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The Gini index, the dual decomposition of aggregation functions, and the consistent measurement of inequality *

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Abstract

In several economic fields, such as those related to health, education or poverty, the individuals' characteristics are measured by bounded variables. Accordingly, these characteristics may be indistinctly represented by achievements or shortfalls. A difficulty arises when inequality needs to be assessed. One may focus either on achievements or on shortfalls but the respective inequality rankings may lead to contradictory results. Specifically, this paper concentrates on the poverty measure proposed by Sen. According to this measure the inequality among the poor is captured by the Gini index. However, the rankings obtained by the Gini index applied to either the achievements or the shortfalls do not coincide in general. To overcome this drawback, we show that an OWA operator is underlying in the definition of the Sen measure. The dual decomposition of the OWA operators into a self-dual core and anti-self-dual remainder allows us to propose an inequality component which measures consistently the achievement and shortfall inequality among the poor.

Keywords: Aggregation functions, dual decomposition, OWA operators, Gini index, consistent measures of achievement/shortfall inequality, Sen index, poverty measures.

JEL Classification: C02, D63, I32.

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1 Introduction

Poverty reduction is without doubt a goal of development policy in most countries. To evaluate the evolution of poverty over time in some particular region, the differences of poverty across different countries or the effect of different policies in the alleviation of poverty, one should be first able to measure poverty. According to the 1998 Nobel Prize Laureate A.K. Sen [22], any poverty index should be sensitive to the number of people below the poverty line, to the extent of the income shortfall of the poor from the poverty line, and to the exact pattern of the income distribution of the poor. In other words, every poverty measure should be expressed as a function of these three poverty indicators, showing the incidence, the intensity and the inequality of the poor, respectively. Poverty changes can be more meaningful and easily understandable if poverty indices can be decomposed into these underlying contributing factors. A number of poverty indices[1] and their decompositions have been proposed to explicitly identify these three components[2].

The basic axiom of inequality measurement is the Pigou-Dalton principle which establishes that a transfer of income from a poor individual to a richer one increases inequality. Sen [25] points out that "a transfer of income from a person below the poverty line to anyone who is richer must increase the poverty measure". In other words, the poverty counterpart of the Pigou-Dalton principle should be fulfilled by any poverty index.

Since a transfer of income from a poorer to a richer person entails a transfer of the shortfall from the latter to the former, the poverty measure is bound to decrease if the inequality component involved in the index is defined in terms of either incomes or shortfalls. In fact, in the mentioned decompositions this third component refers to income inequality or to shortfall inequality indistinctly[3]. However, as will be shown below, the choice between income and shortfall inequality is not innocuous and different choices may lead to contradictory results. This difficulty arises not only in poverty measurement but also in different economic fields in which bounded variables are involved. Recent papers (among them Clarke *et al.* [6], Erreygers [7] and Lambert and Zheng [18]) deal with this issue in health measurement. The results derived by Lambert and Zheng [18] may have a straightforward application to the measurement of the inequality among the poor. They introduce a property of consistency which requires that achievement and shortfall inequality rankings should not be reversed, and show that all relative and intermediate inequality indices fail their requirement. Accordingly, whenever a relative or intermediate inequality index is involved in the decomposition of a poverty index, the inequality component is not consistent. We think this is a serious drawback which may distort the conclusions in the

[1]For comprehensive surveys on poverty and inequality measures see Silber [25] and Chakravarty [4].

[2]Besides Clark *et al.* [5], Osberg and Xu [21], Xu and Osberg [29] and Aristondo *et al.* [1], some of them may be found in Kakwani [17].

[3]For instance, whereas in the original proposals of Sen [22] and Shorrocks [24] the "Gini index of the poor income" takes part in the decompositions, Osberg and Xu [21] and Xu and Osberg [29] derive alternative decompositions in which the "Gini index of the gaps" is included. Similarly, the "inequality among the poor" is captured in term of gaps in the TIP curves introduced by Jenkins and Lambert [15] and in the decomposition for the FGT indices (Foster *et al.* [9]) proposed by Aristondo *et al.* [1].

analysis of the poverty trends and, consequently, the poverty decompositions are found wanting in displaying changes in the inequality among the poor, one of their main points.

This paper concentrates on the Sen index [22].[4] Two different decompositions of this index have been proposed (Sen [22] and Xu and Osberg [29]). The inequality among the poor is captured by the Gini index, applied either to the poor income or to the shortfall of the poor. However, as Lambert and Zheng [18] show, no relative inequality index offers consistent results.

In this paper a different point of view is proposed. We show that the Sen poverty index may be interpreted as an OWA operator (Yager [30]). Consequently, the dual decomposition of aggregation functions into a self-dual core and anti-self-dual remainder proposed by García-Lapresta and Marques Pereira [12] may be used. We show that these two terms can be interpreted as measures of the intensity and the inequality among the poor respectively. The anti-self-duality of the remainder component guarantees that inequality among the poor does not change if one focus either on incomes or on shortfalls. These inequality components will allow policy makers to determine in a consistent way if inequality among the poor has increased or decreased.

The paper is organized as follows. Basic notation and properties of aggregation functions are introduced in Section 2. Section 3 explores the dual decomposition of an aggregation function into a self-dual core and an associated anti-self-dual remainder, paying special attention to the case of OWA operators. Section 4 analyzes poverty measures and includes our proposal for decomposing poverty into incidence, intensity, and inequality. An illustrative example is described. Finally, section 5 contains some concluding remarks.

2 Aggregation functions

In this section we present notation and basic definitions regarding aggregation functions on the domain $[0,1]^n$, with $n \geq 2$ throughout the text.

Notation. Points in $[0,1]^n$ are denoted $\boldsymbol{x} = (x_1, \ldots, x_n)$, with $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{0} = (0, \ldots, 0)$. Accordingly, for every $x \in [0,1]$, we have $x \cdot \mathbf{1} = (x, \ldots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i \in \{1, \ldots, n\}$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in [0,1]^n$, the increasing and decreasing reorderings of the coordinates of $\boldsymbol{x}$ are indicated as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \ldots, x_n\} = x_{[n]}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\} = x_{[1]}$. In general, given a permutation σ on $\{1, \ldots, n\}$, we denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Finally, the arithmetic mean is denoted $\mu(\boldsymbol{x}) = (x_1 + \cdots + x_n)/n$.

[4] In fact our proposal works for a number of poverty indices in which the inequality is captured by the Gini index [13], such as the Thon index [28], the index introduced by Kakwani [16], the Takayama proposal [27], and the Sen index modified by Shorrocks [24].

We begin by defining standard properties of real functions on $[0,1]^n$. For further details the interested reader is referred to Fodor and Roubens [8], Calvo *et al.* [3], Beliakov *et al.* [2], and Grabisch *et al.* [14].

