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Social evaluation functions used in policy impact analysis can be viewed as real-valued functionals of the underlying outcome distributions. Influence functions may be used to identify the sources of variation in social outcomes in terms of individual or household characteristics. This paper sets forth in clear terms the definition of the influence function and recentered influence function, and catalogs these functions for a wide range of distributional statistics, including measures of central tendency, inequality and poverty and also measures of the degree of pro-poorness of a shock- or policy-induced change in income levels.

**Keywords**: Influence function, robust statistic, distributional statistic, inequality, poverty, social evaluation.
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* The authors wish to thank Daniel Dugger and Phillipe Van Kerm for assistance in the preparation of this paper.
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# Influence functions for distributional statistics

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## Abstract

Social evaluation functions used in policy impact analysis can be viewed as real-valued functionals of the underlying outcome distributions. Influence functions may be used to identify the sources of variation in social outcomes in terms of individual or household characteristics. This paper sets forth in clear terms the definition of the influence function and recentered influence function, and catalogs these functions for a wide range of distributional statistics, including measures of central tendency, inequality and poverty and also measures of the degree of pro-poorness of a shock- or policy-induced change in income levels.

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## 1. Introduction

Distributional considerations play a crucial role in public policymaking. Indeed policy implementation entails a redistribution of socioeconomic burdens and advantages, and policymakers are particularly interested in knowing whether or not an intervention had (or is likely to have) the intended effects on policy-relevant socioeconomic groups, and why. Policy impact analysis is therefore an exercise in *social impact evaluation* understood as an assessment of variation in individual and social outcomes attributable to a socioeconomic shock or the implementation of public policy. Sen (1995) explains that any evaluative approach is characterized by its informational basis which clearly identifies the information required in passing judgments within the chosen approach. Two key elements underpin the informational basis. The focal space defines individual advantage (desirable outcome at the individual level), while the focal combination is used to assess social progress (desirable social outcome). The focal combination is essentially a rule for combining individual outcomes into an aggregative indicator of the prevailing social state. In other words, the focal combination is a social evaluation function (e.g. a social welfare function or a poverty measure) used to rank social states represented by distributions of individual outcomes.

The above considerations suggest that social evaluation functions used in policy impact analysis are nothing but distributional statistics, which can be viewed as real-valued and continuous functionals of the relevant outcome distributions. In the context of robust statistics, a distributional statistic that is continuous is said to be qualitatively robust (Wilcox, 2005). If this statistic is also differentiable, then it is said to be infinitesimally robust. Its first-order directional derivative is known as its *influence function* (Hampel, 1974). In other words, the influence function of a distributional statistic (and hence of a social evaluation function) measures the relative effect of a small perturbation in the underlying outcome distribution on the statistic of interest. One can therefore build an infinitesimal approach to policy impact analysis on the notion of influence function. Within that approach and under the assumption that the distributional change in question is due to policy implementation, the influence function of a social evaluation criterion may be viewed as a local measure of the distributional impact of policy. This is analogous to what Rothe (2010) calls a *distributional policy effect*.[1]

For policymaking purposes, it is not enough to measure the impact of policy on the outcome distribution; policymakers are also interested in identifying the forces that drive the observed outcomes. Individual outcomes from participation in a policy intervention ultimately depend on individual endowments, behavior and the circumstances that determine the returns to those endowments in any socioeconomic transaction. This perspective creates a need for linking social evaluation functions to individual (or household) characteristics depending on the unit of analysis. Recent work of Firpo et al. (2009), has developed a regression technique using the *recentered influence function* (RIF), as a straightforward way to establish such a link. These authors define the RIF as the first two leading terms of the von Mises (1947) linear

[1] The influence function has also been used to quantify the impact of data contamination upon various distributional statistics, see for example Cowell (1999) and Cowell and Victoria-Feser (1996, 2002).

approximation of the corresponding social evaluation function. Assuming that the outcome variable is observed along with relevant covariates representing individual characteristics, they use the fact that the expected value of the influence function is equal to zero and the law of iterated expectations to express the social evaluation function of interest as the conditional expectation of the RIF given the covariates. This conditional expectation is what they call RIF regression.

