Working Paper Series

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Mª Casilda Lasso de la Vega
Ana Urrutia
Henar Diez

ECINEQ WP 2009 – 109

ECINEQ 2009-109
March 2009

www.ecineq.org

The Bourguignon and Chakravarty Multidimensional Poverty Family: A Characterization*

Mª Casilda Lasso de la Vega
Ana Urrutia
Henar Diez

University of the Basque Country†

Abstract

The family of multidimensional poverty indices introduced by Bourguignon and Chakravarty (Journal of Economic Inequality, 2003) has attracted a great deal of interest in the field of poverty measurement. In this note we explore a number of properties fulfilled by the members of this family, related to both the way to aggregate, for each individual, the deprivations in the various attributes, and the procedure for combining the individuals' overall deprivations. Then we show that the properties we highlight characterize the functional form of the family.

Key Words: multidimensional poverty indices, Bourguignon and Chakravarty family, deprivation.
JEL classification: I32

* This research has been partially supported by the Spanish Ministerio de Educación y Ciencia under project SEJ2006-05455, cofunded by FEDER, and by the Basque Government under the project GIC07/146-IT-377-07.
We are very grateful to Professors Jean-Yves Duclos, Peter Lambert and Jacques Silber for their useful suggestions. Preliminary versions of this paper were presented in the 9th International Meeting of the Society for Social Choice and Welfare (Montreal, 2008) and in the 4th Winter School on Inequality and Social Welfare Theory (Canazei, 2009). We also wish to thank Koen Decanq, Mª Ana Lugo, Erwin Ooghe, Claudio Zoli and the rest of participants in these events for their suggestions and comments.
† **Address of correspondence**: Mª Casilda Lasso de la Vega casilda.lassodelavega@ehu.es

1. INTRODUCTION

This paper tackles the problem of measuring multidimensional poverty. A number of multidimensional poverty indices have been proposed in the literature trying to measure this complex phenomenon.[1] Specifically the family of poverty measures introduced by Bourguignon and Chakravarty (2003),[2] henceforth B-Ch family, has attracted a great deal of interest from both a theoretical and an empirical point of view. Some theoretical papers have been published analysing the properties of these indices and also many empirical studies have been carried out taking into consideration the measures of this family.[3] Nevertheless, to our knowledge, no characterization is known of the members of this class. This is the aim of our paper. We explore a number of properties fulfilled by these measures which allow us to better understand the way these indices behave. Then we show that these properties characterize the family.

There exist in the literature two different forms of aggregation often used to derive indicators for measuring either deprivation or standard of living in a multidimensional framework. The first combines different elements of deprivation (resp. the standard of living) for each individual, which are then aggregated over individuals to form a summary index of the overall deprivation (resp. the standard of living) of the society. The second summarizes an index across individuals for each attribute to construct, then, an indicator of all the attributes.[4]

Dutta et al. (2003) and Pattanaik et al. (2007) analyze these two approaches in depth, referring to them as *row-first* and *column-first two-stage procedures* respectively. They show that the indices derived from the latter are unable to to satisfy some basic and attractive properties, among them the sensitivity to the correlation between dimensions, and "must lead

[1]See, among others: (UNDP (1997), Bourguignon and Chakravarty (2003), Chakravarty, Mukherjee and Ranade (1998), Tsui (2002), Chakravarty, Deutsch and Silber (2008), Maasoumi and Lugo (2008).
[2] Actually there exist previous versions of this paper.
[3] Among them Atkinson (2003) deserves a special mention.
[4]Among the indices mentioned above, only the poverty human indices introduced by UNDP (1997) follow the second procedure. All the rest are constructed with the first method.

to possibly untenable conclusions".[5] Therefore, only the row-first two-stage procedure should be adopted to construct multidimensional indicators.

Consequently, to derive a multidimensional poverty index the first problem we face is to aggregate, for each individual, their deprivations in the different attributes. For doing so different ways have been introduced in the literature. In this paper we explore some appealing properties fulfilled by the B-Ch family in this stage and show that these properties characterize the way of aggregation they propose.

