Working Paper Series

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Ma Casilda Lasso de la Vega
Ana Urrutia
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ECINEQ WP 2008 – 99

ECINEQ 2008-99
November 2008

www.ecineq.org

Characterizing multidimensional inequality measures which fulfil the Pigou-Dalton bundle principle*

Ma Casilda Lasso de la Vega†
Ana Urrutia
Amaia de Sarachu
Department of Applied Economics,
University of the Basque Country, Bilbao

Abstract

In the unidimensional setting, the well known Pigou-Dalton transfer principle is the basic axiom to order distribution in terms of inequality. This axiom has a number of generalizations to the multidimensional approach which have been used to derive inequality measures. However, up to now, none of them has assumed the Pigou-Dalton bundle dominance criterion, introduced by Fleurbaey and Trannoy (2003). This principle captures the basic idea of the original Pigou-Dalton transfer principle, demanding that also in the multidimensional context "a transfer from a richer person to a poorer one decreases inequality". Assuming this criterion the aim of this paper is to characterize multidimensional inequality measures. For doing so, firstly we derive the canonical forms of multidimensional relative aggregative inequality measures which fulfil this property. Then we identify sub-families from a normative approach. The inequality measures we derive share their functional forms with other parameter families already characterized in the literature, the major difference being the restrictions upon the parameters. Nevertheless, we show that it is not necessary to give up any of the usual requirements to assume the Pigou-Dalton bundle criterion. Thus, in empirical applications it makes sense to choose measures that also fulfil this principle.

Keywords: Multidimensional inequality, Social welfare, Pigou-Dalton transfer principle.
JEL classification: D63.

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* We are very grateful to Professor John A. Weymark for his helpful comments and suggestions. A preliminary version of this paper was presented in the 9th International Meeting of the Society for Social Choice and Welfare (Montreal, 2008) and we wish to thank the participants in this conference for their comments. This research has been partially supported by the Spanish Ministerio de Educación y Ciencia under project SEJ2006-05455, cofunded by FEDER, by the Basque Departamento de Educación e Investigación under the project GIC07/146-IT-377-07, and by the University of the Basque Country under the project UPV05/117 1

† **Address of correspondence**: University of the Basque Country, Dep. Economía Aplicada IV, Av. Lehendakari Aguirre, 83, 48015-Bilbao (Spain). casilda.lassodelavega@ehu.es, amaia.sarachu@ehu.es, anamarta.urrutia@ehu.es

1 Introduction

This paper deals with the inequality measurement. Given different distributions, the concern of inequality measurement is essentially to establish when one distribution is more unequal than another, that is, to define criteria for ranking distributions. As is well-known, when only income is considered, the basic criterion for ordering distributions is the Pigou-Dalton transfer principle.[1] Nevertheless, in recent years there has been considerable agreement that inequality is a multidimensional problem and other attributes apart from income should also be taken into consideration (Kolm (1977), Atkinson and Bourguignon (1982), Maasoumi (1986, 1999), Sen (1992), Tsui (1995, 1999), Savaglio (2006), Weymark (2006)).

The straightforward generalization of the Pigou-Dalton transfer principle for one attribute to any number of attributes may be established as follows: a transfer from a richer person to a poorer one preserving the order diminishes the inequality. Fleurbaey and Trannoy (2003) have formalized this extension as the *Pigou-Dalton bundle dominance*,[2] henceforth PDB. However this is not the only generalization of the Pigou-Dalton transfer principle. Alternative formulations of this principle (Hardy, Littlewood and Pólya (1934, 1952)), have been used to propose generalizations in the multivariate framework, and among them the Uniform Majorization criterion -UM from now on- proposed by Kolm (1977) is one of the most widely used.[3]

Since the rankings obtained by these dominance criteria are not complete, in order to compare any pair of distributions, inequality indices are derived. There already exist in the literature multidimensional inequality indices fulfilling UM (Tsui (1995, 1999),

[1] Recently this principle is being reconsidered in order to tackle the real sense of inequality (Kolm (1999), Chateauneuf and Moyes (2005) for instance).
[2] Specifically, they analyze how this multidimensional version of the Pigou-Dalton transfer principle clashes with the Pareto principle in social welfare dominance in a heterogeneous society. This principle is formally defined in Section 2.2.
[3] This principle is defined in Section 2.2.

