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Generalizing the S-Gini family: Some properties*

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Abstract

In the literature there are a number of generalizations of the Gini coefficient which inherit most of its appealing properties. These families allow the incorporation of different value judgments and all of them are more sensitive to transfers among the poorest individuals in society than to transfers among the richest. Consequently they fail to capture a fact with which perhaps not everybody agrees: it is always good for society to give much more additional income to the richest person than to the poorest one. The aim of this paper is to propose extensions of these generalizations of the Gini coefficient with measures which, preserving their properties, complete the information about all the inequality perceptions.

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JEL classification: D63

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1. INTRODUCTION.

The Gini coefficient is without doubt the most widely used inequality index in applied work. This could be because its geometrical interpretation in relation with the Lorenz curve has a very intuitive appeal, while also having a number of desirable properties. Specifically, and in contrast with other inequality measures, the Gini index accommodates non-positive incomes easily, and inequality as measured by this coefficient depends on the significance of the income gaps in society. Moreover, this relative index implicitly defines a class of social evaluation functions that is also related to the Gini absolute index. However, the Gini coefficient is not additive decomposable into a between group and a within group term (Shorrocks 1980), or even aggregative (Shorrocks 1984). Further disadvantages are insensitivity (given its linear structure) to changes in the income distribution, and the impossibility to incorporate value judgements. The last of these difficulties has been overcome in a series of papers by Weymark (1981) and Donaldson and Weymark (1980, 1983). These authors propose a generalized Gini inequality family that allows different value judgments to be incorporated through a weighting function of incomes. These authors suggest a single-parameter Gini class as the most suitable, since this class retains all of the standard properties of inequality measures as well as all the good properties of the original Gini index mentioned above. This has been known as the single-parameter Ginis, or *S*-Ginis, in the literature.

On the other hand, it seems to be generally accepted that in order to measure inequality one should focus exclusively on the situation of the worst-off in society, as the generalized Gini inequality family does, and thus make use of measures sensitive to transfers among the poorest. In fact, the Atkinson (1970) family depends on an aversion inequality parameter that attaches more weight to lower incomes as the parameter increases (in the standard formulation). However, we can say that this is not the whole truth. Because the concern with inequality (and poverty) stems from the injustice of extremely low incomes, the inequality measure must be more sensitive to what happens to the poor. The other side of the same coin is the injustice of extremely high incomes. This suggests that sometimes it would be of interest to choose an inequality measure sensitive to such incomes in order to obtain a detailed explanation of what really happens. In the literature there are inequality families that contain specific measures sensitive to both high and low incomes, such as the Generalized Entropy family

(Shorrocks 1980) where a parameter indexes the part of the distribution where we want to focus on, or the extended Atkinson family (Lasso de la Vega and Urrutia 2008), which completes the Atkinson family with the consideration of the other tail.

The aim of this paper is to propose an extension of the *S*-Gini inequality family whose members are more sensitive to transfers between rich people. Moreover, given the close relation between the Atkinson (1970) family and the *S*-Gini family as exposed by Yitzhaki (1983), and Araar and Duclos (2003), among others, we explore the connection between the Extended Atkinson family proposed by Lasso de la Vega and Urrutia (2008) and the Extended *S*-Gini family proposed in this paper.

The rest of the paper is structured as follows. Section 2 offers some background and motivation, in addition to introducing the notation. Section 3 summarizes the standard view between inequality and welfare, since the relation between both concepts is of clear relevance in the case of the Gini index. Section 4 makes a proposal to extend the *S*-Gini family and study their properties, while Section 5 provides some concluding comments.

2. BACKGROUND, MOTIVATION AND NOTATION: THE *S*-GINI FAMILY.

We consider a finite population consisting of $n \geq 2$ individuals. Individual i's income is denoted by $y_i \in \mathbb{R}_{++} = (0,\infty)$, $i = 1,2,\ldots,n$. An income distribution is represented by a vector $\mathbf{y} = (y_1, y_2, ..., y_n) \in \mathbb{R}^n_{++}$. We let $D = \bigcup_{n=1}^{\infty} \mathbb{R}^n_{++}$ represent the set of all finite dimensional income distributions and denote the mean and population size of any $\mathbf{y} \in D$ by $\mu(\mathbf{y})$ and $n(\mathbf{y})$ (or μ and n if there is no room for confusion) respectively. An inequality measure is just an application from D into $\mathbb{R}_{+} = [0,\infty)$, $I(\mathbf{y}) : D \to \mathbb{R}_{+}$.