Definition 1 *Let* $A : [0,1]^n \longrightarrow \mathbb{R}$ *be a function.*

1. *A is* symmetric *if* $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$, *for any permutation* σ *on* $\{1,\dots,n\}$ *and all* $\boldsymbol{x} \in [0,1]^n$.
2. *A is* monotonic *if* $\boldsymbol{x} \geq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \geq A(\boldsymbol{y})$, *for all* $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$. *Moreover, A is* strictly monotonic *if* $\boldsymbol{x} > \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) > A(\boldsymbol{y})$, *for all* $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$.
3. *A is* invariant for translations *if* $A(\boldsymbol{x}+t\cdot\mathbf{1}) = A(\boldsymbol{x})$, *for all* $t \in \mathbb{R}$ *and* $\boldsymbol{x} \in [0,1]^n$ *such that* $\boldsymbol{x}+t\cdot\mathbf{1} \in [0,1]^n$. *On the other hand, A is* stable for translations *if* $A(\boldsymbol{x}+t\cdot\mathbf{1}) = A(\boldsymbol{x})+t$, *for all* $t \in \mathbb{R}$ *and* $\boldsymbol{x} \in [0,1]^n$ *such that* $\boldsymbol{x}+t\cdot\mathbf{1} \in [0,1]^n$.
4. *A is* invariant for dilations *if* $A(\lambda\cdot\boldsymbol{x}) = A(\boldsymbol{x})$, *for all* $\lambda > 0$ *and* $\boldsymbol{x} \in [0,1]^n$ *such that* $\lambda\cdot\boldsymbol{x} \in [0,1]^n$. *On the other hand, A is* stable for dilations *if* $A(\lambda\cdot\boldsymbol{x}) = \lambda\, A(\boldsymbol{x})$, *for all* $\lambda > 0$ *and* $\boldsymbol{x} \in [0,1]^n$ *such that* $\lambda\cdot\boldsymbol{x} \in [0,1]^n$.

Definition 2 *Let* $\{A^{(k)}\}_{k\in\mathbb{N}}$ *be a sequence of functions, with* $A^{(k)} : [0,1]^k \longrightarrow \mathbb{R}$ *and* $A^{(1)}(x) = x$ *for every* $x \in [0,1]$. *We say that* $\{A^{(k)}\}_{k\in\mathbb{N}}$ *is* invariant for replications *if it holds that*

$$A^{(mn)}(\overbrace{\boldsymbol{x},\dots,\boldsymbol{x}}^{m}) = A^{(n)}(\boldsymbol{x}) ,$$

for all $\boldsymbol{x} \in [0,1]^n$ *and any number of replications* $m \geq 2$ *of* $\boldsymbol{x}$.

Definition 3 *Consider the binary relation* $\succcurlyeq$ *on* $[0,1]^n$ *defined as*

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \Leftrightarrow \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \;\; \textit{and} \;\; \sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)},$$

for every $k \in \{1,\dots,n-1\}$. *The binary relation* $\succcurlyeq$ *is a partial order on* $[0,1]^n$. *As usual, we write* $\boldsymbol{x} \succ \boldsymbol{y}$ *if* $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ *and* $\boldsymbol{x} \neq \boldsymbol{y}$. *With respect to the binary relation* $\succcurlyeq$, *the notions of* Schur-convexity *(S-convexity) and* Schur-concavity *(S-concavity) of a function* $A : [0,1]^n \longrightarrow \mathbb{R}$ *are defined as follows,*

1. *A is* S-convex *if* $\boldsymbol{x} \succcurlyeq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \geq A(\boldsymbol{y})$, *for all* $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$.
2. *A is* S-concave *if* $\boldsymbol{x} \succcurlyeq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \leq A(\boldsymbol{y})$, *for all* $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$.

Moreover, a function A is strictly S-convex *if* $\boldsymbol{x} \succ \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) > A(\boldsymbol{y})$. *Analogously, a function A is* strictly S-concave *if* $\boldsymbol{x} \succ \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) < A(\boldsymbol{y})$.

Definition 4 *Given $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, we say that $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by a* progressive transfer *if there exist $i, j \in \{1, \ldots, n\}$ and $h > 0$ such that $x_i < x_j$, $y_i = x_i + h \leq x_j - h = y_j$ and $y_k = x_k$ for every $k \in \{1, \ldots, n\} \setminus \{i, j\}$.*

A classical result, see Marshall and Olkin [20, Ch. 4, Prop. A.1], establishes that $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ if and only if $\boldsymbol{y}$ can be derived from $\boldsymbol{x}$ by means of a finite sequence of permutations and/or progressive transfers.

The following are some other important properties of real functions on $[0,1]^n$. As before, see Fodor and Roubens [8], Calvo *et al.* [3], Beliakov *et al.* [2], Grabisch *et al.* [14], plus also García-Lapresta and Marques Pereira [12].

Definition 5 *Let $A : [0,1]^n \longrightarrow [0,1]$ be a function.*

1. *A is* idempotent *if $A(x \cdot \mathbf{1}) = x$, for all $x \in [0,1]$.*
2. *A is* compensative *if $x_{(1)} \leq A(\boldsymbol{x}) \leq x_{(n)}$, for all $\boldsymbol{x} \in [0,1]^n$.*
3. *A is* self-dual *if $I = [0,1]$ and $A(\mathbf{1} - \boldsymbol{x}) = 1 - A(\boldsymbol{x})$, for all $\boldsymbol{x} \in [0,1]^n$.*
4. *A is* anti-self-dual *if $I = [0,1]$ and $A(\mathbf{1} - \boldsymbol{x}) = A(\boldsymbol{x})$, for all $\boldsymbol{x} \in [0,1]^n$.*

Definition 6 *A function $A : [0,1]^n \longrightarrow [0,1]$ is called an n-*ary aggregation function *if it is monotonic and $A(\mathbf{0}) = 0$, $A(\mathbf{1}) = 1$. An aggregation function is said to be* strict *if it is strictly monotonic. For simplicity, the n-arity is omitted whenever it is clear from the context.*

It is easy to see that every aggregation function is compensative. Self-duality and stability for translations are important properties of aggregation functions. In turn, anti-self-duality and invariance for translations are incompatible with idempotency, one of the defining properties of aggregation functions. Nevertheless, anti-self-duality and invariance for translations play an important role in this paper as far as they are properties of important functions associated with aggregation functions, such as we shall discuss later.

3 Dual decomposition

In this section we briefly recall the so-called *dual decomposition* of an aggregation function into its self-dual core and associated anti-self-dual remainder, due to García-Lapresta and Marques Pereira [12]. First we introduce the concepts of self-dual core and anti-self-dual remainder of an aggregation function, establishing which properties are inherited in each case from the original aggregation function. Particular emphasis is given to the properties of stability for translations (self-dual core) and invariance for translations (anti-self-dual remainder).