Bourguignon (1999), List (1999)). Nevertheless, Diez et al. (2007) show that in general these indices fail to rank any pair of multidimensional distributions when one is derived from another by a transfer of part of one attribute from a richer person to a poorer one.[4] As an illustration consider a society of three individuals each endowed with two attributes. The bundle for each individual is $(9,10)$, $(4,8)$ and $(10,4)$ respectively. Notice that since individual one has more of the two attributes than individual two, individual one is richer. Let's assume that individual one transfers 2 units of the first attribute to individual two and the new bundles are now $(7,10)$, $(6,8)$ and $(10,4)$ respectively. As regards the original idea behind the Pigou-Dalton transfer principle the latter society represents less inequality than the original one. However these two distributions can not be compared using UM, since UM orders pairs of distributions only when one is obtained from another by transferring all the attributes in the same proportions and this is not the case. Moreover, many of the mentioned indices fulfilling UM establish that the second distribution is more unequal than the first one.[5] Apart from this, other difficulties arise with UM. Firstly, the reasons for transferring all the attributes in the same proportions are not clear. Secondly, not all the attributes can be considered as transferable. In fact the idea of a transfer is not necessarily meaningful and desirable for all the attributes, for instance for educational level or health status. Finally, if the transfers of all the attributes are made between any two people not necessarily one richer than the other, for instance individuals two and three in the example, the motivations for the new distribution being considered more equal are not evident.

[4] In fact the paper shows the relationships between PDB and other dominance criteria that exist in the literature.
[5] This issue is shown in Diez et al. (2007). They check some of the most important multidimensional inequality measures in order to show whether or not PDB is satisfied.

By contrast PDB gets over the difficulties mentioned above. First of all, transfers only take place between two people, one unambiguously richer than the other. Second, it is not necessary to transfer all the attributes in the same proportions, and finally the attributes considered as transferable can be selected. Although this appealing dominance criterion seems to lead to one of the coarsest inequality orderings not all the inequality indices are consistent with it. In this paper, we characterize classes of relative aggregative multidimensional inequality measures which are PDB consistent.

To derive inequality indices two different approaches have generally been used in the literature. The first one considers a suitable set of axioms and derives indices fulfilling these axioms, without explicitly specifying the underlying social evaluation functions. The second approach characterizes social evaluation functions satisfying certain dominance criteria and these functions are used to derive the indices (Kolm (1969), Atkinson (1970)).

Following the first of these procedures, in Section 2, we characterize multidimensional inequality indices that fulfil PDB. A similar exercise is carried out by Tsui (1999) assuming UM instead of PDB .

In a previous paper, Tsui (1995) proposes ethical multidimensional inequality indices consistent with UM, that is, indices obtained from social evaluation functions which fulfil UM. In Section 3, taking this work as a reference and demanding PDB instead of UM, we obtain relative inequality measures which are a multidimensional generalization of the Atkinson-Kolm-Sen indices.

One significant difference between the families derived in this paper and those derived by Tsui, which can be of interest in empirical applications, is that the restrictions upon the parameter values in our families are far less complicated.

Should anyone consider acceptable the standard multidimensional generalization of the Pigou-Dalton transfer principle, UM, and PDB, it would be interesting to have measures fulfilling both. In this respect one interesting result is that fortunately there exist measures shared by the classes derived in this paper and those derived by Tsui, in such a way that UM and PDB consistent measures can be chosen.

The rest of the paper is structured as follows. The first two sections below are devoted to the derivation of inequality measures from the two mentioned approaches respectively. Each of these sections begins the notation and basic definitions, then PDB is introduced in each field, our characterization results are presented and some conclusions are provided. The paper finishes with some conclusions Most of the proofs of our paper follow both Tsui's (1995, 1999) papers and the relevant results by Shorrocks (1984) as well.

2 Multidimensional inequality measures which fulfil PDB

2.1 Notations and basic axioms of multidimensional inequality measures

We consider a population consisting of $\mathrm{n} \geq 2$ individuals endowed with a bundle of $\mathrm{k} \geq 2$ attributes, such as income, health, education and so on. An $n \times k$ real matrix X represents a multidimensional distribution among the population. The *ij*th entry of *X*, denoted x_{ij}, represents the *i*th individual's amount of the *j*th attribute. The *i*th row is denoted $\underline{x}_i$. For each attribute *j*, $\mu_j(X)$ represents the mean value of the *j*th attribute and $\underline{\mu}(X) = (\mu_1(X), \ldots, \mu_k(X))$ is the vector of the means of the attributes.

We denote $M(\mathrm{n,k})$ the class of $n \times k$ real matrices over the positive real elements and *D* the set of all such matrices, that is, $D = \bigcup_{\mathrm{n} \in \mathbb{N}_+} \bigcup_{\mathrm{k} \in \mathbb{N}_+} M(\mathrm{n,k})$.

Comparisons of the bundles of attributes are denoted as follows: $\underline{x}_q \geq \underline{x}_p \ if \ x_{qj} \geq x_{pj}$ for all $j = 1,...,k$, $\underline{x}_q > \underline{x}_p \ if \ x_{qj} \geq x_{pj}$ and $\underline{x}_p \neq \underline{x}_q$.