The starting point of the paper and its motivation is twofold. On the one hand, two families of relative inequality measures are widely used in the literature. First, the Generalised Entropy class (Cowell and Kuga 1981a; 1981b), henceforth the GE family, which is given by:

$$I_\alpha^{GE}(\mathbf{y}) = \begin{cases} \frac{1}{n}.\frac{1}{\alpha^2-\alpha}\sum_{i=1}^{n}\left[\left(\frac{y_i}{\mu}\right)^{\alpha}-1\right] & \alpha \in \mathbb{R}, \alpha \neq 0,1 \\ -\frac{1}{n}\sum_{i=1}^{n}\log\frac{y_i}{\mu} & \alpha = 0 \\ \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{\mu}\log\frac{y_i}{\mu} & \alpha = 1 \end{cases} \tag{1}$$

As is well known, the GE family includes the mean logarithmic deviation, when α is equal to 0, and the entropy inequality measure, when α is equal to 1 (Theil 1967). Moreover, the members of this family are the only additively decomposable measures in the sense that overall inequality can be decomposed as the sum of the between component and a weighted sum of the subgroup inequality levels (Shorrocks 1980).

Second, the other prominent class of inequality measures is the Atkinson (1970) family given by

$$I_\alpha^{A}(\mathbf{y}) = \begin{cases} 1-\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_i}{\mu}\right)^{\alpha}\right]^{\frac{1}{\alpha}} & \alpha < 1, \alpha \neq 0 \\ 1-\prod_{i=1}^{n}\left(\frac{y_i}{\mu}\right)^{\frac{1}{n}} & \alpha = 0 \end{cases} \tag{2}$$

In the first place, it's noteworthy that whereas the GE family is defined for all real values of the α parameter (i.e. it has two tails, one focusing on the "poor" and the other on the "rich"), the Atkinson family makes sense only for parameter values less than 1 (that is, it has only one tail, the "poor" one). This is so because only for $\alpha < 1$ is the inequality measure *S*-convex.[1]

In the second place, it may be interesting to observe that, as is well-known, the Atkinson family is ordinally equivalent to one tail of the GE family, the one with the same values for the α parameter. The question of: "*what happens with the other tail?*" was answered by Lasso de la Vega and Urrutia (2008). These authors used the result from Shorrocks (1984) that any inequality measure fulfils the Aggregative Principle if

[1] *S*-convexity implies symmetry and in is the requirement that $I(\mathbf{y})$ agrees with the (weak) Lorenz quasi-ordering, or in other words, satisfies the Pigou-Dalton condition.

and only if it is an increasing transformation of one of the GE family, and proposed an extension of the Atkinson family that is ordinally equivalent to both tails of the GE family. Hence, for one member of the GE family we have a corresponding member of the extended Atkinson family and both order any distributions in exactly the same way. This new family was given by

$$I_{\alpha}^{EA}(\mathbf{y}) = \begin{cases} I_{\alpha}^{A}(\mathbf{y}) & \alpha < 1 \\ 1 - \left(\left[\dfrac{1}{n} \sum_{i=1}^{n} \left(\dfrac{y_i}{\mu} \right)^{\alpha} \right]^{\frac{1}{\alpha}} \right)^{-1} & \alpha > 1 \\ 1 - \prod_{i=1}^{n} \left(\dfrac{\mu}{y_i} \right)^{\frac{y_i}{n\mu}} & \alpha = 1 \end{cases} \tag{3}$$

Every member of this family meets the Aggregative Principle and is bounded above by 1. Lasso de la Vega and Urrutia (2008) characterized this new extended family as the only one with a multiplicative decomposition property for any partitioning of the population into mutually exhaustive and disjoint subgroups (see also Lasso de la Vega and Urrutia 2003, 2005 and Goerlich, Lasso de la Vega and Urrutia 2009).

Our second motivation for writing this paper, on the other hand, is as follows. Early in the 20th century Gini (1912) proposed an *ad hoc* measure of income inequality as a measure of the variability in any statistical distribution. He based the coefficient on the average of the absolute differences between all possible pairs of observations and defined it to be the ratio of half of that average to the mean of the distribution. The relation between the Gini coefficient and the Lorenz curve for the corresponding income distribution (Gastwirth 1972, Dorfman 1979) soon became evident, as well as the linear structure of the index (Mehran 1976) and the implicit weighting scheme for individual incomes (Sen 1973). In particular, we can see that the Gini index can be written as

$$I^{G}(\mathbf{y}) = 1 + \frac{1}{n} - \frac{2}{\mu n^2}\left[\tilde{y}_1 + 2\tilde{y}_2 + \ldots + i\tilde{y}_i + \ldots + n\tilde{y}_n \right] = 1 - \frac{\sum_{i=1}^{n}(2i-1)\tilde{y}_i}{\mu n^2} \tag{4}$$

where $\tilde{\mathbf{y}} = (\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n)$ is a permutation of $\mathbf{y}$ such as $\tilde{y}_1 \geq \tilde{y}_2 \geq ... \geq \tilde{y}_n$, and the second equality follows by using the fact that $\mu n = \Sigma_{i=1}^{n} \tilde{y}_i$. This formula makes explicit the weighting scheme behind $I^G(\mathbf{y})$, which involves the first n odd numbers.

