ECINEQ

Society for the Study of Economic Inequality

Working Paper Series

Counting poverty orderings and deprivation curves

Ma. Casilda Lasso de la Vega

ECINEQ WP 2009 – 150

ECINEQ 2009-150
December 2009

www.ecineq.org

Counting poverty orderings and deprivation curves*

Ma. Casilda Lasso de la Vega†
University of the Basque Country

Abstract

Most of the data available for measuring capabilities or dimensions of poverty is either ordinal or categorical. However, the majority of the indices introduced for the assessment of multidimensional poverty behave well only with cardinal variables. The counting approach introduced by Atkinson (2003) concentrates on the number of dimensions in which each person is deprived, and is an appropriate procedure that deals well with ordinal and categorical variables. A method to identify the poor and a number of poverty indices has been proposed taking this framework into account. However, the implementation of this methodology involves the choice of a minimum number of deprivations required in order to be identified as poor. This cut-off adds arbitrariness to poverty comparisons. The aim of this paper is two-fold. Firstly, we explore properties which allow us to characterize the identification method as the most appropriate procedure to identify the poor in a multidimensional setting. Then the paper examines dominance conditions in order to guarantee unanimous poverty rankings in a counting framework. Our conditions are based on simple graphical devices that provide a tool for checking the robustness of poverty rankings to changes in the identification cut-off, and also for checking unanimous orderings in a wide set of multidimensional poverty indices that suit ordinal and categorical data.

Keywords: Multidimensional poverty measurement, deprivation, counting approach, dimension adjusted headcount ratio, ordinal data
JEL Classification: I30, I32, D63

* This research has been partially supported by the Spanish Ministerio de Educación y Ciencia under project SEJ2006-05455, cofunded by FEDER and by Basque Government under the project GIC07/146-IT-377-07. The author gratefully acknowledges the hospitality of the Oxford Poverty and Human Initiative Centre (OPHI) of the University of Oxford where she was visiting when the fist version of this paper was prepared. Preliminary versions were presented in the OPHI seminar and in the LGS6 meeting (Japan, 2009). The author wants to thank Sabina Alkire, James Foster, Masood Awan, Maria Emma Santos, Xiaolin Wang and the rest of the participants in these events for their comments and suggestions, and her colleague Ana Urrutia for her help and support.

† **Address of correspondence:** casilda.lassodelavega@ehu.es

1. INTRODUCTION

In recent years there has been considerable agreement that poverty is a multidimensional phenomenon and great efforts have been made from both a theoretical and an empirical point of view, trying to assess multidimensional poverty.[2] However, since most of the data available to measure capabilities or dimensions of poverty is either ordinal or categorical, only indices that behave well with this sort of variable should be used in empirical applications.

The counting approach introduced by Atkinson (2003) focuses on the number of dimensions in which each person is deprived, and is an appropriate procedure that deals well with ordinal and categorical variables. Based on this framework there are two recent contributions.

On the one hand, Alkire and Foster (2008) propose a new framework for measuring multidimensional poverty that includes an identification procedure and a way of aggregation. The identification step extends the traditional union and intersection approaches and incorporates two cut-offs. The first has to do with the traditional identification of the poor within each dimension using a dimension-specific poverty line. In the second step, a counting approach is used to identify the poor people using a threshold of the number of dimensions in which a person should be deprived in order to be identified as multidimensional poor. Actually Alkire and Foster (2008) have explicitly formulated and analyzed this identification procedure although similar methods had already been used in the literature, for instance Mack and Lindsay (1985), Gordon et al. (2003).A characterization of this identification step is provided in Section 2.

As regards the aggregation step, Alkire and Foster propose the Foster-Greer-Thobercke measures (Foster et al. (1984)) appropriately adjusted to the identification procedure. Specifically, the first of their measures, the *adjusted headcount ratio*, defined as the average of the number of deprivations suffered by the poor, is well suited for working with ordinal data. It also fulfils the dimensional monotonicity property, which means that it will increase if a person already identified as poor becomes deprived in an additional dimension.

The second contribution based on a counting approach is Bossert et al. (2009) that obtain a class of counting measures and generalizes Alkire and Foster indices.

[2] A comprehensive survey on multidimensional poverty can be found in Chakravarty (2009).

In general, the choice of either the identification cut-offs, or the indices, adds arbitrariness to poverty comparisons, and different selections can lead to contradictory results. For this reason it may be of interest to investigate conditions to guarantee that comparisons be unanimous to the different choices.[3] There exists a branch of the literature devoted to establishing dominance criteria which provide unanimous orderings when comparisons are made at a variety of poverty thresholds and measures. Zheng (2000) provides a comprehensive survey of dominance conditions in the poverty unidimensional field. In this paper, we take this literature as a starting point, and more specifically the basic papers by Shorrocks (1983), Foster (1985) and Foster and Shorrocks (1988a, 1988b). In particular we investigate circumstances in which two vectors, which represent the number of deprivations felt by each person, may be unanimously ranked regardless of the identification cut-off and of the poverty measure. In Section 3 we will show that if the ranking provided by the *multidimensional headcount ratio* is unambiguous over all the admissible identification thresholds, then agreement is guaranteed over all counting measures that satisfy monotonicity. A similar result is obtained with respect to the *adjusted headcount ratio*: rankings provided by this latter index are equivalent to agreement over all counting measures that fulfill monotonicity and distribution sensitivity. It is also showed that these orderings coincide with first and second degree stochastic dominance respectively. These results are by no means surprising. Atkinson (1987) derives a similar conclusion as regard the headcount ratio in the unidimensional poverty field. In turn, Foster and Shorrocks (1988a, 1988b) characterize the poverty orderings obtained from the Foster-Greer-Thobercke measures (Foster et al. (1984)), and establish the equivalence between poverty rankings and stochastic dominance.

