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Abstract

This paper introduces a class of intermediate inequality indices, $I_{(\delta,\ \pi)}$, that is at the same time ray-invariant and unit-consistent. These measures permit us to keep some of the good properties of Krtscha's (1994) index while keeping the same "centrist" attitude whatever the income increase is. In doing so, we approach the intermediate inequality concept suggested by Del Río and Ruiz-Castillo (2000) and generalize it in order to extend the range of income distributions which are comparable according to the ray-invariance criterion..

Keywords: Income distribution; Intermediate inequality indices; Unit-consistency; Ray-invariance.
JEL Classification: D63.

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1. Introduction

Seminal works by Shorrocks (1980) and Chakravarty and Tyagarupananda (1998) characterized the set of decomposable inequality measures in the relative and absolute case, respectively, while Chakravarty and Tyagarupananda (2000) undertook a similar analysis for a particular intermediate notion. In a recent paper, Zheng (2007a) goes a step further when characterizing the entire class of decomposable inequality measures without imposing any invariance condition, so that relative, absolute, and intermediate measures are included in this class. In doing so, a new axiom, the unit-consistency axiom, is invoked.[1] This property warranties that inequality rankings between income distributions remain unchanged when all incomes are multiplied by a scalar. In other words, rankings are unaffected by the unit in which incomes are expressed, which makes this axiom very useful for empirical analysis.

One of the reasons why relative indices have been so widely used in empirical research is precisely that this inequality measurement is (cardinally) unaffected by scale changes. However, this imposes value judgments that are incompatible with “centrist” and “leftist” views of inequality (Kolm, 1969, 1976). On the contrary, unit-consistency is an ordinal property and, therefore, it does not impose such strong value judgments on inequality measurement as the scale invariance condition. In this new scenario, not only relative measures, but also absolute and intermediate measures that satisfy the unit-consistency axiom appear as plausible options for empirical research.

Even though the literature on income distribution offers a wide range of relative and absolute inequality measures, this does not happen however with respect to intermediate notions. In this regard, works by Seidl and Pfingsten (1997), Del Río and Ruiz-Castillo (2000), Zoli (2003) and Yoshida (2005) propose several intermediate invariance concepts, but as far as we know, the only intermediate inequality indices proposed in the literature are those based on Bossert and Pfingsten (1990), Kolm (1976), Krtscha (1994) and Zheng (2007a). Unfortunately, the family of indices proposed by the former does not satisfy the unit-consistency axiom, and that of the second only covers a small subset of the whole set of intermediate attitudes. Therefore, only the generalization of

[1] This axiom has also been introduced into poverty measurement by Zheng (2007b).

Krtscha's (1994) "fair compromise" index proposed by Zheng (2007a) appears as an appealing family of intermediate inequality indices.

According to Krtscha's (1994) "fair compromise" notion, to keep inequality unaltered, any extra income should be allocated among individuals as follows: the first extra dollar should be distributed so that 50 cents go to the individuals in proportion to the initial income shares, and 50 cents in equal absolute amounts. The second extra dollar should be allocated in the same manner, starting now from the distribution reached after the first dollar allocation, and so on. Therefore, according to this notion, the set of income distributions with the same intermediate inequality is a parabola. A consequence of this criterion is that the intermediate attitude changes as income rises, approaching the absolute invariance concept. This option seems reasonable if we think that individuals' value judgments constantly change when increasing their income dollar by dollar, but it might appear rather extreme.

Another extreme alternative would be to assume that the "centrist" attitude is ray-invariant, i.e. it is not altered by income growth, as in Seidl and Pfingsten (1997). In other words, given an initial distribution, this "centrist" attitude is the same whether global income increases by \$100 or \$1000. However, their approach does not satisfy horizontal equity, since identical individuals can be treated differently when comparing an initial distribution with another having higher income and the same intermediate inequality level than the former (as pointed out by Zoli, 2003). Del Río and Ruiz-Castillo (2000) go a step further by proposing a ray-invariance notion that does satisfy horizontal equity. Following this approach they also propose an operational method, but not an index, that combines this ray invariance notion with Lorenz dominance criterion.[2] This indirect method permits the ranking of distributions belonging to different "planes", which would be incomparable by using their invariance notion alone.

The aim of this paper is to introduce a class of intermediate inequality indices satisfying at the same time ray-invariance and unit-consistency. These measures permit us to keep some of the good properties of Krtscha's (1994) index—in particular the unit-consistency axiom—while keeping the same "centrist" attitude whatever the income

[2] See Del Río and Ruiz-Castillo (2001) for an empirical implementation.

increase is. In doing so, we first approach the intermediate inequality concept suggested by Del Río and Ruiz-Castillo (2000) and generalize it in order to extend the range of income distributions which are comparable according to the ray-invariance criterion. In other words, we enlarge the "intermediate space" so that it is no longer restricted to a "plane". Second, we propose a class of indices consistent with this new invariance condition, so that each of them allows us to compare any two income distributions in this larger "intermediate space". As far as we know, this is the only class of intermediate inequality indices that is at the same time ray-invariant and unit-consistent. Moreover, when comparing any two income distributions in this space, this approach allows us to determine an interval of intermediate attitudes under which one distribution has a higher inequality than the other.

The paper is structured as follows. In Section 2, we introduce a new intermediate ray-invariance notion that generalizes that of Del Río and Ruiz-Castillo (2000). This notion is proven to satisfy horizontal equity and path-independence. In Section 3, we propose a class of intermediate inequality indices that is ray-invariant, according to the aforementioned notion, and unit-consistent. In the final section we give some suggestions about the implementation of these intermediate indices for the empirical analysis.