Definition 7 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function. The aggregation function* $A^* : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$A^*(\boldsymbol{x}) = 1 - A(\mathbf{1} - \boldsymbol{x})$$

is known as the dual *of the aggregation function* A*.*

Clearly, $(A^*)^* = A$, which means that dualization is an *involution*. An aggregation function A is self-dual if and only if $A^* = A$.

3.1 The self-dual core

Aggregation functions are not in general self-dual. However, a self-dual aggregation function can be associated to any aggregation function in a simple manner. The construction of the so-called *self-dual core* of an aggregation function A is as follows.

Definition 8 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function. The function* $\widehat{A} : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$\widehat{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) + A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) - A(\mathbf{1} - \boldsymbol{x}) + 1}{2}$$

is called the core *of the aggregation function* A*.*

Since $\widehat{A}$ is self-dual, we say that $\widehat{A}$ is the *self-dual core* of the aggregation function A. Notice that $\widehat{A}$ is clearly an aggregation function. It is interesting to note that the self-dual core reduces to the arithmetic mean in the simple case of $n = 2$, but not in higher dimensions.

The following results[5] can be found in García-Lapresta and Marques Pereira [12].

Proposition 1 *An aggregation function* $A : [0,1]^n \longrightarrow [0,1]$ *is self-dual if and only if* $\widehat{A}(\boldsymbol{x}) = A(\boldsymbol{x})$ *for every* $\boldsymbol{x} \in [0,1]^n$*.*

Proposition 2 *The self-dual core* $\widehat{A}$ *inherits from the aggregation function* A *the properties of continuity, idempotency (hence, compensativeness), symmetry, strict monotonicity, stability for translations, and invariance for replications, whenever* A *has these properties.*

[5] Excepting that invariance for replications is inherited by the core (the proof is immediate).

3.2 The anti-self-dual remainder

We now introduce the *anti-self-dual remainder* $\widetilde{A}$, which is simply the difference between the original aggregation function A and its self-dual core $\widehat{A}$.

Definition 9 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. The function $\widetilde{A} : [0,1]^n \longrightarrow \mathbb{R}$ defined as $\widetilde{A}(\boldsymbol{x}) = A(\boldsymbol{x}) - \widehat{A}(\boldsymbol{x})$, that is*

$$\widetilde{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) - A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) + A(\mathbf{1} - \boldsymbol{x}) - 1}{2}$$

is called the remainder *of the aggregation function A.*

Since $\widetilde{A}$ is anti-self-dual, we say that $\widetilde{A}$ is the *anti-self-dual remainder* of the aggregation function A. Clearly, $\widetilde{A}$ is not an aggregation function. In particular, $\widetilde{A}(\mathbf{0}) = \widetilde{A}(\mathbf{1}) = 0$, which violates idempotency and implies that $\widetilde{A}$ is either non monotonic or everywhere null. Moreover, $-0.5 \leq \widetilde{A}(\boldsymbol{x}) \leq 0.5$ for every $\boldsymbol{x} \in [0,1]^n$.

The following results[6] can be found in García-Lapresta and Marques Pereira [12].

Proposition 3 *An aggregation function $A : [0,1]^n \longrightarrow [0,1]$ is self-dual if and only if $\widetilde{A}(\boldsymbol{x}) = 0$ for every $\boldsymbol{x} \in [0,1]^n$.*

Proposition 4 *The anti-self-dual remainder $\widetilde{A}$ inherits from the aggregation function A the properties of continuity, symmetry, invariance for replications, plus also strict S-convexity and S-concavity, whenever A has these properties.*

Summarizing, every aggregation function A decomposes additively $A = \widehat{A} + \widetilde{A}$ in two components: the self-dual core $\widehat{A}$ and the anti-self-dual remainder $\widetilde{A}$, where only $\widehat{A}$ is an aggregation function. The so-called *dual decomposition* $A = \widehat{A} + \widetilde{A}$ clearly shows some analogy with other algebraic decompositions, such as that of square matrices and bilinear tensors into their symmetric and skew-symmetric components.

The following result concerns two more properties of the anti-self-dual remainder based directly on the definition $\widetilde{A} = A - \widehat{A}$ and the corresponding properties of the self-dual core (see García-Lapresta and Marques Pereira [12]).

Proposition 5 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function.*

[6] Excepting that invariance for replications is inherited by the remainder (the proof is immediate) and that strict S-convexity and S-concavity are also inherited by the remainder.

1. $\widetilde{A}(x \cdot \mathbf{1}) = 0$ *for every* $x \in [0, 1]$.

2. *If A is stable for translations, then $\widetilde{A}$ is invariant for translations.*

These properties of the anti-self-dual remainder are suggestive. The first statement establishes that anti-self-dual remainders are null on the main diagonal. The second statement applies to the subclass of stable aggregation functions. In such case, self-dual cores are stable and therefore anti-self-dual remainders are invariant for translations. In other words, if the aggregation function A is stable for translations, the value $\widetilde{A}(\boldsymbol{x})$ does not depend on the average value of the $\boldsymbol{x}$ coordinates, but only on their numerical deviations from that average value. These properties of the anti-self-dual remainder $\widetilde{A}$ suggest that it may give some indication on the dispersion of the $\boldsymbol{x}$ coordinates.

In Maes *et al.* [19], the authors propose a generalization of the dual decomposition framework introduced in García-Lapresta and Marques Pereira [12], based on a family of binary aggregation functions satisfying a form of twisted self-duality condition. Each binary aggregation function in that family corresponds to a particular way of combining an aggregation function A with its dual A^* for the construction of the self-dual core $\widehat{A}$. As particular cases of the general framework proposed in Maes *et al.* [19], one obtains García-Lapresta and Marques Pereira's construction, based on the arithmetic mean, and Silvert's construction, based on the symmetric sums formula (see Silvert [26]). However, the dual decomposition framework introduced in García-Lapresta and Marques Pereira [12] remains the only one which preserves stability under translations, a crucial requirement in the present construction of poverty measures.

3.3 OWA operators

In 1988 Yager [30] introduced OWA operators as a tool for aggregating numerical values in multi-criteria decision making. An OWA operator is similar to a weighted mean, but with the values of the variables previously ordered in a decreasing way. Thus, contrary to the weighted means, the weights are not associated with concrete variables and, therefore, they are symmetric. Because of these properties, OWA operators have been widely used in the literature (see, for instance, Yager and Kacprzyk [31] and Yager *et al.* [32]).