The second stage in the construction of multidimensional indices is to determine the way in which the aggregate deprivations of the individuals are combined. In this case we introduce a new property to be fulfilled by the poverty indices and show that this property also characterizes the method followed by the B-Ch indices.

The note is structured as follows. The next section presents the notation and the definitions and in section 3 we introduce the assumptions and present our results. The paper finishes with some concluding remarks.

2. NOTATION AND BASIC DEFINITIONS.

We consider a population consisting of $n \geq 2$ individuals endowed with a bundle of $k \geq 2$ basic need attributes. A multidimensional distribution among the population is represented by an $n \times k$ real matrix *X*, where the *ij*th entry $x_{ij} \geq 0$ represents the individual *i*'s achievement of the attribute *j*.[6] Regarding the identification of the poor through the specification of a poverty line, let's consider $z_j > 0$ to be the minimum level of subsistence of

[5] We are indebted to Professor Peter Lambert for having introduced us to the Dutta et al. and Pattanaik et al. papers.

[6] For simplicity we assume that any individual attribute should be non negative, although our conclusions can be drawn even if negative values are also considered.

the jth attribute. An individual i is poor as regards attribute j if $x_{ij} < z_j$. Let $\underline{z} = (z_1, z_2, ..., z_k) \in \mathbb{R}^k_{++}$ be the vector of thresholds for all the dimensions.

Poverty is usually measured in terms of deprivations instead of achievements. Given a multidimensional distribution X and a vector of thresholds $\underline{z} \in \mathbb{R}^k_{++}$ a number of deprivation matrices are often considered in order to define poverty indices. One of the most used procedures is to consider the normalized gap $a_{ij} = \max\left(1 - \left(x_{ij}/z_j\right), 0\right)$ as a measure of the deprivation felt by the individual i as regards the attribute j. Specifically the B-Ch family and the multidimensional generalization of the FGT indices (Foster et al. (1984)) proposed by Foster and Alkire (2008) are defined in terms of normalized gaps. A more general deprivation matrix whose elements are also bounded between 0 and 1 is defined by $a_{ij} = \max\left(1 - \left(x_{ij}/z_j\right)^{c_j}, 0\right)$ with $0 < c_j < 1$, (for instance, the indices proposed by Chakravarty et al. (1998)).

Hence, for any multidimensional distribution X and any vector of thresholds $\underline{z} \in \mathbb{R}^k_{++}$, let A be an $n \times k$ deprivation matrix $A = \left(a_{ij}\right)_{n\times k}$ whose typical entry $a_{ij} \in [0,1]$ represents the extent to which the individual i is deprived in the attribute j, where, as usual, 0 indicates the absence of deprivation.[7] The ith row of A is denoted by $\underline{a}_i$ and the jth column is denoted $\overline{a}^j$. We denote by $A(\mathrm{n,k})$ the class of these $n \times k$ deprivation matrices and let $D = \bigcup_{\mathrm{n}\in\mathbb{N}_+} \bigcup_{\mathrm{k}\in\mathbb{N}_+} A(\mathrm{n,k})$.

Once the poverty line is drawn and the deprivations in the different dimensions are quantified an index is needed to measure the extent of the deprivation.

[7] Our conclusions also hold if other intervals different from $[0,1]$ are considered.

Many times there are no reasons to consider one of the attributes more important than others and implicitly we are assuming that the weights associated to each dimension are equal. However, sometimes it may be appropriate to associate different weights to the different dimensions. For allowing this possibility, let's consider $w_j \geq 0$ the weight attached to the attribute *j*. Let $\underline{w} = (w_1, w_2, ..., w_k) \in \mathbb{R}_+^{k*}$ be the vector of weights, where $\mathbb{R}_+^{k*}$ stands for $\mathbb{R}_+^k - \{\mathbf{0}\}$.

In this paper, a *multidimensional deprivation index* is defined as a non-constant function $P: D \times \mathbb{R}_+^{k*} \to \mathbb{R}$ defined on the set of the deprivation matrices whose elements belong to the $[0,1]$ interval and where each row is weighted by a vector $\underline{w} \neq \mathbf{0}$.[8] According to this definition we are implicitly assuming that the deprivation in the social situation *A* depends only on the deprivations of the different individuals in terms of different attributes.

