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Kristof Bosmans
Lucio Esposito
Peter Lambert

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New perspectives on a more-or-less familiar poverty index, with extensions

Kristof Bosmans
Maastricht University

Lucio Esposito
Università degli Studi di Pavia

Peter Lambert*
University of Oregon

Abstract
A particular scale-invariant index of poverty is subjected to careful analysis. This leads to a new perspective, not seen before, on the family of subgroup-consistent and scale-invariant poverty indices. Parametric families of new poverty indices are presented which offer the analyst a degree of flexibility in the choice of transfer sensitivity and distribution sensitivity which has not been available before now.

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* **Addresses of correspondence**: Department of Economics, 1285 University of Oregon, Eugene, OR 97403-1285, USA. *Email:* plambert@uoregon.edu. *Tel:* +1-541-346-4670, *Fax:* +1-541-346-1243.

1. Introduction

In this paper we examine some properties and extensions of one particular poverty index which is already passingly familiar, if not well-known, to poverty analysts. Our investigation leads to a new perspective on the entire family of subgroup-consistent and scale-invariant poverty indices, and to a clarification of the different roles of transfer sensitivity and distribution sensitivity in poverty measurement. New poverty indices are presented which offer the analyst a greater degree of flexibility in these regards than has heretofore been available.

2. The more-or-less familiar poverty index

In a society of n individuals, let $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n_{++}$ be the vector of incomes arranged in non-decreasing order, where y_i is the income of the i^{th} individual, and let $z \in \mathbb{R}_{++}$ be an exogenously given poverty line. The individuals i for whom $y_i < z$ are the poor ones. Let y_q be the largest poor income, so that the headcount ratio is $H = \frac{q}{n}$. The poverty index we are interested in, which we denote P^∞, takes the form:

(1) $$P^\infty(y; z) = \frac{1}{n} \sum_{i=1}^{q} \frac{z}{y_i}.$$

The reason for our notation will become apparent later.

P^∞ may not appear to be very familiar, but it has familiar antecedents, involving the income gap ratio, the Atkinson (1970) inequality index, and what Chakravarty (1983, p. 81) describes as "a fairly natural translation of a relative inequality index of a censored income distribution into a relative poverty index". Let the generic Atkinson (1970) utility of income function be $U_e(x) = \frac{x^{1-e}}{1-e}$ for $0 < e \neq 1$ and $U_e(x) = \ln x$ for $e = 1$. First, inequality among the poor is $I_e^P = 1 - \frac{\xi^P}{\mu^P}$, where ξ^P is the equally distributed equivalent poor income defined by $U_e\left(\xi^P\right) = \frac{1}{q} \sum_{i=1}^{q} U_e(y_i)$, which, for $e = 2$, becomes $\frac{1}{\xi^P} = \frac{1}{q} \sum_{i=1}^{q} \frac{1}{y_i} = \frac{P^\infty}{zH}$; and second, the income gap ratio is $IGR = 1 - \frac{\mu^P}{z}$ where $\mu^P = \frac{1}{q} \sum_{i=1}^{q} y_i$. Putting these two together, we see that P^∞ can be expressed in terms of some well-known ingredients:

(2) $$P^\infty = \frac{H}{(1 - IGR)(1 - I_{e=2}^P)}.$$

P^{∞} is also strongly related to the second family of poverty indices suggested by Clark, Hemming and Ulph (1981). Defining welfare over basic incomes $b_i = \min\{y_i, z\}$, $1 \le i \le n$, and using the generic utility function U_e to do this, the authors define their poverty index as $P_e^{CHU2}(y;z) = 1 - \frac{\xi_b}{z}$ where ξ_b is the equally distributed equivalent basic income.[1] This yields $P_e^{CHU2}(y;z) = 1 - \left\{ \frac{1}{n}\left[n - q + \sum_{i=1}^{q}\left(\frac{y_i}{z}\right)^{1-e} \right] \right\}^{1/(1-e)}$ when $0 < e \ne 1$, where ξ_b is the equally distributed equivalent basic income defined by $U_e(\xi_b) = \frac{1}{n}\sum_{i=1}^{n} U_e(b_i)$. When $e = 2$, we arrive at another way to view our chosen poverty index P^{∞}:

(3) $$P^{\infty} = \frac{1}{1 - P_{e=2}^{CHU2}(y;z)} - (1 - H).$$

Finally, we point out that the poverty contribution function inherent in P^{∞} actually forms a building-block for the entire class of subgroup-consistent and scale-invariant indices of poverty. These are the poverty functions $P(y;z) : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \mathbb{R}_+$ which evaluate aggregate poverty as a normalized sum of individual poverty contributions $p(y_i;z)$:

(4) $$P(y;z) = \frac{1}{n}\sum_{i=1}^{n} p(y_i;z),$$

where $p(y_i;z) = 0$ if $y_i \ge z$ and $p(y_i;z) > 0$ otherwise, $p(y_i;z) = p(\lambda y_i; \lambda z)$ for all $\lambda > 0$, and $p(y_i;z)$ is continuous and non-increasing in y_i for $y_i \in (0, z)$. Our P^{∞} is in this form, as is the Watts (1968) index, for which $p(y_i, z) = \ln\left(\frac{z}{y_i}\right)$, and the ubiquitous

[1] The poverty index of Chakravarty (1983) is also defined in terms of the equally distributed equivalent basic income, and reduces to P_e^{CHU2} when the generic utility function U_e is invoked.

'FGT index' of Foster, Greer and Thorbecke (1984), call it P_α, for which

$p(y_i;z) = \left(\Gamma_i\right)^\alpha$, where $\Gamma_i = \dfrac{z-y_i}{z}$ is person i's normalized poverty gap and $\alpha \in \mathbb{N} \cup 0$.[2]

The individual poverty contributions of all such indices are, in fact, transformations of the poverty contribution of our P^∞. For as Foster and Shorrocks (1991) show, for a poverty index satisfying (4) and the accompanying restrictions,

(5) $$p(y_i;z) = \varphi\left(\frac{z}{y_i}\right)$$

for some continuous and non-decreasing function φ. The 'building block' for this class of poverty indices is thus the individual poverty function $\dfrac{z}{y_i}$ of our P^∞. In the graphical illustration of Figure 1, $\varphi(\dfrac{z}{y_i})$ is plotted as a continuous line when φ is the identity function, namely, for our index P^∞, and dotted lines show the pattern of values $\varphi(\dfrac{z}{y_i})$ takes for the Watts index and for P_α, $\alpha = 0,1,2,3$. The function φ in (5) can be seen as a transformation apt to choose which property or properties inherent in the hyperbolic functional form $\dfrac{z}{y_i}$ are desired, and which not, in an index in the class defined by (4). $\varphi\left(\dfrac{z}{y}\right) = \dfrac{z}{y}$ enjoys upper unboundedness and discontinuity at any finite poverty line. The property of upper unboundedness is rejected for all members of the P_α class but is accommodated by the Watts index; poverty line discontinuity is retained by P_α for $\alpha = 0$ but not by the other members of the P_α class, nor by the Watts index.

[FIGURE 1 ABOUT HERE]

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[2] Although Foster, Greer and Thorbecke (1984) do not specify the nature of the parameter α, thus allowing it to be taken as any non-negative real number, values for the parameter are usually drawn from the set of nonnegative integers, as we shall assume in this paper.

3. A new result, linking P^{∞} with the class of FGT indices

The central insight of this paper comes by a very simple application of the theory of the convergent geometric series, and it links P^{∞} firmly with the P_{α} class. Recalling that $\Gamma_i = \frac{z - y_i}{z}$ is person *i*'s normalized poverty gap, $i < q$, and that $\left(\Gamma_i\right)^{\alpha}$ is that person's contribution to the index P_{α}, observe that:

$$\text{(6)} \qquad \frac{z}{y_i} = (1 - \frac{z - y_i}{z})^{-1} = \frac{1}{1 - \Gamma_i} = \Gamma_i^0 + \Gamma_i^1 + \Gamma_i^2 + \Gamma_i^3 + \cdots = \lim_{M \to \infty} \sum_{\alpha=0}^{M} \left(\Gamma_i\right)^{\alpha} .$$

The poverty contribution function inherent in P^{∞} is thus the infinite sum of those inherent in the FGT indices P_{α} across integer values of α. In turn, one can write P^{∞} itself as an infinite sum of FGT indices:

$$\text{(7)} \qquad P^{\infty} = \frac{1}{n} \sum_{i=1}^{q} \frac{z}{y_i} = \lim_{M \to \infty} \sum_{\alpha=0}^{M} \frac{1}{n} \sum_{i=1}^{q} \left(\Gamma_i\right)^{\alpha} = \lim_{M \to \infty} \sum_{\alpha=0}^{M} P_{\alpha} .$$

There are several implications.