2. A new intermediate ray-invariance notion

Following previous ideas of Dalton (1920), several reports on questionnaires indicate that many people believe that an equiproportional increase in all incomes raises income inequality, whereas an equal incremental decreases it.[3] Kolm (1976) called such attitudes "centrist", and considered "rightist" and "leftist" measures as extreme cases of this more general view.

A "centrist" income inequality attitude can be modeled in various ways, depending on the shape of the set of inequality equivalent income distributions. In this regard, Seidl and Pfingsten (1997) propose a ray-invariance concept, the α-inequality, such that all

[3] See Amiel and Cowell (1992), Ballano and Ruiz-Castillo (1993), Harrison and Seidl (1994) and Seidl and Theilen (1994) among others.

extra income should be distributed in fixed proportions, given by vector α, when comparing any two distributions. This requires the α-rays to be restricted in two ways: they Lorenz-dominate the original distribution, and they are more unequally distributed than translation invariance would require. This “centrist” attitude requires an inequality measure not to change provided any income change is distributed according to the value judgment represented by α. This invariance concept has not, however, a clear economic interpretation and it also violates the horizontal equity axiom, since individuals who have initially the same income level may be treated differently (Zoli, 2003).

Later on, Del Río and Ruiz-Castillo (2000) (DR-RC hereafter) propose another “centrist” attitude which is also ray-invariant, but with the advantage of satisfying the horizontal equity axiom. In this vein, inequality remains unchanged if π100% of the income difference is allocated preserving income shares in the distribution of reference and $(1-\pi)$100% is distributed in equal absolute amounts, which brings economic meaning to this ray-invariance notion. This means that inequality depends now on two parameters, instead of one: the income shares in the distribution of reference, that we denote by vector v, and $\pi \in [0,\ 1]$, which is used to define a convex combination between the relative and absolute rays associated to v.[4] This vector represents an ordered distribution in the n-dimensional simplex. Once these two parameters are fixed, it is possible to calculate the n-dimensional simplex vector $\alpha = \pi v + (1-\pi)\frac{1^n}{n}$, where $1^n \equiv (1,......,1)$, which defines the direction of the inequality equivalence ray. This ray can only be applied to a set of income distributions for which it represents an intermediate attitude with economic meaning.

Let $x = (x_1,.....,x_n) \in \mathbb{R}^n_{++},\ 2 \le n < \infty$, denote an income distribution where $x_1 \le x_2 \le ... \le x_n$, and D the set of all possible ordered income distributions. The set of income distributions for which α represents an intermediate attitude can be written as:

[4] In this paper we have changed their original notation in order to make it clearer. In particular, we have switched vector x by simplex vector v, since only the income shares of the distribution of reference are required to obtain the invariance ray. We also want to reserve letter x for the distribution in which to apply the inequality index.

$\Gamma^{'}(\alpha)=\left\{x\in D:\ \pi_x v_x+(1-\pi_x)\frac{1^n}{n}=\alpha,\ \text{for some } \pi_x\in[0,1]\right\}$, where v_x represents the vector of income shares associated to income distribution x, and therefore it belongs to the n-dimensional simplex. This means that vector $\alpha\in D$ can only be used for income distributions that are not only in the same plane but also weak Lorenz-dominated by α (see Figure 1).

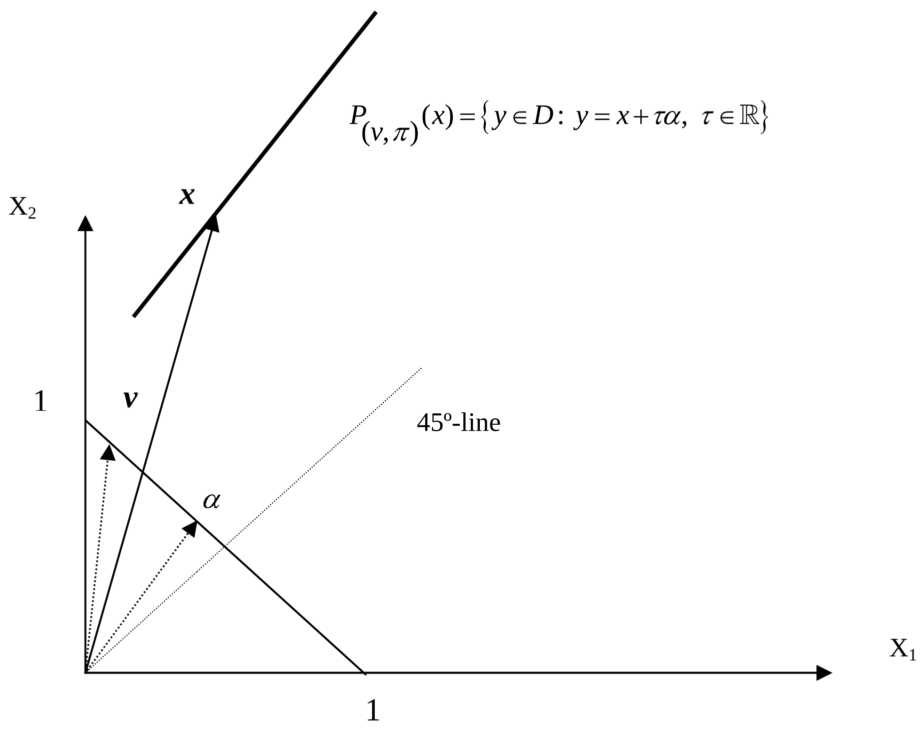

Figure 1. Ray-invariance in DR-RC ($n=2,\ \pi=0.25$).