Within the RIF regression framework, Firpo et al. (2009) define two very useful parameters in policy analysis, the marginal effect of a *ceteris paribus* change in the distribution of covariates, and the unconditional partial effect. If the conditional outcome distribution given the covariates does not change following a small perturbation of the distribution of those covariates, then the marginal effect of this distributional change on the social outcome can be computed on the basis of the RIF regression. Furthermore, the same regression model can be used to compute the partial effect of a small location shift in the distribution of a continuous covariate on the social outcome. Under the assumption that the RIF regression is linear, these effects can be computed using standard OLS. Rothe (2011) proposes a parameter analogous to the unconditional partial effect which he calls the partial distributional policy effect. This parameter is a measure of the effect of a ceteris paribus change in the unconditional distribution of a single covariate on some functional of the unconditional distribution of the outcome variable. In contrast to Firpo et al. (2009), this author presents a fully nonparametric framework for the identification and estimation of this parameter for general changes in the covariate distribution including location shifts.

Declaring a policy outcome socially desirable on the basis of some social evaluation function is a result of aggregate judgment that may hide more than it reveals about the heterogeneity of impacts underlying the aggregate outcome. The possibility to link social evaluation criteria to individual characteristics via RIF regression offers an opportunity to understand this heterogeneity and to design targeted interventions that might enhance the effectiveness of public policy. Furthermore, attribution of outcomes to policy is the hallmark of policy impact evaluation. Variations in individual and social outcomes associated with policy implementation could be driven at least in part by changes in confounding factors in the socioeconomic environment. Looking at the individual outcome as a function of policy and type (base on characteristics), one can resort to counterfactual decomposition of observed distributional change *à la* Oaxaca-Blinder to sort out the part that is due to policy and the part due to confounding factors. RIF regression makes it possible to extend this type of decomposition to variation in social outcomes. However, a key limitation of this approach stems from the fact that RIF regression coefficients provide only a local approximation for the effect of changes in the relevant covariates on the distributional statistic of interest. Nonetheless, the approach based on linearization, which is of course widespread throughout economics, is very attractive in an operational environment.[2]

[2] This limitation of the RIF regression approach is well-recognized. Chernozhukov et al. (2009) directly estimate an exact effect without approximation error. See also Rothe (2010, 2011) in which no shape restriction is imposed on the conditional distribution of the outcome given explanatory variables.

In this paper, we carefully define and catalog the influence functions and recentered influence functions for a wide range of distributional statistics, including measures of central tendency, inequality and poverty and also measures of the degree of pro-poorness of the shock- or policy-induced change in income levels. We include the mean, $p^{th}$ quantile point, variance, Gini coefficient, Atkinson (1970) inequality index, Lorenz curve ordinate, generalized Lorenz curve ordinate (Shorrocks, 1983), Foster et al. (1984) poverty index (henceforth the FGT index), Watts (1969) poverty index, growth incidence curve ordinate (Ravallion and Chen, 2003), TIP curve ordinate (Jenkins and Lambert, 1997), poverty elasticity of the headcount ratio, and pro-poorness measures for the FGT, Watts and headcount indices (Essama-Nssah and Lambert, 2009). Such a catalog has not been available in the literature heretofore.

The organization of the remainder of the paper is as follows. In Section 2, directional derivatives are defined, and are determined for each of the distributional statistics just listed. In Section 3, influence functions and recentered influence functions are determined for each of these distributional statistics. Appendix A contains proofs of all claimed results. For the reader's convenience, Appendix B contains a table in which the definitions of the distributional statistics we have covered, and their recentered influence functions, are specified.

## 2. Directional derivatives of distributional statistics

Suppose that, at the individual or household level, the outcome *y* we are interested in is income. Let *F(y)* be an outcome distribution, and let *T(·)* be a distributional statistic (e.g. a social evaluation function or indicator of a social outcome) that is qualitatively and infinitesimally robust. The influence function is the directional derivative of *T(F)* at *F* and it measures the effect of a small perturbation in *F* on *T(F*), as follows. Let $H$ be some distribution other than F. When the data does not follow $F$ exactly, but a slightly different distribution, one that is "going towards" $H$, the effect is revealed by the *directional derivative of T at F in the direction of H:*

(1) $$\nabla T_{F\to H} = \frac{d}{dt} T\left(tH + (1-t)F\right)\Big|_{t=0} = \lim_{t \downarrow 0} \frac{T\left(tH + (1-t)F\right) - T\left(F\right)}{t}$$

(see Wilcox, 2005).