Note that defining $a_i^G = 2i - 1$, we observe that $\Sigma_{i=1}^{n} a_i^G = \Sigma_{i=1}^{n}(2i-1) = n^2$, therefore normalizing the weights by their sum, $\omega_i = \dfrac{a_i^G}{\Sigma_{i=1}^{n} a_i^G} = \dfrac{2i-1}{n^2}$, we can write

$$I^G(\mathbf{y}) = 1 - \frac{\sum_{i=1}^{n} \omega_i \tilde{y}_i}{\mu} \text{ with } \Sigma_{i=1}^{n} \omega_i = 1.$$

Well known properties of the Gini coefficient are:

G.1 Non-positive incomes are easily accommodated.

G.2 It is bounded by zero and one for non-negative incomes.

G.3 It has a simple geometric interpretation in terms of the Lorenz curve.

G.4 Given its linear structure the weights depend on the rank of each individual in the whole distribution rather than on the specific income levels. As a consequence, the effect on the index of a transfer of income between two individuals depends only on the ranks of these individuals instead of on their income levels.

G.5 It is possible to write the index as proportional to the covariance between income levels and rankings.

$$I^G(\mathbf{y}) = -\frac{2}{\mu} \cdot \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n} \cdot (\tilde{y}_i - \mu) = -\frac{2}{\mu} Cov(q, y) \tag{5}$$

using $\Sigma_{i=1}^{n}(\tilde{y}_i - \mu) = 0$ and where q is the proportion of population receiving incomes greater than or equal to y.

G.6 It satisfies the Pigou-Dalton transfer principle, so a progressive transfer that leaves the rank of the individuals unchanged decreases the value of the index.

G.7 Inequality can be decomposed by income type if the rank-order of incomes does not vary by source of income (Fei, Ranis and Kuo 1978).

If incomes are classified by different sources $j = 1,\ldots,k$, so that $\mathbf{y} = (y_1,\ldots,y_n) = \Sigma_{j=1}^{k}(y_1^j,\ldots,y_n^j) = \Sigma_{j=1}^{k}\mathbf{y}^j$ and the rank order of incomes does not vary by source, the Gini index satisfies an additive decomposition of the form $I^G(\mathbf{y}) = \frac{\Sigma_{j=1}^{k}\mu(\mathbf{y}^j)I^G(\mathbf{y}^j)}{\mu(\mathbf{y})}$.

The three main drawbacks of (4) in comparison with $I_\alpha^{GE}(\mathbf{y})$ and $I_\alpha^{A}(\mathbf{y})$ are the following:

(i) The weights schedule in (4) is completely arbitrary and without a clear justification.

(ii) $I^G(\mathbf{y})$ does not incorporate alternative distributional judgements, or in other words, the absence of any parameter which can be varied to incorporate different ethical concerns and perceptions about the inequality in the income distribution.

(iii) The fact that $I^G(\mathbf{y})$ does not satisfy the Aggregative Principle, which in turn is a consequence of its linear and rank dependence structure.

The first two drawbacks were solved in a series of papers by Weymark (1981), Donaldson and Weymark (1980, 1983) and, from a different perspective, by Yitzhaki (1983). Generalizing the structure of the Gini index, as given by (4), Donaldson and Weymark (1980) defined a single-series Gini index as

$$I^{SSG}(\mathbf{y}) = 1 - \frac{\sum_{i=1}^{n} a_i \tilde{y}_i}{\mu \sum_{i=1}^{n} a_i} \qquad (6)$$

with the restriction of non-decreasing weights starting from one, $1 = a_1 \le a_2 \le \ldots \le a_i \le \ldots \le a_n$. There have been a number of axiomatizations of various subclasses of these indices in the literature (see among others, Ebert 1988, Yaari 1988 or Bossert 1990). Clearly, $I^G(\mathbf{y})$ is obtained from (6) for $a_i = a_i^G = 2i - 1$.

Donaldson and Weymark (1980) proved that if the resulting index from (7) has to satisfy the Principle of Population, as $I^G(\mathbf{y})$ does, then a single-parameter Gini family arises,

$$I_\delta^G(\mathbf{y}) = 1 - \frac{\sum_{i=1}^{n}\left[i^\delta - (i-1)^\delta\right]\tilde{y}_i}{\mu n^\delta} \qquad \delta > 1 \tag{7}$$

This family is known as the *S*-Gini family. Note that $\sum_{i=1}^{n}\left[i^\delta - (i-1)^\delta\right] = n^\delta$. The Gini index is a member of this family for δ = 2, $I_2^G(\mathbf{y}) = I^G(\mathbf{y})$.