Following the Pattanaik et al. framework we consider the following definition:

Definition: A multidimensional deprivation index *P* will be referred to as *derived using a row-first two-stage procedure* if *P* is constructed in two stages according to the following:

- ✓ In the first stage a non-negative function $d: [0,1]^k \times \mathbb{R}_+^{k*} \to \mathbb{R}$ is considered, where $A_i = d(\underline{a}_i, \underline{w})$ represents the overall deprivation of the individual *i* in the social situation A.
- ✓ The second stage uses a function $h: [0,1]^n \to \mathbb{R}$ to combine all the individuals' overall deprivations to derive the multidimensional deprivation index.

[8] Clearly the B-Ch family and the measures proposed by Chakravarty et al. (1998), and Alkire and Foster (2008) can be considered as deprivation indices according to the definition of this paper. However, poverty indices that are not defined in terms of bounded deprivations (for instance Tsui (2002) and the multidimensional extension of the Watts index (Chakravarty et al. (2008)) do not fit our framework.

Thus an index P derived using *a row-first two-stage procedure* can be written in the following way:

$$P\left(A,\underline{w}\right)=h\left(d\left(\underline{\mathrm{a}}_1,\underline{w}\right),...,d\left(\underline{\mathrm{a}}_n,\underline{w}\right)\right) \qquad (1)$$

In the next section we will impose assumptions on d and h in order to add more structure to P.

In this paper we are going to focus on decomposable indices according to the following definition:

Definition: A multidimensional deprivation index P is *decomposable* if

$$P\left(A,\underline{w}\right)=\frac{1}{n}\sum_{i=1}^{n}p\left(\underline{\mathrm{a}}_i,\underline{w}\right) \qquad (2)$$

Some basic properties are fulfilled by these indices. First of all, a decomposable index is clearly invariant under replication of the population.[9] Then, all of them are derived using a *row-first two-stage procedure*. Moreover, the Pattanaik et al. framework allows us to disentangle two different effects on the term $p\left(\underline{\mathrm{a}}_i,\underline{w}\right)$, usually interpreted as the individual i's poverty function. Indeed, consider a hypothetical deprivation matrix A^* with all its rows equal to the individual i's bundle $\underline{\mathrm{a}}_i$. Taking into account (2) and (1) we have:

$$P\left(A^*,\underline{w}\right)=p\left(\underline{\mathrm{a}}_i,\underline{w}\right)=h\left(d\left(\underline{\mathrm{a}}_i,\underline{w}\right),...,d\left(\underline{\mathrm{a}}_i,\underline{w}\right)\right) \qquad (3)$$

This equation tells us that the individual i's poverty function has two sources: on the one hand the aggregation of the deprivations of the individual and, on the other hand, similarly to the unidimensional framework, the way in which this overall deprivation is incorporated to gauge the deprivation of the society.

[9] The strong consequences of this axiom will be discussed in the concluding remarks taking Subramanian (2002) as a basis.

Denoting $h_n(x) = h(x, x, ..., x)$, from (3) equation (2) can be rewritten as

$$P(A, \underline{w}) = \frac{1}{n}\sum_{i=1}^{n} p(\underline{a}_i, \underline{w}) = \frac{1}{n}\sum_{i=1}^{n} h_n(d(\underline{a}_i, \underline{w})) \qquad (4)$$

As already mentioned, the B-Ch family (Bourguignon and Chakravarty (2003)) will play an important role in our paper. Given a multidimensional distribution X, a vector of thresholds $\underline{z} \in \mathbb{R}^k_{++}$, and a vector of weights $\underline{w} \in \mathbb{R}^{k*}_{+}$ the specification of this family is the following:

$$P^{\theta}_{\alpha}(A, \underline{w}) = \frac{1}{n}\sum_{i=1}^{n}\left[\left(w_1 a_{i1}{}^{\theta} + ... + w_k a_{ik}{}^{\theta}\right)^{\frac{1}{\theta}}\right]^{\alpha} \quad \theta > 0 \quad \alpha > 0 \qquad (5)$$

where $a_{ij} = \max\left(1 - \left(x_{ij}/z_j\right), 0\right)$, $\sum_{1 \le j \le k} w_j = 1$, the parameter θ represents the elasticity of substitution between the normalized gaps of the attributes for any person and the α parameter can be interpreted as the aversion of society towards poverty. The higher α, the more sensitive to the poorest P^{θ}_{α} is.[10]