In this paper a multidimensional inequality measure is a function $I : D \to \mathbb{R}$ satisfying the following four properties:

* *Continuity*: I is a continuous function in any individual's attribute.
* *Anonymity*: $I(X) = I(\Pi X)$ for any $X \in D$ and for any $n \times n$ permutation matrix Π.
* *Normalization*: $I(X) = 0$ if all the rows of the matrix X are identical, i.e., all the individuals have exactly the same bundle of attributes.
* *Replication Invariance*: $I(Y) = I(X)$ if Y is obtained from X by a replication.

As regards invariance properties the following is used:

* *Scale Invariance Principle, SI*: I satisfies SI if $I(X) = I(XC)$ for all $X \in D$, where $C = diag(c_1,...,c_k)$, $c_j > 0 \quad j = 1,2,...,k$.

Relative inequality indices are those that are scale invariant.

If the population in which we want to measure inequality is split into groups the aggregative principle allows us to relate inequality in each group to overall inequality:[6]

* *Aggregative Principle*: I is aggregative if there exists a function A such that $I\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) = A\left(I(X_1), \underline{\mu}(X_1), \mathrm{n}_1, I(X_2), \underline{\mu}(X_2), \mathrm{n}_2\right)$ for all $X_1, X_2 \in D$ and A is a continuous and strictly increasing function in the index values $I(X_1)$ and $I(X_2)$.

2.2 The Pigou-Dalton transfer principle for multidimensional inequality measures

[6] In the unidimensional framework this property is proposed by Shorrocks (1984). It is generalized to the multidimensional framework by Tsui (1999).

None of the properties above are sufficient to ensure that the inequality measure be able to capture the essence of multidimensional inequality. For doing so, multidimensional generalizations of the Pigou-Dalton transfer principle are usually used. One generalization proposed by Kolm (1977) widely used is the following:

* *Uniform Majorization principle, UM:* I satisfies UM if $I(Y) < I(X)$ for any $X, Y \in D$ such that $Y = BX$ for some $n \times n$ bistochastic matrix B that is not a permutation matrix.

As already mentioned, given that the effect of transforming all the attributes through the same bistochastic matrix is that the individuals become closer in the attributes space by transferring in the same proportions of all the attributes, some difficulties arise with this principle. Firstly the reasons for transferring the same proportions of all the attributes, that is, using the same bistochastic matrix, are not evident. In addition, this criterion warrants transfers of different directions for different attributes, and is not limited to cases when one individual is richer than another, being not obvious that these transfers are inequality reducing.

These drawbacks are got over by PDB proposed by Fleurbeay and Trannoy (2003), which extends the proper idea behind the Pigou-Dalton transfer principle, that is, a transfer from a richer individual to a poorer one, which preserves the order, diminishes the inequality. This principle may be established as follows:

Definition: Let $X, Y \in D$. Distribution Y is derived from X by a *PDB transfer* if there exist two individuals p, q such that:

i) $\underline{x}_q > \underline{x}_p$

ii) $\underline{y}_m = \underline{x}_m \quad \forall m \neq p, q$

iii) $\underline{y}_q = \underline{x}_q - \underline{\delta}$ and $\underline{y}_p = \underline{x}_p + \underline{\delta}$ where $\underline{\delta} = (\delta_1, ..., \delta_k) \in \mathbb{R}^k_+$ with at least one $\delta_j > 0$

iv) $\underline{y}_q \geq \underline{y}_p$

The first condition implies that, in the initial distribution, individual q is richer than individual p in all the attributes, whereas the fourth requirement is that this ranking is preserved by the transfer. The corresponding notion for the inequality measures is the following:

* *Pigou-Dalton bundle principle, PDB: I* satisfies PDB if $I(Y) < I(X)$ for any $X \in D$ and for all Y matrices derived from X by a finite sequence of PDB transfers of attributes between individuals.

If the Pigou-Dalton transfer principle has been accepted in the univariate case, then everyone will agree that in the multidimensional case PDB should also be satisfied.

2.3 Multidimensional inequality measures which fulfil PDB

In this section we characterize classes of relative aggregative multidimensional inequality which fulfil PDB.

The following result is similar to that established by Tsui (1999) in his theorem 3. The only difference is that demanding PDB instead of UM leads to simpler restrictions upon the parameter families.

Proposition 1: *A relative aggregative inequality measure $I: D \to \mathbb{R}$ satisfies PDB if and only if there exists a continuous increasing function $F: \mathbb{R} \to \mathbb{R}_+$, with $F(0) = 0$, such that either:*

$$F(I(X)) = \frac{1}{\mathrm{n}} \sum_{1 \leq i \leq n} \left[\prod_{1 \leq j \leq k} \left(x_{ij} \big/ \mu_j \right)^{\alpha_j} - 1 \right] \qquad [1]$$

where either $\alpha_j > 1$ for all j or $\alpha_j < 0$ for all j, *or* $$F\left(I\left(X\right)\right)=\frac{1}{\mathrm{n}}\sum_{1\le i\le n}\left[\sum_{1\le j\le k}\beta_j \log\left(\mu_j \big/ x_{ij}\right)\right] \qquad [2]$$

where $\beta_j > 0$ *for all j.*

Proof: In the appendix.