First, in view of what has gone before, one can now think of the FGT class as actually providing the *building block class* for all scale-invariant and subgroup-consistent poverty indices as in (4). Second, a discussion of transfer sensitivity is now in order. The poverty index in (4) does not necessarily even satisfy the transfer principle as written. For that, P would have to be such that if $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_{++}$ is obtained from *y* by an income transfer τ from poor individual *j* to poor individual *h*, where $y_j < y_h$ and where $0 < \tau < z - y_h$, then $P(y;z) < P(x;z)$.[3] If $p(y_i;z)$ is (infinitely) differentiable in y_i for $y_i \in (0,z)$, this requires that $\frac{\partial p}{\partial y_i} < 0$ and $\frac{\partial^2 p}{\partial y_i^2} > 0$; the subsequent degree of transfer sensitivity in poverty measurement theory (represented by the so-called *Transfer Sensitivity Axiom*) requires that $\frac{\partial^3 p}{\partial y_i^3} < 0$. The requirements for further degrees of transfer

[3] This version of the transfer principle accords with the *Minimal Transfer Axiom* of Zheng (1997, p. 132).

transitivity follow the scheme $\frac{\partial^\beta p}{\partial y_i^{\ \beta}} < 0$ for $\beta = (1,3,5,7,...)$ and $\frac{\partial^\beta p}{\partial y_i^{\ \beta}} > 0$ for $\beta = (2,4,6,8,...)$. Despite the lack of expressed transfer properties for the general scale-invariant poverty index in (4), clearly our building block poverty index P^∞ satisfies transfer sensitivity of *all* degrees – or, *infinite transfer sensitivity* as we might say, hence our notation – and in the FGT class, P_α possesses $(\alpha-1)^{\text{th}}$ degree transfer sensitivity but no higher (hence for $\alpha = 0,1$ there is no transfer sensitivity, for $\alpha = 2$ the Pigou-Dalton condition is satisfied, for $\alpha = 3$ the Transfer Sensitivity Axiom is met, etc.). If $\alpha_1 > \alpha_2$, then P_{α_1} accommodates a higher level of transfer sensitivity than P_{α_2}: there is no upper limit to the assignation of transfer sensitivity in the FGT class.[4] Here is a link, then, between scale invariance and transfer sensitivity for subgroup-consistent poverty indices, unremarked upon in previous literature. Clearly the transformation function φ in (5) conditions the transfer sensitivity, or not, of an index in the class defined by (4), just as it conditions the upper (un)boundedness and poverty line (dis)continuity properties of the index.

Finally, it is instructive to examine the *distribution sensitivity* of P^∞ and of P_α in the sense of Zheng (2000), who sets this notion on a sound footing *à la* Pratt-Arrow in the case of inequality aversion. Zheng's measure, which for the poverty index in (4) takes the form $s_p(x;z) = -\frac{\partial^2 p / \partial y^2}{\partial p / \partial y}$ $(x < z)$ and quantifies the sensitivity of $P(y;z)$ to income transfers in (relative) marginal terms, is $s^\infty(y;z) = \frac{2}{y}$ for P^∞ and $s_\alpha(y;z) = \frac{\alpha - 1}{z - y}$ for P_α.

[4] Note, however, that in the limit, as $\alpha \to \infty$, no transfer property is met because the poverty index "approaches a Rawlsian measure which considers only the position of the poorest poor" (Foster et al. 1984, p. 763). The interest in higher degrees of transfer sensitivity was introduced into the poverty literature by Kakwani (1980). Elegant theory for strengthened versions of the Pigou-Dalton principle is fully expounded by Fishburn and Willig (1984), and essentially postulates that, "any combination of a socially desirable transfer [or series of transfers] with its inverse at uniformly higher levels of income will have positive social benefit" (p. 323). Zheng (1999) observes, in regard to the required continuity of higher and higher derivatives of the poverty contribution function, that "even if one is persuaded [in respect of the first derivative] ... it becomes harder to argue ... all the way up to the (k-2)$^{\text{th}}$ degree" (p. 367).