We can interpret this formulation from equation (4): in the first step the normalized gaps for each individual are aggregated using a weighted mean of order θ, a specific CES functional form. The second step proposes to combine the aggregate deprivations of the individuals using the same functional as in the FGT family (Foster et al. (1984)). In the next section we shall analyze these two issues separately.

[10] In common with the FGT family, if α is raised ceteris paribus, measured poverty in any distribution falls. But in comparisons, the situation of the poorest becomes more important.

3. ROW-FIRST TWO-STAGE PROCEDURE TO DERIVE MULTIDIMENSIONAL INDICES.

Let's consider a multidimensional deprivation index $P: D \times \mathbb{R}_+^{k*} \to \mathbb{R}$ derived using a row-first two-stage procedure.

3.1 First stage: Aggregating deprivations for each individual.

In this section we propose a set of intuitive and appealing properties all fulfilled by the B-Ch indices. These conditions allow the characterization of a family of aggregation functions that aggregate individual deprivations in the same way as the B-Ch family does.

Let's consider a non-negative function $d: [0,1]^k \times \mathbb{R}_+^{k*} \to \mathbb{R}$ that aggregates the deprivations felt by the individual *i* in terms of all the weighted attributes. The names of the conditions follow Aczél's designation (Aczél (1966)).

Assumption 1. Symmetry: $d\left(a_1, a_2; w_1, w_2\right) = d\left(a_2, a_1; w_2, w_1\right)$

As usual, in measuring deprivation the names of the dimensions are irrelevant.

There are two normalization requirements as regards both the attributes and the weights:

Assumption 2. Reflexivity: $d(0,0;w_1,w_2) = 0$ and $d(1,1;w_1,w_2) = 1$

Assumption 3. Internality: $d(0,1;1,0) = 0$, $d(0,1;0,1) = 1$ and $d(0,1;w_1,w_2) < 1$ with $w_1, w_2 > 0$.

These two conditions only refer to two attributes and to extreme situations. The first property requires that if the individual is either rich or totally deprived in both attributes, then the overall deprivation should be 0 or 1 respectively. In turn, assumption 3 considers a mixed situation: the individual is rich with respect to one attribute and totally deprived in the other. If no weight is attached to one of the dimensions, the overall deprivation depends only on the weighted dimension. Moreover, in any other case, the overall deprivation will be less than 1.

There follow two monotonicity assumptions also with respect to both the attributes and the weights:

Assumption 4. Increasing in the individual deprivations (second variable): $d\left(a_1, a_2; w_1, w_2\right) < d\left(a_1, a_2^*; w_1, w_2\right)$ with $a_2 < a_2^*$

This property together with symmetry is known as *monotonicity* in other frameworks and demands that if the deprivation felt by the individual in any attribute increases, then the aggregate deprivation also increases.

Assumption 5. Increasing in the second weight: $d\left(0,1; w_1, w_2\right) < d\left(0,1; w_1, w_2^*\right)$ with $w_2 < w_2^*$.

This assumption means that if the individual is deprived in only one attribute, if the weight on this attribute increases, the overall deprivation should increase.

The sixth condition requires that if the weights on every attribute are modified in the same proportion, the aggregate deprivation does not change:

Assumption 6. *Homogeneity of 0th degree in the weights*: $d\left(0,1; w_1 t, w_2 t\right) = d\left(0,1; w_1, w_2\right)$ for all values $w_1, w_2 \geq 0$ and for $w_1 + w_2, t > 0$

Finally, we assume a rule that allows us to carry out multilevel decompositions by subgroups of attributes. This property ensures that the computation of the deprivation level can be carried out in several steps without changes in the final result:

Assumption 7. Aggregativity:

$$d\left[d\left(a_1, a_2; w_1, w_2\right), a_3; w_1 + w_2, w_3\right] = d\left[a_1, d\left(a_2, a_3; w_2, w_3\right); w_1, w_2 + w_3\right]$$

This condition plays a similar role to the *population substitution principle* introduced by Blackorby and Donaldson (1984) and really imposes the functional form in the aggregator.