The implementation of these conditions is based on two different types of curves we call *dimension deprivation curves*, introduced in Section 3. The first one, which we call the *FD* curve, represents the multidimensional headcount ratio for all the admissible cut-offs.[4] The second type of curve, henceforth *SD* curves, represent in the same picture the *headcount*

[3] The robustness of poverty measures as regards the number of deprivations chosen to identify the poor has already been addressed by Batana (2008) and Subramanian (2009). The former proposes to follow the procedure introduced by Davidson and Duclos (2006) and already used by Batana and Duclos (2008), to check the robustness of the adjusted headcount ratio. They suggest a statistical dominance test based on the empirical likelihood ratio. In turn, S. Subramanian introduces the *deprivation distribution profile,* a graphical device to check the robustness of the headcount ratio when different cut-offs are selected. The differences of this work with our paper will be discussed later.

[4] This curve is quite similar to the *deprivation distribution profile* proposed by Subramanian (2009), although, in our opinion, two main differences can be pointed out. On the one hand and following the traditional procedure, since the distributions we are concerned about are discontinuous, we propose to represent this cumulative curve as a step function which is right-continuous. On the other hand, to our knowledge, S. Subramanian does not derive dominance conditions in his paper.

ratio, the *adjusted headcount ratio* and the *average deprivation share* according to Alkire and Foster (2008)'s proposal.

Since the Lorenz curve was introduced in the literature, a number of cumulative curves have been widely used to check unanimous orderings in the inequality, poverty, and polarization fields.[5] In this connection, we will show that the curves proposed in this paper become a powerful tool for checking unanimous orderings according to a wide class of counting measures. They also avoid the choice of an arbitrary identification cut-off and offer a useful way to determine the bounds of the number of dimensions for which multidimensional comparisons are robust. As the *multidimensional headcount ratio* and the *adjusted headcount ratio* behave particularly well with ordinal and categorical data, the *dimension deprivation curves* play a key role in making poverty comparisons when data is ordinal. The paper finishes with some concluding remarks.

2. A COUNTING POVERTY APPROACH

1.1. *Notation and basic definitions*

We consider a population of $\mathrm{n} \geq 2$ individuals endowed with a bundle of $d \geq 2$ attributes considered as relevant to measure poverty. The number of dimensions is given and fixed. In a counting approach, poverty is measured taking into consideration the number of dimensions in which people are deprived. In this framework it is implicitly assumed that comparing each person's achievements with the respective dimension poverty lines, determines whether the individual is deprived or not in each attribute. We assume that the dimensions are represented by binary variables and characteristics of individual *i* are identified by a *deprivation vector* $g_i \in \{0,1\}^d$, whose typical component *j* is defined by $g_{ij} = 1$ when individual *i* is deprived in attribute *j* and $g_{ij} = 0$ otherwise. For simplicity we assume that all the dimensions are equally weighted, although similar conclusions may be derived if different fixed weights are attached to the different dimensions.

[5] Among them the curves proposed by Foster and Shorrocks (1988a, 1988b), the TIP curves introduced by Jenkins and Lambert (1997), the polarization curve introduced by Foster and Wolfson (1992) and more recently the proposal of Shorrocks (2009) to derive unemployment indices.

Let's denote by $c_i \in \{0,1,..,d\} = C$ the number of dimensions in which person i is deprived, that is, $c_i = \sum_{1\le j\le d} g_{ij}$. The vector $c = (c_1,...,c_n) \in C^n$ is referred to as the *vector of deprivation counts*. This vector plays an important role in the poverty measurement when ordinal data are involved. In fact this vector is invariant if the achievement levels and the poverty lines are transformed under the same monotonic transformations, and this is a crucial property when the achievements or capabilities are measured with ordinal variables. We will denote by $\overline{c}$ the permutation of c in which the number of deprived dimensions have been arranged in decreasing order, that is, $\overline{c}_i \ge \overline{c}_{i+1}$ for $i = 1,...,n$. Hence people are ranked from the most deprived to the least. Let $G = \bigcup_{n\ge 1} C^n$ be the set of all admissible vectors of deprivation counts.

