Robust inequality comparisons

Rolf Aaberge
Magne Mogstad

Robust inequality comparisons[*]

Rolf Aaberge[†]
Magne Mogstad
Research Department, Statistics Norway

Abstract

This paper is concerned with the problem of ranking Lorenz curves in situations where the Lorenz curves intersect and no unambiguous ranking can be attained without introducing weaker ranking criteria than first-degree Lorenz dominance. To deal with such situations Aaberge (2009) introduced two alternative sequences of nested dominance criteria for Lorenz curves which was proved to characterize two separate systems of nested subfamilies of inequality measures. This paper uses the obtained characterization results to arrange the members of two different generalized Gini families of inequality measures into subfamilies according to their relationship to Lorenz dominance of various degrees. Since the various criteria of higher degree Lorenz dominance provide convenient computational methods, these results can be used to identify the largest subfamily of the generalized Gini families and thus the least restrictive social preferences required to reach unambiguous ranking of a set of Lorenz curves. From the weight-functions of these inequality measures we obtain intuitive interpretations of higher degree Lorenz dominance, which generally has been viewed as difficult to interpret because they involve assumptions about third and higher derivatives. To demonstrate the usefulness of these methods for empirical applications, we examine the time trend in income and earnings inequality of Norwegian males during the period 1967-2005.

Keywords: The Lorenz curve, Lorenz dominance, rank-dependent measures of inequality, the Gini coefficient, generalized Gini families of inequality measures.
JEL Classification: D31, D63.

1 Introduction

Ranking Lorenz curves in accordance with first-degree Lorenz dominance means that the higher of non-intersecting Lorenz curves is preferred. The normative significance of this criterion follows from the fact that the higher of two non-intersecting Lorenz curves can be obtained from the lower Lorenz curve by means of rank-preserving income transfers from richer to poorer individuals, which means that the criterion of first-degree Lorenz dominance is consistent with the Pigou-Dalton principle of transfers. When one Lorenz curve lies above another Lorenz curve, the higher Lorenz curve therefore displays less inequality than the lower Lorenz curve. However, since Lorenz curves may intersect, which is often the case in applied research, other approaches than first-degree Lorenz dominance are needed to reach conclusions.

Although the theoretical literature offers more general dominance criteria for ranking Lorenz curves,[1] these methods are generally viewed as hard to implement and the results difficult to interpret because they involve assumptions about third and higher derivatives (see e.g. Atkinson, 2003). Thus, most empirical studies rely exclusively on one or a few summary measures of inequality to achieve rankings of intersecting Lorenz curves. A concern is, however, that the conclusions reached are sensitive to the more or less arbitrary choice of inequality measures. It is, therefore, due time to start bridging the wide gap between the theoretical and the empirical strand of the literature concerned with rankings of intersecting Lorenz curves. That is the focus of this paper.

Generalized Lorenz dominance approach. As demonstrated by Aaberge (2009) two alternative dominance criteria emerge as attractive generalizations of first-degree Lorenz dominance; one that aggregates the Lorenz curve from below (second-degree upward Lorenz dominance) and the other that aggregates the Lorenz curve from above (second-degree downward Lorenz dominance). Since first-degree Lorenz dominance implies second-degree upward as well as downward Lorenz dominance, it follows that both methods preserve first-degree Lorenz dominance and thus are consistent with the Pigou-Dalton principle of transfers.

However, the transfer sensitivity of these criteria differ in the sense that second-degree upward Lorenz dominance place more emphasis on transfers occurring in the lower rather than in the upper part of the income distribution, whereas second-degree downward Lorenz dominance is most sensitive to transfers that occur in the upper part of the income distribution. This means that the criterion of second-degree upward Lorenz dominance requires a transfer of money from a richer to a poorer person to be more equalizing the lower it occurs in the income distribution, provided that the proportion of individuals

between the donors and receivers is fixed. By contrast, the criterion of second-degree downward Lorenz dominance requires this type of transfer to be more equalizing the higher it occurs in the income distribution.

For situations where neither upward nor downward Lorenz dominance provide unambiguous rankings of Lorenz curves, Aaberge (2009) introduced two hierarchical sequences of Lorenz dominance criteria and moreover explored what restrictions various Lorenz dominance criteria place on the weight-functions of the general Mehran-Yaari family of rank-dependent measures of inequality. Furthermore, by introducing appropriate general principles of transfer sensitivity, Aaberge (2009) demonstrated that these criteria can be given a normative justification.

Inequality measures approach. An alternative and more common strategy for achieving rankings of intersecting Lorenz curves is to apply summary measures of inequality, which explains why numerous alternative measures of inequality are introduced in the literature. The most widely used measure of inequality is the Gini coefficient, which is equal to twice the area between the Lorenz curve and its equality reference. But since no single measure can reflect all aspects of inequality exhibited by the Lorenz curve, the importance of using alternative measures to the Gini coefficient is universally acknowledged.

As proposed by Mehran (1976), we may use weighed sums of the income share deviations as alternative rank-dependent measures of inequality to the Gini coefficient. The family of rank-dependent measures of inequality, which includes the Gini coefficient, can alternatively be expressed as a weighted area between the Lorenz curve $(L(u))$ and its equality reference (u). The chosen specification of the rank-dependent weight-function, which may be considered as the preference function of a social planner, clarifies whether concern about inequality is particularly related to the lower, the central or the upper part of the income distribution. Thus, the functional form of the weight-function exhibits the inequality aversion profile of the corresponding measure of inequality. Roughly speaking, we may say that a rank-dependent measure of inequality exhibits downside inequality aversion when the weight-function gives more emphasis to the deviation between the Lorenz curve and its equality reference, $u - L(u)$, for small u than for large u. By contrast, when the weight-function gives more weight to the deviation $u - L(u)$ for large u than for small u, we may say that the corresponding inequality measure exhibits upside inequality aversion.

---

[1] See e.g. Kolm (1976), Shorrocks and Foster (1987), Davies and Hoy (1995), Zoli (1999) and Aaberge (2000, 2009).