The members of this class fulfil all the properties mentioned above for the Gini index, *G*.1 to *G*.7. The covariance formula mentioned in *G*.5 is now slightly different, since it involves a function of the weights rather than the rankings themselves

$$I_\delta^G(\mathbf{y}) = -\frac{1}{\mu}.\frac{1}{n}\sum_{i=1}^{n}\left[\frac{i^\delta - (i-1)^\delta}{n^{\delta-1}}\right].\left(\tilde{y}_i - \mu\right) = -\frac{1}{\mu}Cov\left(\phi(\delta), y\right) \tag{8}$$

defining the function $\phi(\delta)$ that represents the weight attached to *y*.

But in addition, they solve two of the drawbacks mentioned above. That is, the arbitrariness of the weighting schedule and the inability to incorporate distributional value judgments. To be specific:

G.8 The weights are determined by the properties imposed on the inequality family. The form of the weights is determined by imposing the Population Principle, whereas the restriction for the parameter value is equivalent to requiring $I_\delta^G(\mathbf{y})$ to be *S*-convex, i.e. symmetric and satisfying the Pigou-Dalton transfer principle.

G.9 Distributional value judgments are introduced by means of the parameter $\delta > 1$. Given that incomes are ranked from the highest to the lowest, more significance is attached to the incomes of the poorer individuals (increasing weights) and all the indices in this family are more sensitive to transfers at the lower end of the distribution. As δ increases, $I_\delta^G(\mathbf{y})$ becomes more sensitive to transfers at the lower end than at the upper end and the middle part of the distribution. For $\delta = 1$

we have a totally insensitive measure, $I_\delta^G(\mathbf{y}) = 0$, so we can say that we are not concerned at all with the distribution of income. On the other side the limiting case is as $\delta \to \infty$, giving $I_\delta^G(\mathbf{y}) \to 1 - \frac{\min_i \{y_i\}_{i=1}^n}{\mu} = 1 - \frac{\tilde{y}_n}{\mu}$, which only takes into account the situation of the poorest individual in society, so for very high values of δ only transfers to the very lowest income group matters.

Hence, all of the Gini's useful properties are inherited by the *S*-Gini which also introduces distributional judgements in a parametric way. Moreover, as we shall see in the next section, there is a clear relation between inequality and welfare.

From (7), and defining the single-parameter weight function as $\omega_i(\delta) = \frac{i^\delta - (i-1)^\delta}{n^\delta}$ for $\delta \geq 1$, we can write the *S*-Gini family as $I_\delta^G(\mathbf{y}) = 1 - \frac{\sum_{i=1}^n \omega_i(\delta)\tilde{y}_i}{\mu}$ with $\Sigma_{i=1}^n \omega_i(\delta) = 1$, where we can see a close similarity in the structure of the Atkinson (1970) family and the *S*-Gini family. The first is the distance from one of the ratio of an α-order mean to the arithmetic mean of the distribution, where more weight is attached to the lower tail of the distribution, $\alpha < 1$ in (2). The second is the distance from one of the ratio of a weighted mean to the arithmetic mean, where again more weight is attached to the poor, since the lower the income the higher the weight in the index, $\delta > 1$. In both cases the parameter restriction assures that the inequality measure is *S*-convex and for the limiting values, $\alpha = 1$ and $\delta = 1$, we get a null value of the index in both cases. In fact, Yitzhaki (1983, section 4) shows that the *S*-Gini family has most of the properties of the Atkinson (1970) index, even if the latter satisfies the Aggregative Principle but the former does not. As a consequence, the *S*-Gini family has one tail, the "poor" one.

Given the extended Atkinson family (3), it is therefore natural to ask: "*what happens to the other tail?*", "*can we extend the S-Gini family in a similar way as to the Atkinson family, so that it now has two tails?*" We shall answer these questions in section 4, but in the next section we summarize the relation between inequality and welfare that will play a role in the sequel.

3. INEQUALITY AND SOCIAL EVALUATION FUNCTIONS.

An index of inequality is called ethical if it implies, and is implied by a social evaluation function, $W(\mathbf{y}): D \to \mathbb{R}$. Define the equally distributed equivalent income (*EDE*), $\xi(\mathbf{y})$, corresponding to a given income distribution $\mathbf{y} \in \mathbb{R}^n_{++}$, (simply ξ if there is no room for confusion) as the income level that if equally distributed among the population would generate the same value of the *W* function, so society would be indifferent between the actual $\mathbf{y}$ or $\xi.\mathbf{1}$, where $\mathbf{1}$ is an *n*-dimensional vector of ones. Hence the value of ξ is implicitly defined by

$$W(\xi.\mathbf{1}) = W(\mathbf{y}) \tag{9}$$

Assuming that *W* is increasing along the ray of equality, then ξ can be solved from (10) and constitutes a particular numerical representation of *W*.

