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Unit-consistency and polarization of income distributions*

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Abstract

Most of the polarization measures proposed in the literature, and likewise the inequality and poverty indices, assume some invariance condition, be that scale, translation or intermediate, which imposes value judgements on the measurement.

In the inequality and poverty fields, B. Zheng suggests rejecting these invariance conditions as axioms and proposes replacing them with the unit-consistency axiom (Economica 2007, Economic Theory 2007 and Social Choice and Welfare 2007). This property demands that the inequality or poverty rankings, rather than their cardinal values, are not altered when income is measured in different monetary units.

Following Zheng's proposal we explore the consequences of the unit-consistency axiom in the polarization field and propose and characterize a class of intermediate polarization orderings which is unit-consistent.

JEL Classification*:* D63
Keywords: Polarization Orderings, Unit-consistency

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1. Introduction

In recent years, the concern about to what extent a society is polarized has attracted a great deal of attention, since polarization has been proved to be closely linked to the generation of tension and social unrest. This interest has led several authors to try to define measures to capture the phenomenon. In independent studies, Foster and Wolfson (1992), Esteban and Ray (1994), and Wolfson (1994, 1997) have conceptualized the notion of polarization and developed the corresponding measures. After these seminal papers a number of studies have been devoted to the proposal of different polarization indices and the introduction of criteria to order distributions in terms of polarization and in most of these proposals some invariance condition, be that scale, translation or intermediate, is assumed[1].

Zheng (2007a, 2007b, 2007c) highlights the limitations that any of these invariance conditions impose on the inequality or poverty indices and argues that there is no justification for any of them being assumed to derive a measure. On the other hand, in empirical applications which take into account different countries or different periods of time, the units in which income is measured should not affect the ranking results. That is, it makes no sense that inequality or poverty orderings, and in general income distribution comparisons vary when the units in which income is measured change. Zheng introduces a new principle, the unit-consistency axiom, which requires that the inequality or poverty rankings, rather than the inequality or poverty levels, should not be affected by the units in which incomes are expressed[2]. In addition, Zheng (2007c) explores the consequences of the unit-consistency axiom in the inequality

[1] For instance: Davis and Huston (1992), Jenkins (1995), Milanovic (2000), Wang and Tsui (2000), Chakravarty and Majumder (2001), Kanbur and Zhang (2001), Duclos et al. (2004), Lasso de la Vega and Urrutia (2006), Chakavarty et al. (2007), Deutsch et al. (2007), Chakravarty and D’Ambrosio (2008).

[2] Diez et al. (2008) have generalized some of Zheng’s results to the multidimensional setting.

orderings, proposes an appealing and comprehensive notion of intermediateness for inequality orderings through the Lorenz curve and characterize the intermediate orderings which are unit-consistent. He shows that the Krtscha-type dominance (Krtscha (1994)) is the only Lorenz ordering which is both intermediate and unit-consistent. These contributions are the background taken to achieve our results.

More precisely, to start with, in Section 2 this paper proposes a straightforward extension of the unit-consistency axiom to the polarization indices, analyses the implications of this property on the polarization measures often used in the literature, and proposes a new family of intermediate polarization indices, based in the Krtscha-type notion, which is unit-consistent.

Section 3 is devoted to the polarization orderings. Since the polarization comparisons may be sensitive to the choice of the polarization measure, a standard procedure in order to avoid any conflict is to demand unanimous agreement among classes of polarization measures. In the inequality field, the Lorenz curve is usually used to test whether one distribution is unambiguously more unequal than another providing that the Pigou-Dalton transfer principle is accepted. The polarization curve proposed by Foster and Wolfson (1992) plays a similar role to the Lorenz curve and has already been used as a tool for ordering distributions in terms of bipolarization (for instance Chakavarty et al. (2007) and Chakravarty and D'Ambrosio (2008)). Taking these papers as a reference and following Zheng (2007c) we propose a general class of intermediate Foster-Wolfson orderings and show that the only unit-consistent members are those related to Krtscha-type intermediate notion.

The paper finishes with some concluding remarks. Most of the proofs of our paper follow both Zheng (2007c) and Chakravarty et al. (2007).