In this vein, an intermediate inequality index $I_{(v,\pi)}:D\rightarrow\mathbb{R}$ is defined by DR-RC as ($v,\ \pi$)-invariant in the set of income distributions $\Gamma^{'}(\alpha)$ if for any $x\in\Gamma^{'}(\alpha)$ the following expression holds:

$$I_{(v,\pi)}(x)=I_{(v,\pi)}(y),\ \text{for any } y\in P_{(v,\pi)}(x),$$

where $P_{(v,\pi)}(x)=\left\{y\in D:\ y=x+\tau\left(\pi v+(1-\pi)\frac{1^n}{n}\right)=x+\tau\alpha,\ \tau\in\mathbb{R}\right\}$ represents the inequality invariance ray (see Figure 1). Note first that this ray is obtained as a convex combination between the "leftist" and "rightist" views associated to vector v (and also to any other distribution that has the same income shares than v). Second, vector α itself is the key element to construct any invariance ray in the aforementioned set—even though vector v and parameter π do intervene in the economic interpretation of this invariance notion.

However, if vector x is not on the plane given by vectors α and $\frac{1^n}{n}$—which includes set $\Gamma^{'}(\alpha)$—the criterion invariance proposed by DR-RC is not sufficient. In fact, when comparing two income distributions belonging to different planes, they propose instead an empirical method that makes use of the Lorenz dominance since their invariance notion would require to define a different index for each plane (plane defined by vectors v and $\frac{1^n}{n}$).[5]

In what follows we propose a new intermediate inequality concept consistent with the invariance criterion suggested by DR-RC but with the advantage of allowing comparisons between distributions in different planes. In doing so, we generalize the (v, π)-invariance notion to a (δ, π)-invariance. In this vein, we propose to fix the "distance", δ, between vectors α and $\frac{1^n}{n}$, rather than the invariance vector α itself. This distance characterizes a different vector α for each plane in which we want to evaluate the index, but in any of these planes the distance between both vectors is always the same. Let $\tilde{\Gamma}(x) = \left\{ y \in D: \ y = \beta_1 v_x + \beta_2 \frac{1^n}{n}, \text{ for some } \beta_1, \beta_2 \ \in \mathbb{R} \right\}$ represent

[5] This method allowed Del Río y Ruiz-Castillo (2001) to compare income distributions in Spain between 1980 and 1990. They concluded that for those people whose opinions are closer to the relative inequality notion (that is, if $\pi \in [0.87, \ 1]$), inequality would have decreased in Spain during that decade. However, for people more scored towards the left side of the political spectrum (that is, if $\pi \in [0, \ 0.71]$), it would be the opposite.

the plane defined by vectors v_x and $\frac{1^n}{n}$, or equivalently by vectors x and $\bar{x}1^n$, where $\bar{x}$ represents the arithmetic mean of $x \in D$ (see Figure 2).[6]

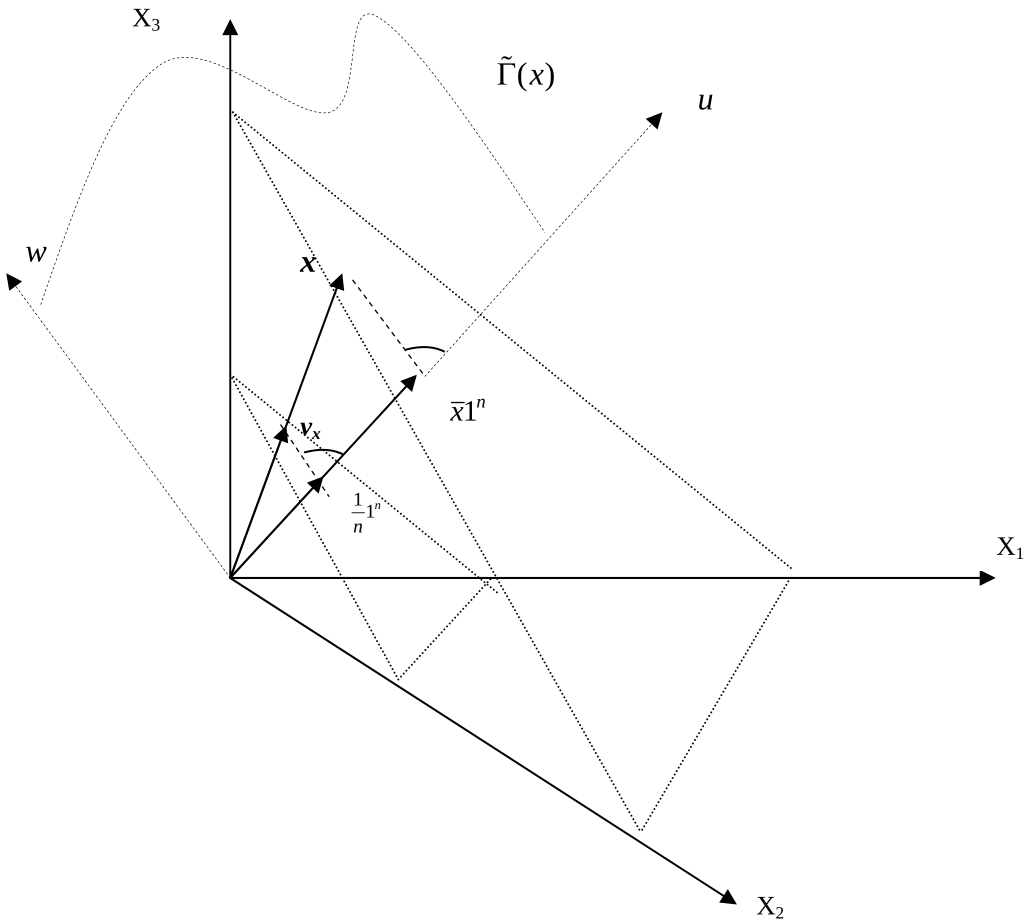

Figure 2. Plane $\tilde{\Gamma}(x)$ ($n=3$).