Thus P^{∞} has declining distribution sensitivity and P_{α} has increasing distribution sensitivity.[5] P^{∞} is more distribution-sensitive at low (poor) income levels whilst P_{α} is more so at high (poor) income levels. The crossover is at $y_{\alpha} = \dfrac{2z}{\alpha+1}$ which, of course, approaches zero as $\alpha \to \infty$. For the general poverty index in (4), with poverty contribution function $p(y;z) = \varphi\left(\dfrac{z}{y}\right)$ as in (5), $s_p(y;z) = s^{\infty}(y;z) + \dfrac{z}{y^2} \cdot \dfrac{\varphi''\left(z/y \right)}{\varphi'\left(z/y \right)}$,

whence its distribution sensitivity relative to that of the building block index P^{∞} is conditioned by the concavity/convexity property of φ.

4. Parametric extensions of P^{∞}

Here we introduce two extended families of poverty indices, we shall call them P_{γ}^{∞} and $P_{\gamma,\omega}^{\infty}$ where γ and $\omega > \gamma$ are non-negative integers. Just as P^{∞} has been revealed to be the sum *ad infinitum* of the FGT indices P_{α} for integer values of α, these new indices are *truncated* sums of FGT indices, the one an infinite sum and the other a finite sum. These introduce enhanced flexibility relative to the FGT family, as will be seen.

The following is our formulation for the poverty contribution function p_{γ}^{∞} of a new poverty index P_{γ}^{∞}:

(8) $$p_{\gamma}^{\infty}(y_i;z) = \left(\Gamma_i\right)^{\gamma} \frac{z}{y_i} = \frac{\left(\Gamma_i\right)^{\gamma}}{1-\Gamma_i} = \Gamma_i^{\gamma} + \Gamma_i^{\gamma+1} + \Gamma_i^{\gamma+2} + \Gamma_i^{\gamma+3} + \cdots = \lim_{M\to\infty} \sum_{\alpha=\gamma}^{M} \left(\Gamma_i\right)^{\alpha}.$$

This results from the product of an FGT poverty contribution function, that inherent in P_{γ}, and the poverty contribution function of our P^{∞} and bears comparison with Sen's

[5] As Zheng (2000, p. 123) points out, satisfaction of the transfer sensitivity axiom is a necessary but not a sufficient condition for a poverty index to exhibit diminishing distribution sensitivity. In Esposito and Lambert (2008), it is argued that transfer and distribution sensitivity in poverty measurement stem not from an *egalitarian* view, valuing equality *per se*, but from a *prioritarian* attitude.

(1976) Axiom N, which states that when transfer properties are *not* of interest – i.e. in the case of a perfectly egalitarian distribution of incomes below the poverty line – aggregate poverty should be given by a multiplicative functional form between two intuitively appealing distribution insensitive indices such as the headcount index and the income gap ratio. The multiplicative form we have in (8), albeit between poverty contribution functions and not between full-blown poverty indices, may be thought of as playing an analogous role *when poor incomes differ*.[6]

The aggregate poverty measure corresponding to (8) is an infinite sum of FGT indices, beginning with P_γ:

(9) $$P_\gamma^\infty(y;z) = \lim_{M\to\infty} \sum_{\alpha=\gamma}^{M} P_\alpha(y;z) = P_\gamma(y;z) + P_{\gamma+1}(y;z) + \cdots.$$

The integer γ can be interpreted as the degree of transfer sensitivity of the least transfer-sensitive summand included in the measure.

Finally, we introduce the further extended class of poverty indices

(10) $$P_{\gamma,\omega}^\infty(y;z) = \frac{1}{n}\sum_{i=1}^{n} p_{\gamma,\omega}^\infty(y_i;z) = \frac{1}{n}\sum_{i=1}^{n}\left\{p_\gamma^\infty(y_i;z) - p_\omega^\infty(y_i;z)\right\}$$

in which γ and $\omega > \gamma$ are both non-negative integers. This index is a *finite* sum of FGT indices:

(11) $$P_{\gamma,\omega}^\infty(y;z) = P_\gamma(y;z) + P_{\gamma+1}(y;z) + \cdots + P_{\omega-1}(y;z)$$

and it embraces both of the following two properties:

i) for $\omega = \gamma + 1$ the members of the P_α class are generated;

ii) for $\omega \to \infty$ we get the members of the P_γ^∞ class.