Theoretical contributions. The purpose of the theoretical part of this paper is to characterize the relationship between dominance criteria and measures of inequality and explore what restrictions various Lorenz dominance criteria place on the weight-functions of two alternative generalized Gini families of rank-dependent measures of inequality, as well as to provide a normative justification of these criteria by relying on appropriate general principles of transfer sensitivity introduced by Aaberge (2009). As is demonstrated in Section 2, second-degree Lorenz dominance forms a natural basis for the construction of two separate hierarchical sequences of partial orderings (dominance criteria), where one sequence places emphasize on changes that occur in the lower part of the Lorenz curve whereas the other places emphasize on changes that occur in the upper part of the Lorenz curve. Both sequences of dominance criteria turn out to depart from the Gini coefficient; one requires higher degree of downside inequality aversion and the other higher degree of upside inequality aversion than what is exhibited by the Gini coefficient. This means that that the Gini coefficient in general favors neither the lower nor the upper part of the Lorenz curves. Section 2 further shows that if we restrict the ranking problem to Lorenz curves with equal Gini coefficients, second-degree upward and downward dominance coincide in the sense that a Lorenz curve L_1 that second-degree upward dominates a Lorenz curve L_2 is always second-degree downward dominated by L_2. Moreover, Section 2 clarifies the relationship between ordering conditions for the Gini coefficient and the family of rank-dependent measures of inequality in the case of singly intersecting Lorenz curves.

As demonstrates in Section 3, the hierarchical and nested structure of the dominance criteria allow us to identify the lowest degree of dominance required to reach unambiguous rankings of Lorenz curves. Moreover, Section 3 shows that the two hierarchical sequences of Lorenz dominance criteria can be used to divide two generalized Gini families of rank-dependent measures of inequality into two corresponding hierarchical systems of nested subfamilies that offer two different inequality aversion profiles; one exhibits successively higher degrees of downside inequality aversion whereas the other exhibits successively higher degrees of upside inequality aversion. Since the criteria of Lorenz dominance provide convenient computational methods, these results will be used to identify the largest subfamily of the generalized Gini families and thus the least restrictive social preferences required to reach unambiguous ranking of any given set of Lorenz curves. From the weight-functions of these inequality measures, we obtain intuitive interpretations of higher degree Lorenz dominance, which generally has been viewed as difficult to interpret. By doing so, we connect the dominance and the inequality measures approach to the ranking of intersecting Lorenz curves.

Empirical application. The evolution of earnings and income inequality in developed countries over the last few decades is one of he most extensively researched topics in economics. In a widely cited review of the literature, Gottschalk and Smeeding (1997) conclude that earnings and income inequality

increased in most OECD countries during the 1980s and early 1990s. Moreover, they argue that many countries with fairly low levels of inequality experienced some of the largest increases in inequality. These conclusions rest on numerous empirical studies relying exclusively on one or a few summary measures of inequality, like the Gini coefficient. A concern is, however, that the conclusions reached are sensitive to the more or less arbitrary choice of inequality measure.

Section 4 examines the time trend in income and earnings inequality of Norwegian males during the period 1967-2005, showing how the dominance and the inequality measures approach to the ranking of intersecting Lorenz curves can be combined in a coherent way in empirical analysis. First, we identify the lowest degree of dominance, and thus the least restrictive social preferences, required to reach unambiguous rankings of the Lorenz curves over this period. Second, we pin down the largest subfamily of the generalized Gini families that is consistent with the actual ranking of these Lorenz curves. Finally, to ease the interpretation of the inequality aversion exhibited in the dominance results, we compute the weight-functions of these inequality measures.

2 The relationship between Lorenz dominance and the Gini coefficient

The Lorenz curve L for a cumulative income distribution F with mean μ is defined by

$$L(u) = \frac{1}{\mu} \int_0^u F^{-1}(t)\, dt, \ 0 \le u \le 1, \tag{1}$$

where $F^{-1}(t) = \inf\{x : F(x) \ge t\}$ is the left inverse of F. Thus, the Lorenz curve L(u) shows the share of total income received by the poorest 100 u per cent of the population.

Under the restriction of equal mean incomes, the problem of ranking Lorenz curves formally corresponds to the problem of choosing between uncertain prospects. This relationship has been utilized by e.g. Atkinson (1970) to characterize the criterion of non-intersecting Lorenz curves in the case of distributions with equal mean incomes. This was motivated by the fact that in cases of equal mean incomes the criterion of non-intersecting Lorenz curves is equivalent to second-degree stochastic dominance[2], which means that the criterion of non-intersecting Lorenz curves obeys the Pigou-Dalton principle of transfers. *The Pigou-Dalton principle of transfers* states that an income transfer from a richer to a poorer individual reduces income inequality, provided that their ranks in the income distribution are unchanged.

---

[2] For a proof see Hardy, Littlewood and Polya (1934) or Atkinson (1970).

To perform inequality comparisons with Lorenz curves we can deal with distributions with equal means, or alternatively simply abandon the assumption of equal means and consider distributions of relative incomes. The latter approach normally forms the basis of empirical studies of income inequality.

The standard criterion of non-intersecting Lorenz curves, called *first-degree Lorenz dominance*, is based on the following definition.[3]

Definition 2.1 *A Lorenz curve L_1 is said to first-degree dominate a Lorenz curve L_2 if*

$$L_1(u) \geq L_2(u) \quad for \ all \ \ u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$.

A social planner who prefers the dominating one of non-intersecting Lorenz curves favors transfers of incomes which reduce the differences between the income shares of the donor and the recipient, and is therefore said to be *inequality averse*.

In order to examine the relationship between various Lorenz dominance criteria and the measurement of inequality, we will rely on the *family of rank-dependent measures of inequality[4]*, defined by

$$J_P(L) = 1 - \int_0^1 P'(u) \, dL(u) = 1 - \frac{1}{\mu} \int_0^1 P'(u) F^{-1}(u) \, du \,, \tag{2}$$

where L is the Lorenz curve of the income distribution F with mean μ, and the weight-function P' is the derivative of a continuous, differentiable and concave function P defined on the unit interval where $P(0) = 0$ and $P(1) = 1$. To ensure that J_P has the unit interval as its range, the condition $P'(1) = 0$ is imposed on P. As demonstrated by Yaari (1987, 1988) and Aaberge (2001), the J_P-family represents a preference relation defined either on the class of distribution functions (F) or on the class of Lorenz curves (L), where P can be interpreted as a preference function of a social planner. The preference function P assigns weights to the incomes of the individuals in accordance with their rank in the income distribution. Therefore, the functional form of P reveals the attitude towards inequality of a

---

[3] Note that most analyses of Lorenz dominance apply a definition that excludes the requirement of strict inequality for some u.

[4] Mehran (1976) introduced the J_P-family by relying on descriptive arguments. For alternative normative motivations of the J_P-family and various subfamilies of the J_P-family we refer to Donaldson and Weymark (1980), Weymark (1981), Yaari (1988) and Aaberge (2001).

social planner who employs J_P to judge between Lorenz curves. The most well-known member of the J_P-family is the Gini coefficient, which is obtained by inserting for $P(u) = 2u - u^2$ in (2).