Within this plane, we can define a simplex vector $\alpha \equiv \alpha_{\tilde{\Gamma}(x)} \in \tilde{\Gamma}(x)$ such that,[7]

$$\delta = n^{\frac{1}{2}} \sqrt{\sum_i \left(\alpha_i - \frac{1}{n} \right)^2}, \tag{1}$$

[6] We assume that distribution x is not equally distributed and, therefore, v_x and $\frac{1^n}{n}$ are linear independent vectors.

[7] To simplify notation from now on we denote $\alpha_{\tilde{\Gamma}(x)}$ by α, but we should keep in mind that there is a different α for each plane $\tilde{\Gamma}(x)$.

where $0 < \delta < (n-1)^{1/2}$, and α_i represents the *i*-component of vector α (see Figure 3).[8] Note that the second term on the RHS of expression (1) is the Euclidean distance between vectors α and $\frac{1^n}{n}$.[9]

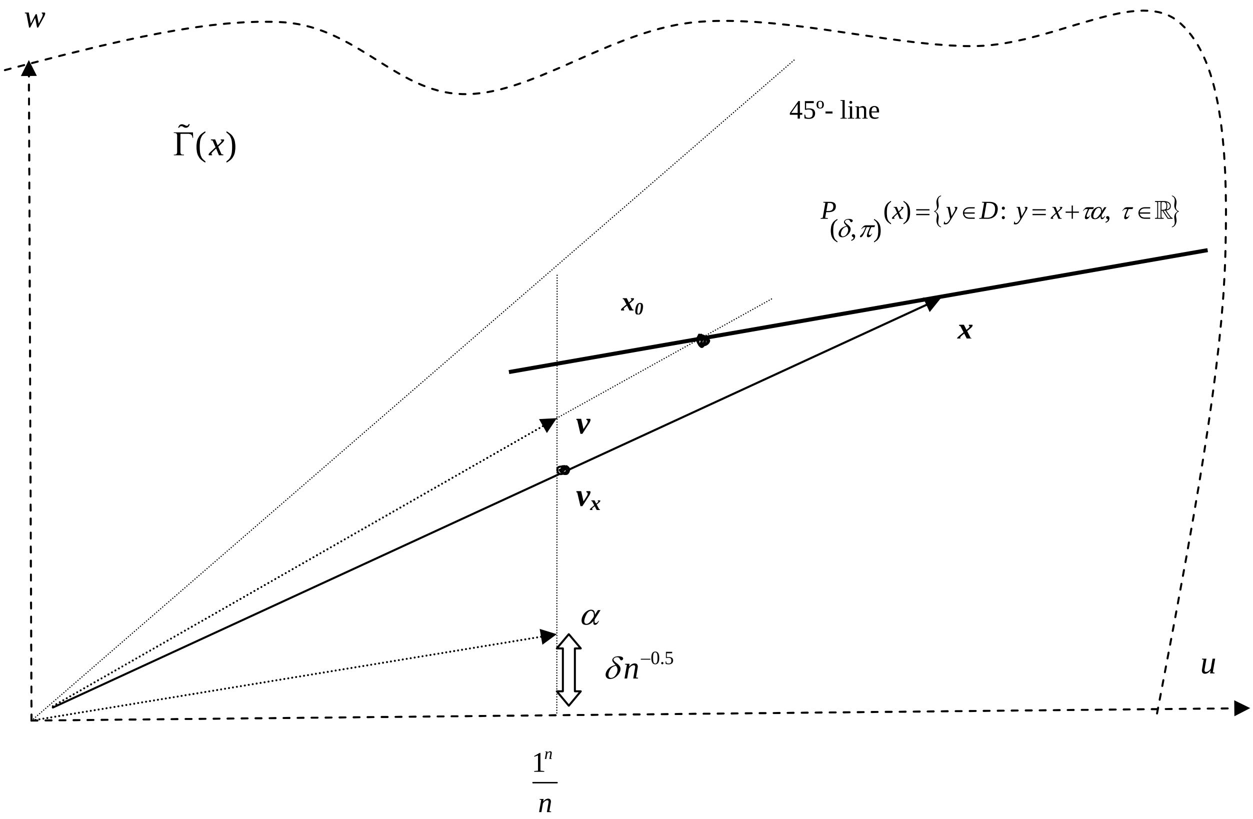

Figure 3. (δ, π) - invariance in plane $\tilde{\Gamma}(x)$ $(\pi = 0.25)$

In order to warranty that this "α" represents an intermediate notion for income distribution $x \in D$, we assume that $\delta < n^{1/2}\sqrt{\sum_i \left(v_{x_i} - \frac{1}{n}\right)^2}$, i.e. α must Lorenz-dominate distribution x. Therefore, we can write the following:

[8] Since we are always in a two-dimensional subspace, this "distance" characterizes a single simplex vector α (see Appendix). The upper bound for parameter δ comes from the upper bound for the Euclidean distance between the simplex vector corresponding to maximal inequality and $\frac{1^n}{n}$.