Definition 10 *Given a weighting vector* $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$ *satisfying* $\sum_{i=1}^{n} w_i = 1$, *the* OWA operator associated with $\boldsymbol{w}$ *is the aggregation function* $A_w : [0,1]^n \longrightarrow [0,1]$ *defined as follows,*

$$A_w(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{[i]} \; ,$$

where $x_{[1]} \geq \cdots \geq x_{[n]}$ *as usual in the literature on OWA operators.*

Simple examples of OWA operators are

$$A_w(\boldsymbol{x}) = \begin{cases} \max\{x_1, \ldots, x_n\}\,, & \text{when } \boldsymbol{w} = (1, 0, \ldots, 0)\,, \\ \min\{x_1, \ldots, x_n\}\,, & \text{when } \boldsymbol{w} = (0, \ldots, 0, 1\,)\,, \\ \mu(\boldsymbol{x})\,, & \text{when } \boldsymbol{w} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})\,. \end{cases}$$

In general, OWA operators are continuous, idempotent (hence, compensative), symmetric, and stable for translations. Moreover, an OWA operator A_w is self-dual if and only if $w_{n+1-i} = w_i$ for every $i \in \{1, \ldots, n\}$, see García-Lapresta and Llamazares [11, Proposition 5].

The following is a classical result, see for instance Chakravarty [4, p. 28].

Proposition 6 *Consider the OWA operator* $A_w : [0,1]^n \longrightarrow [0,1]$ *associated with a weighting vector* $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$. *If the weights are non increasing,* $w_1 \geq \cdots \geq w_n$, *then the OWA operator* A_w *is S-convex. Instead, if the weights are non decreasing,* $w_1 \leq \cdots \leq w_n$, *then* A_w *is S-concave. Both results extend naturally to the strict case: decreasing weights* $w_1 > \cdots > w_n$ *imply strict S-convexity, and increasing weights* $w_1 < \cdots < w_n$ *imply strict S-concavity.*

Proof. Consider two points $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ with $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$, that is, $\sum_{i=1}^n (x_i - y_i) = 0$ and $\sum_{i=1}^k (x_{[i]} - y_{[i]}) \geq 0$ for every $k \in \{1, \ldots, n-1\}$. Then, in the case $w_1 \geq \cdots \geq w_n$, one obtains $A_w(\boldsymbol{x}) - A_w(\boldsymbol{y}) \geq 0$ in the following way,

$$\begin{aligned}
& w_1(x_{[1]} - y_{[1]}) + w_2(x_{[2]} - y_{[2]}) + w_3(x_{[3]} - y_{[3]}) + \cdots + w_n(x_{[n]} - y_{[n]}) \geq \\
& w_2(x_{[1]} - y_{[1]}) + w_2(x_{[2]} - y_{[2]}) + w_3(x_{[3]} - y_{[3]}) + \cdots + w_n(x_{[n]} - y_{[n]}) = \\
& w_2(x_{[1]} + x_{[2]} - y_{[1]} - y_{[2]}) + w_3(x_{[3]} - y_{[3]}) + \cdots + w_n(x_{[n]} - y_{[n]}) \geq \\
& w_3(x_{[1]} + x_{[2]} - y_{[1]} - y_{[2]}) + w_3(x_{[3]} - y_{[3]}) + \cdots + w_n(x_{[n]} - y_{[n]}) = \\
& w_3(x_{[1]} + x_{[2]} + x_{[3]} - y_{[1]} - y_{[2]} - y_{[3]}) + \cdots + w_n(x_{[n]} - y_{[n]}) \geq \cdots \\
& w_n(x_{[1]} + x_{[2]} + x_{[3]} + \cdots + x_{[n]} - y_{[1]} - y_{[2]} - y_{[3]} - \cdots - y_{[n]}) = 0\,.
\end{aligned}$$

Analogously, in the case $w_1 \leq \cdots \leq w_n$ one obtains $A_w(\boldsymbol{x}) - A_w(\boldsymbol{y}) \leq 0$. In the strict case, we consider two points $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ with $\boldsymbol{x} \succ \boldsymbol{y}$, which means that at least one of the cumulative differences $\sum_{i=1}^k (x_{[i]} - y_{[i]})$ is positive. We then use the appropriate strict monotonicity of the weights. ■

In general, the self-dual core $\widehat{A}_w$ and the anti-self-dual remainder $\widetilde{A}_w$ of an OWA operator A_w can be written as

$$\widehat{A}_w(\boldsymbol{x}) = \sum_{i=1}^n \frac{w_i + w_{n-i+1}}{2}\, x_{[i]} \quad \text{and} \quad \widetilde{A}_w(\boldsymbol{x}) = \sum_{i=1}^n \frac{w_i - w_{n-i+1}}{2}\, x_{[i]}\,.$$

As we know, the self-dual core $\widehat{A}_w$ is an aggregation function. Moreover, since

$$\sum_{i=1}^{n} \frac{w_i + w_{n-i+1}}{2} = 1\,,$$

the self-dual core $\widehat{A}_w$ is again an OWA operator, that is $\widehat{A}_w = A_{\widehat{w}}$ with $\widehat{w}_i = (w_i + w_{n-i+1})/2$ for every $i \in \{1, \ldots, n\}$. Notice that $\widehat{A}_w$ reduces to the arithmetic mean in the simple case $n = 2$, but not in higher dimensions.

On the other hand, the anti-self-dual remainder $\widetilde{A}_w$ is not an aggregation function. Notice, in particular, that $\widetilde{A}_w(\mathbf{1}) = 0$, since

$$\sum_{i=1}^{n} \frac{w_i - w_{n-i+1}}{2} = 0\,.$$

The self-dual core and the anti-self-dual remainder can be equivalently written as follows,

$$\widehat{A}_w(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, \frac{x_{[i]} + x_{[n-i+1]}}{2} \quad \text{and} \quad \widetilde{A}_w(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, \frac{x_{[i]} - x_{[n-i+1]}}{2}\,.$$

These expressions show clearly that the self-dual core is a weighted average of pairwise averages of $\boldsymbol{x}$ coordinates (*quasi-midranges*), whereas the anti-self-dual remainder is a weighted average of pairwise differences of $\boldsymbol{x}$ coordinates (*quasi-ranges*). The anti-self-dual remainder is therefore independent of the overall average of the coordinates of $\boldsymbol{x}$ and constitutes a form of dispersion measure.

Finally, one can show that $w_1 \geq \cdots \geq w_n$ implies $\widetilde{A}_w(\boldsymbol{x}) \geq 0$ and $w_1 \leq \cdots \leq w_n$ implies $\widetilde{A}_w(\boldsymbol{x}) \leq 0$. In fact, the anti-self-dual remainder can be written as follows,

$$\widetilde{A}_w(\boldsymbol{x}) = \frac{1}{2} \sum_{i=1}^{n} \left(w_i - w_{n-i+1}\right) x_{[i]} = \frac{1}{4} \sum_{i=1}^{n} \left(w_i - w_{n-i+1}\right) \left(x_{[i]} - x_{[n-i+1]}\right)\,.$$

Then, in the first case, we have $\widetilde{A}_w(\boldsymbol{x}) \geq 0$ because in each term of the summation both factors are non negative for $i \leq n-i+1$ and non positive for $i \geq n-i+1$. Analogously, in the second case, we have $\widetilde{A}_w(\boldsymbol{x}) \leq 0$ because in each term of the summation for $i \leq n-i+1$ the first factor is non positive while the second factor is non negative, and vice-versa for $i \geq n-i+1$.

