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The Impossibility of a Just Pigouvian

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Abstract

An income inequality measure satisfies the Pigou-Dalton transfer principle if progressive transfers decrease income inequality. When transfers cause transaction costs, one can trace out the maximum leakage such that the transfer pays at the margin. An income inequality measure is leaky-bucket consistent if the transaction costs of a transfer are neither negative nor do they exceed the amount of the transfer. We show that the Pigou-Dalton transfer principle and leakybucket consistency are not reconcilable.
Experimental research has shown that subjects' behavior exhibit graded compensating justice, that is compensating income changes which maintain the degree of income inequality and point in the same direction should provide less income compensation for richer than for poorer income recipients. We also show that the Pigou-Dalton transfer principle and graded compensating justice are not reconcilable. Moreover, we show that only constant income inequality measures satisfy graded compensating justice.

Keywords: Income distribution; Intermediate inequality indices; Unit-consistency.
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1 Introduction: Transfers with Transaction Costs or Income Changes which Preserve Income Inequality

Okun (1975, pp. 91-95) investigated the Pigou-Dalton transfer principle when transfers incur transaction costs. He observed that "the money must be carried from the rich to the poor in a leaky bucket. Some of it will simply disappear in transit, so the poor will not receive all the money that is taken from the rich." (Okun 1975, p. 91.) Okun raised the question to the observer of how much leakage he or she would accept and still support the Tax-and-Transfer Equalization Act. Stating his own attitude, he would stop at a leakage of 60 percent (Okun 1975, p. 94), but recognizes that some people might wish to take money away from the super-rich even if not one cent reached the poor.

To pin down Okun's problem, we have to answer the question as to the maximum amount of transaction costs such that a transfer would be considered as justified at the margin in the eyes of the beholder. This can be expressed in terms of the beholder's social welfare function (that is, welfare should be maintained after the respective move), or in terms of the beholder's inequality perception (that is, the degree of income inequality is maintained after the respective move)[1]. As social welfare functions and income inequality measures are but two sides of the same coin (Blackorby and Donaldson (1978), Dagum

[1]Although Okun is not explicit about the measure to be applied to determine the maximum leakage, we will not go astray in assuming that he had the observer's judgment on the social welfare of the society in mind. In this view, he followed Pigou (1950, p. 89) who remarked: "... it is evident that any transference of income from a relatively rich man to a relatively poor man of similar temperament, since it enables more intense wants to be satisfied at the expense of less intense wants, must increase the aggregate sum of satisfaction. ... Any cause which increases the absolute share of real income in the hands of the poor, provided that it does not lead to a contraction in the size of the national dividend from any point of view, will, in general, increase economic welfare." [Cf. also Pigou (1912, p. 24).] Although Pigou (1950, pp. 90-91) surmised that the rich are, from the nature of their upbringing and training, capable of obtaining considerably more satisfaction from a given income than the poor, he recognized (pp. 91-92) that these differences will fade away in the long run: "in the long run differences of temperament and taste between rich and poor are overcome by the very fact of a shifting of income between them." Dalton, like Pigou, considered the economic welfare derived from income as the decisive economic category, but Dalton (1920, p. 351) took the alternative route, as he associated the transfer principle in the first place with diminishing income inequality. Incidentally, note that Dalton (1920, pp. 349-350) anticipated both Atkinson's and Theil's income inequality measures.

(1990, 1993)), we shall focus on the beholder's income inequality perception.

Income transfers are just a special case of income inequality perceptions. They illustrate but one way of how changes in income distributions can come about. More generally, we will consider income changes of some specified income recipients and ask for the necessary change of the incomes of some specified other income recipients such that the former degree of income inequality is recovered.[2] We call this view *compensating justice*.

Well-known examples of compensating justice are scale and translation invariant income inequality measures, respectively. Suppose that the incomes of some, but not all, income recipients are decreased or increased by the same percentage. This changes the indicator of income inequality. Its original value is restored for a scale invariant inequality measure if the incomes of the rest are also changed by the same percentage. Translation invariance repeats this story for equal absolute changes of incomes. If they apply to some, but not to all, income recipients, then the original value of a translation invariant inequality measure is restored if the incomes of the rest are also changed by the same amount.

Experimental research by Camacho-Cuena et al. (2006) has shown that compensating justice in a specific form governs subjects' behavior also for cases in which only two incomes change. If the income of a poor income recipient increases (decreases), then the income of a richer income recipient should also increase (decrease), however less (more), in order to restore the original degree of perceived income inequality. This pattern is more pronounced the greater the income difference between the involved income recipients is. We call this behavioral pattern *graded compensating justice*. For our theoretical considerations it can easily be generalized in terms of an axiom for any finite set of income recipients.