[9] The term $n^{1/2}$ is included in the above expression to warranty the inequality index satisfies the replication invariance axiom. For a given n-dimension space, fixing the Euclidean distance between both vectors is equivalent to fixing parameter δ.

$$\alpha = \pi' v_x + (1-\pi')\frac{1^n}{n} \text{ for some } \pi' \in [0,\ 1] \ . \qquad (2)$$

Lemma 1. *The vector* α*, corresponding to plane* $\tilde{\Gamma}(x)$ *and "distance"* δ*, can be written as a function of* x *and* δ *as follows:*

$$\alpha = \frac{\delta n^{-\frac{1}{2}}}{\sqrt{\sum_i \left(v_{x_i} - \frac{1}{n}\right)^2}} v_x + \left(1 - \frac{\delta n^{-\frac{1}{2}}}{\sqrt{\sum_i \left(v_{x_i} - \frac{1}{n}\right)^2}}\right)\frac{1^n}{n}.$$

Proof. By equation (2) we know that

$$\alpha_i - \frac{1}{n} = \pi'\left(v_{x_i} - \frac{1}{n}\right).$$

Substituting this into (1), and reordering, we obtain

$$\pi' = \frac{\delta}{n^{\frac{1}{2}}\sqrt{\left(v_{x_i} - \frac{1}{n}\right)^2}}.$$

Hence, substituting this value for π' into expression (2) leads immediately to the desired result. □

Therefore, once we have an income distribution we can plot the map of ray invariances of the corresponding plane. We should keep in mind, however, that in spite of the above formulation, vector α is only characterized by plane $\tilde{\Gamma}(x)$ and distance δ, as mentioned above, so that by considering any other Lorenz-dominated distribution in the same plane we would obtain the same vector.

As in DR-RC, we want α to represent a "centrist" attitude such that invariant rays are constructed as convex combinations between the relative ray of the reference distribution and the absolute ray. In other words, we want α to be written as

$$\alpha = \pi v + (1-\pi)\frac{1^n}{n},$$

where $v \in \tilde{\Gamma}(x)$ would represent again the income shares of our distribution of reference, and $\pi \in (0,\ 1)$ (Figure 3). Therefore, we also consider rays so that π% of any extra income is allocated to individuals according to income shares in the distribution of reference (v) and $(1-\pi)$% in equal absolute amounts.

From the above expression we can obtain v as a function of α and π:

$$v = \frac{\alpha}{\pi} - \frac{1-\pi}{\pi}\frac{1^n}{n} \quad . \tag{3}$$

This vector is also in the n-dimensional simplex. On the other hand, since α is univocally determined on plane $\tilde{\Gamma}(x)$, so is v (once parameter π is fixed). As we show later on, this vector plays an important role in the analysis, not only because it brings α economical meaning, but also because it offers a common reference with which to compare the different invariant rays constructed on plane $\tilde{\Gamma}(x)$.

Definition. Let $0 < \delta < (n-1)^{1/2}$, and $\pi \in (0,\, 1)$ be two parameters. Two income distributions $x, y \in \tilde{\Gamma}(x)$, for which δ represents a "centrist" inequality attitude, i.e. $x, y \in D_\delta = \left\{ z \in D : \delta < n^{1/2}\sqrt{\sum_i \left(v_{z_i} - \frac{1}{n}\right)^2} \right\}$, are ($\delta,\ \pi$)-inequality invariant according to index $I_{(\delta,\pi)} : D_\delta \to \mathbb{R}$ if and only if:

$$I_{(\delta,\pi)}(x) = I_{(\delta,\pi)}(y), \text{ for any } y \in P_{(\delta,\pi)}(x),$$

where

$$P_{(\delta,\pi)}(x) = \left\{ y \in D_\delta :\ y = x + \tau\alpha = x + \tau \left[\frac{\delta n^{-\frac{1}{2}}}{\sqrt{\sum_i \left(v_{x_i} - \frac{1}{n}\right)^2}} v_x + \left(1 - \frac{\delta n^{-\frac{1}{2}}}{\sqrt{\sum_i \left(v_{x_i} - \frac{1}{n}\right)^2}} \right) \frac{1^n}{n} \right],\ \tau \in \mathbb{R} \right\}$$

denotes the inequality invariance ray.

This invariance notion has two good properties: it satisfies the horizontal equity and path independence axioms.

Horizontal equity. As discussed by Zoli (2003), it seems reasonable for the invariance notion to call for an equal treatment of individuals who enjoy the same income level. Since in our case α can be written as $\alpha = \pi' v_x + (1-\pi')\frac{1^n}{n}$ for some $\pi' \in [0, 1]$ and for every $x \in D_\delta$, if $x_i = x_j$, then $\alpha_i = \alpha_j$. Therefore, the (δ, π)-invariance notion satisfies this axiom. However, Seidl and Pfingsten (1997) does not satisfy this axiom since they do not require α to be a convex combination between v_x and $\frac{1^n}{n}$, but a simplex vector that Lorenz-dominates distribution x. In other words, in their case α does not necessarily belong to plane $\tilde{\Gamma}(x)$. The intermediate notions proposed by Bossert and Pfingsten (1990), Krtscha (1994) and DR-RC do not have this problem and therefore they satisfy this axiom.

Path independence. This axiom requires the inequality level to be the same whether the income growth along the invariance ray has taken place in only one step, or in several increases at different moments (Zoli, 2003). This axiom is certainly satisfied by any inequality index consistent with the (δ, π)-invariance, since given an income distribution $x \in D_\delta$, the corresponding ray verifies:

$$P_{(\delta,\pi)}(x) = P_{(\delta,\pi)}\left(x + (\tau_1 + \tau_2)\alpha\right) = P_{(\delta,\pi)}\left((x + \tau_1\alpha) + \tau_2\alpha\right).$$

It is also easy to see that this property is also satisfied by the invariance notions proposed by Krtscha (1994), Seidl and Pfingsten (1997) and DR-RC.[10]

3. Unit-consistent and ray-invariant intermediate inequality measures: a proposal

In this section we propose a class of intermediate inequality indices that is at the same time ray-invariant—consistent with the above (δ, π)-invariance—and unit consistent. For this purpose, we first characterize the invariance ray in $x \in D_\delta$ through the point where this ray intersects the ray given by vector v, which is the common reference with

[10] In this regard, we should note that Zoli (2003) misunderstands DR-RC approach, since in his interpretation the invariance ray does depend on x.

which we compare each invariance ray on plane $\tilde{\Gamma}(x)$. This point is denoted in Figure 3 by $x_0 \in \tilde{\Gamma}(x)$. Therefore, each invariance ray is characterized by a different cutting point. Second, for each parameter δ, we define the inequality index in $x \in D_\delta$ as the variance of distribution $x_0 \in \tilde{\Gamma}(x)$, which is a function of the Euclidean distance between the above distribution and the equalitarian distribution where any individual has an equal income of $\overline{x}_0$ (the arithmetic mean of x_0). Third, we show that this family of indices satisfies the unit-consistency axiom.