4 Poverty measures

We begin with a brief summary of the basic notions about poverty measures. Notation and definitions follow García-Lapresta *et al.* [10].

We consider a population consisting of n individuals, with $n \geq 2$. An *income distribution* is represented by a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in [0, \infty)^n$, where x_i is the income of individual i.

Since Sen [22], any poverty measure consists of a method to identify the poor together with an aggregative measure. Thus, the first step to define a poverty measure is the identification of the poor people in society. This step requires the specification of a *poverty line* $z \in (0, \infty)$ which represents the necessary income to maintain a minimum level of living. For an income distribution $\boldsymbol{x}$, person i is considered to be *poor* if $x_i < z$. Otherwise the person is *non-poor* or rich.

We denote the set of poor people by

$$Q(\boldsymbol{x}, z) = \{i \in \{1, \ldots, n\} \mid x_i < z\},$$

and by $q(\boldsymbol{x}, z)$ the number of the poor, i.e., $q(\boldsymbol{x}, z) = \#Q(\boldsymbol{x}, z)$.

Once the poor people have been identified, the second step to determine the extent of poverty involves the aggregation scheme. In what follows, a poverty measure is a non-constant function $P(\boldsymbol{x}, z)$ of the income distribution $\boldsymbol{x}$ and the poverty line z.

4.1 Axioms

A number of axioms are usually assumed for a poverty measure.

- *Poverty Focus* (**PF**): For all $\boldsymbol{x}, \boldsymbol{y} \in [0, \infty)^n$ and $z \in (0, \infty)$, if $Q(\boldsymbol{x}, z) = Q(\boldsymbol{y}, z) = Q$ and $x_i = y_i$ for every $i \in Q$, then $P(\boldsymbol{x}, z) = P(\boldsymbol{y}, z)$.
- *Poverty Monotonicity* (**PM**): For all $\boldsymbol{x}, \boldsymbol{y} \in [0, \infty)^n$ and $z \in (0, \infty)$, if $Q(\boldsymbol{x}, z) = Q(\boldsymbol{y}, z) = Q$ and $\boldsymbol{x} = \boldsymbol{y}$ except for $x_i > y_i$ with $i \in Q$, then $P(\boldsymbol{x}, z) < P(\boldsymbol{y}, z)$.

Since poverty measurement is concerned with the deprivations of poor people, these two properties, postulated by Sen [22], are considered as the basic axioms for a poverty measure. Thus, axiom **PF** requires that a poverty index should not depend on the income of the non-poor people, i.e., the poverty level should not vary if the rich incomes change, as long as the set of poor people remains unchanged. On the other hand, axiom **PM** demands that poverty should increase if the income of a poor person decreases.

The following axiom is concerned with inequality among the poor. In the inequality field, the Pigou-Dalton transfer principle establishes that a progressive transfer, that is a transfer from a richer person to a poorer one, should decrease inequality. Accordingly, a progressive transfer among the poor should decrease inequality among the poor. Sen [22] introduces the counterpart of this principle in the poverty field, requiring poverty also to decrease. This is captured by the Transfer Sensitive axiom below.

- *Transfer Sensitivity* (**TS**): For all $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$ and $z \in (0,\infty)$, if $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by a progressive transfer among the poor, then $P(\boldsymbol{y}, z) < P(\boldsymbol{x}, z)$.

A progressive transfer among the poor entails an increment of income for one poor individual, and a decrement for another poor person, the richer of the two. This **TS** axiom goes beyond **PM** and demands that greater weight should be placed on the poorer person and that poverty should decrease if inequality among the poor decreases.

A normalization condition is also usually assumed in the poverty measurement. This property requires that if all the individuals are non-poor, then the society poverty level is equal to 0.

- *Normalization* (**N**): For all $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$ and $z \in (0,\infty)$, $P(\boldsymbol{x}, z) = 0$ if and only if $Q(\boldsymbol{x}, z) = \emptyset$, that is $x_i \geq z$ for every $i \in \{1, \ldots, n\}$.

The two following invariance axioms are also standard requirements for a poverty measure:

- *Poverty Symmetry* (**PS**): For all $\boldsymbol{x} \in [0,\infty)^n$, $z \in (0,\infty)$, and any permutation σ on $\{1, \ldots, n\}$, it holds that $P(\boldsymbol{x}_\sigma, z) = P(\boldsymbol{x}, z)$.
- *Replication Invariance* (**RI**): For all $\boldsymbol{x} \in [0,\infty)^n$ and $z \in (0,\infty)$, if $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by a replication, that is $\boldsymbol{y} = (\overbrace{\boldsymbol{x}, \ldots, \boldsymbol{x}}^{m})$ for some $m \in \mathbb{N}$, then $P(\boldsymbol{y}, z) = P(\boldsymbol{x}, z)$.

The **PS** axiom establishes that no other characteristic apart from income deprivation matters in defining a poverty index. In turn, **RI** allows us to compare populations of different sizes.

The first poverty measure introduced in the literature has been the *headcount ratio* $H : [0,\infty)^n \times (0,\infty) \longrightarrow [0,1]$ defined as

$$H(\boldsymbol{x}, z) = \frac{q(\boldsymbol{x}, z)}{n},$$

which measures the percentage of poor people in the society.

This is a crude index, which is able to capture the incidence of poverty. However, it is able to capture neither the intensity nor the inequality among the poor. In fact it violates both **PM** and **TS**, since it does not change if the income of a poor decreases, and under progressive transfers among the poor.

In most cases, measuring poverty involves gauging the extent of the deprivation felt by each individual, once the income poverty line has been determined. One of the most used procedures to measure individual i's shortfall is to consider the normalized gap of individual i.

Definition 11 *For all* $\boldsymbol{x} \in [0,\infty)^n$ *and* $z \in (0,\infty)$*, the* normalized income gap of individual i *is defined as*

$$g_i = \max\left\{\frac{z - x_i}{z}, 0\right\}.$$

Notice that $g_i \in [0,1]$ for every $i \in \{1,\ldots,n\}$, with $g_i = 0$ iff $x_i \geq z$, and $g_i = 1$ iff $x_i = 0$. Moreover, the normalized income gaps $(g_i,\ldots,g_n)$ are invariant under dilations, that is, uniform scale changes on incomes $(x_i,\ldots,x_n)$ and poverty line z.

On the other hand, a progressive transfer among the poor people leads to an increment in the richer individual gap whereas the poorer person gap decreases. Since the richer gap is smaller than the poorer one, the progressive transfers among the poor incomes are equivalent to the progressive transfers among the poor gaps. Then, according to Marshall and Olkin [20, Ch. 4, Prop. A.1], the **TS** axiom is to be fulfilled whenever the function is strictly S-convex either in incomes or in gaps.