2.4 When correlation increasing transfers principle is assumed

Atkinson and Bourguignon (1982) point out that multidimensional inequality must be sensitive to the correlation between distributions of different attributes, and Tsui (1999) introduces the following majorization criterion, the correlation increasing principle.

Definition: Let $X, Y \in D$. Distribution Y may be derived from distribution X by a *Correlation Increasing transfer* if there exist two individuals p and q such that: i) $\underline{y}_p = \left(\min\left\{x_{p1}, x_{q1}\right\},...,\min\left\{x_{pk}, x_{qk}\right\}\right)$, ii) $\underline{y}_q = \left(\max\left\{x_{p1}, x_{q1}\right\},...,\max\left\{x_{pk}, x_{qk}\right\}\right)$ and iii) $\underline{y}_m = \underline{x}_m \;\; \forall m \neq p, q$. A correlation increasing transfer is *strict* whenever $\underline{x}_p \neq \underline{y}_p$.

* *Correlation Increasing principle, CIM*: A multidimensional inequality measure I satisfies CIM if $I\left(X\right) < I\left(Y\right)$ for any $X \in D$ and for all Y matrices derived from X by a permutation of rows and a finite sequence of correlation increasing transfers, at least one of which is strict.

A weak version of this principle will also play a role in this paper. As long as we replace a strict inequality sign with the inequality sign in the definition the weak Correlation Increasing principle, WCIM, is obtained.

All relative aggregative inequality measures fulfilling PDB satisfy some version of CIM, that is, either CIM or WCIM. This is not the case for the measures that fulfil UM.

Proposition 2: *If a multidimensional inequality measure* I *satisfies the Aggregative principle, PDB and SI then* I *satisfies WCIM.*

Proof: In the appendix.

2.5 Some remarks about the family derived in proposition 1

In proposition 1 we obtain the canonical forms of all relative aggregative multidimensional measures that satisfy PDB which can be considered as a multidimensional generalization of the Generalized Entropy family. Following Tsui (1999) and assuming PDB instead of UM we obtain multidimensional relative inequality indices consistent with PDB. The functional forms being the same, the only difference with respect to Tsui's family is the range of the parameter values, which in our case is less complicated to compute.

Moreover it can be seen that there exist aggregative measures in our family that do not belong to the family derived by Tsui. For instance, if we consider the case of two attributes, when $\alpha_1 > 1$, and $\alpha_2 > 1$ in equation [1] the corresponding measures fulfil PDB and not UM. And vice versa, there exist measures in the family derived by Tsui that do not belong to our family. For instance, when $\alpha_1 + \alpha_2 > 1$, and either $\alpha_1 < 0$ or $\alpha_2 < 0$ in Tsui (1999 equation [4a]) the measures fulfil UM and not PDB. Fortunately there exist measures belonging to both families, it suffices to take $\alpha_1 < 0$ and $\alpha_2 < 0$.

Finally the implications of CIM have also been also considered.

If we restrict the aggregative inequality measures that fulfil UM (theorem3 in Tsui(1999)) to be those that also satisfy WCIM, it can be proved that the obtained subfamily satisfies PDB as well. This subfamily corresponds to measures in proposition 1 according to equation [1] with $\alpha_j < 0$ for all j, and to equation [2] with $\beta_j > 0$ for all j, and we are going to show in the following section that his members are transformations of measures with separable underlying social evaluation functions.

3 Multidimensional inequality measures which fulfil PDB derived from social evaluation functions

The inequality measures in the above section are obtained without taking into consideration the underlying social evaluation functions. The main goal of this section is to provide characterizations of classes of social evaluation functions that satisfy PDB and to derive their corresponding multidimensional inequality indices.

First we have to add some basic axioms.

3.1 Basic axioms of multidimensional social evaluation functions

In the following we assume that a multidimensional social evaluation function is a function $W: D \to \mathbb{R}$ that possesses the following four properties:

* *Continuity*: W is a continuous function in any individual's attributes.
* *Pareto principle*: W is strictly increasing in the elements of X.
* *Anonymity*: $W(X) = W(\Pi X)$ for any $X \in D$ and for any $n \times n$ permutation matrix Π.
* *Homothetic principle, HP*: W is homothetic if for any two distributions $X, Y \in D$ such that $W(X) = W(Y)$ then $W(XC) = W(YC)$ for any $C = diag(c_1, ..., c_k)$, $c_j > 0, \quad j = 1, ..., k$.

With respect to the multidimensional generalization of the Pigou-Dalton transfer principle used in this paper, the following axioms are used.

* *Uniform Majorization principle, UM: W* satisfies UM if $W(Y) > W(X)$ for any $X, Y \in D$ such that $Y = BX$ for some $n \times n$ bistochastic matrix B that is not a permutation matrix.