As demonstrated by Yaari (1988), the J_P-family of inequality measures characterizes the condition of first-degree Lorenz dominance when the functional form of the preference function P is being strictly concave. This means that J_P satisfies the Pigou-Dalton principle of transfers for concave P-functions. To deal with situations where Lorenz curves intersect a weaker principle than first-degree Lorenz dominance is called for. To this end it is normal to employ *second-degree upward Lorenz dominance* defined by

Definition 2.2 *A Lorenz curve L_1 is said to second-degree upward dominate a Lorenz curve L_2 if*

$$\int_0^u L_1(t)\,dt \geq \int_0^u L_2(t)\,dt \quad for\ all\ \ u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$.

The term upward dominance refers to the fact that the Lorenz curves are aggregated from below.[5] The aggregated Lorenz curve can be considered as a sum of weighted income shares, where the weights decrease linearly with increasing rank of the income receiver in the income distribution. Thus, a social planner who prefers the second-degree upward dominating of two intersecting Lorenz curves pays more attention to inequality in the lower than in the upper part of the income distribution. As proposed by Aaberge (2009), an alternative ranking criterion to second-degree upward Lorenz dominance can be obtained by aggregating the Lorenz curve from above.

Definition 2.3 *A Lorenz curve L_1 is said to second-degree downward dominate a Lorenz curve L_2 if*

$$\int_u^1 \left(1 - L_2(t)\right)dt \geq \int_u^1 \left(1 - L_1(t)\right)dt \quad for\ all\ \ u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$.

Note that both downward and upward Lorenz dominance of second degree preserve first-degree Lorenz dominance, since first-degree Lorenz dominance implies second-degree upward as well as

---

[5] Note that second-degree upward Lorenz dominance is equivalent to a normalized version of third-degree inverse stochastic dominance introduced by Muliere and Scarsini (1989).

second-degree downward Lorenz dominance. Consequently, both dominance criteria are consistent with the Pigou-Dalton principle of transfers. The choice between second-degree upward and downward Lorenz dominance clarifies whether or not equalizing transfers between poorer individuals should be considered more important than those between richer individuals.

As recognized by Muliere and Scarsini (1989), there is no simple relationship between third-degree stochastic dominance and second-degree upward Lorenz dominance, but second-degree upward Lorenz dominance is equivalent to third-degree upward inverse stochastic dominance. Thus, a general characterization of second-degree Lorenz dominance or third-degree inverse stochastic dominance in terms of ordering conditions for the utilitarian measures introduced by Kolm (1969) and Atkinson (1970) cannot be obtained. By contrast, the family of rank-dependent measures of inequality, defined by (2), forms a convenient basis for judging the normative significance of second-degree and higher degrees of Lorenz dominance.

To judge the normative significance of criteria for ranking intersecting Lorenz curves, more powerful principles than the Pigou-Dalton principle of transfers are needed. To this end, Kolm (1976) introduced *the principle of diminishing transfers*, which for a fixed difference in income considers a transfer from a richer to a poorer person to be more equalizing the further down in the income distribution it takes place.[6] As indicated by Shorrocks and Foster (1987) and Muliere and Scarsini (1989), the principle of diminishing transfers is, however, not consistent with second-degree upward Lorenz dominance.

Mehran (1976) introduced an alternative version of the principle of diminishing transfers by accounting for the difference in the proportion of individuals between donors and recipients of the income transfers, rather than for the difference in income. This principle, denoted first-degree downside positional transfer sensitivity (DPTS) by Aaberge (2009), proves to characterize second-degree upward Lorenz dominance. By contrast, a social planner who considers a given transfer of money from a richer to a poorer person to be more equalizing the higher it occurs in the income distribution, provided that the proportions of the population located between the receivers and the donors are equal, favors the principle of first-degree upside positional transfer sensitivity (UPTS). The UPTS characterizes second-degree downward Lorenz dominance. To ensure equivalence between second-degree upward Lorenz dominance and J_P-measures as decision criteria, Aaberge (2009) proved

---

[6] For a formal definition, see Kolm (1976).

8

that it is necessary to restrict the preference functions P to be concave with positive third derivatives.[7] By contrast, the condition of negative third derivative of P yields second-degree downward dominance.

An inequality averse social planner that supports the criterion of second-degree upward Lorenz dominance will act in line with the principle of first-degree DPTS and assign more weight to changes that take place in the lower part of the Lorenz curve than to changes that occur in the upper part of the Lorenz curve. By contrast, the criterion of second-degree downward Lorenz dominance emphasizes changes that occur in the upper part of the Lorenz curve. Thus, an inequality averse social planner who employs the criterion of second-degree downward Lorenz dominance acts in favor of the Pigou-Dalton principle of transfers and the principle of first-degree UPTS.

Since the third derivative of the P-function that corresponds to the Gini coefficient is always equal to 0, the Gini coefficient neither preserves second-degree upward Lorenz dominance nor second-degree downward Lorenz dominance apart from the case when the inequality in Definitions 2.2 and 2.3 holds strictly for $u = 1$ and $u = 0$, respectively.[8] Thus, the suggestion of Muliere and Scarsini (1989) that the Gini coefficient is coherent with second-degree upward Lorenz dominance requires a definition of second-degree Lorenz dominance that abandons the condition of strict inequality (for some $u \in \langle 0,1 \rangle$). However, by assuming that the Lorenz curves cross only once the following results hold.[9]

Proposition 2.1A. *Assume that L_1 and L_2 are singly intersecting Lorenz curves and L_1 crosses L_2 initially from above, and let $G(L_1)$ and $G(L_2)$ be the two corresponding Gini coefficients. Then the following statements are equivalent,*

(i) $J_P(L_1) < J_P(L_2)$ *for all P such that $P'(u) > 0, P''(u) < 0$ and $P'''(u) > 0$ for all $u \in (0,1)$*

(ii) $G(L_1) \le G(L_2)$.

(Proof in Appendix)

---

[7] Aaberge (2000) demonstrated that J_P defined by (2) satisfies Kolm's principle of diminishing transfers under conditions that depend on the shape of the preference function P as well as on the shape of the income distribution F.

[8] Aaberge (2000) gave an alternative interpretation of this property by demonstrating that the Gini coefficient attaches an equal weight to a given transfer irrespective of where it takes place in the income distribution, as long as the income transfer occurs between individuals with the same difference in ranks.

[9] Zoli (1999) provided a result similar to Proposition 2.1A for singly intersecting generalized Lorenz curves under the condition of equal means.