2. UNIT-CONSISTENT POLARIZATION INDICES.

2.1 Basic Notions about Polarization Indices.

We consider a population of $n \geq 2$ individuals. Individual i's income is denoted by $x_i \in \mathbb{R}_{++} = (0, \infty)$, i=1,...,n. An income distribution is represented by a vector $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n_{++}$. We let $D = \bigcup_{n=1}^{\infty} \mathbb{R}^n_{++}$ represent the set of all finite dimensional income distributions and denote the mean and the median of any $\mathbf{x} \in D$ by $\overline{\mathbf{x}}$ and $m(\mathbf{x})$ respectively.

In this paper we assume that a polarization measure is a function $P : D \to \mathbb{R}$ which fulfils the following four basic properties:

i) *Continuity:* $P(\mathbf{x})$ is a continuous function of **x**.

ii) *Symmetry:* $P(\mathbf{x}) = P(\Pi \mathbf{x})$ for any $\mathbf{x} \in D$ and for any permutation matrix Π.

iii) *Replication Invariance*: $P(\mathbf{y}) = P(\mathbf{x})$ if **y** is obtained from **x** by a replication.

iv) *Normalization:* $P(\mathbf{x}) = 0$ if all the incomes of the distribution **x** are identical.

The symmetry axiom allows us, without loss of generality, to assume that the income distribution **x** is ranked, that is, $x_1 \leq x_2 \leq ... \leq x_n$. In this case, if $\overline{n} = (n+1)/2$ we define $\mathbf{x}_+$ the sub-vector of $\mathbf{x}$ which contains x_i for $i > \overline{n}$ and $\mathbf{x}_-$ the sub-vector of $\mathbf{x}$ which contains x_i for $i < \overline{n}$. In turn the replication invariance principle permits comparisons of distributions of different sizes.

None of these properties are sufficient to guarantee that the function *P* be able to capture the essence of polarization. Two basic features are practically unquestionable for a polarization measure. Higher within-group homogeneity is bound to increase polarization. Likewise, heterogeneity across groups contributes to an increase in polarization. In the case of bipolarization measures, when the two poles in which population is split are defined according to the median income, these principles are usually known as *Non-Decreasing Spread* and *Non-Decreasing Bipolarity* axioms respectively, and are formulated in the following way:

v) *Non-decreasing Spread:* $P(\mathbf{x}) \leq P(\mathbf{y})$ whenever $\mathrm{m}(\mathbf{x}) = \mathrm{m}(\mathbf{y})$ and **y** is derived from **x** by an increase of the income of an individual above the median or/and by a reduction of the income of an individual below the median.

vi) *Non-decreasing Bipolarity:* $P(\mathbf{x}) \leq P(\mathbf{y})$ whenever $\mathrm{m}(\mathbf{x}) = \mathrm{m}(\mathbf{y})$ and **y** is derived from **x** by a progressive transfer from a richer person to a poorer one both on the same side of the median.

To derive polarization indices, invariance properties are often invoked. A *relative* polarization index remains unchanged with proportional changes in all incomes, whereas an *absolute* polarization index remains unchanged if a common amount is added for all the individuals. These two notions correspond to what in the inequality field are known as the rightists and leftist points of view according to Kolm's designation (1977). There exist other intermediate concepts of invariance which can be adopted to define polarization indices. Chakravarty and D'Ambrosio (2008) incorporate the notion of intermediateness proposed by Pfingsten (1986) which demands that any combination of an equal proportional increase in all incomes and an equal amount increase in all incomes should not change the polarization level. That is, polarization is not affected if **x** is changed by **z** according to the following transformation:

$$\frac{x_i - m(\mathbf{x})}{\mu m(\mathbf{x}) + 1 - \mu} = \frac{z_i - m(\mathbf{z})}{\mu m(\mathbf{z}) + 1 - \mu}$$

for all $i = 1, ..., n$ and for some constant $0 < \mu < 1$, or equivalently:

$$z_i = x_i + \lambda(\mu x_i + 1 - \mu).$$

The μ-parameter indicates the degree of intermediateness.