* *Pigou-Dalton bundle principle, PDB: W* satisfies PDB if $W(Y) > W(X)$ for any $X \in D$ and for all Y matrices derived from X by a finite sequence of PDB transfers of attributes between individuals.

A Counterpart of the *WCIM* principle proposed in the previous section can be assumed for a welfare function.

* *Weak Correlation Increasing principle, WCIM: W* satisfies WCIM if $W(Y) \leq W(X)$ for any $X \in D$ and for all Y matrices derived from X by a permutation of rows and a finite sequence of correlation increasing transfers.

Tsui (1995) introduces the following separability axiom:

* *Separability: W* is separable if for all the subsets of individuals $S \subset \{1, 2, ..., n\}$, such that: $W(X) = W\left(\psi\left(X^S\right), X^C\right)$ where ψ is some continuous function, X^S is the submatrix of *X* including the vector of attributes of the individuals in *S,* and X^C is the complement of X^S.

When $n \geq 3$ Tsui (1995) proves that this axiom guarantees that *W* is ordinally equivalent to a utilitarian social welfare function additively separable in an homogeneous society, that is, W is ordinally equivalent to $\sum_{1 \leq i \leq n} U\left(\underline{x}_i\right)$, where $U : \mathbb{R}^k_{++} \to \mathbb{R}$ is an increasing function. Then additive separability is implicitly assumed with this axiom.

3.2 Multidimensional inequality measures which fulfil PDB derived from social evaluation functions

We focus on the derivation of relative inequality measures following the approach introduced by Kolm (1977) to derive the multidimensional generalization of the

Atkinson-Kolm-Sen inequality indices.[7] Kolm suggests the following multidimensional relative inequality index:

$$I_R(X) = 1 - \delta(X) \qquad [3]$$

where $\delta(X)$ is such that $W(X) = W(\delta(X) X_\mu)$ and the *i*th row of X_μ is equal to $\underline{\mu}(X)$ for all *i*.

The inequality of a multidimensional distribution as measured by this index can be interpreted as the fraction of the amount of each attribute that could be discarded if every attribute were equally redistributed and the resulting distribution were indifferent to the original distribution according to the social evaluation function *W*.

The main result of this section, the multidimensional generalization of the Atkinson inequality indices that satisfy PDB, is presented in the following proposition.

Proposition 3: *Suppose that* $n \geq 3$. *A multidimensional social evaluation function* $W : D \to \mathbb{R}$ *satisfies Separability and PDB if and only if* W *is ordinally equivalent to* $\sum_{1\leq i\leq n} U(\underline{x}_i)$ *where* $U : \mathbb{R}^k_{++} \to \mathbb{R}$ *is a function such that either:*

$$U(\underline{x}_i) = a + b\prod_{1\leq j\leq k} x_{ij}^{\alpha_j} \qquad [4]$$

or

$$U(\underline{x}_i) = a + \sum_{1\leq j\leq k} \beta_j \log x_{ij} \qquad [5]$$

where the parameter a is an arbitrary constant, $b < 0$, $\alpha_j < 0$ *and* $\beta_j > 0$ *for all j.*

The corresponding inequality index is relative, aggregative, satisfies PDB, and has the forms

$$I(X) = 1 - \left[\left(\tfrac{1}{n}\right)\sum_{1\leq i\leq n}\prod_{1\leq j\leq k}\left(\tfrac{x_{ij}}{\mu_j}\right)^{\alpha_j}\right]^{1/\sum_{1\leq j\leq k}\alpha_j} \qquad [6]$$

[7] Tsui (1995) using this approach characterizes the multidimensional inequality indices whose related social evaluation functions, *W*, are separable, homothetic and strictly quasi-concave. He also shows that this property with the anonymity is enough to guarantee that *W* satisfies UM.

or $$I(X) = 1 - \prod_{1\le i\le n}\left[\prod_{1\le j\le k}\left(x_{ij}/\mu_j\right)^{\beta_j/\sum_{1\le j\le k}\beta_j}\right]^{1/n} \quad [7]$$

Proof: In the appendix.

The inequality measures derived in this proposition can be interpreted as a generalization of the Atkinson inequality indices. Since they are aggregative, they are monotonically related to a subfamily of the class obtained in proposition 1. As mentioned in the previous section, this subfamily not only fulfils PDB but also UM and WCIM. In other words, from a normative point of view, assuming PDB is equivalent to requiring UM and WCIM for a relative aggregative measure.

4 Conclusions

This work sheds light on the classes of multidimensional aggregative inequality indices that satisfy PDB. In section 2 we characterize the relative aggregative inequality measures, and in Section 3 we investigate the aggregative inequality measures derived from separable social evaluation functions which fulfil PDB.

We have only focused on relative indices. A similar exercise is also possible invoking the translation invariance principle to derive absolute inequality indices.