Under a minimum set of assumptions on the social evaluation function, $W(\mathbf{y})$ being continuous, symmetric, *S*-concave and homothetic function, there is a one-to-one relationship between $W(\mathbf{y})$ and a relative inequality index (Blackorby and Donaldson 1978). Following the Atkinson (1970), Kolm (1969), and Sen (1973) approach every relative inequality index, $I(\mathbf{y})$, may be associated to a representation of a social evaluation function, $W(\mathbf{y}): D \to \mathbb{R}$, according to the following expression:

$$I(\mathbf{y}) = 1 - \frac{\xi}{\mu} \tag{10}$$

Comparison with the formula (4) for the Gini index shows that for this index the *EDE* is given by

$$\xi^G(\mathbf{y}) = \frac{\sum_{i=1}^{n}(2i-1)\tilde{y}_i}{n^2} = \frac{\sum_{i=1}^{n}(2i-1)\tilde{y}_i}{\sum_{i=1}^{n}(2i-1)} \tag{11}$$

More generally, the single-series Gini family of relative inequality indices, as given by equation (7), implicitly define a class of social evaluation functions which are ordinally equivalent to the following particular numerical representation

$$\xi^{SSG}(\mathbf{y}) = \frac{\sum_{i=1}^{n} a_i \tilde{y}_i}{\sum_{i=1}^{n} a_i} \tag{12}$$

Which for the single-parameter Gini family specializes to

$$\xi_\delta^G(\mathbf{y}) = \frac{\sum_{i=1}^{n} \left[i^\delta - (i-1)^\delta \right] \tilde{y}_i}{n^\delta} \qquad \delta > 1 \tag{13}$$

It is worth noting that any member of this class can be written as a linear function of incomes with the weights being a non-decreasing sequence of numbers. One argument in favour of this class is that their members are also translatable functions (invariant under translation), and so they can be used to derive absolute ethical inequality indices, $A(\mathbf{y})$. Following the approaches by Kolm (1976a, 1976b), and Blackorby and Donaldson (1980), every absolute inequality index, $A(\mathbf{y})$, may be associated to a representation of a social evaluation function, $W(\mathbf{y}): D \to \mathbb{R}$, according to the following expression:

$$A(\mathbf{y}) = \mu - \xi \tag{14}$$

which is simply the difference between per capita income, μ, and the equally distributed equivalent income, ξ.

A nice feature of the Gini index, which is also satisfied by the single-series and single-parameter Gini families, is that:

G.10 The relative and absolute indices imply and are implied by the same class of ordinally-equivalent social-evaluation functions, as shown by (11), (12) and (13).

4. THE EXTENDED SINGLE-PARAMETER *S*-GINI FAMILY.

The motivations for extending the *S*-Gini family were set up at the end of section 2 as a natural counterpart to the extended Atkinson (1970) family, and given the similarities between the Atkinson (1970) and the *S*-Gini families. It is, however, useful to look at a particular numerical example.

As was stressed at the beginning of the paper, the indices of the *S*-Gini family are more sensitive to transfers among the poorer than among the richer. As a consequence, they focus on the part of the distribution corresponding to the worst-off. The following example tries to direct attention towards the implications of using only measures attaching more weight to the lowest incomes, without taking into account the injustice of extremely high incomes. Consider the income distribution $\mathbf{y} = (1, 3, 5, 11, 30)$. Assume that four additional units of income have to be distributed among the individuals of the society, and assume that we increase the lowest income by one and the highest income by three, so that the new distribution is $\mathbf{y}' = (2, 3, 5, 11, 33)$. Although increasing the lowest income reduces inequality, and increasing the highest income increases inequality, the magnitude of these two effects together is not at all clear.[2] Perhaps not everybody would agree that this sharing of the extra income has been undertaken in a way that reduces inequality. Nevertheless, this is the conclusion derived by using all the members of the *S*-Gini family. It is true that the income of the poorest person has increased, and according to these inequality measures inequality should, *ceteris paribus*, decrease. But this is not the end of the story. These measures fail to capture some inequality perceptions sensitive to the fact that more income is given to the richest person than to the poorest. In fact the reduction in inequality is greater the higher the value of the parameter δ, because increasing δ attaches more weight to poorest and less to the richest. Therefore, it might be worth extending such a family in order to have inequality measures that complete the information about the inequality comparisons.

In the case of the Atkinson family the extension was achieved by looking at an ordinally equivalent transformation to the corresponding tail of the GE family (Lasso de la Vega and Urrutia 2008), so for the other values of the parameter defining the extended Atkinson family, $\alpha > 1$, the corresponding inequality measures are again *S*-convex. In practice this is done by inverting the complement to one in the index, as (3) demonstrates.