To achieve the weighted means of order *θ*, that is, the same functional forms as in the B-Ch family we need an additional assumption:

Assumption 8. Homogeneity (1st degree) in the individual's deprivation levels: $d\left(\lambda \underline{a};\underline{w}\right)=\lambda d\left(\underline{a};\underline{w}\right)$ for all $\lambda \in \left(0,1\right]$.

which means that if for each individual, the deprivation with respect to every attribute is modified in the same proportion, then the overall deprivation felt by that individual changes in the same proportion.

If assumptions from (1) through (8) are considered as appealing requirements for a function to aggregate individual's deprivations the only possibility for the function d is to perform according to B-Ch's procedure.

Proposition 1. *The first stage* $d:\left[0,1\right]^{k}\times\mathbb{R}_{+}^{k*}\rightarrow\mathbb{R}$ *to derive a deprivation index satisfies:*

i) *Assumptions 1 through 7 if and only if d is of the form*

$$d\left(\underline{a};\underline{w}\right)=f\left(\sum\nolimits_{1\leq i\leq k}\frac{w_i}{\sum\nolimits_{1\leq j\leq k}w_j}f^{-1}\left(a_i\right)\right) \tag{6}$$

with $f:\left[0,1\right]\rightarrow\mathbb{R}$ *a continuous strictly monotonic function which can be expressed explicitly by* $f\left(t\right)=d\left(0,1;1-t,t\right)$.

ii) *Assumptions 1 through 8 if and only if d is of the form*

$$d\left(\underline{a};\underline{w}\right)=\left(\sum\nolimits_{1\leq i\leq k}\frac{w_i}{\sum\nolimits_{1\leq j\leq k}w_j}a_i^{\theta}\right)^{\frac{1}{\theta}} \text{ with } \theta>0 \text{ is a real parameter.} \tag{7}$$

Proof. See the Appendix.

Different requirements have been used in the literature to characterize the means of order θ (equation (7)). The crucial point in all these characterizations is the domain for which we want to establish the results. For instance, the characterization provided by Blackorby and Donaldson (1982) works with no constraints in the domain, whereas in our case the deprivations are restricted to take values in some closed and bounded interval including values equal to 0. On the other hand, from our point of view, the conditions assumed in this section are quite intuitive and appealing to as requirements for an aggregation function.

One concern in measuring deprivation in a multidimensional framework is the identification of the poor, which is by no means an elementary issue. According to the derived aggregation function (both equation (6) and (7)) an individual is to be considered rich if their overall deprivation is equal to 0, and this happens only if the individual is rich in all the dimensions. In other words, the identification of the poor corresponds to the union procedure. If the monotonicity requirements, assumptions 4 and 5, were weakened the geometric mean would be included in the formulation, and in this case the poor would be identified according to the intersection definition.[11] Also if we changed the normalization condition and the rich individual deprivations took values greater than 0, all the weighted means for all the values of θ would appear in the formulation including the geometric weighted mean.

It may be worth remembering of some properties of these means. When $\theta = 1$, equation (7) coincides with the arithmetic mean. For the rest of values, the θ-order means are sensitive to the inequality among dimensions. Thus if the dimension's deprivations are different and $\theta > 1$ the order mean is greater than the arithmetic mean and the limiting case, when $\theta \to \infty$, tends to the greatest deprivation. In other words, given two deprivation bundles with the same arithmetic means, the greater the difference between the deprivations, the higher is the individual's deprivation level. In contrast, when $\theta < 1$, the aggregate deprivation is always

[11] The union and the intersection procedures correspond to the Duclos et al. (2006) designations and they refer to two well-known methodologies to identify the poor: one individual is poor either they are poor in at least one attribute or in all attributes respectively. Duclos et al. (2006) also introduce an intermediate definition. Recently, Alkire and Foster (2008) propose an alternative methodology to identify the poor that generalizes the union and intersection approaches and is quite appropriate to deal with ordinal data. This fundamental discussion is beyond the scope of this paper, although with a slight modification of our framework and introducing the "adjusted" notion as Alkire and Foster do, all our conclusions hold after having identified the poor according to the procedure they introduce.

less than the arithmetic mean, and the greater the difference between the dimensions, the lower the deprivation level.