Definition 12 *The* aggregate income gap ratio $M : [0,\infty)^n \times (0,\infty) \longrightarrow [0,1]$ *is defined as*

$$M(\boldsymbol{x}, z) = \mu(\boldsymbol{g}_p) = \frac{1}{q}\sum_{i=1}^{q} g_{[i]}\,,$$

where $g_{[1]} \geq \cdots \geq g_{[q]}$ *are the positive normalized poverty gaps generated by the income distribution* $\boldsymbol{x}$*, the remaining normalized poverty gaps* $g_{[q+1]} = \cdots = g_{[n]} = 0$ *being null. Accordingly, it holds that*

$$1 - M(\boldsymbol{x}, z) = \mu(\boldsymbol{x}_p/z) = \frac{1}{q}\sum_{i=1}^{q} x_{(i)}/z\,,$$

where $x_{(1)} \leq \cdots \leq x_{(q)} < z$ *are the poor incomes in the income distribution* $\boldsymbol{x}$*, the remaining incomes* $z \leq x_{(q+1)} \leq \cdots \leq x_{(n)}$ *being non poor.*

This index usually measures the intensity of poverty and gives the minimum cost of eliminating poverty but does not reflect the inequality among the poor.

4.2 The Sen poverty index and an alternative decomposition proposal

We now introduce the *Sen poverty index* [22]. Although this is not Sen's original proposal, the modified expression indicated below is presently the standard reference, see also Sen [23].

Definition 13 *The* Sen poverty index $S : [0,\infty)^n \times (0,\infty) \longrightarrow [0,1]$ *is defined as follows,*

$$S(\boldsymbol{x}, z) = \frac{1}{qn} \sum_{i=1}^{q} (2(q-i)+1)\, g_{[i]}\,, \tag{1}$$

where $g_{[1]} \geq \cdots \geq g_{[q]}$ *are the positive normalized income gaps generated by the distribution* $\boldsymbol{x}$*.*

The summation structure of the S poverty index essentially combines the normalized income gaps of the poor with q positive coefficients which are larger for individuals with larger income gaps: apart from the overall factor $1/qn$, the largest gap has coefficient $2q-1$ and the smallest gap has coefficient 1, with decreasing two unit differences from one coefficient to the next. Actually, as we will see below, the S index is in fact a convex combination of the normalized poverty gaps, multiplied by the headcount ratio q/n. Moreover, the S index satisfies **PF**, **PM**, **TS**, **PS**, and **RI** [22].

In the literature, two alternative decompositions have been proposed of this index. On the one hand, Sen [22] shows that the index satisfies

$$S(\boldsymbol{x}, z) = H(\boldsymbol{x}, z)(M(\boldsymbol{x}, z) + (1 - M(\boldsymbol{x}, z))\, G(\boldsymbol{x}_p))\,, \tag{2}$$

where $G(\boldsymbol{x}_p) \in [0,1]$ is the Gini index of the poor sector of the population,

$$G(\boldsymbol{x}_p) = 1 - \frac{1}{\mu(\boldsymbol{x}_p)} \sum_{i=1}^{q} \frac{2(q-i)+1}{q^2}\, x_{(i)},$$

and $x_{(1)} \leq \cdots \leq x_{(q)} < z$ are the poor incomes in the distribution $\boldsymbol{x}$.

On the other hand, Xu and Osberg [29] propose the following alternative decomposition

$$S(\boldsymbol{x}, z) = H(\boldsymbol{x}, z)(M(\boldsymbol{x}, z) + M(\boldsymbol{x}, z)\, G(\boldsymbol{g}_p))\,, \tag{3}$$

where $G(\boldsymbol{g}_p) \in [0,1]$ is the Gini index of the normalized income gaps of the poor,

$$G(\boldsymbol{g}_p) = 1 - \frac{1}{\mu(\boldsymbol{g}_p)} \sum_{i=1}^{q} \frac{2i-1}{q^2}\, g_{[i]}.$$

The difference in the summation coefficients appearing in the expressions of $G(\boldsymbol{x}_p)$ and $G(\boldsymbol{g}_p)$ above is due only to the two different re-orderings of the index values $i = 1, \ldots, n$ involved.

However, as already mentioned, the choice between the Gini index of the poor incomes and that of the normalized income gaps of the poor is not innocuous. To illustrate this, let us consider two income distributions $\boldsymbol{x}^1 = (4, 5, 25, 35)$ and $\boldsymbol{x}^2 = (3, 4, 22, 32)$. Let us assume that the poverty line is $z = 36$. Then, the corresponding poverty gap distributions are $\boldsymbol{g}^1 =$

$\left(\frac{32}{36}, \frac{31}{36}, \frac{11}{36}, \frac{1}{36}\right)$ and $\boldsymbol{g}^2 = \left(\frac{33}{36}, \frac{32}{36}, \frac{14}{36}, \frac{4}{36}\right)$. The Gini index of the income distributions concludes that the inequality among the poor is higher in the latter than in the former, $G(\boldsymbol{x}^1) = 0.409 < 0.430 = G(\boldsymbol{x}^2)$. Nevertheless, this conclusion is reversed if the Gini index of the gap distributions is computed since $G(\boldsymbol{g}^1) = 0.377 > 0.316 = G(\boldsymbol{g}^2)$.

In what follows we propose an alternative decomposition of the Sen index that overcomes this drawback. We begin by rewriting the S index as

$$S(\boldsymbol{x}, z) = H(\boldsymbol{x}, z) \sum_{i=1}^{q} \frac{2(q-i)+1}{q^2} \, g_{[i]} \, , \tag{4}$$

where, as before, $g_{[1]} \geq \cdots \geq g_{[q]}$ are the positive normalized income gaps generated by the distribution $\boldsymbol{x}$. The summation multiplying the headcount ratio corresponds to an OWA operator $A_G : [0,1]^q \longrightarrow [0,1]$ applied to the normalized poverty gaps,

$$A_G(\boldsymbol{g}_p) = \sum_{i=1}^{q} w_i \, g_{[i]} \, , \qquad w_i = \frac{2(q-i)+1}{q^2} \, , \quad i = 1, \ldots, q \tag{5}$$

with decreasing positive weights $w_1 > \cdots > w_q$, satisfying $\sum_{i=1}^{q} w_i = 1$. This OWA operator A_G satisfies a number of important properties.