We will say that the vector *c'* is obtained from the vector *c* by a *permutation* if $\overline{c}' = \overline{c}$; by a *replication* if $c' = (c, c, ..., c)$; by an *increment* if $c'_i > c_i$ for some i and $c'_j = c_j$ for all $j \ne i$; and by a *deprived dimension (regressive) transfer* if $c'_i > c_i > c_j$, $c'_i + c'_j = c_i + c_j$; $c'_k = c_k$ for all $k \ne i, j$.

2.2. *The identification of the poor*

Since Sen (1976) any poverty measure should consist of a method to identify the poor and an aggregative measure.

Two main methods have been used in the identification step in the multidimensional setting, referred to as the 'union' and the 'intersection' approaches respectively. Whereas the union procedure identifies the poor as someone who is deprived in at least one dimension, the intersection definition requires a poor person to be deprived in all dimensions. These methods present well known drawbacks when the number of poverty dimensions is great. Whereas "almost nobody" is identified as poor with the intersection approach, "almost everybody" is poor with the union identification.

There is an intermediate procedure, formalized by Alkire and Foster (2008), which proposes to identify a person as poor if they are deprived in at least *k* dimensions. According to this method, person *i* is identified as poor if $c_i \ge k$, i.e., the number of dimensions in which they are deprived is at least *k*; and person *i* is non-poor otherwise, that is, if $c_i < k$. For $k = 1$, this method coincides with the union approach, whereas for $k = d$, it is equivalent to the intersection approach. Following Alkire and Foster, we will use ρ_k to denote this procedure. In this framework the identification function is assumed to be the same for all the individuals.

The ρ_k method is simple and intuitive, and it may be of interest to examine the conditions which lead to ρ_k in a multidimensional setting.

For doing so, first of all we will assume that the function that identifies the poor satisfies a property of *dichotomization*, that ensures that identifying a person as poor depends only on each individual's deprivations. This property is formalized as follows:

Dichotomization: An identification procedure ρ is a *dichotomized identification function* if $\rho : \{0,1\}^d \to \{0,1\}$ links g_i, the vector of deprivations of individual *i*, with an indicator variable such that $\rho(g_i) = 1$ if person *i* is identified as poor and $\rho(g_i) = 0$ if person *i* is not poor.

In addition we will introduce a property for a dichotomized identification function. We think that a reasonable assumption is to require that if a person is considered as poor according to an identification method, then any other person deprived in equal or more dimensions should also be considered as poor. We call this property *Poverty Consistency* and it is formulated as follows:

Poverty Consistency. Let ρ be a dichotomized identification function. We say that ρ satisfies the *poverty consistency property* if given a person *i* with $\rho(g_i) = 1$ then $\rho(g_{i'}) = 1$ for all person i' such that $c_i \le c_{i'}$.

The following proposition characterizes the ρ_k identification method.

Proposition 1. *A non trivial dichotomized identification function* ρ *fulfils the poverty consistency property if and only if there exists some* $k \in \{1,...,d\}$ *such that* $\rho(g_i) = 1$ *if* $c_i \ge k$ *and* $\rho(g_i) = 0$ *otherwise.*

Proof. Since ρ is a non trivial identification method, there exists a person *i* such that $\rho(g_i) = 1$. Let be $k = \min_{i'=1,...,n} \{c_{i'} / \rho(g_{i'}) = 1\}$. For definition $\rho(g_{i'}) = 0$ if $c_{i'} < k$ and for poverty consistency, $\rho(g_{i'}) = 1$ if $c_{i'} \ge k$. The sufficiency of the proof is clear. Q.E.D.

As a consequence of this proposition, the only dichotomized identification method that is poverty consistent is ρ_k for some $k \in \{1,...,d\}$. Throughout this paper the poor are identified according to a ρ_k procedure.

Let's denote by Q_k and q_k respectively, the set and number of poor identified using the dimension cut-off *k*. For each vector *c* of deprivation counts, we define the *censored vector of deprivation counts*, denoted by $c(k)$, as follows: $c_i(k) = c_i$ if $c_i \geq k$, and $c_i(k) = 0$ if $c_i < k$.

2.3. *Aggregating deprivations with a counting measure*

The second problem in the poverty measurement is the aggregation of the deprivation of the poor. In what follows, a counting poverty measure P is a non-constant function whose typical image, denoted by $P_k(c)$, represents the level of poverty in a society with a vector of deprivation counts *c* and where the poor are identified according to a ρ_k procedure. The following four properties are the counterparts for a counting measure of the basic properties assumed in the poverty field:

Poverty Focus (PF): For all $k:1,...,d$, P_k remains unchanged if the number of deprived dimensions of a non-poor person decreases.

Dimensional Monotonicity (MON): For all $k:1,...,d$, $P_k(c) < P_k(c')$ if *c'* is obtained from *c* by an increment of a poor person.

Symmetry (SYM): For all $k:1,...,d$, $P_k(c) = P_k(c')$ if *c'* is obtained from *c* by a permutation.

Replication Invariance (RI): For all $k:1,...,d$, $P_k(c) = P_k(c')$ if *c'* is obtained from *c* by a replication.