It is usual in the literature to interpret the θ parameter as a measure of the degree of substitutability between dimensions: when $\theta > 1$ the attributes are considered complements whereas for $\theta < 1$ they are substitutes. However this classification should hold for all the dimensions at the same time, and when more than two attributes are involved the conclusions seem to be quite limited.

Although we have implicitly assumed that the function *d* is invariant with respect to the individuals, this assumption can easily be relaxed, allowing different aggregation functions for different individuals. This generalization would encompass a more broad formulation of the B-Ch family which allows the β parameter to depend on the level of deprivation of each individual.

3.2. Second stage: Combining Individual Deprivations.

Let's consider a deprivation index derived through a first-row two-stage procedure. Let's assume that individual's deprivation $A_i \in [0,1]$.[12] The second stage to construct a deprivation index establishes the procedure to combine the overall deprivations for all the individuals to compute the deprivation in society using a function $h:[0,1]^n \to \mathbb{R}$. We denote by $\underline{A} = (A_1, A_2, ..., A_n) \in [0,1]^n$ the vector of the aggregate deprivations of the individuals.

First of all we are going to assume some very basic assumptions:

Assumption 9. Symmetry: the names of the individuals are irrelevant.

Assumption 10. Normalization: $h(0,0,...,0) = 0$ and $h(1,1,...,1) = 1$

[12] This is the case if the first stage satisfies Assumptions 1 through 7. Otherwise, the results can be generalized taking into account any bounded and closed interval in $\mathbb{R}$.

If all the individuals are rich, deprivation in society is equal to 0. In contrast, if all the individuals are completely deprived, deprivation in society takes the highest value.

Assumption 11. Monotonicity: h is strictly increasing in its arguments.

Assumption 12. Continuity: h is a continuous function in its arguments.

These four requirements are quite reasonable and compelling. Now let´s take a look at the B-Ch family. Using the aggregate deprivation for each individual A_i, the second stage can be rewritten as

$$P_\alpha^\theta\left(A;\underline{w}\right)=h\left(\underline{A}\right)=\frac{1}{n}\sum_{i=1}^{n}A_i^\alpha$$

Note that if the aggregate deprivations for all the individuals are multiplied by the same constant $\lambda\in\left(0,1\right]$, then the overall deprivation level is multiplied by λ to the α-power, that is:

$$h\left(\lambda\underline{A}\right)=\frac{1}{n}\sum_{i=1}^{n}\left(\lambda A_i\right)^\alpha=\frac{\lambda^\alpha}{n}\sum_{i=1}^{n}A_i^\alpha=\lambda^\alpha h\left(\underline{A}\right)$$

We attempt to generalize this property. Let's consider two deprivation matrices, A and B, such that the overall deprivation in the first society is less than in the second. Let's suppose that in both societies the aggregate deprivations of all the individuals are modified in the same proportion. Then it seems intuitive to demand that this modification should not affect the deprivation rankings, that is, deprivation in the first society should remain less than in the second. We have called this property *Increasing Deprivation Consistency Axiom* and it is articulated as follows[13]

Increasing Deprivation-Consistency Axiom: *(IDC):* The second stage $h:\left[0,1\right]^n\to\mathbb{R}$ to derive a row-first two-stage index satisfies *IDC* if for any two vectors of individual's deprivations $\underline{A},\underline{B}\in\left[0,1\right]^n$ and for all $\lambda\in\left(0,1\right]$: $h\left(\underline{A}\right)<h\left(\underline{B}\right)$ implies $h\left(\lambda\underline{A}\right)<h\left(\lambda\underline{B}\right)$.

[13] We have taken the "unit consistency axiom" proposed by Zheng in both the inequality (2007a) and the poverty (2007b) fields as a basis.