Proposition 7 *The OWA operator A_G defined above satisfies continuity, idempotency (hence, compensativeness), symmetry, strict monotonicity, stability for translations, invariance for replications, and strict S-convexity.*

Proof. In general, OWA operators are continuous, idempotent (hence, compensative), symmetric and stable for translations. In the case of A_G, positivity of the weights implies strict monotonicity. Moreover, the fact that weights are decreasing implies strict S-convexity, as explained in Subsection 3.3. Finally, invariance for replications can be derived as follows. Let $\boldsymbol{gg}_p$ denote the duplicated vector of normalized income gaps, with $j = 1, 2, \ldots, 2q-1, 2q$ components given by $gg_{[2i-1]} = gg_{[2i]} = g_{[i]}$ for $i = 1, \ldots, q$. Then,

$$A_G(\boldsymbol{gg}_p) = \sum_{j=1}^{2q} \frac{2(2q) - 2j + 1}{(2q)^2} \; gg_{[j]} =$$

$$= \sum_{i=1}^{q} \left(\frac{2(2q) - 2(2i-1) + 1}{(2q)^2} + \frac{2(2q) - (2i) + 1}{(2q)^2} \right) \; g_{[i]} =$$

$$= \sum_{i=1}^{q} \left(\frac{8q - 8i + 4}{4q^2} \right) \; g_{[i]} = A_G(\boldsymbol{g}_p) \; .$$

The proof easily extends to higher order replications. ■

A straightforward application of the previous section allows us to compute the self-dual core and the anti-self-dual remainder of A_G. By Propositions 2 and 7, the core $\widehat{A}_G$ is idempotent, symmetric, strictly monotonic, and stable for translations. The strictly monotonicity axiom implies that is increasing in the income gap of a poor person. The stability for translations means that equal absolute changes in all poor gaps lead to the same absolute change in $\widehat{A}_G$. These properties can be regarded as basic properties of a poverty intensity index. In the particular case of the Sen index, Proposition 8 below shows that the core $\widehat{A}_G$ coincides with the aggregate income gap ratio, as already mentioned the archetypical measure of the poverty intensity.

Proposition 8 *The self-dual core of the OWA operator A_G is given by*

$$\widehat{A}_G(\boldsymbol{g}_p) = \mu(\boldsymbol{g}_p) \; .$$

Proof. Since $w_i = (2(q-i)+1)/q^2$ and $w_{q-i+1} = (2i-1)/q^2$, we obtain

$$\widehat{A}_G(\boldsymbol{g}_p) = \sum_{i=1}^{q} \frac{w_i + w_{q-i+1}}{2} \, g_{[i]} = \sum_{i=1}^{q} \frac{1}{q} \, g_{[i]} = \mu(\boldsymbol{g}_p) \; . \quad \blacksquare \tag{6}$$

On the other hand, the antiself-dual remainder $\widetilde{A}_G$ is symmetric, fulfills that $\widetilde{A}_G(\boldsymbol{g}_p) = 0$ if and only if $g_1 = \cdots = g_q$, and Propositions 4 and 7 ensure that it is S-convex, and consequently the Pigou-Dalton transfer principle is satisfied. Hence, obtains a direct interpretation of $\widetilde{A}_G$ as a measure of inequality among the poor individuals. What is more interesting in our discussion, is that $\widetilde{A}_G$ is anti-self-dual, that is, inequality among the poor does not change if we focus on poverty gaps, or on achievements as measured by $\boldsymbol{x}_p/z$. This component is also invariant if the units in which income is measured change.

Moreover, $\widetilde{A}_G$ is invariant for translations (Proposition 5), thus it measures inequality from an absolute point of view and remains invariant if the gaps of all the poor are increased by the same amount. Propositions 9 and 10 below show that the anti-self-dual remainder of the OWA operator A_G associated with the Sen index corresponds to the absolute Gini index of the normalized poverty gaps, $G_A(\boldsymbol{g}_p) = \mu(\boldsymbol{g}_p)\, G(\boldsymbol{g}_p)$. Or, equivalently, to the absolute Gini index of the poor incomes normalized by the poverty line, $G_A(\boldsymbol{x}_p/z) = \mu(\boldsymbol{x}_p/z)\, G(\boldsymbol{x}_p/z)$.

Proposition 9 *The anti-self-dual remainder of the OWA operator A_G is given by*

$$\widetilde{A}_G(\boldsymbol{g}_p) = G_A(\boldsymbol{g}_p) \; .$$

Proof. Again, since $w_i = (2(q-i)+1)/q^2$ and $w_{q-i+1} = (2i-1)/q^2$, we obtain

$$\widetilde{A}_G(\boldsymbol{g}_p) = \sum_{i=1}^{q} \frac{w_i - w_{q-i+1}}{2}\, g_{[i]} = \sum_{i=1}^{q} \frac{q-2i+1}{q^2}\, g_{[i]} =$$

$$\sum_{i=1}^{q} \frac{1}{q}\, g_{[i]} - \sum_{i=1}^{q} \frac{2i-1}{q^2}\, g_{[i]} = \mu(\boldsymbol{g}_p)\, G(\boldsymbol{g}_p) = G_A(\boldsymbol{g}_p)\ . \quad \blacksquare$$

Therefore, the dual decomposition of the OWA operator A_G involved in the Sen index S, as in equation (5), leads to

$$A_G(\boldsymbol{g}_p) = \widehat{A}_G(\boldsymbol{g}_p) + \widetilde{A}_G(\boldsymbol{g}_p) = \mu(\boldsymbol{g}_p) + G_A(\boldsymbol{g}_p)\ . \tag{7}$$

Now, changing focus from shortfalls to achievements, that is from normalized poverty gaps to poor incomes normalized by the poverty line, we can prove the following result.

Proposition 10 *The absolute Gini index $G_A(\boldsymbol{g}_p)$ of the normalized poverty gaps coincides with the absolute Gini index $G_A(\boldsymbol{x}_p/z)$ of the poor incomes normalized by the poverty line,*

$$G_A(\boldsymbol{g}_p) = G_A(\boldsymbol{x}_p/z)\ .$$

Proof. In the previous result we have shown that $G_A(\boldsymbol{g}_p)$ corresponds to the anti-self-dual remainder of the OWA operator A_G. Then, by anti-self-duality, we immediately obtain

$$G_A(\boldsymbol{g}_p) = G_A(\mathbf{1} - \boldsymbol{g}_p) = G_A(\boldsymbol{x}_p/z)\ . \quad \blacksquare$$