Note that graded compensating justice lies outside the corridor of intermediate income inequality measures (see, e.g., Kolm (1976), Bossert and Pfingsten (1990), Seidl and Pfingsten (1997), del Rio and Ruiz-Castillo (2000)) as marked by scale invariance on the one side and translation invariance on the other. Graded compensating justice can be considered as an attitude of infra-translation invariance, or, to put it in Kolmian (1976, pp. 417-425) terms, it defines an ultra-leftist income inequality measure. This might

[2]This stance corresponds to the view of Lambert and Lanza (2006).

sound strange to some readers, but recall that it is overwhelmingly supported by real subjects' behavior.

Section 2 provides some preliminaries of income inequality relationships, Section 3 presents impossibility theorems of the Pigou-Dalton transfer principle on the one hand, and compensating justice and leaky-bucket consistency, respectively, on the other. Finally, we show that only constant income inequality measures satisfy graded compensating justice. Section 4 concludes.

2 Theory of Income Inequality Relationships

We allow only for strictly positive and finite incomes, which are different for different income recipients. The number of income recipients is assumed to be finite. The anonymity axiom holds. The income inequality measures are assumed to be continuously differentiable.

Thus, we consider only income distributions $\mathbf{y} = \{y_i \mid i = 1, \ldots, n;\ 1 < n < \infty\}$ such that, by the anonymity axiom, we can focus on an increasing arrangement of incomes, $0 < y_1 < y_2 < \ldots < y_n < \infty$. μ denotes mean income. The set of such income distributions is denoted by Y. Income inequality measures are denoted by $I : \mathbb{R}^n_{++} \to \mathbb{R}_+$.

Definition 1 (Invariance): A *relative* inequality measure satisfies scale invariance, that is $I(\mathbf{y}) = I(\lambda \mathbf{y})$ for all $\lambda > 0$. An *absolute* inequality satisfies translation invariance, that is $I(\mathbf{y}) = I(\mathbf{y} + \alpha \mathbf{e})$ for all $\alpha \in \mathbb{R}$ such that $y_1 + \alpha > 0$, where $\mathbf{e}$ denotes the unit vector.

Definition 2 (Transfer Principle): $I(\cdot)$ satisfies the Pigou-Dalton transfer principle if

$$I(y + \tau \mathbf{e_j} - \tau \mathbf{e_k}) < I(y) \text{ and } I(y - \tau \mathbf{e_j} + \tau \mathbf{e_k}) > I(y) \text{ for all } y_j < y_k,$$

where $\mathbf{e_i}$ denotes an $n-$dimensional vector with a 1 on the $i-$th position and zeros everywhere else, and τ is such that $0 < \tau < \min_{i,\ell \in \{1,\ldots,n\}} \mid y_i - y_\ell \mid$.

Definition 3 (Inequality Aversion): A differentiable income inequality measure $I(\cdot)$

is inequality averse if

$$\frac{\partial I}{\partial y_j} < \frac{\partial I}{\partial y_k} \text{ for } y_j < y_k.$$

Note that the observance of the Pigou-Dalton transfer principle or of inequality aversion rules out constant income inequality measures.

Lemma 4: If a differentiable income inequality measure $I(\cdot)$ is scale invariant, then

$$\sum_{i=1}^{n} \frac{\partial I}{\partial y_i} y_i = 0.$$

Proof: Consider a scale variation $\lambda \mathbf{y}$, $\lambda > 0$ of $\mathbf{y}$. Then $I(\lambda \mathbf{y}) \equiv I(\mathbf{y})$ for all $\lambda > 0$ implies

$$\frac{\partial I(\lambda \mathbf{y})}{\partial \lambda} = \sum_{i=1}^{n} \frac{\partial I(\lambda \mathbf{y})}{\partial \lambda y_i} y_i \equiv 0 \text{ for all } \lambda > 0. \text{ Hence, } \lim_{\lambda \to 1} \frac{\partial I(\lambda \mathbf{y})}{\partial \lambda} = \sum_{i=1}^{n} \frac{\partial I(\mathbf{y})}{\partial y_i} y_i = 0.$$

Q.E.D.