Since poverty measurement is concerned with the deprivations of the poor people, the first two properties, postulated by Sen (1976) in the unidimensional setting, are considered as basic axioms for a poverty measure. Thus, PF requires that a poverty index should not depend on the non poor people's deprivations, and MON demands that poverty should increase if the number of deprived dimensions suffered by a poor person increases. It may be worth noting that PF ensures that $P_k(c) = P_k(c(k))$.

SYM and RI are also standard requirements for a poverty measure. SYM establishes that no other characteristic apart from the number of dimensions in which a person is deprived matters in defining a counting poverty index. In turn, RI allows us to compare populations of different sizes.

According to Sen (1976), a poverty measure should be sensitive to inequality among the poor, then the counterpart of the Pigou-Dalton transfer principle for a counting measure may be introduced as follows:

Transfer Sensitivity (TS): $P_k(c) < P_k(c')$ if *c'* is obtained from *c* by a deprived dimension transfer between two people that are poor before and after the transfer.

We define the following two inclusive classes of counting poverty measures:

$$\mathbf{P_1} = \{P\ counting\ poverty\ measure / P\ satisfies\ PF, MON, DDC, SYM,\ and\ RI\}$$

$$\mathbf{P_2} = \{P\ counting\ poverty\ measure / P\ satisfies\ PF, MON, DDC, SYM, RI\ and\ TS\}$$

Clearly $\mathbf{P_2} \subset \mathbf{P_1}$, and as will be shown, the inclusion is strict.

The first counting poverty measure introduced in the literature is the *multidimensional headcount ratio*, denoted by *H*, which is the percentage of poor people in the society. In other words, for each *k* identification cut-off, $H_k = q_k/n$ is the percentage of the population deprived in at least *k* dimensions. There are some advantages to this index, usually used to measure the incidence of poverty. One of them is that it can be used with ordinal and categorical data. There are also some shortcomings, since it is able to capture neither the intensity nor the inequality among the poor and violates MON, that is, it does not change if a person already identified as poor becomes deprived in an additional dimension in which the person was not poor previously.

The *adjusted headcount ratio, M,* introduced by Alkire and Foster (2008) is defined as the average of the number of deprivations suffered by the poor, that is, $M_k(c) = \frac{\sum_{1\le i\le n} c_i(k)}{nd}$. This index overcomes the drawbacks of the headcount ratio since it satisfies MON. However *M* does not belong to class $\mathbf{P_2}$, since although it satisfies a weaker version of TS, it violates TS as proposed in this paper.

It will be useful to note that $M_k(c)$ can be decomposed in the following way

(1) $$M_k(c)=\frac{1}{d}\left[d\,H_d+\left(H_{d-1}-H_d\right)(d-1)+\ldots+\left(H_k-H_{k+1}\right)(k)\right]$$

More information about poverty can be incorporated using the *average deprivation share across the poor* denoted by *A*, also introduced by Alkire and Foster (2008), which is defined as the mean among the poor, of the number of deprivations suffered by the poor, that is, $A_k=\frac{\sum_{1\le i\le n}c_i(k)}{q_k d}$. This index captures the intensity of poverty. Furthermore, *M* can be computed as the product of the *multidimensional headcount ratio* and the *average deprivation share across the poor*.

3. COUNTING POVERTY ORDERINGS

This section is concerned with how two vectors of deprivation counts are ranked in order to evaluate whether poverty is higher in one society than in another. Poverty rankings may be reversed depending on the identification threshold, or on the measure selected. Thus, in order to avoid contradictory results, poverty orderings require unanimous rankings for a set of identification cut-offs, or a class of poverty measures. As it is impossible to check unanimity for infinite pairwise comparisons, ordering conditions are derived to characterize unanimous agreement. Following the existing literature, given a counting poverty measure *P* we define the *partial ordering with respect to P*, denoted by $\precsim_P$, in the set of vectors of deprivation counts, by the rule[6]

$$c'\precsim_{\mathrm{P}} c \text{ if and only if } P_k(c')\ge P_k(c) \text{ for all } k:1,\ldots,d.$$

In this section we will examine the partial poverty orderings with respect to the *multidimensional headcount ratio, H*, and the *adjusted headcount ratio, M*.

[6] We follow Atkinson (1987) and adopt the weak definition of a partial ordering. Although not all the results derived in this paper hold for the other two levels (the semi-strict and the strict ones) similar conditions could be also obtained in these two cases.