Assuming this notion of intermediateness Chakravarty and D'Ambrosio (2008) introduce two new classes of bipolarization intermediate measures:

$$F_\mu(\mathbf{x}) = \frac{2\left(\overline{\mathbf{x}}_+ - \overline{\mathbf{x}}_- - A_G(\mathbf{x})\right)}{\mu m(\mathbf{x}) + 1 - \mu} \tag{1}$$

where $A_G(\mathbf{x})$ is the absolute Gini coefficient of income distribution **x**; and

$$C_\mu(\mathbf{x}) = \frac{\left(n^{-1}\sum_{1\le i\le n} |x_i - m(\mathbf{x})|^r\right)^{\frac{1}{r}}}{\mu m(\mathbf{x}) + 1 - \mu}, \quad \text{with } 0 < r \le 1 \tag{2}$$

On the other hand, the intermediate inequality notion proposed by Krtscha (1994), and characterized and generalized by Yoshida (2005), may also be applied in the polarization field. In this case, distributions **x** and **z** are polarization equivalents if and only if:

$$\frac{x_i - m(\mathbf{x})}{\left(m(\mathbf{x})\right)^\lambda} = \frac{z_i - m(\mathbf{z})}{\left(m(\mathbf{z})\right)^\lambda}$$

for all $i = 1,...,n$ and for some constant $0 < \lambda < 1$ which can be regarded as the degree of intermediateness or, equivalently, if and only if **x** is transformed to **z** through:

$$z_i = \gamma^\lambda x_i + \left(\gamma - \gamma^\lambda\right) m(\mathbf{x})$$

for some constant $\gamma > 1$.

Analogously, it is not difficult to prove that the expressions below can be interpreted as classes of Krtscha-type intermediate bipolarization indices:

$$P_\lambda(\mathbf{x}) = \frac{2\left(\overline{\mathbf{x}}_+ - \overline{\mathbf{x}}_- - A_G(\mathbf{x})\right)}{\left(m(\mathbf{x})\right)^\lambda} \tag{3}$$

where $A_G(\mathbf{x})$ is again the absolute Gini coefficient of income distribution **x**; and

$$P_\lambda(\mathbf{x}) = \frac{\left(n^{-1}\sum_{1\le i\le n}\left|x_i - m(\mathbf{x})\right|^r\right)^{\frac{1}{r}}}{\left(m(\mathbf{x})\right)^\lambda} \quad \text{with } 0 < r \le 1 \tag{4}$$

Zoli (1998, 2008) has generalized the intermediate notion of both Pfingsten- and Krtscha-type. This two-parameter generalization in the polarization field can be established as follows: polarization should remain without changes if distribution **x** is transformed to **z** through:

$$\frac{x_i - m(\mathbf{x})}{\left(\mu m(\mathbf{x}) + 1 - \mu\right)^\lambda} = \frac{z_i - m(\mathbf{z})}{\left(\mu m(\mathbf{z}) + 1 - \mu\right)^\lambda}$$

for all $i = 1,...,n$ and for some constants μ and λ with $0 < \mu, \lambda < 1$.

Obviously this condition becomes Pfingsten's condition when $\lambda = 1$, and the generalization of Krtscha's condition proposed by Yoshida (2005) when $\mu = 1$.

Moreover from this condition it is not difficult to prove that the following class of intermediate polarization indices may be considered as a class of Zoli-type intermediate polarization indices:

$$F_{\lambda\mu}(\mathbf{x}) = \frac{2\left(\overline{\mathbf{x}}_+ - \overline{\mathbf{x}}_- - A_G(\mathbf{x})\right)}{\left(\mu m(\mathbf{x}) + 1 - \mu\right)^\lambda} \tag{5}$$

There exist in the literature other possible answers as to how to distribute a given amount of income among all the individuals without altering the inequality level which could be incorporated to the polarization field. However, that discussion is beyond the aim of this paper.

2.2 Unit-Consistency Axiom for Polarization Measures.

As already mentioned, Zheng (2007a, 2007b) has analysed in depth the implications of the invariance conditions usually assumed to define inequality and poverty measures and has proposed a new axiom of unit-consistency which requires that the inequality or poverty rankings between two distributions should not be affected by the unit in which income is expressed. This axiom has a straightforward generalization to the polarization field:

vii) *Unit-Consistency Axiom*: A polarization measure P is *unit-consistent* if for any two distributions $\mathbf{x}$ and $\mathbf{y} \in \mathrm{D}$ such that $P(\mathbf{x}) < P(\mathbf{y})$ then $P(\theta\mathbf{x}) < P(\theta\mathbf{y})$ for any $\theta > 0$.