This trick will also work in extending the *S*-Gini family to values of the parameter less than one in the sense that the corresponding inequality measures are again *S*-convex. However, inversion will destroy linearity of the new indices and therefore will

[2] Lambert and Lanza (2006) discuss this issue in more depth.

not preserve the nice properties of the *S*-Gini family we want to keep for the new tail. As a consequence we extend the *S*-Gini family for $0 < \delta < 1$ simply as $-I_{\delta}^{G}(\mathbf{y})$.

The extension we propose is the following:

$$I_{\delta}^{EG}(\mathbf{y}) = \begin{cases} 1 - \dfrac{\sum_{i=1}^{n} \left[i^{\delta} - (i-1)^{\delta} \right] \tilde{y}_i}{\mu n^{\delta}} & \delta > 1 \\ \dfrac{\sum_{i=1}^{n} \left[i^{\delta} - (i-1)^{\delta} \right] \tilde{y}_i}{\mu n^{\delta}} - 1 & 0 < \delta < 1 \end{cases} \tag{15}$$

The relative *S*-Gini inequality family corresponds to the first expression, when $\delta > 1$, and most of the useful properties of this family are inherited by the extension proposed in this paper, that is, when $0 < \delta < 1$.[3]

For instance, it is straightforward to prove that all the members of this family are relative and that the restriction over the parameter values guarantees that the inequality indices fulfil the Pigou-Dalton transfer principle as the following proposition shows.

Proposition 1: *For each* $\delta \in \mathbb{R}_{++} - \{1\}$, $I_{\delta}^{EG}(\mathbf{y})$ *is an inequality measure which satisfies Symmetry, the Pigou-Dalton Transfers Principle, Normalization, Replication Invariance and the Scale Invariance Principle.*

Proof. Since the *S*-Gini indices satisfy all these properties it suffices to prove that they hold for the members of this family for $0 < \delta < 1$. But the proof is trivial since $-I_{\delta}^{G}(\mathbf{y})$ is *S*-convex for this range of parameter values, takes a value of zero when all individuals have the same income, μ, and the negation preserves the Replication and Scale Invariance Principles. ■

Obviously non-positive incomes can be easily accommodated to measure inequality and the covariance formula works as before. However, even if the lower bound for the indices is zero the upper bound is one only when the maximum income is lower than twice the mean income in the distribution, which is not usually the case for real distributions.

[3] It is worth mentioning that the extension for $0 < \delta < 1$ is mentioned in Lambert and Lanza (2006, footnote 11, page 274) but they neither justify nor explore the properties of these new indices.

The effect of a transfer between individuals depends only on the ranks of these individuals in the income distribution. As mentioned above, more significance is attached to the incomes of the poorer individuals as δ increases from 1. In contrast, as δ decreases from 1, more significance is attached to the incomes of the richer individuals and the indices in the new tail are more sensitive to transfers at the upper end of the distribution. In this case as $\delta \to 0$, $I_{\delta}^{G}(\mathbf{y}) \to \frac{\max_i \{y_i\}_{i=1}^{n}}{\mu} - 1 = \frac{\tilde{y}_1}{\mu} - 1$ which only considers transfers to the richest income group.

Regarding the social evaluation framework of section 3, the underlying social evaluation function of this family is ordinally equivalent to the following particular numerical representation:

$$\xi_{\delta}^{EG}(\mathbf{y}) = \begin{cases} \dfrac{\sum_{i=1}^{n}\left[i^{\delta} - (i-1)^{\delta}\right]\tilde{y}_i}{n^{\delta}} & \delta > 1 \\ 2\mu - \dfrac{\sum_{i=1}^{n}\left[i^{\delta} - (i-1)^{\delta}\right]\tilde{y}_i}{n^{\delta}} & 0 < \delta < 1 \end{cases} \tag{16}$$

The class corresponding to δ > 1 in (16) is the *S*-Gini social evaluation function. Note that any member of this class can be written as a linear function with non-decreasing positive weights. This property implies that they are monotone (non-decreasing) in individuals' incomes. Therefore, the increase in inequality which results from increasing one person's income, while holding other incomes constant, must be balanced by the increase in total income. One significant difference between the social evaluation function of the two tails is that the members of the new extension, the class corresponding to 0 < δ < 1, are not monotone. In this case, the inequality resulting from the increase in one person's income may surpass the effect of the increase in total income, so social evaluation becomes negative. As we will show later, by no means do we think that this is a drawback of the new family, since it tells us that more income is preferred only if inequality is not extremely high (in particular when $I_{\delta}^{EG}(\mathbf{y}) > 1$ for 0 < δ < 1). Thus, any increase in income that does not reduce inequality by a sufficiently high amount implies a reduction in the social valuation. Nevertheless, all the members in the extended *S*-Gini family are increasing along rays (homogeneous of degree one).