Proposition 2.1B. *Assume that L_1 and L_2 are singly intersecting Lorenz curves and L_2 crosses L_1 initially from above, and let $G(L_1)$ and $G(L_2)$ be the two corresponding Gini coefficients. Then the following statements are equivalent,*

(i) $J_P(L_1) < J_P(L_2)$ *for all P such that $P'(u) > 0, P''(u) < 0$ and $P'''(u) < 0$ for all $u \in (0,1)$*

(ii) $G(L_1) \leq G(L_2)$.

The proof of Proposition 2.1B can be achieved by following the line of reasoning used in the proof of Proposition 2.1A. Note that Proposition 2.1A can be considered as a dual version of the results of Shorrocks and Foster (1987) and Dardanoni and Lambert (1988) that clarify the relationship between third-degree (upward) stochastic dominance ordering conditions for the coefficient of variation and transfer sensitive measures of inequality. Proposition 2.2 gives results for the case of equal Gini coefficients. In this case, second-degree upward and downward Lorenz dominance "coincide" in the sense that a Lorenz curve L_1 that second-degree upward dominates a Lorenz curve L_2 will always be second-degree downward dominated by L_2. Thus, L_1 can be attained from L_2 by transfers that obey the principle of first-degree DPTS, whereas L_2 can be attained from L_1 by transfers that obey the principle of first-degree UPTS.

Proposition 2.2. *Let L_1 and L_2 be Lorenz curves with equal Gini coefficients. Then the following statements are equivalent,*

(i)   *L_1 second-degree upward dominates L_2*

(ii)   *L_2 second-degree downward dominates L_1*

(iii)   *$J_P(L_1) < J_P(L_2)$ for all P with $P'(u) > 0$ and $P''(u) < 0$ being such that J_P obeys the principle of first-degree DPTS.*

(iv)   *$J_P(L_2) < J_P(L_1)$ for all P with $P'(u) > 0$ and $P''(u) < 0$ being such that J_P obeys the principle of first-degree UPTS.*

(Proof in Appendix).

3 The relationship between general criteria of Lorenz dominance and generalized Gini families of inequality measures

To deal with situations where second-degree (upward or downward) Lorenz dominance does not provide unambiguous ranking of Lorenz curves, Aaberge (2009) introduced two hierarchical sequences of nested

Lorenz dominance criteria that proved to divide the family of rank-dependent measures of inequality (2) into two corresponding hierarchical systems of nested subfamilies. In this section, we use these results to divide two generalized Gini families of rank-dependent measures of inequality into two corresponding hierarchical systems of nested subfamilies that offer two different inequality aversion profiles; one exhibits successively higher degrees of downside inequality aversion whereas the other exhibits successively higher degrees of upside inequality aversion. As explained in Aaberge (2009), application of the criteria of upward Lorenz dominance requires social preferences with higher degree of aversion to downside inequality the higher is the degree of upward Lorenz dominance. A similar relationship holds between downward Lorenz dominance and aversion to upside inequality aversion. The highest degree of downside inequality aversion is achieved when focus is exclusively turned to the situation of the worst-off income recipient. Thus, the most downside inequality averse J_P-measure that is obtained as the preference function approaches

$$P_d(t) = \begin{cases} 0, & t = 0 \\ 1, & 0 < t \leq 1, \end{cases} \tag{3}$$

can be considered as the J_P-measure that exhibits the highest degree of downside inequality aversion. Inserting (3) in (2) yields

$$J_{P_d}(L) = 1 - \frac{F^{-1}(0+)}{\mu}. \tag{4}$$

Hence, the inequality measure J_{P_d} corresponds to *the Rawlsian maximin criterion*. As J_{P_d} is compatible with the limiting case of upward Lorenz dominance, the Rawlsian (relative) maximin criterion preserves all degrees of upward Lorenz dominance and rejects downward Lorenz dominance. By contrast, the J_P-measure that is obtained as P approaches[10]

$$P_u(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & t = 1, \end{cases} \tag{5}$$

exhibits the highest degree of upside inequality aversion. Inserting (5) in (2) yields

$$J_{P_u}(L) = 1 + \frac{F^{-1}(1)}{\mu}. \tag{6}$$

Thus, J_{P_u}, which we will denote *the relative minimax criterion*, is "dual" to the Rawlsian (relative) maximin criterion in the sense that it is compatible with the limiting case of downward Lorenz

---

[10] Note that the normalization condition $P'(0)=1$ is ignored in this case.

dominance. When the comparison of Lorenz curves is based on the relative minimax criterion, the preferred Lorenz curve is the one for which the largest relative income is smallest. The only transfers that decrease inequality are those from the richest unit to anyone else.

As will be demonstrated below, the extreme inequality aversion measure (4) is associated with *the extended Gini family of inequality measures* introduced by Donaldson and Weymark (1980) and Yitzhaki (1983). This family is defined by

$$G_i(L) = 1 - i(i+1)\int_0^1 (1-u)^{i-1} L(u)\,du = 1 - \frac{(i+1)}{\mu}\int_0^\infty (1-u)^i F^{-1}(u)\,du,\ i \geq 1. \qquad (7)$$

In order to introduce an alternative "generalized" Gini family of inequality measures, Aaberge (2000, 2007) draws on the practice from the statistical literature as motivation for using the moments of the Lorenz curve as a basis for defining *the Lorenz family of inequality measures*[11]

$$D_i(L) = 1 - (i+1)\int_0^1 u^{i-1} L(u)\,du = 1 - \frac{i+1}{\mu i}\int_0^\infty (1-u^i) F^{-1}(u)\,du,\ i = 1,2,\dots, \qquad (8)$$

which is associated with the relative minimax criterion (6). Note that $D_1 = G_1$ is equal to the Gini coefficient. As was demonstrated by Aaberge (2007), the Lorenz family of inequality measures $\{D_i : i = 1,2,\dots\}$ uniquely determines the Lorenz curve. Thus, we can, without loss of information, restrict the examination of inequality in an income distribution F to the family $\{D_i : i = 1,2,\dots\}$ of inequality measures.

As shown by the following expression

$$G_i(L) = 1 + (i+1)\sum_{k=1}^i (-1)^k \binom{i}{k} \frac{k}{k+1}(1 - D_k(L)),\quad i = 1,2,\dots. \qquad (9)$$

there is a one-to-one correspondence between the Lorenz family and the integer subfamily $\{G_i(L) : i = 1,2,\dots,r\}$ of the extended Gini family of inequality measures. Thus, the extended Gini subfamily $\{G_i(L) : i = 1,2,\dots,r\}$ is uniquely determined by the corresponding Lorenz subfamily

---

[11] The Lorenz family of inequality measures proves to be a subclass of the "illfare-ranked single-series Ginis" discussed by Donaldson and Weymark (1980) and Bossert (1990).