Lemma 5: If a differentiable income inequality measure $I(\cdot)$ is translation invariant, then

$$\sum_{i=1}^{n} \frac{\partial I}{\partial y_i} = 0.$$

Proof: By translation invariance $I(\mathbf{y}) = I(\mathbf{y} + \alpha \mathbf{e})$ for all $\alpha \in \mathbb{R}$ such that $y_1 + \alpha > 0$. This implies

$$\frac{\partial I(\mathbf{y} + \alpha \mathbf{e})}{\partial \alpha} = \sum_{i=1}^{n} \frac{\partial I(\mathbf{y} + \alpha \mathbf{e})}{\partial y_i} \equiv 0 \text{ for all } \alpha \in \mathbb{R},\ y_1 + \alpha > 0.$$

Hence

$$\lim_{\alpha \to 0} \frac{\partial I(\mathbf{y} + \alpha \mathbf{e})}{\partial \alpha} = \sum_{i=1}^{n} \frac{\partial I(\mathbf{y})}{\partial y_i} = 0.$$

Q.E.D.

Theorem 6: For differentiable inequality averse relative and absolute income inequality measures there exists a benchmark y^*, $y_1 < y^* < y_n$ such that $\frac{\partial I}{\partial y_i} < 0$ for all $y_i < y^*$, and $\frac{\partial I}{\partial y_i} > 0$ for all $y_i > y^*$.

PROOF: As all $y_i > 0$, Lemmata 5 and 6 demonstrate that not all $\frac{\partial I}{\partial y_i}$ can be positive. By inequality aversion $\frac{\partial I}{\partial y_i}$ increases monotonically as y_i increases. Hence, there exists a benchmark, y^*, such that $\frac{\partial I}{\partial y_i} < 0$ for all $y_i < y^*$ and $\frac{\partial I}{\partial y_i} > 0$ for all $y_i > y^*$. Q.E.D.

It seems that the existence and properties of benchmarks were first noticed and analyzed by Seidl (2001) and Hoffmann (2001). The most comprehensive study is by Lambert and Lanza (2006).

DEFINITION 7 (GRADED COMPENSATING JUSTICE): Let $(\delta_1^+, \ldots, \delta_n^+)$ denote a decreasing series whose members are all positive, let $(\delta_1^-, \ldots, \delta_n^-)$ denote a decreasing series whose members are all negative, and let $(\varepsilon_1, \ldots, \varepsilon_n)$ denote a vector whose components are either 0 or 1, where at least two ε_i's assume the value 1. Then graded compensating justice holds if for any feasible vector $(\varepsilon_1, \ldots, \varepsilon_n)$ there exists some $(\delta_1^+, \ldots, \delta_n^+)$ and some $(\delta_1^-, \ldots, \delta_n^-)$ such that $I(y_1, \ldots, y_n) = I(y_1 + \varepsilon_1\delta_1^+, \ldots, y_n + \varepsilon_n\delta_n^+) = I(y_1 + \varepsilon_1\delta_1^-, \ldots, y_n + \varepsilon_n\delta_n^-)$.

Note that this definition allows that the δ_i's may depend on the vector $(\varepsilon_1, \ldots, \varepsilon_n)$. However, this is immaterial for our proofs, as the general pattern, viz. that the δ_i's form a decreasing series, remains intact. In other words, the decreasing series need not be unique. Simple compensating justice means that the series $(\delta_1^+, \ldots, \delta_n^+)$ has only positive members which need not be decreasing, and that the series $(\delta_1^-, \ldots, \delta_n^-)$ has only negative members which need not be decreasing.

The next definition returns to Okun's problem and considers a simple progressive transfer from a richer to a poorer income recipient. It states *leaky-bucket consistency*, that is simply that transaction costs should neither exceed the transfer nor should the transaction costs become negative (which would be tantamount to a transfer subsidy).

DEFINITION 8 (LEAKY-BUCKET CONSISTENCY): An income inequality measure $I(\cdot)$ is leaky-bucket consistent if the transaction costs t of a transfer τ, which together maintain the same degree of income inequality, satisfy the condition $0 \leq t \leq \tau$.

3 Impossibility Theorems

This section shows three impossibility theorems and a possibility theorem. First, we show that graded compensating justice and the Pigou-Dalton transfer principle are not reconcilable for differentiable income inequality measures. This can be shown irrespective of conventional invariance conditions such as scale and translation invariance. In other words, one must abandon either graded compensating justice or the Pigou-Dalton transfer principle. This is all the more remarkable, since experimental research has not yielded convincing support for the Pigou-Dalton transfer principle,[3] whereas graded compensating justice enjoys much greater behavioral support (Camacho et al. (2006)).