It is clear that the scale invariance principle implies unit-consistency, and hence, every relative polarization measure is unit-consistent. Also the absolute index proposed by Chakravarty et al. (2007) is unit-consistent. In fact:

$$\mathrm{Q}(\theta\mathbf{x}) = 2\left(\overline{\theta\mathbf{x}}_{+} - \overline{\theta\mathbf{x}}_{-} - \mathrm{A}_{\mathrm{G}}(\theta\mathbf{x})\right) = 2\theta\left(\overline{\mathbf{x}}_{+} - \overline{\mathbf{x}}_{-} - \mathrm{A}_{\mathrm{G}}(\mathbf{x})\right) = \theta\mathrm{Q}(\mathbf{x})$$

The Krtscha-type bipolarization indices introduced in the previous section, equations (3) and (4), also fulfil the unit-consistency axiom:

$$\mathrm{P}_{\lambda}(\theta\mathbf{x}) = \frac{\theta}{\theta^{\lambda}} \frac{2\left(\overline{\mathbf{x}}_{+} - \overline{\mathbf{x}}_{-} - \mathrm{A}_{\mathrm{G}}(\mathbf{x})\right)}{\left(\mathrm{m}(\mathbf{x})\right)^{\lambda}} = \frac{\theta}{\theta^{\lambda}} \mathrm{P}_{\lambda}(\mathbf{x})$$

$$\mathrm{P}_{\lambda}(\theta\mathbf{x}) = \frac{\theta}{\theta^{\lambda}} \frac{\left(\mathrm{n}^{-1} \sum_{1 \le i \le n} \left|\mathrm{x}_{i} - \mathrm{m}(\mathbf{x})\right|^{r}\right)^{\frac{1}{r}}}{\left(\mathrm{m}(\mathbf{x})\right)^{\lambda}} = \frac{\theta}{\theta^{\lambda}} \mathrm{P}_{\lambda}(\mathbf{x})$$

However the two classes of intermediate bipolarization indices proposed by Chakravarty and D'Ambrosio (2008), equations (1) and (2), violate the unit-consistent axiom except for the two polar cases, that is, when $\mu = 1$, which coincides with the relative Wolfson polarization index and for $\mu = 0$, the absolute index proposed by Chakravarty et al. (2007). Consider the two distributions $\mathbf{x} = (1,2,3,4,5)$ and $\mathbf{y} = (0.1, 0.1, 0.6, 1, 2)$. Computing the index according to equation (1) with $\mu = 0.01$ we get $F_{0.01}(\mathbf{x}) = 4.3137 > F_{0.01}(\mathbf{y}) = 1.0321$ whereas multiplying **x** and **y** by $\theta = 1000$ we find $F_{0,01}(\theta\mathbf{x}) = 141.9812 < F_{0,01}(\theta\mathbf{y}) = 147.06$. With respect to the indices in equation (2), for the same values of the parameters with $r = 1$, we obtain $C_{0.01}(\mathbf{x}) = 0.980 > C_{0.01}(\mathbf{y}) = 0.5622$ and $C_{0.01}(\theta\mathbf{x}) = 32.268 < C_{0.01}(\theta\mathbf{y}) = 80.114$. In general examples can be found for the rest of the members of these two families.

Moreover, following Zheng (2007, pg 522) it is not difficult to prove that the Zoli-type polarization indices, apart from the members which hold Krtscha's condition, are not unit-consistent.

3. Unit-Consistent Polarization Orderings.

3.1 Basic Notions about Polarization Orderings.

In order to establish unanimous polarization rankings of income distributions, generalizations of the curve proposed by Foster and Wolfson (1992) have been introduced in the literature.

Foster and Wolfson (1992) define, for any ranked income distribution **x**, a relative polarization curve, which we refer to as FWC_R curve, which displays to what extent the current distribution is different from the hypothetical situation in which everybody enjoys the median income[3]. The ordinate corresponding to the k/n-percentage of population is computed according to the following expression:

$$\mathrm{FW}_R(\mathbf{x};k)=\begin{cases}\dfrac{1}{n}\sum\limits_{k\le i\le \bar{n}}\dfrac{\left(m(\mathbf{x})-x_i\right)}{m(\mathbf{x})} & \text{if} \quad 1\le k\le \bar{n}\\ \dfrac{1}{n}\sum\limits_{\bar{n}\le i\le k}\dfrac{\left(x_i-m(\mathbf{x})\right)}{m(\mathbf{x})} & \text{if} \quad \bar{n}\le k\le n\end{cases}$$

This FWC_R-curve allows the introduction of a dominance criterion as follows: for any two distributions $\mathbf{x}$ and $\mathbf{y}\in D$, $\mathbf{x}$ relative FW dominates $\mathbf{y}$, and we denote $\mathbf{x}\succsim_{FW_R}\mathbf{y}$, if and only if

$$\mathrm{FWC}_R(\mathbf{x};p)\ge \mathrm{FWC}_R(\mathbf{y};p)$$

for all $p\in[0,1]$ and the strict inequality holds at least once. This dominance criterion establishes an unambiguous ranking of income distributions among a class of polarization relative measures in the sense that all the indices in this family unanimously agree on their polarization ordering of a pair of distributions when their FWC_R-curves do not cross.

