a_2 - a_1^2}$ has real roots, that is $4a_0 a_2 - a_1^2 > 0$. Since:

$$\begin{aligned} 4a_0 a_2 - a_1^2 = 4\left[\Delta s^2 (1-\tau_l)(1-\tau_k)(\rho^* + \beta^*)\omega l_i\right]^2 * \\ * \left\{\Phi_{i,i}\Theta_{i,i} + \Phi_{i,i}\Gamma/\left[\Delta s^2 (1-\tau_k)^2 (\rho^* + \beta^*)^2\right] - \Omega_{ii}^2\right\} \end{aligned}$$

and therefore:

$$4a_0 a_2 - a_1^2 > 0 \Leftrightarrow \Phi_{i,i}\left\{\Theta_{i,i} + \Gamma/\left[\Delta s^2 (1-\tau_k)^2 (\rho^* + \beta^*)^2\right]\right\} - \Omega_{i,i}^2 > 0,$$

that is (see Eq. (31)):

$$\sum_{j=1}^{F} \phi_{i,j}^2 \left(\sum_{j=1}^{F} \theta_{i,j}^2 + \Gamma/\left[\Delta s^2 (1-\tau_k)^2 (\rho^* + \beta^*)^2\right]\right) > \left(\sum_{j=1}^{F} \theta_{i,j}\phi_{i,j}\right)^2.$$

The latter always holds because:

$$\sum_{j=1}^{F} \phi_{ij}^2 \sum_{j=1}^{F} \theta_{ij}^2 \geq \left(\sum_{j=1}^{F} \theta_{ij}\phi_{ij} \right)^2 .$$

given the Cauchy inequality, see Hardy et al. (1954).

QED

E Proof of Proposition 8

The proof follows the same steps of proof of Proposition 2 in Appendix D. When $N \to \infty$ from Eqq. (32) and (54) we have:

$$\mu(p_i) = z_0 - z_1 p_i; \tag{79}$$

$$\sigma^2(p_i) = a_2 p_i^2, \tag{80}$$

where:

$$z_0 = s\left[(1-\tau_l)\,\omega^* l_i + \tau_k \rho^* \bar{p} + \tau_l \omega^* \bar{l}\right] - \chi;$$
$$z_1 = \nu - s\,(1-\tau_k)\,\rho^*;$$
$$a_2 = \Delta s^2 (1-\tau_k)^2 (\rho^* + \beta)^2\, \Theta_{i,i} + \Gamma.$$

The marginal distribution satisfies:

$$f^{SW}(p_i) = \left(\frac{B}{\sigma^2(p_i)} \right) e^{2\int dp_i \mu(p_i)/\sigma^2(p_i)}, \tag{81}$$

where B is a constant. Therefore:

$$f^{SW}(p_i) = \frac{\mathcal{N}^{SW}}{a_2 p_i^{2(1+z_1/a_2)}} e^{-\left(\frac{z_0}{a_2 p_i}\right)},$$

where $\mathcal{N}^{SW}$ is such that $\int_{-\infty}^{\infty} f^{SW}(p_i)\,dp_i = 1$.

QED

F Proof of Proposition 9

Condition (56) ensures that $\bar{p}$ can grow forever, in fact from Eq. (37) we have that for large value of $\bar{p}$ $d\bar{p}/dt > 0 \forall \bar{p} > 0$. If also Condition (57) holds then $d\bar{p}/dt > 0$ for $\bar{p} = 0$. Therefore for the concavity of $g(.)$ $d\bar{p}/dt > 0$ for all $\bar{p} \geq 0$. This case is reported in Figure 6. Otherwise if Condition (59) holds, then $d\bar{p}/dt > 0$ for $\bar{p} = 0$. This means that there exists one value of p,

$\bar{p}^*$, such that $d\bar{p}/dt = 0$ (see Figure 7). Economy will be growing in the long run if the initial value of per capita wealth is higher than $\bar{p}^*$, otherwise $\bar{p}$ converges towards zero. Finally, Eqq. (60) and (61) are directly derived by Eqq. (15) and (16), taking into account Eqq. (18) and (19).

QED

G Proof of Proposition 10

Given the definition of u_i we have that:

$$\frac{du_i}{dt} = \frac{dp_i}{dt}/\bar{p} - u_i \frac{d\bar{p}}{dt}/\bar{p}.$$

From Eq.(38) and (60) we have that:

$$\lim_{\bar{p}\to\infty} \frac{d\bar{p}}{dt}/\bar{p} = (s\rho^* - v),$$

given that $\lim_{\bar{p}\to\infty} g\left(\bar{p}/\bar{l}\right) / \left(\bar{p}/\bar{l}\right) = \lim_{\bar{p}\to\infty} g'\left(\bar{p}/\bar{l}\right)$. Morevover, from Eq. (32) we have that:

$$\lim_{\bar{p}\to\infty} \frac{dp_i}{dt}/\bar{p} = s\left[(1-\tau_k)\rho^* u_i + \tau_k \rho^*\right] - \nu u_i + \eta_i/\bar{p},$$

given that $\lim_{\bar{p}\to\infty} \omega^* = 0$ (see Eq. (61), taking into account that $\lim_{\bar{p}\to\infty} g\left(\bar{p}/\bar{l}\right) / \left(\bar{p}/\bar{l}\right) = \lim_{\bar{p}\to\infty} g'\left(\bar{p}/\bar{l}\right)$). Therefore:

$$\lim_{\bar{p}\to\infty} \frac{du_i}{dt} = s(1-\tau_k)\rho^* + \tilde{\eta}_i,$$

where $\tilde{\eta}_i = \eta_i/\bar{p}$. The derivation of $\lim_{\bar{p}\to\infty} \lim_{N\to\infty} H_{i,i'}[\vec{u}]$ follows the same steps reported in Proposition 1. In fact:

$$E\left[\tilde{\eta}_i(t)\,\tilde{\eta}_{i'}(t')\right] = \frac{1}{\bar{p}(t)\,\bar{p}(t')} E\left[\eta_i(t)\,\eta_{i'}(t')\right] = \frac{H_{i,i'}[\vec{p}]}{\bar{p}(t)\,\bar{p}(t')}\delta(t-t'),$$

where $H_{i,i'}[\vec{p}]$ is the same as in Proposition 1, taken into account that in the limit $\bar{p}\to\infty$ only the quadratic terms in $p_i = \bar{p}u_i$ survives.

QED