Second, we show that graded compensating justice and the Pigou-Dalton transfer principle are not reconcilable for relative and absolute income inequality measures. This is not unexpected for two reasons: recall that scale invariance and translation invariance are two different varieties of compensating justice. As graded compensating justice, as defined in Definition 7, represents another form of compensating justice, impossibility is around the corner. Graded compensating justice lies, on the other hand, outside the corridor of intermediate income inequality measures between the borders marked by scale and translation invariance. Being an ultra-leftist income inequality measure, it is, therefore, at variance both with scale and translation invariance (as well as with other intermediate income inequality measures). However, this has to be formally proved.

The most intriguing of these three impossibility theorems seems to be the third one. In this impossibility theorem, we draw on leaky-bucket consistency rather than on graded compensating justice and show that it is not reconcilable with the Pigou-Dalton transfer principle. This is nothing else but a devastating blow against Okun's conjecture. For many transfer situations there do not exist transaction costs which are neither negative nor do they exceed the amount of the transfer itself. In other words, for many transfer situations there do not exist feasible transaction costs such that the transfer "pays at the margin". This result is also fatal for leaky-bucket experiments which overlook this circumstance.

Notice that our impossibility theorems are stated in terms of the Pigou-Dalton transfer

[3] See, for instance, Amiel and Cowell (1992, 1998, 2000), Ballano and Ruiz-Castillo (1993), Harrison and Seidl (1994 a,b).

principle, whereas in our proofs we rely on inequality aversion. This is immediate, given Theorem 15 in the Appendix, which shows equivalence between inequality aversion and the Pigou-Dalton transfer principle. Note, moreover, that Theorem 15 in the Appendix also allows us to extend our impossibility theorems to S-convexity and Lorenz consistency.

Finally, we take up the challenge of characterizing income inequality measures which satisfy compensating justice. Alas, we arrive at the result that only constant income inequality measures satisfy compensating justice, which means that subjects' perceptions cannot be captured by nontrivial income inequality measures.

Theorem 9 (Impossibility of a Just Pigouvian): There is no differentiable income inequality measure which satisfies graded compensating justice and the Pigou-Dalton transfer principle.

Proof: By inequality aversion there are either $\frac{n}{2}$ incomes for which the $\frac{\partial I}{\partial y_i}$ are non-positive or non-negative. In the first case select a subset of the respective ε_i's, say K, $\sharp K \geq 2$, and set them equal to one and the other ε_i's equal to zero.

Then we have for $\{\delta_i^+ \mid i \in K\}$:

$$I(y_1 + \delta_1^+ \varepsilon_1, \ldots, y_i + \delta_i^+ \varepsilon_i, \ldots, y_n + \delta_n^+ \varepsilon_n) \approx I(y_1, \ldots, y_n) + \sum_{i \in K} \frac{\partial I}{\partial y_i} \delta_i^+ < I(y_1, \ldots, y_n)$$

for inequality aversion. Yet by Theorem 15 in the Appendix inequality aversion is equivalent to the Pigou-Dalton transfer principle.

For $\{\delta_i^- \mid i \in K\}$ we have:

$$I(y_1 + \delta_1^- \varepsilon_1, \ldots, y_i + \delta_i^- \varepsilon_i, \ldots, y_n + \delta_n^- \varepsilon_n) \approx I(y_1, \ldots, y_n) + \sum_{i \in K} \frac{\partial I}{\partial y_i} \delta_i^- > I(y_1, \ldots, y_n)$$

for inequality aversion, which is equivalent to the Pigou-Dalton transfer principle.

The proof for nonnegative $\frac{\partial I}{\partial y_i}$ is obvious. Q.E.D.

Theorem 10 (Impossibility of a Scale or Translation Invariant Just Pigouvian): There is no relative or absolute inequality measure which satisfies graded compensating justice and the Pigou-Dalton transfer principle.

Proof: Let $\varepsilon_i = 1$ for all $i = 1, 2, ..n$. Consider

$$\lambda \doteqdot 1 + \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} y_i} \text{ and } \alpha \doteqdot \frac{1}{n} \sum_{i=1}^{n} \delta_i,$$

where the δ_i's can stand either for δ_i^+ or for δ_i^-.

Scale invariance implies $I(\mathbf{y}) = I(\lambda \mathbf{y})$; translation invariance implies $I(\mathbf{y}) = I(\mathbf{y} + \alpha \mathbf{e})$.