This result is central to the consistent measurement of inequality on the bounded scale $[0,1]$, irrespectively of whether one focus on achievements $\boldsymbol{x}_p/z$ or shortfalls $\boldsymbol{g}_p = \mathbf{1} - \boldsymbol{x}_p/z$ of the poor sector of the population. The crucial fact that the absolute Gini index G_A contains the appropriate weighting mechanism in order to provide a common synthesis of the two descriptions is well illustrated in the following explicit derivation,

$$\begin{aligned}
&G_A(\boldsymbol{g}_p) = \mu(\boldsymbol{g}_p)\, G(\boldsymbol{g}_p) = \sum_{i=1}^{q} \frac{1}{q}\, g_{[i]} - \sum_{i=1}^{q} \frac{2i-1}{q^2}\, g_{[i]} = \\
&\sum_{i=1}^{q} \frac{q-2i+1}{q^2}\, g_{[i]} = \sum_{i=1}^{q} \frac{2(q-i)+1}{q^2}\, g_{[i]} - \sum_{i=1}^{q} \frac{1}{q}\, g_{[i]} = \\
&\sum_{i=1}^{q} \frac{2(q-i)+1}{q^2} \left(1 - \frac{x_{(i)}}{z}\right) - \sum_{i=1}^{q} \frac{1}{q} \left(1 - \frac{x_{(i)}}{z}\right) = \\
&\sum_{i=1}^{q} \frac{1}{q}\, \frac{x_{(i)}}{z} - \sum_{i=1}^{q} \frac{2(q-i)+1}{q^2}\, \frac{x_{(i)}}{z} = \mu(\boldsymbol{x}_p/z)\, G(\boldsymbol{x}_p/z) = G_A(\boldsymbol{x}_p/z)\ .
\end{aligned} \tag{8}$$

Finally, the next result provides a decomposition of the Sen index in three components: incidence, intensity and inequality. The interest of this result is that the inequality component, expressed in terms of the absolute Gini index, provides a consistent measure of either achievements (incomes normalized with respect to the poverty line) or shortfalls (normalized income gaps) of the poor.

Proposition 11 *The Sen index satisfies the following decomposition*

$$S(\boldsymbol{x}, z) = H(\boldsymbol{x}, z)\left(M(\boldsymbol{x}, z) + G_A(\boldsymbol{g}_p)\right) = H(\boldsymbol{x}, z)\left(M(\boldsymbol{x}, z) + G_A\left(\boldsymbol{x}_p/z\right)\right),$$

where intensity is expressed by the aggregate income gap ratio $M(\boldsymbol{x}, z) = \mu(\boldsymbol{g}_p) = 1 - \mu(\boldsymbol{x}_p/z)$ *and inequality is consistently measured by the absolute Gini index*

$$G_A(\boldsymbol{g}_p) = \mu(\boldsymbol{g}_p)\, G(\boldsymbol{g}_p) = \mu(\boldsymbol{x}_p/z)\, G(\boldsymbol{x}_p/z) = G_A(\boldsymbol{x}_p/z)\ .$$

Proof. Straightforward from the two standard decompositions (2) and (3), plus equation (7) and Proposition 10. ■

4.3 An illustrative example

We now illustrate the possibilities of the decomposition of the Sen poverty measure proposed in this paper. First, consider seven income distributions and their corresponding normalized gaps for the poverty line $z = 1800$ in the first and second columns of Table 1, respectively. Notice that distributions $\boldsymbol{x}^1$, $\boldsymbol{x}^2$, $\boldsymbol{x}^3$, $\boldsymbol{x}^5$ and $\boldsymbol{x}^7$ share the same average income gap of the poor and the others two are close to this amount. However their poverty levels are quite different and the decomposition of the S poverty index in its three contribution components allow us to determine where the differences stem from. The poverty measure S and the three components, H, $\widehat{A}_G$ and $\widetilde{A}_G$, can be seen in the four columns of Table 2. For instance, notice that $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ have the same inequality and the same intensity, and the difference in their poverty levels arises from the different percentages of poor people. Distributions $\boldsymbol{x}^2$, $\boldsymbol{x}^3$ and $\boldsymbol{x}^7$ have the same headcount ratio and the poverty intensity level, nevertheless income among the poor are more equally distributed in $\boldsymbol{x}^7$ than in $\boldsymbol{x}^2$ and $\boldsymbol{x}^3$. By contrast, $\boldsymbol{x}^3$ and $\boldsymbol{x}^4$ share the headcount ratio and the inequality levels, being different their poverty intensity.

In general, we may compare any pair of distributions and analyze its poverty components to better understand their differences. For example, if we concentrate on distributions $\boldsymbol{x}^5$, $\boldsymbol{x}^6$ and $\boldsymbol{x}^7$, we may conclude that $\boldsymbol{x}^5$ exhibits the lowest headcount ratio, while $\boldsymbol{x}^6$ and $\boldsymbol{x}^7$ have the lowest values of inequality and intensity respectively.

Table 1: Incomes and gaps

Incomes	Gaps for $z = 1800$
$\boldsymbol{x}^1 = (122, 778, 1100, 1200)$	$\boldsymbol{g}^1 = (0.932, 0.568, 0.389, 0.333)$
$\boldsymbol{x}^2 = (300, 800, 1300, 2400)$	$\boldsymbol{g}^2 = (0.833, 0.556, 0.278, 0.000)$
$\boldsymbol{x}^3 = (100, 800, 1500, 2400)$	$\boldsymbol{g}^3 = (0.944, 0.556, 0.167, 0.000)$
$\boldsymbol{x}^4 = (300, 1000, 1700, 3800)$	$\boldsymbol{g}^4 = (0.833, 0.444, 0.278, 0.000)$
$\boldsymbol{x}^5 = (178, 1422, 1900, 2500)$	$\boldsymbol{g}^5 = (0.901, 0.210, 0.000, 0.000)$
$\boldsymbol{x}^6 = (40, 520, 1520, 1620)$	$\boldsymbol{g}^6 = (0.978, 0.711, 0.156, 0.100)$
$\boldsymbol{x}^7 = (460, 940, 1000, 2600)$	$\boldsymbol{g}^7 = (0.744, 0.478, 0.444, 0.000)$

Table 2: Decomposition of the Sen poverty measure

i	$S(\boldsymbol{x}^i, z)$	$H(\boldsymbol{x}^i, z)$	$\widehat{A}_G(\boldsymbol{g}^i)$	$\widetilde{A}_G(\boldsymbol{g}^i)$
1	0.679	1	0.556	0.123
2	0.509	0.75	0.556	0.123
3	0.546	0.75	0.556	0.173
4	0.463	0.75	0.444	0.173
5	0.364	0.5	0.556	0.173
6	0.685	1	0.486	0.199
7	0.467	0.75	0.556	0.067

5 Concluding remarks

We have investigated the structure of the Sen poverty index within the framework of the dual decomposition of aggregation functions. The Sen index can be written as a product of the standard headcount ratio and an OWA operator applied to the poverty gaps. This OWA operator decomposes into a self-dual core, corresponding to the the average poverty gap, and an anti-self-dual remainder which corresponds to the classical Gini index of the normalized incomes of the poor. In this new decomposition of the Sen poverty index, therefore, the self-dual core and the anti-self-dual remainder measure (respectively) the intensity and the inequality of poverty within the given income distribution. The central result is thus that the dual decomposition of the Sen poverty index contains an inequality measure which is naturally achievement/shortfall consistent.

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