Although recently the Pigou-Dalton transfer principle is being reconsidered, it is the corner stone of the inequality measurement theory. In this sense we hope that this paper provides a greater understanding of this concept in the multidimensional framework.

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Appendix

The following result by Tsui (1999), that is a generalization to the multidimensional setting of one result in Shorrocks (1984), is used in the next two lemmas which give some clues for the proof of our characterizations.

Tsui ((1999), Lemma 1): A *multidimensional inequality measure* $I : D \to \mathbb{R}$ *satisfies the aggregative principle if and only if there exist continuous functions* ϕ *and F such that, for every* $X \in D$ *with mean vector* $\underline{\mu}(X) = (\mu_1(X), ..., \mu_k(X))$,

$$F(I(X), \underline{\mu}) = \left(1/n\right) \sum_{1 \le i \le n} \left(\phi(\underline{x}_i) - \phi(\underline{\mu})\right) \qquad [8]$$

where F is strictly increasing in $I(X)$ *and* $F(0, \underline{\mu}) = 0$.

Lemma 1: *If a multidimensional inequality measure* $I : D \to \mathbb{R}$ *satisfies the aggregative principle and PDB then equation* [8] *holds with* $\phi : \mathbb{R}^k \to \mathbb{R}$ *strictly convex in each component.*

Proof: Suppose that *I* satisfies the aggregative principle and PDB. By the lemma above equation [8] holds. It suffices to prove that if PDB is also satisfied then $\phi : \mathbb{R}^k \to \mathbb{R}$ is strictly convex in each component. That is

$$\phi\left(x_1, \ldots, \frac{x_j + y_j}{2}, \ldots, x_k\right) < \frac{1}{2}\phi(x_1, \ldots, x_j, \ldots, x_k) + \frac{1}{2}\phi(x_1, \ldots, y_j, \ldots, x_k) \text{ for all } j = 1, ..., k .$$

Let us consider the distribution matrices *X* and *Y* which represent two person societies with *k* attributes:

$$X = \begin{pmatrix} x_1 & ... & x_j & ... & x_k \\ x_1 & ... & y_j & ... & x_k \end{pmatrix} \text{ with } x_j > y_j \text{ and } Y = \begin{pmatrix} x_1 & ... & \dfrac{x_j + y_j}{2} & ... & x_k \\ x_1 & ... & \dfrac{x_j + y_j}{2} & ... & x_k \end{pmatrix}$$

Then according to PDB $I(Y) < I(X)$. Consequently for the increasing transformation F, we have that $F\left(I(Y), \underline{\mu}\right) < F\left(I(X), \underline{\mu}\right)$ and from the specific forms of the distribution *X* and *Y* and equation [8] we get the result. *Q.E.D.*

Lemma 2: *Let $I : D \rightarrow \mathbb{R}$ be a multidimensional inequality measure which satisfies the aggregative principle.*

I satisfies PDB if and only if for all Y distributions derived from any distribution X by a PDB transfer of an attribute l, δ_l, from individual q, the richer, to individual p, the poorer, there exists a continuous function ϕ such that the following expression holds:

$$\phi\left(x_{q1},\ldots,x_{ql}-\delta_l,\ldots,x_{qk}\right)+\phi\left(x_{p1},\ldots,x_{pl}+\delta_l,\ldots,x_{pk}\right)<\phi\left(x_{q1},\ldots,x_{ql},\ldots,x_{qk}\right)+\phi\left(x_{p1},\ldots,x_{pl},\ldots,x_{pk}\right) \quad [9]$$

Proof: In view of Tsui ((1999), lemma 1), if *I* satisfies the aggregative principle equation [8] holds.

Let *Y* be a distribution matrix derived from any distribution *X* by a PDB transfer of any attribute *l*, δ_l, from a richer individual *q* to a poorer individual *p* where $x_{ql} > \left(x_{ql}-\delta_l\right) \geq \left(x_{pl}+\delta_l\right) > x_{pl}$. According to PDB, *I* satisfies PDB if and only if for any increasing transformation *F*, $F\left(I(Y), \underline{\mu}\right) < F\left(I(X), \underline{\mu}\right)$. From equation [8] and taking into account the specific form of distributions *X* and *Y*, it follows that *I* satisfies PDB if and only if $\left(1/n\right)\left(\phi\left(\underline{y}_q\right)+\phi\left(\underline{y}_p\right)\right) < \left(1/n\right)\left(\phi\left(\underline{x}_q\right)+\phi\left(\underline{x}_p\right)\right)$. Operating we get the result. *Q.E.D.*

Proof of the Proposition 1*:* If the aggregative principle is satisfied equation [8] holds. Tsui ((1999), theorem 3) also proves that adding SI to equation [8] inexorably leads to functional forms [1], [2] and to the following:

$$F\left(I\left(X\right)\right)=\left(\frac{1}{n}\right)\sum_{1\le i\le n}\left(\frac{x_{ih}}{\mu_h}\right)\left[\sum_{1\le j\le k}a_{hj}\log\left(\frac{x_{ij}}{\mu_j}\right)\right] \qquad [10]$$

In order to establish the implications of PDB for the parameter values in these functional forms, we analyse each one separately.