For any $x \in D_\delta$, there always exists a single point $x_0 \in \tilde{\Gamma}(x)$, where the invariance ray, summarized by $\alpha \in \tilde{\Gamma}(x)$, and the ray given by $v \in \tilde{\Gamma}(x)$ cut, since these lines are in the same plane and v is always Lorenz-dominated by α. In what follows this point is determined.

Lemma 2. *The invariance ray* $P_{(\delta,\pi)}(x)$ *and the ray containing vector* v *intersect at point* x_0*:*

$$x_0 \equiv \frac{-x_1 + \alpha_1\left(\sum_i x_i\right)}{\alpha_1 - v_1} v \,.$$

Proof. See Appendix.

Lemma 3. *The variance of distribution* x_0 *can be written as*

$$Var[x_0] = \left(\frac{1}{1-\pi}\right)^2 \left[\sqrt{\frac{\sum_i \left(x_i - \overline{x}\right)^2}{n}} - \delta\overline{x}\right]^2 .$$

Proof. See Appendix.

Theorem. *The family of measures* $I_{(\delta,\pi)} : D_\delta \subset D \to \mathbb{R}$ *defined as*

$$I_{(\delta,\pi)}(x) = \left(\frac{1}{1-\pi}\right)^2 \left[\sqrt{\frac{\sum_i \left(x_i - \overline{x}\right)^2}{n}} - \delta\overline{x}\right]^2 \qquad (4)$$

represents a class of inequality indices consistent with the (δ, π)-invariance and, therefore, it satisfies

$$I_{(\delta,\pi)}(x) = I_{(\delta,\pi)}(y), \text{ for any } y \in P_{(\delta,\pi)}(x).$$

Proof. From Lemmas 2 and 3, it follows that $I_{(\delta,\pi)}$ measures the variance of x_0, which is the distribution that characterizes each (δ, π)-invariance ray on plane $\tilde{\Gamma}(x)$ (see Figure 3). Notice that if we worked on a single plane, we would not need to define an index in order to compare any two given income distributions in the "intermediate space", since by using the norm of the corresponding vector x_0 we could rank these income distributions in terms of inequality (in the proof of Lemma 2 this norm is denoted by λ). However, if we want to compare any two distributions that are in different planes, the norm alone is not enough, since to characterize vector x_0 we need this norm but also vector v, which differs among planes.

Any member of this class represents an inequality index since it satisfies:

a) Symmetry, since $I_{(\delta,\pi)}(x)$ is a symmetric function of x.

b) Continuity, since $I_{(\delta,\pi)}(x)$ is a continuous function of x.

c) Differentiability, since $I_{(\delta,\pi)}(x)$ has continuous first-order and second-order partial derivatives.

d) Replication invariance, since $I_{(\delta,\pi)}(\underbrace{x,...,x}_{k}) = I_{(\delta,\pi)}(x)$.

e) Strict Schur concavity, since by assumption $\delta < n^{1/2}\sqrt{\sum_i \left(v_{x_i} - \frac{1}{n}\right)^2}$ for any $x \in D_\delta$ and, therefore, $\sqrt{\frac{\sum_i \left(x_i - \overline{x}\right)^2}{n}} > \delta\overline{x}$, which makes the index decrease when there is a progressive transference (the LHS of the above expression decreases and the RHS remains unaltered).

Note that the domain of our family of inequality indices is not the whole income distribution set D, but the "intermediate space" D_δ, where the corresponding δ represents an intermediate attitude—i.e. D_δ is the subset of income distributions that are Lorenz-dominated by the vector α associated to δ. This means that these indices are not defined for those distributions where all individuals have the same income. Thus, within this approach the normalization property does not make sense. In any case, we should note that $I_{(\delta,\pi)}(x)$ tends to zero when $x \in D_\delta$ tends to $\bar{x}1^n$, since the value of parameter δ which allows this analysis is necessarily close to zero.

Next, we show that this class is consistent with the (δ, π)-invariance. In order to prove this, we only require that distributions $x \in D_\delta$, and $y \in P_{(\delta,\pi)}(x)$ have associated the same x_0. This is obvious by substituting x with $x + \tau\alpha$ in the expression given in Lemma 2. Thus, the index for both income distributions is the same, which completes the proof. □

A desirable property for the measurement of income inequality is the aforementioned unit-consistency axiom, proposed by Zheng (2007a), which warranties independence on the unit of measurement without imposing scale invariance:[11]

Unit consistency. For any two distributions x and y, and any inequality measure I, if $I(x) < I(y)$, then $I(\theta x) < I(\theta y)$ for any $\theta \in \mathbb{R}_{++}$.

Certainly, any relative inequality measure satisfies the above property since they are defined as those such that $I(\theta x) = I(x)$ for any $\theta \in \mathbb{R}_{++}$. However, we should keep in mind that there are other unit-consistent indices, apart from the scale invariant ones. In this vein, as shown by Zheng (2007a), the variance and the "fair compromise" measure proposed by Krtscha (1994) are an absolute and intermediate index, respectively, that satisfy this property.