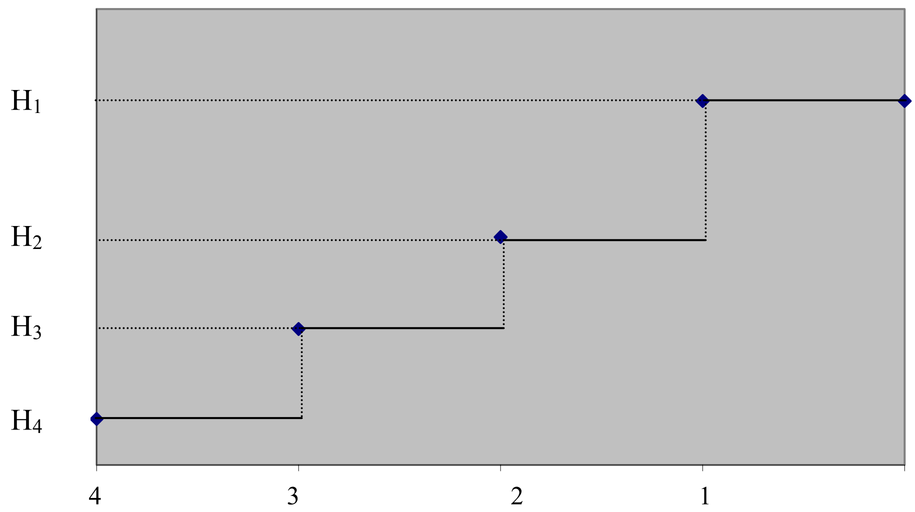
2.1. *Poverty ordering with respect to the multidimensional headcount ratio, H, and the FD-curve*

As already mentioned, given a cut-off of the number of dimensions, *k*, the multidimensional headcount ratio, H_k, gauges the percentage of people deprived in at least *k* dimensions. For any vector of deprivation counts it is possible to consider the graph of H as a function of the cut-off of the number of dimensions, with the number of dimensions ranked in decreasing order. We will refer to this curve as the *FD curve* associated with the vector of deprivation counts *c*. For any vector of deprivation counts *c,* the ordinates of the *FD* curve are computed as follows:

$$FD(c;d-k+\theta)=H_k,\ k=1,...,d\ ,\ \theta\in[0,1)$$

The following example helps to clarify this. Let's consider the vector of deprivation counts $c=(4,3,3,2,2,1,1,1,0,0)$ in a society of 10 individuals endowed with 4 attributes. The *FS* curve for this vector is displayed in Figure 1.

Figure 1. *Plotting the identification cut-offs and the headcount ratio: the FD curve.*



Some interesting properties of this curve may be mentioned. The *FS* curve is an increasing step function which is right-continuous. The horizontal axis displays the identification cut-offs ranked in decreasing order, and in the vertical axis, by definition, the multidimensional headcount ratio, H_k, is recovered. Two limiting curves correspond with the extreme situations: if nobody is deprived, the curve coincides with the horizontal axis;

whereas, if everybody is deprived in all dimensions, the curve becomes the parallel line to the horizontal axis through the point $(0,1)$.

It is clear from the graph that, for two vector of deprivation counts with the same population size, $c, c' \in C^n$, if $c' \precsim_H c$, that is, if $H_k(c') \geq H_k(c)$ for all $k: 1, ..., d$, then the *FD* curve of *c* must be below or to the left of the *FD* curve of *c'*. And the reverse is also true: if the *FD* curve of *c* is below or to the left of the *FD* curve of *c'* then $H_k(c') \geq H_k(c)$ for all $k: 1, ..., d$, and $c' \precsim_H c$. We get the following proposition:

Proposition 2. *For any* $c, c' \in C^n$ *vectors of deprivation counts, the following statements are equivalent:*

i) $FD(c'; p) \geq FD(c; p)$ *for all* $\mathrm{p} \in [0,1]$*;*

ii) $H_k(c) \leq H_k(c')$ *for all* $k = 1, ..., d$*;*

iii) $c_i \leq c'_i$ *for all* $i = 1, ..., n$*;*

iv) *c' may be obtained from c by a finite sequence of increments;*

v) $\sum_{1 \leq i \leq n} \varphi(c_i) \leq \sum_{1 \leq i \leq n} \varphi(c_i')$ *for all continuous, increasing functions* $\varphi : [0, d] \longrightarrow \mathbb{R}$

Proof. It is similar to the proof shown in Foster and Shorrocks (1988b) in Lemma 1 and Theorem 2.

This proposition shows that when the *FD* curve of a vector of deprivation counts lies above or to the right of the curve of other with the same population size, or equivalently, when these two vectors can be ordered with respect to *H*, then one may be obtained from the other by a sequence of increments. Consequently, any poverty measure belonging to class $\mathbf{P_1}$ will rank these two vectors exactly in the same way. Moreover, since both, H and the measures belonging to class $\mathbf{P_1}$, are replication invariant, the result also holds for vectors with different population sizes.

The reverse is also true. In fact, consider the class of counting measures: $P(c,k) = \frac{1}{n} \sum_{1 \leq i \leq n} \psi(c_i(k))$, with $\psi : [0, d] \longrightarrow \mathbb{R}$ a continuous strictly increasing convex function. It is quite simple to show that *P* belongs to class $\mathbf{P_1}$. Given any continuous increasing function $\varphi:[0,d]\longrightarrow\mathbb{R}$ and $\varepsilon>0$, then the measures $P_\varepsilon(c,k)=\frac{1}{n}\sum_{1\le i\le n}(\varepsilon\psi+\varphi)(c_i(k))$ also belong to class $\mathbf{P_1}$. Consequently, given two vectors *c* and *c'* with $P_\varepsilon(c,k)\le P_\varepsilon(c',k)$, when $\varepsilon\to 0$ we get statement v) in Proposition 2 and have the following result, that links the ordering with respect to *H* with first degree stochastic dominance:

Proposition 3. *For any* $c,c'\in G$ *vectors of deprivation counts:*

$$FD(c';p)\ge FD(c;p) \textit{ for all } \mathrm{p}\in[0,1]$$

if and only if $P_k(c')\ge P_k(c)$ *for all* $P\in\mathbf{P_1}$ *and for all identification cut-off* $k\in\{1,\ldots,d\}$.