Proposition 2. *A second stage* $h:[0,1]^n \to \mathbb{R}$ *to derive a decomposable deprivation index is a symmetric, normalized, strictly increasing, continuous function and satisfies IDC if and only if, up to a positive constant*

$$h(A_1, A_2, ..., A_n) = \frac{1}{n}\sum\nolimits_{1 \le i \le n} A_i^{\alpha} \quad \textit{with } \alpha > 0 \qquad \textit{(8)}$$

Proof. See the Appendix.

Corollary 3. *P is a decomposable deprivation index such that:*

i) the first stage satisfies assumptions 1 through 8,

ii) the second stage is a symmetric, normalized, strictly increasing, continuous function that satisfies IDC,

if and only if, up to a constant:

$$P(A, \underline{w}) = \frac{1}{n}\sum_{i=1}^{n}\left[\left(w_1 a_{i1}{}^{\theta} + ... + w_k a_{ik}{}^{\theta}\right)^{\frac{1}{\theta}}\right]^{\alpha} \quad \theta > 0 \quad \alpha > 0 \qquad (9)$$

Proof. It is straightforward from Proposition 1 and 2. *Q.E.D.*

Depending on the procedure to build the deprivation matrix *A*, equation (9) corresponds to the B-Ch family or generalizations of these indices. Absolute gaps also have room in this formulation as long as the bounds of the deprivation levels for all the attributes are the same. Yet a mixture of absolute and relative gaps is possible, following the García-Diaz (2003) proposal, provided all the deprivation numbers for all the attributes lie in the same interval.

The α parameter in equations (8) and (9) is a measure of the sensitivity towards poverty. For $\alpha = 0$, the index may be interpreted as the multidimensional headcount ratio. When

$\alpha = 1$, it becomes just a mean of the deprivation of the individuals. The higher the value of α, the more sensitive the index is to extreme deprivation.

An interesting particular case appears when $\alpha = \theta$.[14] This subfamily fulfils some interesting additional properties: they are the only indices which can be alternatively derived by the *column-first two-stage procedure* (Dutta et al. (2003)). These indices may be quite interesting for some particular political purposes when the aim is to reduce deprivation in specific dimensions. Moreover for $\alpha = \theta = 1$ equation (9) is a generalization of the family introduced by Chakravarty et al. (1998).

None of the properties required so far is able to capture inequality among the poor, one of the crucial issues that a deprivation index should be sensitive to. A broad number of properties have been introduced in the multidimensional poverty field as generalizations of the Pigou-Dalton transfer principle and this discussion is beyond the aim of this paper. Anyway, Bourguignon and Chakravarty (2003) discuss the relationship required between α and θ for these properties to be fulfilled by the members of their family.

CONCLUDING REMARKS

The aim of this note is to point out some properties fulfilled by the B-Ch family with a view to better understanding the behaviour of these indices and we think this goal is achieved. Nevertheless we have only characterized the functional form of the family and several choices remain open in this formulation. Policy makers should choose not only the poverty lines, the methodology to identify the poor, and the gauge of the deprivation felt by each individual with respect to any dimension, but also the weight attached to any dimension and the values of the α and θ parameters.

[14] This is the choice in Alkire and Foster (2008) after having identified the poor according to the procedure they introduce..

One strong constraint we have assumed is that the entries of the deprivation matrix should belong to the same interval. This allows the possibility of mixing relative and absolute gaps as already mentioned, but only with quite restrictive conditions. The option of exploring different intervals for different attributes could be an interesting generalization of the results.

Moreover we have taken decomposable indices as a basis, according to the usual definition. Thus we are implicitly assuming the replication invariance principle. However, as Subramanian (2002) points out, some difficulties arise in the measurement of poverty to interpret the notion of "the extent of poverty" and two possible ways are open. On the one hand, assuming the replication invariance principle leads to the usual interpretation of poverty in the literature. Nevertheless, another way is possible: giving up this invariance principle and assuming two others very basic and appealing proposed by Subramanian (2002). So we could take this choice and to examine deprivation indices according to this proposal.

Finally the paper has been focused on deprivation measures. However, all the results can be extended to the measurement of standards of living. The only change needed is the interpretation of the elements of the matrices. In this alternative framework matrix entries indicate the level of achievement of some individual in terms of some attribute, with a higher number denoting a higher level of achievement.