Chakravarty et al. (2007) introduce the absolute polarization curve, the FWC_A-curve, scaling up the FWC_R-curve by the median, according to the following expression:

$$\mathrm{FW}_A(\mathbf{x};k)=\begin{cases}\dfrac{1}{n}\sum\limits_{k\le i\le \bar{n}}\left(m(\mathbf{x})-x_i\right) & \text{if} \quad 1\le k\le \bar{n}\\ \dfrac{1}{n}\sum\limits_{\bar{n}\le i\le k}\left(x_i-m(\mathbf{x})\right) & \text{if} \quad \bar{n}\le k\le n\end{cases}$$

[3] In this paper, for the sake of simplicity and whenever it is not confusing we use the same notation for an income distribution and its ranked permutation fulfilling $x_1\le x_2\le \ldots \le x_n$.

They establish the corresponding dominance criterion: **x** absolute FW dominates **y**, and we denote $\mathbf{x} \succsim_{FW_A} \mathbf{y}$, if and only if

$$FWC_A(\mathbf{x};p) \geq FWC_A(\mathbf{y};p)$$

for all $p \in [0,1]$ and the strict inequality holds at least once. They also identify the class of bipolarization indices which agree on their ranking when their FWC_A-curves do not intersect.

Similarly to the Lorenz curve in the inequality framework, a family of intermediate polarization curves may be defined. If the dominance condition proposed by Zoli (1998, 2008) corresponding to the generalized intermediate notion of both Pfingsten- and Krtscha-type is taken into account, the ordinates of the polarization curve are the following:

$$FW_Z(\mathbf{x};\mu,\lambda,k) = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{k \leq i \leq \bar{n}} \dfrac{(m(\mathbf{x}) - x_i)}{(\mu m(\mathbf{x}) + 1 - \mu)^{\lambda}} & \text{if} \quad 1 \leq k \leq \bar{n} \\ \dfrac{1}{n}\displaystyle\sum_{\bar{n} \leq i \leq k} \dfrac{(x_i - m(\mathbf{x}))}{(\mu m(\mathbf{x}) + 1 - \mu)^{\lambda}} & \text{if} \quad \bar{n} \leq k \leq n \end{cases}$$

for some constant μ and λ ($0 < \mu$, $\lambda < 1$).

One of the extreme cases included, when $\lambda = 1$, coincides with the Pfingsten-type curve proposed by Chakravarty and D'Ambrossio (2008) and the other one, when $\mu = 1$, can be considered the Krtscha-type polarization curve. In addition, since the class of bipolarization indices in expression (5) corresponds to the area under the FWC_Z-curve for the different values of the λ and μ parameters, the indices in expressions (1) and (3) also coincide with the areas under the respective curves.

The dominance criterion related to this curve can be formulated as follows: for any two distributions $\mathbf{x}$ and $\mathbf{y} \in \mathrm{D}$, $\mathbf{x}$ Zoli(μ,λ)-type FW dominates $\mathbf{y}$, and we denote $\mathbf{x} \succsim_{\mathrm{FW_Z}} \mathbf{y}$, if and only if

$$\mathrm{FWC_Z}(\mathbf{x};\mu,\lambda,\mathrm{p}) \geq \mathrm{FWC_Z}(\mathbf{y};\mu,\lambda,\mathrm{p})$$

for all $\mathrm{p} \in [0,1]$ and the strict inequality holds at least once.

The particular case when $\mu = 1$ will be referred to in this paper as the Krtscha(λ)-type FW dominance.

Finally Zheng (2007c) proposes the most comprehensive notion of intermediateness[4]. Capturing the essence of what 'intermediateness' means in relation to extremes, and since the natural extremes for inequality orderings are the absolute and the relative Lorenz orderings, he stresses that "it is reasonable for the intermediate notion to be sensitive to and only to the changes in these bounds".