This proposition reveals that, although *H* fails to satisfy MON, the ordering with respect to *H* is equivalent to agreement over all counting measures satisfying MON. Consequently, if the *FS* curves of two vectors of deprivation counts do not intersect, then all poverty counting measures satisfying MON will lead to the same verdict.

By contrast, when the curves intersect, there are two possibilities in order to obtain unanimous ranking: either restricting the set of measures, as shown in section 2.2, or limiting the admissible cut-offs, as will be developed in section 2.3.

2.2. *Poverty ordering with respect to the adjusted headcount ratio, M, and the SD curve*

One interesting feature of the *FS* curve introduced in the previous section is that, given a vector *c* and a threshold *k*, it is straightforward to proof from the decomposition shown in equation (1) that the area beneath the curve of the censored vector, $FS(c(k))$, is equal to $d\,M_k$. Thus, even if a conclusive poverty verdict could not be reached with the *H* ordering, it would be possible to get unanimous rankings with respect to *M*.

As usual, we propose constructing the *SD* curve, for any vector *c*, plotting the headcount ratio against the adjusted headcount ratio, that is, pairs of points (H_k,M_k). We also plot two extreme points $(0,0)$ as the start of the curve, and $(1,M_1)$, as the end of the curve. Then we join the dots. Figure 2 shows the *SD* curve associated with the vector *c* in the previous example:

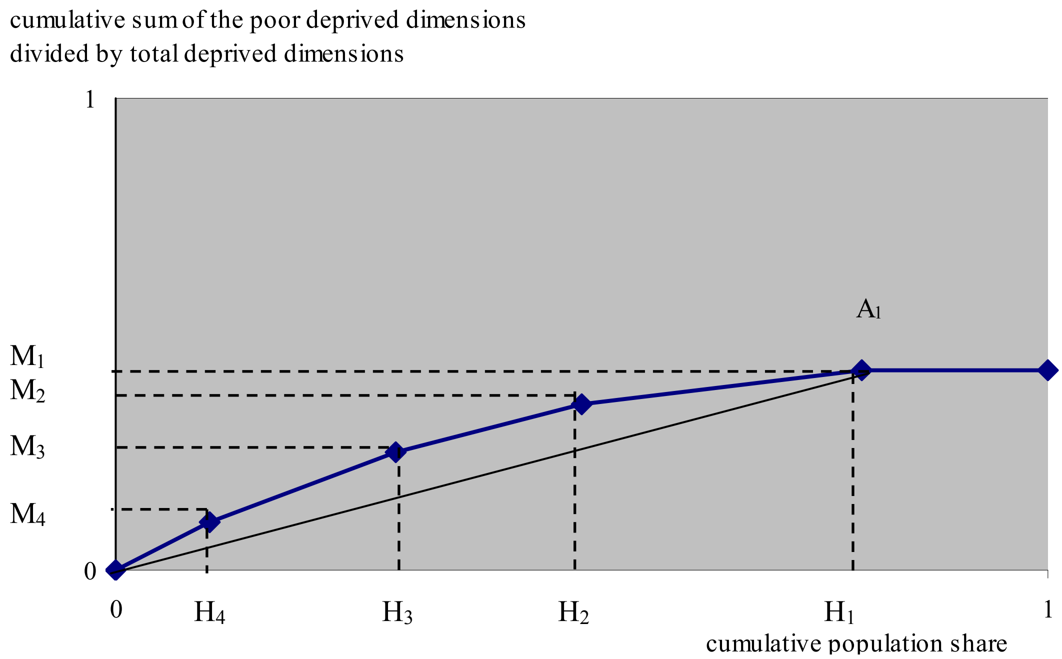
It may be worth noting that, for any vector of deprivation counts *c*, ranked from the most deprived to the least, the *SD* curve can equivalently be defined in the following way: for each integer $p = 0,...,n-1$ the ordinate of the curve is computed as the cumulative of the sum of the total number of deprivations experienced by the first *p* people divided by the total number of deprivations that could possibly experienced by all people. At intermediate points the curve is determined by linear interpolation. Thus, the ordinates of the *SD* curve are computed as follows:

$$SD(c;0) = 0$$

$$SD\left(c;\frac{p}{n}\right) = \frac{1}{nd}\sum_{1\le i\le p} c_i \ , \ p = 1,...,n$$

$$SD\left(c;\frac{p+\theta}{n}\right) = \frac{1}{nd}\sum_{1\le i\le p}\left(c_i + \theta c_{p+1}\right), \ p = 0,...,n-1, \ \theta \in [0,1]$$

Figure 2. *Plotting the headcount ratio and the adjusted headcount ratio: the SD curve.*



Some interesting properties of this curve may be mentioned. First of all, the ordinates of this curve are replication invariant, and are also invariant to permutation of *c*. The graph, as displayed in Figure 2, begins at the origin, and is a continuous non-decreasing concave function.