[11] Zoli (2003) also proposes an analogous property named "weak currency independence".

In our case, we can easily show that $I_{(\delta,\pi)}$ is also a unit-consistent measure, since $I_{(\delta,\pi)}(\theta x)=\theta^2 I_{(\delta,\pi)}(x)$, for any $\theta\in\mathbb{R}_{++}$.

Remark 1. To define our intermediate inequality index we have used an absolute inequality index, the variance. Alternatively, we could use another absolute index, such as that proposed by Chakravarty and Tyagarupananda (1998), which ranks income distributions in the same way that Kolm-Pollak. We have chosen the variance since it is unit-consistent, while the other is not. In any case, we should note that the variance attaches equal weights to a progressive transfer at all income levels, while the class of indices proposed by Chakravarty and Tyagarupananda (1998) weights transfers in a different way depending on a parameter. Thus, the judgment values when using one of these two types of absolute measures or the other are different.

Remark 2. Any differentiable strictly increasing function of the above measure, $\varphi(I_{(\delta,\pi)})$, also represents a class of inequality indices consistent with the (δ, π)-invariance. In this regard, we could drop $\left(\frac{1}{1-\pi}\right)^2$ from expression (4) and raise the index to 0.5, so that new index $\varphi\left(I_{(\delta,\pi)}(x)\right)=\sqrt{\frac{\sum_i\left(x_i-\overline{x}\right)^2}{n}}-\delta\overline{x}$, would be quite similar to that proposed by Krtscha (1994). Moreover, as suggested in that paper, we could also consider function $\varphi\left(I_{(\delta,\pi)}\right)=\frac{I_{(\delta,\pi)}}{1+I_{(\delta,\pi)}}$, since this index would be bounded between 0 and 1, and therefore it would be more attractive for empirical analysis.

Remark 3. Note that $I_{(\delta,\pi)}$ is only a truly inequality index within the "intermediate space", D_δ, on which it is defined. This index can certainly be evaluated outside this space, but this would not be correct since it would not satisfy some basic inequality axioms.

4. Final comments

When undertaking empirical analysis, we should keep in mind in which contexts it is more adequate to use some inequality indices against others. In this vein, let us consider a situation where income distribution y has at the same time a higher absolute inequality level and a lower relative inequality level than x, which can be assumed to have a lower mean without loss of generality. In this context, the use of any intermediate inequality index with which to go deeper in the analysis of inequality seems the most appropriate.

In particular, Krtscha's index would allow us to conclude whether distribution y represents an improvement in terms of inequality with respect to x by using an adaptive intermediate notion. On one hand, this index offers two important advantages: it is not only decomposable—which can be very helpful for some analyses—but it is also defined on the whole set of income distributions, which makes it unnecessary to limit the intermediate space.[12] On the other hand, however, the price we have to pay for this wide domain is that distributions sharing the same inequality level as a given distribution must have income shares closer to total equity as mean income increases—an assumption that some people may find rather extreme. It is precisely the flexibility of this approach, imposing an intermediate notion based on marginal changes with respect to each distribution, which brings at the same time a wide domain for the index and a challenging "centrist" attitude. Note that the fact that the invariance set is not a straight line, but a parabola approaching the absolute view, makes it rather difficult for inequality to decrease when analyzing an economy over time.

The (δ, π)-inequality concept we propose is more conservative, since being ray-invariant the "centrist" attitude remains constant. Therefore, it does not allow any change in individuals' value judgments about inequality when increasing aggregate income, what seems plausible for analysis in the short and medium run. It follows then that this approach brings a complementary perspective to the former. The price we have

[12] As shown by Zheng (2007a), only the "first compromise" index proposed by Krtscha (1999) and the generalization proposed by the former are intermediate indices satisfying at the same time decomposability and unit-consistency. However, so far there is no clear economic motivation for the latter.

to pay is its narrower domain, since each of these indices is only well-defined for those distributions where (δ, π) represents an intermediate notion. Note, however, that this does not contradict the fact that this family covers the whole set of intermediate attitudes when changing the value of parameters δ and π. Moreover, the class of $I_{(\delta,\pi)}$ indices offers a clear economic interpretation, while sharing with Krtscha's index a desirable property in empirical analysis: unit-consistency, which prevents comparisons from being affected by the unit of measurement.

In implementing this approach, we could first define vector v as the income shares of x, and choose the parameter π reflecting our inequality-invariance value judgments. Next, we would build vector α as the convex combination between vectors v and $\frac{1^n}{n}$ given by π, and would find the parameter δ consistent with the above choice. In choosing parameter π we have to take into account that the correspondent vector α must also represent an intermediate attitude for distribution y (i.e. $x, y \in D_\delta$). By using this benchmark, we could determine whether distribution y has lower inequality than the distribution we would have reached if π% of the income surplus had been distributed according to income shares in x and the remaining $(1-\pi)$% in equal amounts among the individuals. Moreover, we could use different π's so as to obtain the range of values under which distribution y has lower inequality than x, and therefore, to determine the set of value judgments that would make society to prefer one distribution against the other.

Appendix

Characterizing vector α *in plane* $\tilde{\Gamma}(x)$ *from parameter* δ

$\tilde{\Gamma}(x)$ is a vector subspace of dimension 2, and therefore it is isomorphic to $\mathbb{R}^2$. Thus, it suffices to show that vector α can be univocally determined in $\mathbb{R}^2$: $\alpha = \alpha_1 e_1 + \alpha_2 e_2$, where e_1 and e_2 represent the canonical base. From expression (1) we know that $\left(\alpha_1 - \frac{1}{2}\right)^2 + \left(\alpha_1 - \frac{1}{2}\right)^2 = \frac{\delta^2}{2}$ (since now $n = 2$). In other words, α must lie in a circumference centered on point $\left(\frac{1}{2}, \frac{1}{2}\right)$. On the other hand, α is in the two-dimensional simplex and therefore, it must lie on line $\alpha_1 + \alpha_2 = 1$.