APPENDIX

Proof of Proposition 1.[15]

i) It is straightforward from Azcél (1966, p.242). Moreover we get that:

$$f(0) = d(0,1;1,0) = 0 \text{ and } f(1) = d(0,1;0,1) = 1 \qquad (10)$$

ii) Since the sufficiency of this part is obvious it is enough to show that d defined in equation (6) is of the form in (7) if assumption 8 is also fulfilled. We can follow the proof of theorem

[15] We want to thank Mikel Bilbao for having helped us in this proof.

2.2.1 in Eichhorn (1978, p.32) to show that, under these requirements, *f* must satisfy an equation like

$$f(\lambda x) = \alpha(\lambda) f(x) + \beta(\lambda) \tag{11}$$

with $(\lambda, x) \in (0,1] \times (0,1]$, $\alpha : (0,1] \to \mathbb{R}$ strictly monotonic, $\beta : [0,1) \to \mathbb{R}$ and $\alpha(\lambda) \neq 0$ for all λ. From (10) we also get that $\beta(\lambda) = 0$ and $\alpha(\lambda) = f(\lambda)$, and hence (11) can be rewritten:

$$f(\lambda x) = f(\lambda) f(x) \text{ with } (\lambda, x) \in [0,1] \times [0,1] \tag{12}$$

Defining $\tilde{f}(y) = \begin{cases} f(y) & \text{if } 0 \leq y \leq 1 \\ 1/f(1/y) & \text{if } y > 1 \end{cases}$ we find that $\tilde{f}$ is a continuous extension of *f* to $\mathbb{R}_+$ fulfilling:

$$\tilde{f}(xy) = \tilde{f}(x)\tilde{f}(y) \text{ for all } x, y \in \mathbb{R}_+ \tag{13}$$

Resorting to Azcél (1966, pp. 145 and 41) it can be proved that the general continuous solution of equation (13) is $\tilde{f}(t) = t^{\theta}$ with $\theta > 0$ an arbitrary real constant. Then we have the result.

Q.E.D.

We need a previous lemma to prove Proposition 2.

Lemma 1. *A second stage* $h : [0,1]^n \to \mathbb{R}$ *to derive a deprivation index is a symmetric, normalized, strictly increasing, continuous function that satisfies IDC if and only if there exists a continuous function* $f(.,.)$ *which is increasing in the second argument such that*

$$h(\lambda \underline{A}) = f[\lambda, h(\underline{A})] \tag{14}$$

for all vectors of individual's deprivations $\underline{A} \in [0,1]^n$ *and for all* $\lambda \in (0,1]$.

Proof. The proof is straightforward following that of Proposition 1 in Zheng (2007a).

Q.E.D.

Proof of Proposition 2. We can follow the proof of Proposition 6 of Zheng (2007b) to get the following functional equation:

$$f_\lambda(y_1 + y_2) = f_\lambda(y_1) + f_\lambda(y_2) \tag{15}$$

where $y_1 = h_n(A_1)$, $y_2 = h_n(A_2)$ and $f_\lambda(x) = f(\lambda, h_n(x)) = f(\lambda, h(x,...,x))$ whose existence is assured by Lemma 1. Equation (15) holds for all $y_1, y_2 \in [0,1]$. The solution to this functional equation (15) (Aczél, 1966, p.66) is

$$f_\lambda(x) = ax \text{ for some constant } a \neq 0. \quad (16)$$

Taking into account that $h_n(x) = h(x,...,x)$ and substituting (16) in equation (14) we get

$$h_n(\lambda A_i) = a(\lambda) h_n(A_i) \quad (17)$$

which is a Peixeder equation that holds for all $A_i, \lambda \in [0,1]$. In a similar way to for equation (12) in Lemma 1 this equation can be extended to hold in $\mathbb{R}_+$. Then the general solutions are the following (Azcél (1966, pp. 145 and 41)

$$h_n(t) = ct^\alpha \text{ and } a(t) = t^\alpha$$

with $\alpha, c > 0$ real constants. Taking into account (4) we have the result.

Q.E.D.

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