Note that, by construction, $(\lambda \mathbf{y})$, $(\mathbf{y} + \alpha \mathbf{e})$, and $(y_1 + \delta_1, y_2 + \delta_2, \ldots, y_n + \delta_n)$ all have the same mean. Yet $(y_1 + \delta_1, y_2 + \delta_2, \ldots, y_n + \delta_n)$ can be attained from $(\lambda \mathbf{y})$ or $(\mathbf{y} + \alpha \mathbf{e})$ by a series of progressive transfers. This means

$$I(y_1 + \delta_1, y_2 + \delta_2, \ldots, y_n + \delta_n) < I(\lambda \mathbf{y}) = I(\mathbf{y}), \text{ or}$$

$$I(y_1 + \delta_1, y_2 + \delta_2, \ldots, y_n + \delta_n) < I(\mathbf{y} + \alpha \mathbf{e}) = I(\mathbf{y}),$$

which establishes a contradiction to graded compensating justice. Q.E.D.

Note that scale invariance—and, more generally, unit consistency as defined by Zheng (2007, p. 102)—is of an ambiguous character. On the one hand, it makes an income inequality measure immune to inflation and to the unit of measurement. It would indeed be absurd if the degree of income inequality in Spain or in Germany had changed as a consequence of the transition from Pesetas and Deutschmarks, respectively, to Euros. On the other hand, if all *real* incomes had changed in the same proportion, this might well matter for the degree of income inequality in a given country.[4] Considerations like this have given rise to the development of intermediate income inequality measures, graded compensating justice being analogous to them.

Theorem 11 (Impossibility of a Leaky-Bucket-Consistent Pigouvian): There is no differentiable relative or absolute income inequality measure which satisfies the Pigou-Dalton transfer principle and leaky-bucket consistency.

[4]Dalton (1920, p. 356) argued that both equal proportional and equal absolute additions to all incomes diminish inequality. He restricted that to *real* income increases and required that the units of money income in any two cases to be compared must have approximately equal purchasing power (Dalton (1920, p. 356, footnote 2). Following Mill, Pigou (1950, p. 90) argued that it is the *relative* position of an income recipient in the income gamut which determines his or her satisfaction, rather than the actual level of income.

PROOF: By Theorem 6 there is at least one income recipient above and one below the benchmark. Consider first $y_j < y^* < y_k$. Maintenance of the degree of income inequality requires

$$I(y_1, \ldots, y_n) = I(y_1, \ldots, y_j + (\tau - t), \ldots, y_k - \tau, \ldots, y_n) \approx I(y_1, \ldots, y_n) + \frac{\partial I}{\partial y_j}(\tau - t) - \frac{\partial I}{\partial y_k}\tau. \tag{1}$$

However, this is a contradiction because

$$-\frac{\partial I}{\partial y_k}\tau < 0 \text{ and } \frac{\partial I}{\partial y_j}(\tau - t) \leq 0 \text{ for } 0 \leq t \leq \tau.$$

Consider, second, that both income recipients are above the benchmark. Then equation (1) gives us another contradiction because

$$-\frac{\partial I}{\partial y_k}\tau + \frac{\partial I}{\partial y_j}(\tau - t) < 0 \text{ for } 0 \leq t \leq \tau.$$

From Theorem 15 of the Appendix we know that inequality aversion is equivalent to the Pigou-Dalton transfer principle, which completes the proof. Q.E.D.

For the first case in the proof of Theorem 11, the degree of income inequality could only be restored for transaction costs which exceed the amount of the transfer, that is, the poor should also suffer an income loss resulting from the "transfer" in order to restore the former degree of income inequality.

For the second case in the proof of Theorem 11, the degree of income inequality could only be restored for negative transaction costs, that is, for transfer subsidies. The (poorer) transfer recipient would have to get more than the amount of the transfer in order to restore the former degree of income inequality.

Theorem 11 is, in particular, fatal for leaky-bucket experiments which confine subjects' responses to transaction costs satisfying the constraint $0 \leq t \leq \tau$. An experimental design constrained as such abandons either scale or translation invariance or the Pigou-Dalton transfer principle.

Finally, our results provoke the question whether an income inequality measure exists which satisfies graded compensating justice and how it looks like. The next theorem gives the answer.

Theorem 12 (Possibility of a Graded-Compensating-Justice Income Inequality Measure): Only a constant differentiable income inequality measure satisfies graded compensating justice.