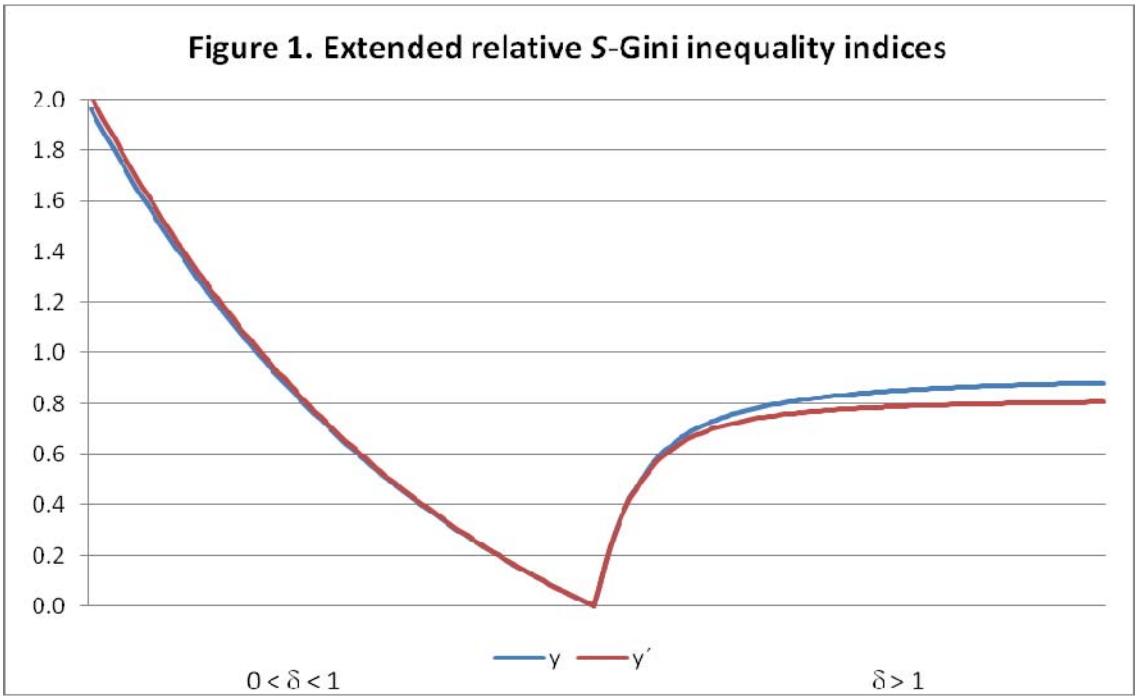
Another interesting property shared by all the members of the extended *S*-Gini family of social evaluation functions (16) is that they are not only homothetic but also translatable, and consequently (as already mentioned in section 3) they may be used to derive not only relative but also absolute inequality indices. Hence, using (14) the family of absolute indices for the extended *S*-Gini family is given by

$$A_\delta^{EG}(\mathbf{y}) = \begin{cases} \mu - \dfrac{\sum_{i=1}^{n}\left[i^\delta - (i-1)^\delta\right]\tilde{y}_i}{n^\delta} & \delta > 1 \\ \dfrac{\sum_{i=1}^{n}\left[i^\delta - (i-1)^\delta\right]\tilde{y}_i}{n^\delta} - \mu & 0 < \delta < 1 \end{cases} \qquad (15')$$

Therefore, the relative and absolute *S*-Gini indices of the new tail, the second expression of (15) and (15′), imply and are implied by the same class of ordinally equivalent social welfare functions, the second expression of (16).

In addition, another interesting feature of the absolute *S*-Gini family defined in (15′) is that they all fulfil the unit-consistency axiom proposed by Zheng (2005, 2007). This appealing principle requires that the inequality rankings, rather than the inequality levels, not be affected by the units in which incomes are measured. Obviously any relative index is unit-consistent but not many absolute indices satisfy this axiom. To verify that unit-consistency is satisfied by the members of (15′) it is enough to note that $A_\delta^{EG}(\lambda\mathbf{y}) = \lambda A_\delta^{EG}(\mathbf{y})$, for $\lambda > 0$, and consequently the inequality ranking of two given distributions (as measured by any of these absolute indices) does not change when the units in which income is measured vary.

Up to now we have only shown that most of the properties of the *S*-Gini family are also fulfilled by this extended family. It is therefore natural to ask: "*what is actually the contribution of the new tail?*" As we have just mentioned, our aim is to propose measures that are able to capture that perhaps not everybody agrees with the fact that giving much more additional income to the richest person than to the poorest one is always good for society. Going back to our example at the beginning of this section, Figure 1 displays the extended *S*-Gini indices for a range of values of the parameter δ for the two distributions **y** and **y**′.