$\{D_i(L): i=1,2,...,r\}$ for any integer r. Note that expressions (7) and (8) show that G_i places more weight on changes in the lower and D_i on changes in the upper part of the Lorenz curve as i increases.

Based on Theorems 3.1 A, 3.1B, 3.2A and 3.2B in Aaberge (2009), we shall now demonstrate how the various Lorenz dominance criteria can be applied to evaluate the ranking properties of the Lorenz and the extended Gini families of inequality measures. Note that $\{G_i : i > 0\}$ is a subfamily of J_P formed by the following family of P-functions,

$$P_{1k}(t) = 1 - (1-t)^{k+1}, k \geq 0. \tag{10}$$

Differentiating P_{1k} defined by (10), we find that

$$P_{1k}^{(j)}(t) = \begin{cases} (-1)^{j-1} \dfrac{(k+1)!}{(k-j+1)!}(1-t)^{k-j+1}, & j=1,2,...,k+1 \\ 0 & , \quad j=k+2,k+3,... \end{cases} \tag{11}$$

As can be observed from (11) the weight function P_{1k} of the extended Gini family of inequality measures is a member of the family P_{1i}^* (and P_{1i}^{**}) in Aaberge (2009) for $k = i, i+1,...$. Thus, we get the following result from Theorem 3.2A of Aaberge (2009),[12]

Proposition 3.1. *Let L_1 and L_2 be members of $\mathbf{L}$. Then*

    (i) L_1 i^{th}-degree upward dominates L_2

implies

    (ii) $G_k(L_1) < G_k(L_2)$ for $k = i, i+1, i+2,...$

Equation (11) implies that $P_{1k}''(t) < 0$ for all $t \in \langle 0,1 \rangle$ when $k > 0$ and thus that the G_k-measures satisfy the Pigou-Dalton principle of transfers for $k > 0$. Moreover, $P_{1k}'''(t) > 0$ for all $t \in \langle 0,1 \rangle$ when $k > 1$. Hence all G_k for $k > 1$ preserve second-degree upward Lorenz dominance. Further, the derivatives of P_k alternate in sign up to the $(k+1)^{th}$ derivative and $P_{1k}^{(j)}(1) = 0$ for all $j \leq k$. Thus, it follows from Theorem 3.2A of Aaberge (2009) that G_k preserves upward Lorenz dominance of degree k and obeys the principles of DPTS up to and including $(k-1)^{th}$-degree. The highest degree of downside inequality averse behavior occurs as $k \to \infty$, which corresponds to the inequality averse behavior of the Rawlsian (relative) maximin criterion. Thus, G_k satisfies all degrees of upward Lorenz dominance as $k \to \infty$. At the other extreme, as $k = 0$, the preference function $P_0(t) = t$, which means that J_{P_0}

---

[12] Muliere and Scarsini (1989) gave an alternative proof of Corollary 3.1A

does not obey the Pigou-Dalton principle of transfers or any principle of DPTS. The stated properties of the G_k-measures are summarized in the following proposition,

Proposition 3.2. *The extended Gini family of inequality measures defined by (7) has the following properties,*

(i) G_k *preserves upward Lorenz dominance of degree k and all degrees lower than k,*

(ii) G_k *obeys the Pigou-Dalton principle of transfers for $k > 0$,*

(iii) G_k *obeys the Pigou-Dalton principle of transfers and the principles of DPTS up to and including $(k-1)^{th}$ degree for k>1,*

(iv) *The sequence $\{G_k\}$ approaches 0 as $k \to 0$.*

(v) *The sequence $\{G_k\}$ approaches the Rawlsian relative maximin criterion as $k \to \infty$.*

Note that P_{1k} has negative derivatives (of any order) when $0 < k < 1$. Thus, G_k for $0 < k < 1$ preserves downward Lorenz dominance of all degrees.

As demonstrated by Aaberge (2000), the Lorenz family of inequality measures is a subfamily of J_P formed by the following family of P-functions,

$$P_{2k}(t) = \frac{1}{k}\left((k+1)t - t^{k+1}\right), \quad k = 1, 2, \ldots \tag{12}$$

Differentiating P_{2k} defined by (24) yields

$$P_{2k}^{(j)}(t) = \begin{cases} \dfrac{k+1}{k}(1-t^k), & , j=1 \\ -(k+1)(k-1)(k-2)\ldots(k-j+2)t^{k-j+1}, & j=2,3,\ldots,k+1 \\ 0 & , j=k+2,k+3,\ldots \end{cases} \tag{13}$$

By noting from (13) that the weight-function P_{2k} of the Lorenz family of inequality measures is a member of the family $\mathbf{P}_{2i}^*$ in Aaberge (2009) for $k = i, i+1, \ldots$, we obtain the following result from Theorem 3.2B of Aaberge (2009),

Proposition 3.3. *Let L_1 and L_2 be members of $\mathbf{L}$. Then*

(i) L_1 i^{th}-*degree downward dominates L_2*

implies

(ii) $D_k(L_1) < D_k(L_2)$ *for* $k = i, i+1, i+2, \ldots$

14

$$and \quad \frac{F_1^{-1}(1)}{\mu_1} < \frac{F_2^{-1}(1)}{\mu_2}.$$

The results of a similar evaluation of the Lorenz family of inequality measures as that carried out for the extended Gini family are summarized in the following corollary.

Proposition 3.4. *The Lorenz family of inequality measures defined by (8) has the following properties,*

(i) D_k *preserves downward Lorenz dominance of degree k and all degrees lower than k,*

(ii) D_k *obeys the Pigou-Dalton principle of transfers for k > 0 ,*

(iii) D_k *obeys the Pigou-Dalton principle of transfers and the principles of UPTS up to and including (k-1)*th *degree for k>1,*

(iv)   *The sequence* $\{D_k\}$ *approaches 0 as* $k \to \infty$.

(v)   *The sequence* $\{kD_k + 1\}$ *approaches the relative minimax criterion as* $k \to \infty$.

Note that the derivatives of P_{2k} alternate in sign when $-1 < k < 1$. Thus, D_k for $-1 < k < 1$ preserves upward Lorenz dominance of all degrees and approaches the Rawlsian relative maximin as k approaches -1. As demonstrated by Aaberge (2000), D_k approaches the Bonferroni coefficient as $k \to 0$.

Proposition 3.1 shows that the various degrees of upward Lorenz dominance are preserved by sub-families of the extended Gini measures of inequality, which divide the integer subscript subclass of the extended Gini family into nested subfamilies. Thus, the hierarchical sequence of nested upward Lorenz dominance criteria offers a convenient computational method for identifying the largest subfamily of the integer subscript extended Gini family of inequality measures that is consistent with the actual ranking of Lorenz curves. As demonstrated by Proposition 3.2, the various degrees of downward Lorenz dominance divide the Lorenz family of inequality measures into a similar sequence of nested subfamilies.