Proof: From Definition 7 we assume without loss of generality that

$$I(y_1,\ldots,y_n) \doteq I(y_1,\ldots,y_j+\delta_j^+,\ldots,y_k+\delta_k^+,\ldots,y_n), \quad \delta_j^+>0,\, \delta_k^+>0.$$

Hence,

$$I(y_1,\ldots,y_j+\delta_j^+,\ldots,y_k+\delta_k^+,\ldots,y_n)-I(y_1,\ldots,y_n) \approx \frac{\partial I}{\partial y_j}\delta_j^+ + \frac{\partial I}{\partial y_k}\delta_k^+ \doteq 0. \qquad (2)$$

As $\delta_j^+>0$ and $\delta_k^+>0$, $\frac{\partial I}{\partial y_j}$ and $\frac{\partial I}{\partial y_k}$ must have opposite signs. Let us, without loss of generality, assume $\frac{\partial I}{\partial y_j} \leq 0$ and $\frac{\partial I}{\partial y_k} \geq 0$.

Let us now assume that we change k's and i's incomes, where $i \neq j$. Then we have

$$I(y_1,\ldots,y_i+\delta_i^+,\ldots,y_k+\delta_k^+,\ldots,y_n)-I(y_1,\ldots,y_n) \approx \frac{\partial I}{\partial y_i}\delta_i^+ + \frac{\partial I}{\partial y_k}\delta_k^+ \doteq 0. \qquad (3)$$

Because of $\delta_i^+>0$ and $\delta_k^+>0$, $\frac{\partial I}{\partial y_i}$ and $\frac{\partial I}{\partial y_k}$ must have opposite signs. As we assumed $\frac{\partial I}{\partial y_k} \geq 0$, we must have $\frac{\partial I}{\partial y_i} \leq 0$.

Suppose finally that we change i's and j's incomes by $\hat{\delta}_i^+ > 0$ and $\hat{\delta}_j^+ > 0$. Then compensating justice should give us

$$I(y_1,\ldots,y_i+\hat{\delta}_i^+,\ldots,y_j+\hat{\delta}_j^+,\ldots,y_n)-I(y_1,\ldots,y_n) \approx \frac{\partial I}{\partial y_i}\hat{\delta}_i^+ + \frac{\partial I}{\partial y_j}\hat{\delta}_j^+ \doteq 0. \qquad (4)$$

Then $\frac{\partial I}{\partial y_j} \leq 0$ yields $\frac{\partial I}{\partial y_i} \geq 0$. Recalling $\frac{\partial I}{\partial y_i} \leq 0$ from above implies $\frac{\partial I}{\partial y_i} = 0$. Then, by equation (3), also $\frac{\partial I}{\partial y_k} = 0$, and by equation (2) we have $\frac{\partial I}{\partial y_j} = 0$.

Consequently $\frac{\partial I}{\partial y_i} = \frac{\partial I}{\partial y_k} = \frac{\partial I}{\partial y_k} = 0$ and $I(\cdot)$ is a constant function. Q.E.D.

This result is amazing. In an experimental study, Camacho-Cuena et al. (2006) observed that subjects' perceptions of maintaining the degree of income inequality when the income of some subject is changed exhibits a behavior of graded compensating justice. Yet it is not possible to capture this empirically observed perception of the maintenance of the degree of income inequality in a society in terms of a nonconstant differentiable income inequality measure. Note that this proof applies to simple compensating justice (without the grading pattern) as well.

Readers might wonder why we termed this result a *possibility* theorem, as only trivial income inequality measures pass the compensating-justice test. In our choice we followed Arrow's (1951, p. 59) lead who reserved the epithet "possibility" for imposed or dictatorial social welfare functions.

4 Conclusion

This paper starts with Okun's investigation of the Pigou-Dalton transfer principle in the presence of transaction costs which cause a leakage in transferring income. Okun raised the question as to the maximum amount of transaction costs such that a transfer is considered as justified at the margin.

More generally, consider income changes of some specified income recipients. The problem of changing the incomes of some other specified income recipients which restore the original degree of income inequality is called *compensating justice*. Experimental research suggests that people support a particular variety of compensating justice, viz. *graded compensating justice*. It demands that all compensating income changes point in the same direction; the triggering income changes (all having the same sign) imply that the compensating income changes of richer (poorer) income recipients are less than the income changes of poorer (richer) income recipients. This means that graded compensating justice is an ultra-leftist inequality attitude.

Section 2 sets the formal frame of our analysis. It defines invariance, the transfer principle, inequality aversion, graded compensating justice, and leaky-bucket consistency (i.e. that the transaction costs should be non-negative and should not exceed the transfer). Moreover, we establish the existence of a benchmark separating negative and positive partial derivatives of an income inequality measure.