The following proposition both defines the notation used in the rest of the paper, and determines the directional derivatives for the distributional statistics we have listed. The density functions associated with *F(x)* and *H(x)* are *f(x)* and *h(x),* and *z* is an exogenous poverty line. Some explanatory comments follow the statement of the proposition.

### Proposition 1

For each of the following distributional statistics *T(·),* the directional derivative $\nabla T_{F\to H}$ is as shown:

*(a)* The mean, $\mu_F = T_\mu(F) = \int x f(x)dx$: $\nabla T_{\mu,\, F\to H} = \mu_H - \mu_F$

*(b)* The $p^{th}$ quantile point, $v_p = T_{vp}(F) = F^{-1}(p)$: $\nabla T_{vp,\, F\to H} = \dfrac{\left[p - H(v_p)\right]}{f(v_p)}$

*(c)* The variance, $\sigma_F^2 = T_{\sigma^2}(F) = \int (x-\mu)^2 f(x)dx$ : $\nabla T_{\sigma^2, F\to H} = \sigma_H^2 - \sigma_F^2 + (\mu_H - \mu_F)^2$

*(d)* The Gini coefficient, $G_F = T_G(F) = \frac{1}{\mu_F}\int F(x)\left[1 - F(x)\right]dx$:

$$\nabla T_{G,\, F\to H} = \frac{\mu_F - \mu_H}{\mu_F} G_F + \frac{1}{\mu_F}\int \left[H(x) - F(x)\right]\left[1 - 2F(x)\right]dx$$

*(e)* The Atkinson inequality index $I_F(e) = T_{ATK}(F) = 1 - \dfrac{\xi}{\mu_F}$, $e > 0$, where

$$U(x) = \begin{cases} x^{1-e}\big/1-e & 0 < e \neq 1 \\ \ln(x) & e = 1 \end{cases}$$ and $U(\xi) = \int U(x) f(x)dx$:

$$\nabla T_{ATK,F\to H}(F) = \begin{cases} \left\{\dfrac{e}{1-e} + \dfrac{\mu_H}{\mu_F}\right\}\left[1 - I_F\right] - \dfrac{\left[1 - I_F\right]^e \int x^{1-e} h(x)dx}{(1-e)\mu_F^{\,1-e}} & 0 < e \neq 1 \\ (1 - I_F)\left\{\ln\left\{\mu_F\left(1 - I_F\right)\right\} - \int \ln(x)h(x)dx + \left(\dfrac{\mu_H}{\mu_F} - 1\right)\right\} & e = 1 \end{cases}$$

*(f)* The generalized Lorenz ordinate at $p \in [0,1]$, $GL_F(p) = T_{GLp}(F) = \int_0^{v_p} x f(x)dx$ :

$$\nabla T_{GLp,F\to H} = \int_0^{v_p} x h(x)dx + v_p\left[p - H(v_p)\right] - T_{GLp}(F)$$

*(g)* The Lorenz ordinate at $p \in [0,1]$, $L_F(p) = T_{Lp}(F) = \int_0^{v_p} x f(x)dx \Big/ \mu_F$ :

$$\nabla T_{Lp,F\to H} = \frac{\int_0^{v_p} x h(x)dx + v_p\left[p - H(v_p)\right]}{\mu_F} - T_{Lp}(F).\frac{\mu_H}{\mu_F}$$

*(h)* The FGT index for poverty line *z*, $T_{FGT\alpha}(F) = \int_0^z \left(1 - \frac{x}{z}\right)^\alpha f(x)dx$ :

$$\nabla T_{FGT\alpha,F\to H} = \int_0^z \left(1 - \frac{x}{z}\right)^\alpha h(x)dx - T_{FGT\alpha}(F)$$