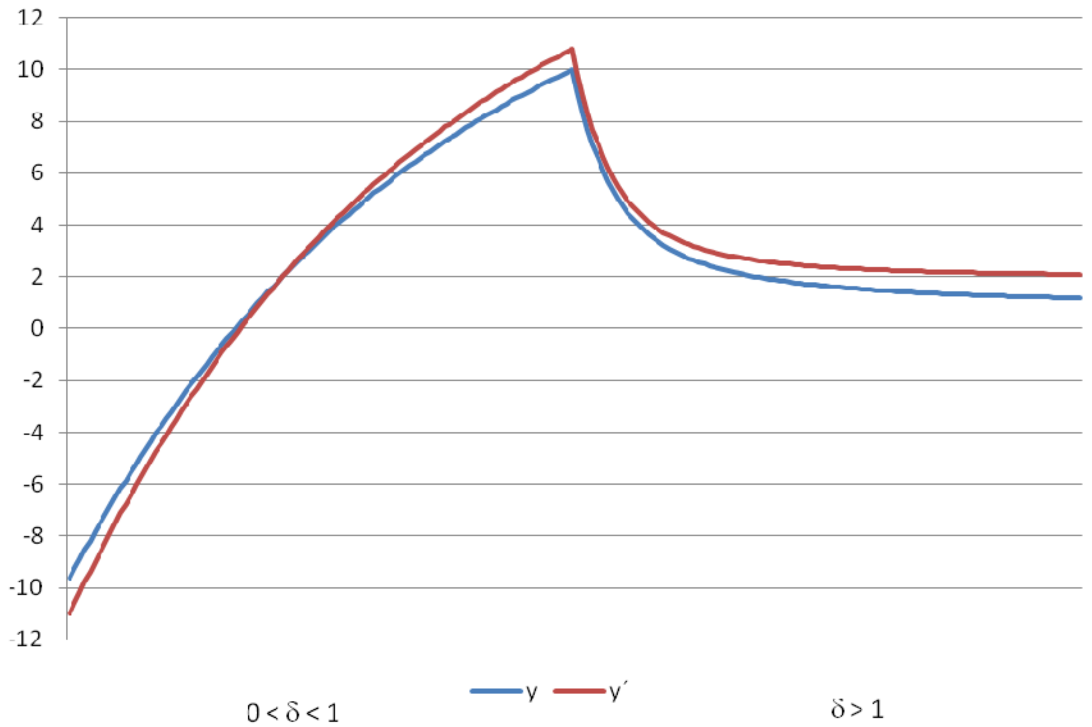


As is shown in Figure 1, all the relative S-Gini indices (the right hand side of the graph) establish that inequality decreases in going from $\mathbf{y}$ to $\mathbf{y}'$, but regarding the members of the new tail in the extended family (the left hand side of the graph), inequality rises in going from $\mathbf{y}$ to $\mathbf{y}'$. It is clear that in this case, the S-Gini family alone does not enable us to order these two distributions according to all inequality perceptions and that the information provided by this extended family can be completed using the other tail - hence the extended S-Gini family.

The mirror image of Figure 1 is the social evaluation function (16), as depicted in Figure 2. For all members of the S-Gini family, which are positively linear in incomes, social valuation is bound to increase. However, for most of the members of the new tail in the extended family, in particular for $\delta < 0.43$, the result is just the opposite and in this case we conclude that social valuation has decreased. It is clear that in this case the information provided by the S-Gini family can be completed using the new tail.

In this sense the extended S-Gini family (15), which contains specific measures sensitive to both high and low incomes, may be of great interest if we are really concerned about poor people.

Figure 2. Extended *S*-Gini social evaluation function



5. CONCLUDING REMARKS.

In this paper we have proposed an extended version of the *S*-Gini family (Donaldson and Weymark 1980, Weymark 1981) which preserve most of its properties and allow consideration of more inequality perceptions. The *S*-Gini family, a generalization of the well known Gini index, introduces the inequality aversion in a parametric way as a method to capture the views of an ethical observer (Donaldson and Weymark 1980, 1983; Yitzhaki 1983). However, as is usual in the literature, this generalization is unable to capture the whole spectrum of attitudes sensitive to what happens with high incomes, especially when we have to compare growing economies. To some extent the paper illustrates that these incomes are also important if we are committed to improving the situation of the worst-off.

Following this kind of reasoning, the paper extends the *S*-Gini family in a similar way to how the Atkinson family was previously extended (Lasso de la Vega and Urrutia 2008). The aim is to provide a whole picture of what is really happening (maybe by graphical means), since the δ-parameter is a measure of the degree of relative sensitivity to transfers at different rank orders of individuals in society.

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