In Section 3 we present three impossibility theorems and a possibility theorem. First, we show that graded compensating justice and the Pigou-Dalton principle of transfers are not reconcilable. Second, we show that this impossibility result extends to relative and absolute income inequality measures at large. Third, we show that leaky-bucket consistency is not reconcilable with the Pigou-Dalton transfer principle for relative and absolute income inequality measures. Fourth, we show that the only differentiable income inequality measures which satisfy (graded) compensating justice are constant income inequality measures.

Appendix

Definition 13 (Lorenz Consistency): $I(\cdot)$ is Lorenz-consistent if Lorenz-domination of $\mathbf{y}$ over $\mathbf{y}'$ implies $I(\mathbf{y}) < I(\mathbf{y}')$.

Definition 14 (S-convexity): $I(\cdot)$ is strictly S-convex if $I(\mathbf{yP}) < I(\mathbf{y})$ for all bistochastic matrices $\mathbf{P}$ except permutation matrices.

Theorem 15 (Equivalence Theorem): The following statements are equivalent:

(i) $I(\cdot)$ is inequality averse.

(ii) $I(\cdot)$ satisfies the transfer principle.

(iii) $I(\cdot)$ is strictly S-convex.

(iv) $I(\cdot)$ is Lorenz-consistent.

Proof: (i) $\Rightarrow$ (ii). Inequality aversion implies $\dfrac{\partial I}{\partial y_k} < \dfrac{\partial I}{\partial y_j}$ for all $y_k < y_j$. Hence, $\dfrac{\partial I}{\partial y_k} - \dfrac{\partial I}{\partial y_j} < 0$, which implies for $0 < \tau < \min_{i,\ell \in \{1,\ldots,n\}} \mid y_i - y_\ell \mid$:

$$0 > \left(\frac{\partial I}{\partial y_k} - \frac{\partial I}{\partial y_j} \right) \tau := \Delta I \text{ for all } y_k < y_j \,.$$

But this implies the transfer principle.

(ii) $\Rightarrow$ (iii). Define $\kappa := \dfrac{1}{\tau}(y_j - y_k)$, where $y_k < y_j$. Because of

$$0 < \tau < \min_{i,\ell \in \{1,\ldots,n\}} \mid y_i - y_\ell \mid$$

we have $\kappa > 1$. Then $(y + \tau \mathbf{e_k} - \tau \mathbf{e_j})$ can equivalently be achieved by multiplying y by the n–dimensional bistochastic matrix

$$\mathbf{A} := \mathbf{E} - \frac{1}{\kappa}\mathbf{E}^{jj}_{kk} + \frac{1}{\kappa}\mathbf{E}^{jk}_{kj},$$

where $\mathbf{E}$ denotes the unit matrix, $\mathbf{E}_{kk}^{jj}$ denotes a matrix whose elements $\psi_{i\ell}$ are $\psi_{i\ell} = 1$ for $i = \ell = j$ and for $i = \ell = k$, and 0 otherwise, and $\mathbf{E}_{kj}^{jk}$ denotes a matrix whose elements $\varphi_{i\ell}$ are $\varphi_{i\ell} = 1$ for $i = j$, $\ell = k$ and for $i = k$, $\ell = j$, and 0 otherwise. By the transfer principle we have

$$I(y) > I(y + \tau\mathbf{e_k} - \tau\mathbf{e_j}) = I(yA),$$

which implies strict S-convexity of $I(\cdot)$, as there exists a bistochastic matrix for any sequence of progressive transfers.

(iii) $\Rightarrow$ (iv). Strict S-convexity of $I(\cdot)$ implies

$$I(\mathbf{y}) > I(\mathbf{yP}) = I(\mathbf{yP\Pi})$$

for all bistochastic matrices $\mathbf{P}$ and all permutation matrices $\mathbf{\Pi}$. Let $(z_1, \ldots, z_n)$ denote a permutation of $\mathbf{yP}$ such that $z_1 \leq z_2 \leq \ldots \leq z_n$. Then

$$y_1 \leq \min_{\ell \in \{1,\ldots,n\}} \{\sum_{i=1}^{n} y_i p_{i\ell}\} := z_1, \text{ and } y_n \geq \max_{\ell \in \{1,\ldots,n\}} \{\sum_{i=1}^{n} y_i p_{i\ell}\} := z_n.$$

Because $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i$ and $y \neq z$ (as $\mathbf{P}$ is not a permutation matrix), there exists a unique k, such that $y_i \leq z_i$ for all $i \leq k$, and $y_k < z_k$, $y_{k+1} > z_{k+1}$. Hence, $\mathbf{z}$ Lorenz-dominates $\mathbf{y}$.