4. Empirical Application: Income and Earnings Inequality in Norway, 1967-2005

Below, we combine the dominance and inequality measures approach to examine the time trend in income and earnings inequality of Norwegian males during the period 1967-2005.

Data. Our data are based on administrative registers from Statistics Norway covering the entire resident population in Norway between 1967 and 2004. The unique individual identifier allows

merging information about individual characteristics, like age, with data on annual income and earnings taken from tax registers in each year. In the analysis, we employ two measures of income. First, we use a measure of annual pre-tax income including all taxable income after deductions. Secondly, we use a measure of earnings including all market income, from wages and self-employment. Individuals with missing observation on income or earnings are excluded. In each year 1967--2004, we include the entire population of males aged 20--65 who were alive and resident in that year. The reason for focusing on males is their role of breadwinner and primary wage-earners over most of this period.

It should be noted that we in the empirical analysis exclude individuals with incomes lower than the 1 percentile or higher than the 99 percentile, to avoid crossing of Lorenz curves due to measurement error and noise at the tails of the distributions.

Empirical results. Figure 1 reports Gini coefficients for income and earnings, suggesting that Norway has experienced a large increase in inequality since the early 1980s, like most other OECD countries. Both income and earnings inequality decrease in the first period until the early 1980s, bottoming out at a level below .24. The trends then turn sharply upwards, approaching .32 in the early 1990s. While income inequality increase further, earnings inequality stabilizes over the last decade – a likely reason is the large increase in capital income in the upper part of the income distribution during this period (Aaberge and Atkinson, 2009). It should be noted that the spike in inequality in the early 1990s was associated with a tax-reform, and that inequality is likely to have increased more steadily in the absence of this reform (see e.g. Fjærli and Aaberge, 2000). The same is true for the year 2005, when another tax-reform was implemented.

Figure 1. Gini-coefficient in income and earnings, 1970-2005
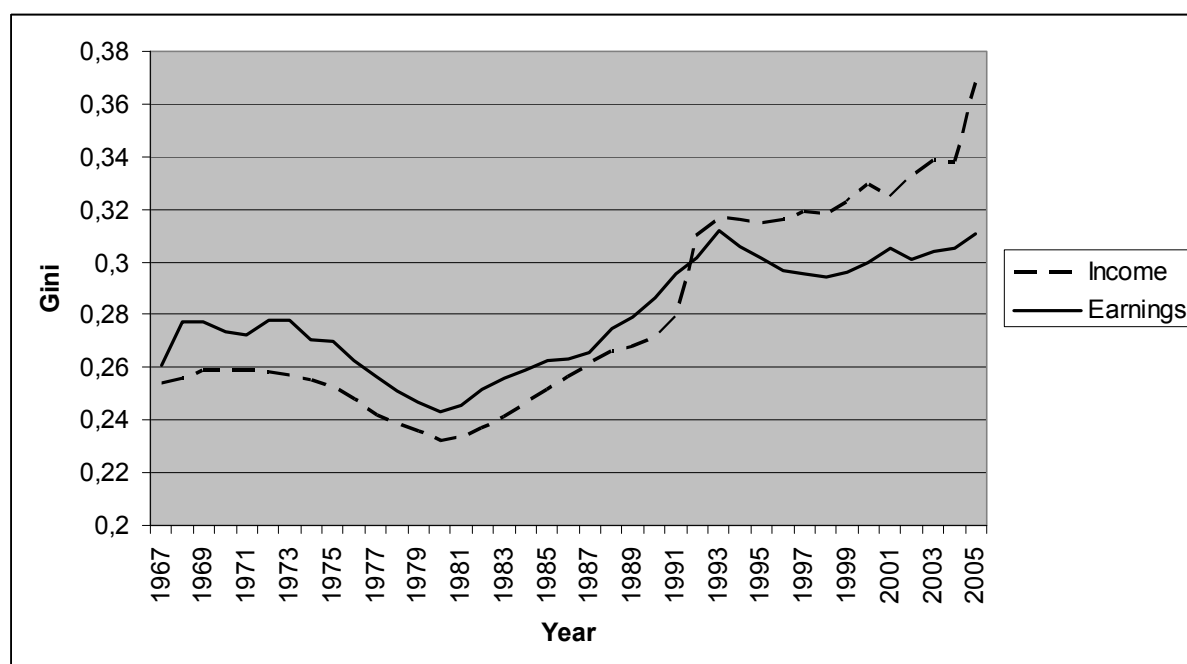
A concern with the above conclusions is they might be sensitive to the arbitrary choice of inequality measure. In Table 1, we therefore report dominance results for the Lorenz curves in income and earnings for the years 1970, 1980, 1990, and 2005. To ease the interpretation of the inequality aversion exhibited in the dominance results, Table 2 computes the weight-functions of the two different generalized Gini families of inequality measures according to their relationship to Lorenz dominance of various degrees. Specifically, Table 2 reports the ratios of the weights of the median individual compared to the 1 percent poorest, the 5 percent poorest, the 30 percent poorest, the 30 percent richest, and the 5 percent richest. We immediately see that the extended Gini family of inequality measures, G_k defined by (7), becomes increasingly sensitive to transfers that occur in the lower rather than the upper part of the distribution, as k increases. It is also evident that the Lorenz family of inequality measures, D_k defined by (8), becomes increasingly sensitive to transfers that occur in the upper rather than the lower part of the distribution, as k increases. Hence, the choice between upward and downward dominance depends on to what extent equalizing transfers between poorer individuals should be considered more important than those between richer individuals.

Consider first Panel 1 in Table 1, where 1970 is used as the base-year. We see that first degree dominance is sufficient to rank 1970 as more equal than 2000 and 2005, both for earnings and income. As stated above, first-degree dominance implies that all inequality measures that obey the Pigou-Dalton principle of transfer yield an unambiguous ranking of these Lorenz curves. However, when considering the years 1980 and 1990 more general dominance criteria are required.

17

For example, when focusing on earnings we see that upward dominance of second degree is necessary to consistently rank 1970 as more equal than 1990, and further that upward dominance of (at least) 22 degree is required to unambiguously rank 1970 as more equal than 1980. From Proposition 3.1 we know that upward dominance of second degree implies that G_k will consistently rank 1970 as more equal than 1990 for k greater or equal to 2. And further, that G_k will consistently rank 1970 as more equal than 1980 for k greater or equal 22. Table 2 shows that the weight-function associated with G_{22} implies an extreme degree of downside inequality aversion, whereas G_2 is more sensitive to income differences in the central and upper part of the distribution. For instance, while G_2 assigns about twice as much weight to the 30 percent poorest compared to the median individual, G_{22} weights the former individual 1640 times more than the latter individual. Hence, far more restrictive social preferences are required to unambiguously rank 1970 and 1980 compared to 1970 and 1990 according to upward dominance.