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[6] Sen's own acknowledgment that "the multiplicative form chosen in Axiom N though simple, is arbitrary" (p. 227) is appeased by the work of Basu (1985), in which Axiom N is shown equivalent to three more elementary properties whose desirability is more readily appreciable. It is interesting to check the performance of P_γ^∞ with respect to Basu's three axioms. While Axioms 2 and 3, which concern monotonicity in first differences of an assumed function of the headcount and income gap ratio, are met in the case of our product function, Axiom 1, concerning extreme values, is not because $p_\gamma(y_i;z) \in [0,1]$ whilst $\frac{z}{y} \in [1,\infty)$.

These properties of $P^{\infty}_{\gamma,\omega}$ are illustrated in Table 1.

[TABLE 1 ABOUT HERE]

The new classes of indices permit the important distinction between transfer sensitivity and distribution sensitivity to be made much clearer. The measure $P^{\infty}_{\gamma,\omega}$ takes the transfer sensitivity of the *final* FGT measure in sum (11), while its distribution sensitivity is an average of the transfer sensitivities of *all* the FGTs in sum (11). By consequence, the measure allows to some extent to combine relatively high degrees of transfer sensitivity with relatively low degrees of distribution sensitivity or vice versa. The simple FGT indices do not allow this much flexibility, nor do any other poverty indices we know of.

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Figure 1: Poverty contribution functions of P^{∞}, the Watts index and P_{α} for $\alpha = 0,1,2,3$

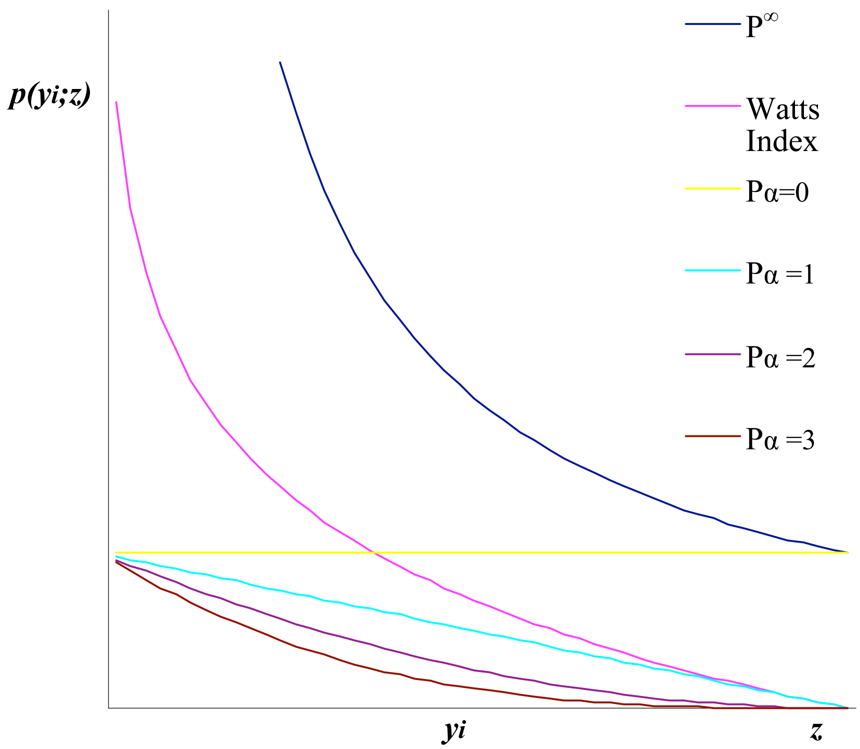

Table 1: Different combinations of parameters ω and γ, $\omega > \gamma$.

ω γ	0	1	2	3	4	∞
0	X	$P_{\alpha=0}$	$P_{\alpha=0}+P_{\alpha=1}$	$P_{\alpha=0}+P_{\alpha=1}+P_{\alpha=2}$	$P_{\alpha=0}+P_{\alpha=1}+P_{\alpha=2}+P_{\alpha=3}$	$P^{\infty}_{\gamma=0}$
1	X	X	$P_{\alpha=1}$	$P_{\alpha=1}+P_{\alpha=2}$	$P_{\alpha=1}+P_{\alpha=2}+P_{\alpha=3}$	$P^{\infty}_{\gamma=1}$
	X	X	X	$P_{\alpha=2}$	$P_{\alpha=2}+P_{\alpha=3}$	$P^{\infty}_{\gamma=2}$
3	X	X	X	X	$P_{\alpha=3}$	$P^{\infty}_{\gamma=3}$
4	X	X	X	X	X	$P^{\infty}_{\gamma=4}$
∞	X	X	X	X	X	X