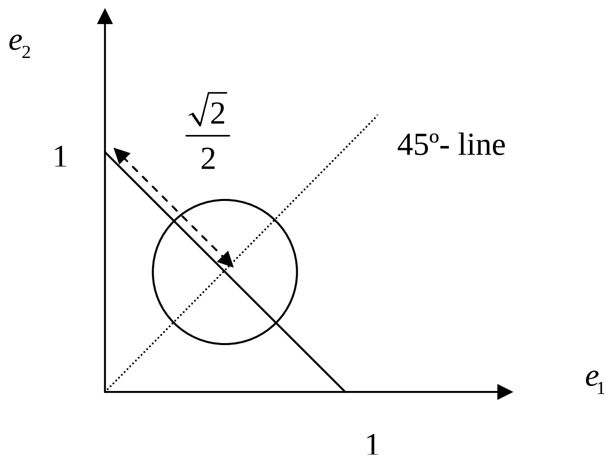

Figure A1. Characterizing vector α

Since $n = 2$ and $0 < \delta < (n-1)^{1/2} = 1$, the maximum Euclidean distance between α and $\frac{1^n}{n}$ is $\sqrt{\sum_i \left(\alpha_i - \frac{1}{n}\right)^2} = \delta n^{-\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, which is precisely the maximum radius that makes the circumference and the above line to cut in the first quadrant (Figure A1). Thus, we have two intersections in the first quadrant, but only that of the left is the one we look for, since we only work with ordered income distributions ($\alpha \in D$), which completes the proof. □

Proof of Lemma 2:

As mentioned in the previous section, given (δ, π) and $x \in D_\delta$ we can define simplex vectors $\alpha \in \tilde{\Gamma}(x)$ and $v \in \tilde{\Gamma}(x)$. The intermediate invariance ray $P_{(\delta,\pi)}(x)$ and the scale invariance ray associated with v are two lines that always intersect in the first quadrant, since α Lorenz-dominates v and v_x. Thus, any distribution in $P_{(\delta,\pi)}(x)$ can be written as

$$x = x_0 + \tau\alpha \ , \tag{A1}$$

where x_0 denotes the cutting point and $\tau \in \mathbb{R}$ (Figure 3). On the other hand, this cutting point can be expressed as

$$x_0 = \lambda v \ , \tag{A2}$$

for some $\lambda \in \mathbb{R}_+$. From (A1) and (A2), and taking into account that v belongs to the n-dimensional simplex, it follows that:

$$\sum_i x_i = \lambda + \tau \ . \tag{A3}$$

By using expressions (A1) to (A3) we can write that:

$$x_1 = x_{0_1} + \tau\alpha_1 = \lambda v_1 + \tau\alpha_1 = \left(\sum_i x_i - \tau\right)v_1 + \tau\alpha_1 = \left(\sum_i x_i\right)v_1 + \tau\left(\alpha_1 - v_1\right).$$

Therefore,

$$\tau = \frac{\left(\sum_i x_i\right)v_1 - x_1}{v_1 - \alpha_1}. \tag{A4}$$

From (A3) and (A4), we can obtain

$$\lambda = \sum_i x_i - \frac{\left(\sum_i x_i\right)v_1 - x_1}{v_1 - \alpha_1}.$$

By simple calculations the above expression can be rewritten as:

$$\lambda = \frac{-x_1 + \alpha_1\left(\sum_i x_i\right)}{\alpha_1 - v_1}. \tag{A5}$$

From (A2) and (A5) the desired result follows. Note that invariance ray $P_{(\delta,\pi)}(x)$ is characterized by x_0, or equivalently by λ. Any distribution in that line shares the same λ, and two different lines have different λ's. □

Proof of Lemma 3:

Simple calculations lead to $\bar{x}_0 = \frac{1}{n}\frac{-x_1+\alpha_1\left(\sum_i x_i\right)}{\frac{1-\pi}{\pi}\left(\frac{1}{n}-\alpha_1\right)}$. From this it follows that

$$Var[x_0] = \frac{1}{n}\sum_i\left(x_{0_i}-\bar{x}_0\right)^2 = \frac{1}{n}\sum_i\left\{\left(\frac{-x_1+\alpha_1\left(\sum_i x_i\right)}{\frac{1-\pi}{\pi}\left(\frac{1}{n}-\alpha_1\right)}\right)^2\left(\frac{\alpha_i}{\pi}-\frac{1}{\pi n}\right)^2\right\}.$$

By substituting α, as given in Lemma 1, into the above expression we obtain that

$$Var[x_0] = n\left[\frac{\frac{-x_1}{n}+\bar{x}\left(\frac{\delta n^{-1/2}}{\sqrt{\sum_i\left(v_{x_i}-\frac{1}{n}\right)^2}}\left(v_{x_1}-\frac{1}{n}\right)+\frac{1}{n}\right)}{(1-\pi)\left(\frac{1}{n}-v_{x_1}\right)}\right]^2\sum_i\left(v_{x_i}-\frac{1}{n}\right)^2.$$

On the other hand, $\sum_i\left(v_{x_i}-\frac{1}{n}\right)^2 = \frac{1}{n^2\bar{x}^2}\sum_i\left(x_i-\bar{x}\right)^2$. Taking this into account, and after some simple calculations, we can complete the proof. □

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