Turing attention to downward dominance in earnings, we see that the picture is reversed: 1970 is actually dominated by 1980 and 1990 of second and seventh degree, respectively. This illustrates that a consistent ranking of 1970, 1980, and 1990 depends crucially on the choice between upward and downward dominance criteria, that is, to what extent concern about inequality is particularly related to inequality in the lower, the central or the upper part of the distribution. From Proposition 3.3 we know that downward dominance of second degree implies that D_k will consistently ranks 1980 as more equal than 1970 for k greater or equal to 2, and 1990 as more equal than 1970 for k greater or equal to 7. From the weights-functions displayed in Table 2, we see that D_7 exhibits considerably more upside inequality aversion than D_2. Consequently, less restrictive social preferences are required to unambiguously rank 1970 and 1980 compared to 1970 and 1990 according to downward dominance.

Moving on to Panel 2, we see that 1980 first-degree dominates 1990, 2000, and 2005 for earnings, Hence, earnings inequality has unambiguously increased since the 1980. By contrast, downward dominance of second degree is necessary to consistently rank 1980 as more equal than 1990, 2000 and 2005 in terms of income. From Panel 3, it is clear that 1990 first-degree dominates 2000, both for earnings and income. Yet for earnings, more restrictive social preferences are required to unambiguously rank 1990 and 2005. The same holds when comparing 2000 and 2005 in Panel 4: While 2000 first-degree dominates 2005 for income, downward dominance of second degree is necessary for 2000 to dominate 2005 and upward dominance of tenth degree is required for 2005 to dominate 2000 in terms of earnings.

Table 1. Dominance results for Lorenz curves in income and earnings, 1970-2005

	Earnings		Income	
	Upward	Downward	Upward	Downward
Panel 1: Base year: 1970				
1970-1980	1970: 22+ degree	1980: 2 degree	1970: 22+ degree	1980: 2 degree
1970-1990	1970: 2 degree	1990: 7 degree	1990: 22+degree	1970: 2 degree
1970-2000	1970: 1 degree	1970: 1 degree	1970: 1 degree	1970: 1 degree
1970-2005	1970: 1 degree	1970: 1 degree	1970: 1 degree	1970: 1 degree
All (max)	22+ degree	7 degree	22+ degree	2 degree
Panel 2: Base year: 1980				
1980-1990	1980: 1 degree	1980: 1 degree	1990: 22+ degree	1980: 2 degree
1980-2000	1980: 1 degree	1980: 1 degree	2000: 22+ degree	1980: 2 degree
1980-2005	1980: 1 degree	1980: 1 degree	2005: 22+ degree	1980: 2 degree
All (max)	1 degree	1 degree	22+ degree	2 degree
Panel 3: Base year: 1990				
1990-2000	1990: 1 degree	1990: 1 degree	1990: 1 degree	1990: 1 degree
1990-2005	2005: 19 degree	1990: 2 degree	1990: 1 degree	1990: 1 degree
All (max)	19 degree	2 degree	1 degree	1 degree
Panel 4: Base year: 2000				
2000-2005	2005: 10 degree	2000: 2 degree	2000: 1 degree	2000: 1 degree
All (max)	10 degree	2 degree	1 degree	1 degree

Notes: Panel 1 compares the Lorenz curve in 1970 to the Lorenz curves in 1980, 1990, 2000, and 2005. Panel 2 compares the Lorenz curve in 1980 to the Lorenz curves in 1990, 2000, and 2005. Panel 3 compares the Lorenz curve in 1990 to the Lorenz curves in 2000 and 2005. Panel 4 compares the Lorenz curve in 2000 to the Lorenz curves in 2005. In each panel, the first column states the years of comparison, columns 2 and 3 report upward and downward dominance results for earnings, and columns 4-5 report dominance results for income. For every dominance results, we first report the year that dominates and then the order of dominance. The rows denoted All(max) report the maximum order of dominance for the all the years compared to the chosen base year. We only report dominance up to a maximum of 22 degree.

Table 2. Distributional weight of inequality measures by degree of upward and downward Lorenz dominance

Upward Lorenz	Panel A: G_k				
dom. of order k	$P'_{1k}(.01)/$ $P'_{1k}(.5)$	$P'_{1k}(.05)/$ $P'_{1k}(.5)$	$P'_{1k}(.3)/$ $P'_{1k}(.5)$	$P'_{1k}(.7)/$ $P'_{1k}(.5)$	$P'_{1k}(.95)/$ $P'_{1k}(.5)$
k = 1	1,98	1,90	1,40	0,60	0,10
k = 2	3,92	3,61	1,96	0,36	0,01
k = 10	926	613	29	0	0
k = 19	433150	197842	598	0	0
k = 22	3362282	1356998	1640	0	0
Downward Lorenz	Panel B: D_k				
dom. of order k	$P'_{2k}(.01)/$ $P'_{2k}(.5)$	$P'_{2k}(.05)/$ $P'_{2k}(.5)$	$P'_{2k}(.3)/$ $P'_{2k}(.5)$	$P'_{2k}(.7)/$ $P'_{2k}(.5)$	$P'_{2k}(.95)/$ $P'_{2k}(.5)$
k = 1	1,98	1,90	1,40	0,60	0,10
k = 2	1,33	1,33	1,21	0,68	0,13
k = 7	1,01	1,01	1,01	0,92	0,30
k = 22	1,00	1,00	1,00	1,00	0,68

Notes: Panel A considers upwards Lorenz dominates, whereas Panel B considers downward Lorenz dominance. In each panel, the first column states the degree of dominance, and columns 2-5 compute the weight-functions of the generalized Gini families of inequality measures according to their relationship to Lorenz dominance of various degrees. Specifically, column 2 reports the ratio of the weights of the median individual compared to the 1 percent poorest, column 3 reports the ratio of the weights of the median individual compared to the 5 percent poorest, column 4 reports the ratio of the weights of the median individual compared to the 30 percent poorest, column 5 reports the ratio of the weights of the median individual compared to the 30 percent richest, and column 6 reports the ratio of the weights of the median individual compared to the 5 percent richest.. The weight-function $P'_{1k}(u)$ of G_k is defined in equation (11) for j = 1, while the weight function $P'_{2k}(u)$ of D_k is defined in equation (13) for j = 1.