Let Y be a distribution matrix derived from a distribution X by a PDB transfer of an attribute l, δ_l, from individual q to individual p where

$x_{ql}>\left(x_{ql}-\delta_l\right)\ge\left(x_{pl}+\delta_l\right)>x_{pl}$.

i) We start with the first of these functional forms, equation [1]. For this functional form, equation [8] holds with $\phi\left(\underline{x}_i\right)=\rho\prod_{1\le j\le k}\left(x_{ij}\right)^{\alpha_j}$. Moreover from lemma 1 a necessary condition for I to hold PDB is function ϕ be strictly convex in each component, so it should be $\rho\alpha_j\left(\alpha_j-1\right)>0$ for all j, that is, either $\rho>0$ and $\alpha_j<0$, or $\rho>0$ and $\alpha_{\mathrm{j}}>1$, or $\rho<0$ and $0<\alpha_j<1$.

From lemma 2, and rewriting expression [9] for this functional form, we get that PDB is satisfied if the following inequality holds

$$\rho\left(x_{ql}-\delta_l\right)^{\alpha_l}\prod_{\substack{1\le j\le k\\ j\ne l}}\left(x_{qj}\right)^{\alpha_j}+\rho\left(x_{pl}+\delta_l\right)^{\alpha_l}\prod_{\substack{1\le j\le k\\ j\ne l}}\left(x_{pj}\right)^{\alpha_j}<\rho\left(x_{ql}\right)^{\alpha_l}\prod_{\substack{1\le j\le k\\ j\ne l}}\left(x_{qj}\right)^{\alpha_j}+\rho\left(x_{pl}\right)^{\alpha_l}\prod_{\substack{1\le j\le k\\ j\ne l}}\left(x_{pj}\right)^{\alpha_j} \qquad [11]$$

Now we analyse separately the different cases which fulfil the necessary condition.

- $\rho>0$, $\alpha_j<0$ for all *j*.

Since $x_{pl}<x_{pl}+\delta_l\le x_{ql}-\delta_l<x_{ql}$ and the fact that in this case the function x^{α_j} is decreasing and convex we get

$$\left(\prod_{\substack{1\le j\le k\\ j\ne l}}\left(x_{qj}\right)^{\alpha_j}\right)\left[\left(x_{ql}-\delta_l\right)^{\alpha_l}-\left(x_{ql}\right)^{\alpha_l}\right]<\left(\prod_{\substack{1\le j\le k\\ j\ne l}}\left(x_{qj}\right)^{\alpha_j}\right)\left[\left(x_{pl}\right)^{\alpha_l}-\left(x_{pl}+\delta_l\right)^{\alpha_l}\right] \qquad [12]$$

and given that in this case $0<\prod_{\substack{1\le j\le k\\ j\ne l}}\left(x_{qj}\right)^{\alpha_j}\le\prod_{\substack{1\le j\le k\\ j\ne l}}\left(x_{pj}\right)^{\alpha_j}$ it follows that

$$\left(\prod_{\substack{1\le j\le k\\ j\ne l}}\left(x_{qj}\right)^{\alpha_j}\right)\left[\left(x_{ql}-\delta_l\right)^{\alpha_l}-\left(x_{ql}\right)^{\alpha_l}\right]<\left(\prod_{\substack{1\le j\le k\\ j\ne l}}\left(x_{pj}\right)^{\alpha_j}\right)\left[\left(x_{pl}\right)^{\alpha_l}-\left(x_{pl}+\delta_l\right)^{\alpha_l}\right]$$

Operating, rearranging and multiplying by $\rho>0$ we get the expression [11], so PDB is satisfied.

- $\rho>0$, $\alpha_j>1$ for all *j*.

For this case the proof is similar to the previous one taking into consideration that for these parameter values the function x^{α_j} is increasing and convex.

- $\rho<0$, $0<\alpha_j<1$ for all *j*.

We are going to show that in this case PDB is not satisfied in general.

Given x_{ql}, x_{pl}, δ_l, and x_{ph}, since the function x^{α_j} is now increasing, we can always choose x_{qh} big enough such that $\left(\frac{x_{qh}}{x_{ph}}\right)^{\alpha_h}>\left(\frac{\left(x_{pl}+\delta_l\right)^{\alpha_l}-x_{pl}^{\alpha_l}}{x_{ql}^{\alpha_l}-\left(x_{ql}-\delta_l\right)^{\alpha_l}}\right)$ then

$$\left(x_{qh}\right)^{\alpha_h}\left(x_{ql}^{\alpha_l}-\left(x_{ql}-\delta_l\right)^{\alpha_l}\right)>\left(x_{ph}\right)^{\alpha_h}\left(\left(x_{pl}+\delta_l\right)^{\alpha_l}-x_{pl}^{\alpha_l}\right)$$

Assuming that $x_{qj}=x_{pj}\ \forall j\ne l,h$ and operating, we find that expression [11] is verified in the opposite sense.