Using the relative and absolute FW polarization curves instead of the Lorenz ones, we take advantage of Zheng's axiom of intermediateness to propose the following one in the polarization field:

vii) *The Intermediateness Axiom*: For any $\mathbf{x},\mathbf{y} \in \mathrm{D}$, if $\mathbf{x}$ both relative and absolute FW dominates $\mathbf{y}$, then $\mathbf{x}$ should intermediate FW dominates $\mathbf{y}$. Specifically, for all $\mathrm{p} \in [0,1]$, if $\mathrm{FWC_R}(\mathbf{x};\mathrm{p}) \geq \mathrm{FWC_R}(\mathbf{y};\mathrm{p})$ and $\mathrm{FWC_A}(\mathbf{x};\mathrm{p}) \geq \mathrm{FWC_A}(\mathbf{y};\mathrm{p})$ then $\mathrm{FWC_I}(\mathbf{x};\mu,\mathrm{p}) \geq \mathrm{FWC_I}(\mathbf{y};\mu,\mathrm{p})$ for all $\mu \in (0,1)$, where μ is the parameter of intermediateness and $\mathrm{FWC_I}(\mathbf{x};\mu,\mathrm{p})$ denotes any intermediate FW curve.

[4] For a better understanding of what we apply to the polarization field summarizing from Zheng's paper, a thorough reading of Section 3 in Zheng (2007c) is highly recommended.

In addition $\mathrm{FWC_I}(\mathbf{x};\mu,p) > \mathrm{FWC_I}(\mathbf{y};\mu,p)$ if and only if at least one of the above inequalities is strict.

It is easy to prove that both Pfingsten- and Krtscha-type dominances satisfy this axiom.

The following proposition establishes the implication of the intermediateness axiom for a general intermediate polarization curve.

Proposition 3.1.1 *An intermediate polarization curve FWC_I satisfies the intermediateness axiom if and only if there exists a function G which is continuous and increasing in its first two arguments such that*

$$FWC_I(\mathbf{x};\mu,p) = G\left[FWC_R(\boldsymbol{x};p), FWC_A(\boldsymbol{x};k), \mu\right] \tag{7}$$

for all $p \in [0,1]$.

Proof: The proof is straightforward following that of Proposition 3.1 in Zheng (2007c).

Q.E.D

Since intuitively an 'intermediate' curve should lie between the relative and the absolute Lorenz curves, B. Zheng proposes a particular way to consider intermediate curves using the well-known *quasilinear-weighted-means*. In polarization terms this 'intermediate' curve, which we refer to as Zheng-type intermediate polarization curve may be defined as follows:

Definition 3.1.1 *For any distribution* $\mathbf{x} \in D$*, its Zheng-type intermediate polarization curve denoted, the* $\mathrm{FWC}_{\mathrm{Zh}}$ *curve, is defined as a quasilinear-weighted-mean of the relative and the absolute FW curves according to the following expression:*

$$\mathrm{FWC}_{\mathrm{Zh}}(\mathbf{x};\mathrm{p}) = \mathrm{f}\left[\mu \mathrm{f}^{-1}\left(\mathrm{FWC}_{\mathrm{R}}(\mathbf{x};\mathrm{p})\right) + (1-\mu)\mathrm{f}^{-1}\left(\mathrm{FWC}_{\mathrm{A}}(\mathbf{x};\mathrm{p})\right)\right]$$

where $\mu \in (0,1)$ *is the parameter of intermediateness and f is some continuous, invertible and strictly monotonic function.*

This curve allows us to introduce the corresponding dominance criterion: for any two distributions $\mathbf{x}$ and $\mathbf{y} \in \mathrm{D}$ we say that $\mathbf{x}$ Zheng-type FW dominates $\mathbf{y}$, and we denote $\mathbf{x} \succsim_{\mathrm{FW_{Zh}}} \mathbf{y}$, if and only if

$$\mathrm{FWC}_{\mathrm{Zh}}(\mathbf{x};\mathrm{p}) \geq \mathrm{FWC}_{\mathrm{Zh}}(\mathbf{y};\mathrm{p})$$

for all $\mathrm{p} \in [0,1]$ and the strict inequality holds at least once.