(iv) $\Rightarrow$ (i). For $0 < \tau < \min_{i,\ell \in \{1,\ldots,n\}} \mid y_i - y_\ell \mid$ the income distribution $(\mathbf{y} + \tau\mathbf{e_k} - \tau\mathbf{e_j})$, $k < j$, Lorenz-dominates the income distribution $\mathbf{y}$. For a Lorenz-consistent income inequality measure we have, thus, $I(\mathbf{y} + \tau\mathbf{e_k} - \tau\mathbf{e_j}) < I(\mathbf{y})$, $k < j$. This implies $\frac{\partial I}{\partial y_k} - \frac{\partial I}{\partial y_j} < 0$ and, thus, $\frac{\partial I}{\partial y_k} < \frac{\partial I}{\partial y_j}$, which is inequality aversion.

Q.E.D.

References

Amiel, Y. and Cowell, F. A. (1992). Measurement of Income Inequality: Experimental Test by Questionnaire. *Journal of Public Economics*, 47:3-26.

Amiel, Y. and Cowell, F. A. (1998). Distributional Orderings and the Transfer Principle: A Reexamination. *Research on Economic Inequality*, 8:195-215.

Amiel, Y. and Cowell, F. A. (2000). *Attitudes to Risk and Inequality: A New Twist on the Transfer Principle.* STICERD Distributional Analysis Discussion Paper No. 56. London School of Economics, London.

Arrow, K. J. (1951). *Social Choice and Individual Values.* John Wiley & Sons, New York.

Ballano, C. and Ruiz-Castillo, J. (1993). Searching by Questionnaire for the Meaning of Income Inequality. *Revista Española de Economía*, 10:233-259.

Blackorby, C. and Donaldson, D. (1978). Measures of Relative Equality and their Meaning in Terms of Social Welfare. *Journal of Economic Theory*, 18:59-80.

Bossert, W. and Pfingsten, A.(1990). Intermediate Inequality: Concepts, Indices, and Welfare Implications. *Mathematical Social Sciences* 19:117-134.

Camacho-Cuena, E., Neugebauer, T. and Seidl, C. (2006). Leaky Buckets versus Compensating Justice: An Experimental Investigation. Discussion paper; submitted.

Dagum, C. (1990). On the Relationship Between Income Inequality Measures and Social Welfare Functions. *Journal of Econometrics*, 43:91-102.

Dagum, C. (1993). The Social Welfare Bases of Gini and other Income Inequality Measures. *Statistica*, 53:3-30.

Dalton, H. (1920). The Measurement of the Inequality of Incomes. *The Economic Journal*, 30:348-361.

del Rio, C. and Ruiz-Castillo, J. (2000). Intermediate Inequality and Welfare. *Social Choice and Welfare*, 17:223-239.

Harrison, E. and Seidl, C. (1994a). Acceptance of Distributional Axioms: Experimental Findings. In Eichhorn, W., editor, *Models and Measurement of Welfare and Inequality*, pages 67-99. Springer-Verlag, Berlin.

Harrison, E. and Seidl, C. (1994b). Perceptional Inequality and Preferential Judgments: An Empirical Examination of Distributional Judgements. *Public Choice*, 19:61-81.

Hoffmann, R. (2001). Effect of the Rise of a Person's Income on Inequality, *Brazilian Review of Econometrics*, 21:237-262.

Kolm, S.-C. (1976). Unequal Inequalities, I and II. *Journal of Economic Theory*, 12:416-442 and 13:82-111.

Lambert, P. J. and Lanza, G. (2006). The Effect on Inequality of Changing One or Two Incomes. *Journal of Economic Inequality*, 4:253-277.

Lorenz, M. C. (1905). Methods of Measuring the Concentration of Wealth. *Quarterly Publications of the American Statistical Association*, 9:209-219.

Okun, A. M. (1975). *Equality and Efficiency: The Big Trade-Off.* Brookings Institution, Washington DC.

Pigou, A. C. (1950). *The Economics of Welfare.* Fourth edition. Macmillan, London. [First edition: *Wealth and Welfare.* Macmillan, London 1912.]

Seidl, C. (2001). Inequality Measurement and the Leaky-Bucket Paradox. *Economics Bulletin*, 4(6):1-7.
`http://www.economicsbulletin.com/2001volume/4-01D30001A.pdf`.

Seidl, C. and Pfingsten, A. (1997). Ray Invariant Inequality Measures. In Zandvkili, S., editor, *Research on Economic Inequality*, Vol. 7, pages 107-129. JAI Press, Greenwich, Conn.

Zheng, B. (2007). Unit-Consistent Decomposable Inequality Measures. *Economica* 74:97-111.