- $\rho>0\ \ \exists l,h\in\{1,\ldots,k\}/\alpha_h<0,\alpha_l>1$

Since x^{α_h} is decreasing we can get the same conclusion as in the previous case, choosing x_{qh} big enough such that $\left(\frac{x_{ph}}{x_{qh}}\right)^{\alpha_h}>\left(\frac{x_{ql}^{\alpha_l}-\left(x_{ql}-\delta_l\right)^{\alpha_l}}{\left(x_{pl}+\delta_l\right)^{\alpha_l}-x_{pl}^{\alpha_l}}\right)$ and taking into consideration that x^{α_l} is increasing.

ii) As regards the second functional form, equation [2], equation [8] holds with $\phi\left(\underline{x}_i\right)=\sum_{1\le j\le k}-\beta_j\log\left(x_{ij}\right)$**.** From lemma 2 and rewriting expression [9] for this functional form we get that PDB is satisfied if

$$\beta_l \log\left(x_{ql}-\delta_l\right)+\beta_l \log\left(x_{pl}+\delta_l\right)>\beta_l \log\left(x_{ql}\right)+\beta_l \log\left(x_{pl}\right)$$

Taking into account that the logarithmic function is concave it is easy to verify that this inequality holds if $\beta_l > 0$, so by anonymity we get that PDB is satisfy if $\beta_j > 0$ for all *j*.

iii) With respect to the third functional form, equation [10], equation [8] holds with $\phi\left(\underline{x}_i\right)=x_{ih}\sum_{1\le j\le k} a_{hj}\log\left(x_{ij}\right)$. From lemma 2, if we rewrite expression [9] for this functional form we get that *I* satisfies PDB if the following inequality holds

$$x_{qh}a_{hl}\log\left(x_{ql}-\delta_l\right)+x_{ph}a_{hl}\log\left(x_{pl}+\delta_l\right)<x_{qh}a_{hl}\log\left(x_{ql}\right)+x_{ph}a_{hl}\log\left(x_{pl}\right) \qquad [13]$$

Moreover from lemma 1, a necessary condition for *I* to hold PDB is function ϕ be strictly convex in each component so it should be $a_{hl} < 0 \ \forall l \neq h$.

For such parameter restrictions, in general, functional form [10] does not verify PDB, as the following example shows.

Given x_{ql}, x_{pl}, δ_l, and x_{ph}, since the logarithmic function is increasing, we can always choose x_{qh} big enough such that $\dfrac{x_{qh}}{x_{ph}}>\dfrac{\log\left(x_{pl}+\delta_l\right)-\log\left(x_{pl}\right)}{\log\left(x_{ql}\right)-\log\left(x_{ql}-\delta_l\right)}$

Then $x_{ph}\left(\log\left(x_{pl}+\delta_l\right)-\log\left(x_{pl}\right)\right)<x_{qh}\left(\log\left(x_{ql}\right)-\log\left(x_{ql}-\delta_l\right)\right)$.

Assuming that $x_{qj}=x_{pj} \ \forall j\neq l,h$ and operating, it turns out that that expression [13] is verified in the opposite sense.

The proof of the sufficiency is straightforward taking into account that if equation [1] holds, the aggregative principle, and SI are satisfied. Moreover if *Y* is derived from *X* by a PDB transfer for these parameter values we get: $I\left(Y\right)<I\left(X\right)$. A similar reasoning can be used for equation [2]. *Q.E.D.*

Proof of the Proposition 2: The proof is straightforward given that all the functions ϕ corresponding to measures in equation [1] and [2] fulfil $\phi''_{hl} \geq 0$ for all $h, l = 1, \ldots, k$ such that $h \neq l$ and this is a necessary and sufficient condition in order to verify WCIM (Tsui (1999)). *Q.E.D.*

Proof of the Proposition 3: If W satisfies the separability axiom by Tsui (1995) for every $X \in D$ $W(X)$ is ordinally equivalent to $\sum_{1 \leq i \leq n} U(\underline{x}_i)$ where $U : \mathbb{R}^k_{++} \to \mathbb{R}$ is a strictly increasing function. Tsui ((1995), theorem 1) also proves that adding HP this expression inexorably leads to the two functional forms [4] and [5] and once the functional forms of U are determined their corresponding inequality measures, equations [6] and [7], may be easily derived using [3].

The parameter restrictions are derived in a similar way as in proposition 1. *Q.E.D.*