It is clear that the Zheng-type curve is a particular case of equation (6). In addition the quasilinear–weighted-means have been widely used both in the economics and in the mathematics literature, and by no means is their formulation as complicated as it seems at first sight. For instance, when $\mathrm{f}(\mathrm{t}) = \mathrm{t}$, we find that the ordinates of the Zheng-type curve are just a weighted arithmetic mean of the ordinates of the relative and absolute FW curves. The Zheng-type curve becomes the Krtscha-type one for $\mathrm{f}(\mathrm{t}) = \mathrm{e}^{\mathrm{t}}$ and the Pfingsten-type curve for $\mathrm{f}(\mathrm{t}) = 1/\mathrm{t}$.

Furthermore, they fulfil six properties which not only are reasonable for an intermediate polarization curve but also allow us to characterize (Aczél (1966)) the quasilinear–weighted-means. Now we are going to explore these six properties:

First of all, the two polar cases, that is, when $\mu = 0$ and $\mu = 1$ correspond with the absolute and the relative FW polarization curves, and for any other value the Zheng-type curve lies between these extremes. This property is known as *internality* according to Aczél's designation.

Second, if the relative and absolute FW curves coincide, then the Zheng-type curve is also the same: *reflexivity*.

By definition the Zheng-type curve is *homogeneous of 0 degree in the weights*.

In addition the curve is *increasing in both variables* which means that if either the relative or the absolute FW curve ordinates increase then, with the same level of intermediateness μ, the Zheng-type curve ordinates also increase.

The curve is *also increasing in the weight associated to the variable whose value is higher*. This means that when the relative curve lies above the absolute one, if the level of intermediateness increases, then the intermediate curve ordinates also increase. And the same happens when the absolute curve lies above the relative and the μ-parameter decreases.

The sixth requirement has to do with the consistency of the procedure followed in the construction of the curve. This property ensures that the construction of the intermediate curve can be carried out in several steps without changes in the final result. This condition and symmetry in the sense that the roles played by the relative and the absolute curves are interchangeable is the *bisymmetry* requirement.

If the above six conditions are considered as appealing requirements for a general intermediate curve FWC_I, the only possibility for FWC_I is to be a curve according to definition 3.1.1, that is, a Zheng-type curve.

Proposition 3.1.2 *Internality, reflexivity, homogeneity of 0 degree in the (2^{nd}) weight, increasing in the (2^{nd}) variable, increasing in the weight associated to the variable whose value is higher and bisymmetry on the function G (equation 6) are necessary and sufficient conditions for the intermediate polarization curve FWC_I to be Zheng-type polarization curve.*

Proof. It is straightforward from Azcél (1966, p.242). Q.E.D.

3.2 Unit-Consistency Axiom for Polarization Orderings.

For polarization orderings and dominance to make sense in order to carry out comparisons between countries and over time, they should not depend on the units in which income is measured. The unit-consistency axiom for polarization orderings is the following:

viii) *Unit-Consistency Axiom*: A polarization dominance criterion $\succsim_P$ is *unit-consistent* if for any two distributions $\mathbf{x}$ and $\mathbf{y} \in D$ such that $\mathbf{x} \succsim_P \mathbf{y}$ then $\theta\mathbf{x} \succsim_P \theta\mathbf{y}$ for any $\theta > 0$.

Analogously to the previous section as regards the polarization indices, it is clear also in this case that both the relative and the absolute FW dominance criteria are unit-consistent, as well as the Krtscha-type FW dominance.

The class of Pfingsten-type polarization orderings proposed by Chakravarty and D'Ambrosio (2008) violates the unit-consistent axiom. Consider the same distributions as in the previous example, that is, $\mathbf{x} = (1,2,3,4,5)$ and $\mathbf{y} = (0.1,0.1,0.6,1,2)$, and the same values of the parameters, i.e., $\mu = 0.01$ and $\theta = 1000$. The ordinates of the intermediate polarization curve for $\mathbf{x}$ are $(0.6, 0.2, 0, 0, 0.2, 0.6)$ and the corresponding ones for $\mathbf{y}$ are $(0.2, 0.1, 0, 0, 0.1, 0.4)$. Thus $\mathrm{FWC_Z}(\mathbf{x}; 0.01, 1, \mathrm{p}) \geq \mathrm{FWC_Z}(\mathbf{y}; 0.01, 1, \mathrm{p})$ for all $\mathrm{p} \in [0,1]$. In contrast, multiplying the two distributions by $\theta = 1000$ we find $(19.4, 6.5, 0, 0, 6.5, 19.4)$ and $(28.6, 14.3, 0, 0, 11.4, 51.5)$ respectively. Therefore $\mathrm{FWC_Z}(\theta\mathbf{x}; 0.01, 1, \mathrm{p}) \leq \mathrm{FWC_Z}(\theta\mathbf{y}; 0.01, 1, \mathrm{p})$

Similarly to the previous section, and again following Zheng (2007, pg 522), it is straightforward to prove that the Zoli-type orderings are not in general unit-consistent. Only the members which hold the Krtscha-type condition fulfil the unit-consistency axiom.