There are two bounding curves which correspond with the extreme situations of minimum and maximum deprivation. If nobody is deprived, the curve coincides with the horizontal axis. By contrast, if everybody is deprived in all dimensions, the curve becomes the diagonal line.

In general, the slope of the curve is to change *d* times, as many as the number of dimensions considered. Each point p/n at which the curvature of the curve changes, yields the percentage of people deprived in at least as many dimensions as person *p*. If we call this number *k*, we find that the adjusted headcount ratio, H_k, is recovered in these points. In contrast, the vertical axis displays, by definition, the dimension adjusted headcount ratio M_k. Thus, the first time the slope changes corresponds to the headcount ratio according to the intersection procedure, H_d. The last time, when the curve becomes horizontal, yields the headcount ratio as regards the union procedure, H_1. At this point the curve reaches its maximum value which corresponds to the ratio between the sum of the total number of deprivations experienced by all the people, and the total number of deprivations that could possibly be experienced, that is, M_1 according to Alkire and Foster (2008) designation.

The *average deprivation share across the poor*, A_k, is also represented in the graph by the slope of the ray from (0,0) to $\left(p, DD(p)\right)$.

The following proposition is based on the results established by Marshall and Olkin (1979, propositions 4.A.2 and A.B.2) for vectors with the same number of components:

Proposition 4. *For any* $c, c' \in C^n$ *vectors of deprivation counts, the following statements are equivalent:*

i) $SD(c'; p) \geq SD(c; p)$ *for all* $\mathrm{p} \in [0,1]$*;*

ii) $M_k(c) \leq M_k(c')$ *for all* $k = 1, \ldots, d$ *;*

iii) $\sum_{1\le i\le p} \overline{c}_i \le \sum_{1\le i\le p} \overline{c}_i^{'}$ *for all* $p = 1,...,n$ *;*

iv) *c' may be obtained from c by a finite sequence of permutations, increments and/or deprived dimension transfers;*

v) $\sum_{1\le i\le n} \varphi(c_i) \le \sum_{1\le i\le n} \varphi(c_i^{'})$ *for all continuous, increasing and convex functions* $\varphi : [0,d] \longrightarrow \mathbb{R}$.

In this proposition it is established that when the *SD* curve of a vector *c'* lies above the curve of another, *c*, with the same population size, or equivalently, when these two vectors can be ordered with respect to *M*, then one may be obtained from the other by a sequence of increments and/or permutations. Consequently, any poverty measure belonging to class $\mathbf{P_2}$ will rank these two vectors in exactly the same way. In addition, as the deprivation curves are invariant under replication, and the same holds for any measure $P \in \mathbf{P_2}$, the result also holds for vectors with different population sizes.

The reverse is also true and the proof is completely similar to the corresponding result in the previous section. So we get

Proposition 5. *For any* $c, c' \in G$ *vectors of deprivation counts:*

$$SD(c'; p) \ge SD(c; p) \text{ for all } \mathrm{p} \in [0,1]$$

if and only if $P_k(c') \ge P_k(c)$ *for all* $P \in \mathbf{P_2}$ *and for all identification cut-off* $k \in \{1,...,d\}$.

Then, this result reveals that although *M,* the dimension adjusted headcount ratio, violates TS, if two vectors of deprivation counts can be unanimously ranked by M_k at all cut-offs, then all poverty counting measures satisfying TS will rank societies in the same way. The equivalence between the ordering with respect to *M* and the second degree stochastic dominance is also established.

3.3. *Poverty ordering when the curves intersect.*

When the dimension deprivation curves introduced in the two previous sections intersect, it is still possible to establish dominance conditions by restricting the set of identification cut-offs. In fact, even if the curves of two vectors cross, there exists a threshold $k^* \in \{1,...,d\}$ that corresponds with the identification cut-off after which the intersection occurs. In other words, k^* ensures that the curves do not intersect for all $k = k^*,...,d$ and $C' = \{k^*,...,d\}$ becomes the relevant set for the cut-offs. A simple way to establish dominance conditions in these cases is to base comparisons on the censored vectors, and to develop the previous sections as shown in the following proposition. Taking into consideration the respective censored vectors, denoted by $c(k^*)$ and $c'(k^*)$, we get the following proposition:

Proposition 6. *For any* $c, c' \in G$ *vectors of deprivation counts:*

i) $$FD\big(c'(k^*);p\big) \geq FD\big(c(k^*);p\big) \text{ for all } p \in [0,1]$$

if and only if $P_k(c') \geq P_k(c)$ *for all* $P \in \mathbf{P_1}$ *and for all identification cut-off* $k \in \{1,...,k^*\}$.

ii) $$SD\big(c'(k^*);p\big) \geq SD\big(c(k^*);p\big) \text{ for all } p \in [0,1]$$

if and only if $P_k(c') \geq P_k(c)$ *for all* $P \in \mathbf{P_2}$ *and for all identification cut-off* $k \in \{1,...,k^*\}$.