5. Conclusion

Because Lorenz curves often intersect, other approaches than first-degree Lorenz dominance are called for. Although the theoretical literature offers more general dominance criteria for ranking Lorenz curves,[13] these methods are generally viewed as hard to implement and the results difficult to interpret because they involve assumptions about third and higher derivatives (see e.g. Atkinson, 2003). Thus, most empirical studies rely exclusively on one or a few summary measures of inequality to achieve rankings of intersecting Lorenz curves. A concern is, however, that the conclusions reached are sensitive to the more or less arbitrary choice of inequality measures. The purpose of this paper has

---

[13] See e.g. Kolm (1976), Shorrocks and Foster (1987), Davies and Hoy (1995), Zoli (1999) and Aaberge (2000, 2009).

been to bridge the wide gap between the theoretical and the empirical strand of the literature concerned with rankings of intersecting Lorenz curves.

We first arrange the members of generalized Gini families of inequality measures into subfamilies according to their relationship to Lorenz dominance of various degrees. Since the various criteria of higher degree Lorenz dominance provide convenient computational methods, these results can be used to identify the largest subfamily of the generalized Gini families and thus the least restrictive social preferences required to reach unambiguous ranking of a set of Lorenz curves. From the weight-functions of these inequality measures we obtain intuitive interpretations of higher degree Lorenz dominance. To demonstrate the usefulness of these methods for empirical applications, we examine the time trend in income and earnings inequality of Norwegian males during the period 1967-2005.

Appendix

Proof of Proposition 2.1A. The statement (i) implies (ii) follows from Theorem 2.2A of Aaberge (2009).

To prove the converse statement assume that (ii) holds and that L_1 and L_2 cross at $u = a$. Then the following inequalities hold,

$$\int_0^a \left(L_1(u) - L_2(u)\right) du > 0 \tag{A1}$$

and

$$\int_0^1 \left(L_1(u) - L_2(u)\right) du = \frac{1}{2}\left(G(L_2) - G(L_1)\right) \geq 0 . \tag{A2}$$

Since L_1 and L_2 cross only once (A1) and (A2) imply that

$$\int_0^u \left(L_1(u) - L_2(u)\right) du \geq 0 \quad \text{for all} \quad u \in [0,1]$$

and the inequality holds strictly for some u, and the desired result is obtained by applying Theorem 2.2A of Aaberge (2009) .

Proof of Proposition 2.2. The equivalence between (i) and (ii) follows by noting that

$$\int_u^1 \left[\left(1 - L_1(t)\right) - \left(1 - L_2(t)\right)\right] dt = \int_u^1 \left(L_2(t) - L_1(t)\right) dt =$$

$$\int_0^1 \left(L_2(t) - L_1(t)\right) dt + \int_0^u \left(L_1(t) - L_2(t)\right) dt = \int_0^u \left(L_1(t) - L_2(t)\right) dt$$

when L_1 and L_2 have equal Gini coefficients.

As noted above the motivation for introducing the principles of DPTS (UPTS) was to successively strengthen the emphasis of transfers taking place lower down (higher up) in the income distribution.

References

Aaberge, R. (2000): "Characterizations of Lorenz curves and income distributions", *Social Choice and Welfare,* **17**, 639-653.

Aaberge, R. (2001): "Axiomatic characterization of the Gini coefficient and Lorenz curve orderings", *Journal of Economic* Theory, **101**, 115-132.

Aaberge, R. (2007): "Gini's Nuclear Family", *Journal of Economic Inequality*, **5**, 305-322.

Aaberge, R. (2009): "Ranking Intersecting Lorenz Curves", *Social Choice and Welfare*, **33**, 235–259.

Aaberge, R. and A.B. Atkinson (2009): "Top Incomes in Norway", Statistics Norway, Discussion Paper, no. 552.

Atkinson, A.B. (1970): "On the measurement of inequality", *Journal of Economic Theory*, **2**, 244-263.

Atkinson, A. B. (2003): "Multidimensional Deprivation: Contrasting Social Welfare and Counting Approaches", *Journal of Economic Inequality*, **1**, 51-65

Dardanoni, V. and P.J. Lambert (1988): "Welfare rankings of income distributions: A role for the variance and some insights for tax reforms", *Social Choice and Welfare,* **5**, 1-17.

Davies, J.B. and M. Hoy (1995): "Making inequality comparisons when Lorenz curves intersect", *American Economic Review*, **85**, 980-986.

Donaldson, D. and J. A. Weymark (1980): "A single parameter generalization of the Gini indices of inequality", *Journal of Economic Theory*, **22**, 67-86.

Donaldson, D. and J. A. Weymark (1983): "Ethically flexible indices for income distributions in the continuum", *Journal of Economic Theory*, **29**, 353-358.

Gottschalk, P. and T. Smeeding (1997): "Cross-National Comparisons of Earnings and Income Inequality", *Journal of Economic Literature*, **3**5, 633-687.

Fjærli, E. and R. Aaberge (2000): "Tax Reforms, Dividend Policy and Trends in Income Inequality Empirical Evidence based on Norwegian Data", Statistics Norway, Discussion Paper, no. 284.

Hardy, G. H., J.E. Littlewood and G. Polya (1934): "Inequalities", Cambridge University Press, Cambridge.

Kolm, S. C. (1969): "The optimal production of social justice", in "Public Economics", (J. Margolis and H. Guitton, Eds.), Macmillan, New York/London.

Kolm, S. C. (1976): "Unequal inequalities II", *Journal of Economic Theory*, **13**, 82-111.

Mehran, F. (1976): "Linear measures of inequality", *Econometrica*, **44**, 805-809.

Muliere, P. and M. Scarsini (1989): "A note on stochastic dominance and inequality measures", *Journal of Economic Theory*, **49**, 314-323.

Shorrocks, A.F. and J.E. Foster (1987): "Transfer sensitive inequality measures", *Review of Economic Studies*, **14**, 485-497

Weymark, J. (1981): "Generalized Gini inequality indices", *Mathematical Social Sciences*, **1**, 409-430.

Yaari, M.E. (1987): "The dual theory of choice under risk", *Econometrica*, **55**, 95-115.

Yaari, M.E. (1988): "A controversial proposal concerning inequality measurement", *Journal of Economic Theory*, **44**, 381-397.

Yitzhaki, S. (1983): "On an extension of the Gini inequality index", *International Economic Review*, **24**, 617-628.

Zoli, C. (1999): "Intersecting generalized Lorenz curves and the Gini index", *Social Choice and Welfare*, **16**, 183-196.