Finally, if we take into consideration the Zheng-type dominance criterion, we get the following result, similar to proposition 3.3 he proves in the inequality ordering field:

Proposition 3.2.1 *The Zheng-type intermediate polarization orderings are unit-consistent if and only if* $f(t) = \frac{1}{\beta} e^{t/\alpha}$ *for some constants* $\alpha \neq 0$ *and* $\beta > 0$.

Proof: The proof is straightforward following that of Proposition 3.3 in Zheng (2007c).

Q.E.D

Substituting $\mathrm{f}(\mathrm{t}) = \frac{1}{\beta} \mathrm{e}^{\mathrm{t}/\alpha}$ in Zheng-type polarization curves according to definition 3.1.1 leads to the Krtscha-type curves.

3.3 The Krtscha-type Unit-Consistent Bipolarization Orderings.

In this last section we seek the class of bipolarization indices which rank two given distributions according to the Krtscha-type FW dominance, the only intermediate orderings among the wide class of Zheng-type ones which are unit-consistent.

Proposition 3.3.1 *For any two distributions* $\mathbf{x}$ *and* $\mathbf{y} \in \mathrm{D}$, $\mathbf{x}$ *Krtschaλ-type FW dominates* $\mathbf{y}$ *if and only if* $\mathrm{P}(\mathbf{x}) \geq \mathrm{P}(\mathbf{y})$ *for all* $\mathrm{P} : \mathrm{D} \to \mathbb{R}$ *Krtschaλ-type bipolarization indices which fulfil the non-decreasing spread and the non-decreasing bipolarization axioms.*

Proof. The proof is straightforward following that of the similar theorems in Chakravarty et al. (2007) defining $z_i = \left(\frac{m(\mathbf{x})}{m(\mathbf{y})}\right)^{\lambda} y_i + \left(\frac{m(\mathbf{x})}{m(\mathbf{y})} - \left(\frac{m(\mathbf{x})}{m(\mathbf{y})}\right)^{\lambda}\right) m(\mathbf{y})$, when $m(\mathbf{x}) > m(\mathbf{y})$ and $t_i = \left(\frac{m(\mathbf{y})}{m(\mathbf{x})}\right)^{\lambda} x_i + \left(\frac{m(\mathbf{y})}{m(\mathbf{x})} - \left(\frac{m(\mathbf{y})}{m(\mathbf{x})}\right)^{\lambda}\right) m(\mathbf{x})$, when $m(\mathbf{x}) \leq m(\mathbf{y})$ and taking into account the index $P_{\lambda}(\mathbf{x}) = \frac{1}{n} \frac{\sum_{1 \leq i \leq n} |x_i - m(\mathbf{x})|}{(m(\mathbf{x}))^{\lambda}}$ for the sufficiency part.

Q.E.D

Concluding Remarks

If it is true that a good measure of how much you like a proposal is how much you try to imitate it, it should be obvious that we really appreciate the unit-consistency axiom proposed by B. Zheng. As he stresses: “Unit consistency, in the sense that the use of different measuring units should not lead to contradictory conclusions, is important to all scientific studies. Recognizing this importance, it is surprising that until recently the issue has not been appropriately addressed in the inequality measuring literature" (Zheng (2007c), pg 536). Replacing inequality with polarization this assertion holds. And it may be true that is our contribution in this paper: replacing 'inequality' with 'polarization'.

However, since polarization theory is developing, we think it is time to incorporate this principle. We have proved that for the moment the absolute and, of course, the relative indices proposed in the literature satisfy this axiom. If other value judgements are to be incorporated through intermediate measures, only the Krstcha-type ones are allowed in the sense that with other intermediateness notions income distribution comparisons, between across countries or over time, may change when the units in which income is measured vary.

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