The implication of this proposition is that, even when the *dimension deprivation* curves intersect, they allow us to obtain robust conclusions in a wide set of counting measures restricting the set of identification cut-offs. Since not all the admissible cut-offs are equally meaningful in poverty measurement, this result may be quite useful in empirical applications: when two deprivation vectors can not be unanimous ranked for all cut-offs, concentrating on the poorest people can lead to conclusive verdict.

CONCLUDING REMARKS

A counting approach based on the number of deprivations suffered by the poor is quite an appropriate framework to measure multidimensional poverty with ordinal or categorical data.

The choice of a cut-off to identify the poor, and a poverty measure to aggregate the data are two sources of arbitrariness and different selections may lead to contradictory conclusions. In this paper we derive dominance conditions in order to obtain unanimous rankings in a wide set of counting measures, and a set of identification cut-offs. Specifically we show that the orderings obtained from the *multidimensional headcount ratio* and from the *adjusted headcount ratio* are equivalent to agreement in two a wide set of counting poverty measures and correspond to what in the literature are known as first and second degree dominance conditions respectively.

The implementation of these conditions is based on two different types of *dimension deprivation* curves, which guarantee unanimous rankings of vectors of deprivation counts when they do not intersect. And, even if the curves cross, additional results are derived that lead to conclusive verdicts by restricting the admissible cut-offs in the identification of the poor. Thus, these curves become a useful way to determine the bounds of the number of dimensions for which counting poverty comparisons are robust and have been shown to play a key role in making poverty comparisons when the data is ordinal.

Empirical applications and the implementation of statistical inference tests are left for future research.

REFERENCES

Alkire, S. and J. Foster (2008), "Counting and multidimensional poverty measurement", OPHI Working Paper 2008-7.5.

Atkinson, A.B. (1987), "On the measurement of Poverty", *Econometrica,* 55, 244-63.

Atkinson, A.B. (2003), "Multidimensional deprivation: contrasting social welfare and counting approaches", *Journal of Economic Inequality,* 1, 51-65.

Batana, Y. M. (2008), "Multidimensional Measurement of Poverty in Sub-Saharan Africa", OPHI Working Paper 2008-13.

Batana, Y. M. and J. Y. Duclos, (2008), "Multidimensional Poverty Dominance: Statistical Inference and an Application to West Africa", CIRPÉE Working Paper 08-08, CIRPÉE.

Bossert, W., S.R. Chakravarty and C. D'Ambrossio (2009), "Measuring Multidimensional Poverty: the Generalized Counting Approach", ECINEQ Working Paper, 2009-129.

Chakravarty, S.R., (2009), *Inequality, Polarization and Poverty: Advances in Distributional Analysis,* Economic Studies in Inequality, Social Exclusion and Well-Being, **6** Silber, J. (Ed.), Springer, New York

Davidson, R. and J. Y. Duclos, (2006), "Testing for Restricted Stochastic Dominance", IZA Discussion Paper 2047, IZA.

Foster, J.E. (1985), "Inequality Measurement" in H. Peyton Young (ed.), *Fair Allocation*, Providence, RI: American Mathematical Society.

Foster, J.E. and A. Shorrocks, (1988a), "Poverty Orderings", *Econometrica,* 56, 173-177.

Foster, J.E. and A. Shorrocks, (1988b), "Poverty Orderings and Welfare Dominance", *Social Choice and Welfare,* 5, 179-198.

Foster, J.E. and M.C. Wolfson, (1992), "Polarization and Decline of the Middle Class: Canada and the U.S.". Discussion paper Vanderbilt University.

Foster, J.E., J. Greer and E. Thorbecke, (1984), "A Class of Decomposable Poverty Measures", *Econometrica*, 52, 761-766.

Gordon, G., S. Nandy, C. Pantazis, S. Pemberton and P. Townsend, (2003), *Child Poverty in the Development World*, The Policy Press, Bristol.

Jenkins, S. and P. Lambert, (1997), "Three 'I's of poverty curves, with an analysis of UK poverty trends", Oxford Economic Papers, 49, 317-327.

Mack, J. and S. Lindsay, (1985), *Poor Britain*, George Allen and Unwin Ltd., London.

Marshall A. W. and I. Olkin, (1979), *Inequalities: Theory of Majorization and its Applications*, Academic, New York.

Sen, A.K., (1976), "Poverty: An Ordinal Approach to Measurement", *Econometrica* 44, 219-231.

Shorrocks, A. (1983), "Ranking Income Distributions", *Economica*, 50, 3-17.

Shorrocks, A. (2009), "On the measurement of unemployment", *Journal of Economic Inequality*, 7 (3), 311-327.

Subramanian, S. (2009), "The deprivation distribution profile: A graphical device for comparing alternative regimes of multidimensional poverty", mimeo.

Zheng, B. (2000), "Poverty Orderings", Journal of Economic Surveys, 14 (4), 427-466.