*(i)* The Watts index for poverty line *z*, $T_W(F) = \int_0^z \ln\left(\frac{z}{x}\right) f(x)dx$ :

$$\nabla T_{W,F\to H} = \int_0^z \ln\left(\frac{z}{x}\right) h(x)dx - T_W(F)$$

*(j)* The TIP curve ordinate for poverty line *z* at $p \in [0,1]$,

$$TIP_F(p) = T_{TIPp}(F) = \begin{cases} \int_0^{v_p} (z-x) f(x)dx & v_p \le z \\ \int_0^z (z-x) f(x)dx & v_p \ge z \end{cases} :$$

$$\nabla T_{TIPp,F\to H}(F) = \begin{cases} \int_0^{v_p} (z-x)h(x)dx + (z-v_p)\left[p - H(v_p)\right] - T_{TIPp}(F) & v_p \le z \\ \int_0^z (z-x)h(x)dx - T_{TIPp}(F) & v_p \ge z \end{cases}$$

*(k)* The growth incidence curve ordinate at p, $GIC_F(p) = T_{GICp}(F) = \gamma q(v_p)$ where $\gamma$ is the aggregate growth rate: $\Delta T_{GICp,F\to H} = \frac{\mu_H}{\mu_F} T_{GICp}(F) + \gamma q'(v_p).\left[\frac{p - H(v_p)}{f(v_p)}\right]$

*(l)* Pro-poorness for the FGT index, $T_{ppFGT\alpha}(F) = \frac{\alpha}{z}\int_0^z x\left(1 - \frac{x}{z}\right)^{\alpha-1} [q(x)-1] f(x)dx$:

$$\nabla T_{ppFGT\alpha,F\to H} = \frac{\alpha}{z}\int_0^z x\left(1-\frac{x}{z}\right)^{\alpha-1} \left\{\gamma q(x)\left[q(x) + xq'(x)\right] - 1\right\} h(x)dx - \left[1 + \frac{\mu_H}{\mu_F}\right] T_{ppFGT\alpha}(F)$$

*(m)* Pro-poorness for the Watts index, $T_{ppW}(F) = \int_0^z [q(x)-1] f(x)dx$:

$$\nabla T_{ppW,F\to H}(F) = \int_0^z \left\{\gamma q(x)\left[q(x) + xq'(x)\right] - 1\right\} h(x)dx - \left[1 + \frac{\mu_H}{\mu_F}\right] T_{ppW}(F)$$

*(n)* Pro-poorness for the headcount ratio: $T_{ppHC}(F) = z\big(q(z)-1\big) f(z)$ :

$$\nabla T_{ppHC,F\to H} = -zh(z)\left\{1 + \gamma q(z)\big(q(z) + zq'(z)\big)\right\} - \left[1 + \frac{\mu_H}{\mu_F}\right] T_{ppHC}(F) - zf(z)\frac{\mu_H}{\mu_F}$$

*(o)* The poverty elasticity of the headcount ratio, $T_{EHC}(F) = -\left( zq(z) f(z) \big/ F(z) \right)$:

$$\nabla T_{EHC,F\to H} = -T_{EHC}(F)\left[\frac{H(z)}{F(z)} + \frac{\mu_H}{\mu_F} + \frac{\gamma h(z)}{f(z)}\left[q(z) + zq'(z)\right]\right]$$

All of these results follow using calculus and/or limiting arguments. Each is proven in Appendix A, part I. Note that the median income value is case *(b)* with $p = \frac{1}{2}$. For the growth incidence curve and pro-poorness measures, *q(x)* is the income growth pattern, an elasticity function telling by what percentage income *x* grows when the overall income growth is 1%, as in Essama-Nssah and Lambert (2009), where pro-poorness for a poverty index $P = \int_0^z \psi(x \mid z) f(x)dx$, in which the poverty contribution function $\psi(x \mid z)$ is convex, decreasing and equals zero for $x \ge z$, is defined as

$T_{ppP}(F) = \int_0^z \{-x\psi'(x \mid z)\}[q(x)-1]f(x)dx$ (and pro-poorness for the headcount ratio *F(z)* is separately defined as in *(n)*).

## 3. Influence functions and recentered influence functions

For *y* in the domain of *F*, let $H = \Delta_y$ be the cumulative distribution function for a probability measure which gives mass 1 to *y*. That is, $H(x) = \Delta_y(x) = \begin{cases} 0 & x < y \\ 1 & x > y \end{cases}$. The density function *h(x)* is zero everywhere except for an infinite spike at *x* = *y*. In particular, $\int_0^\infty \Delta_y(x) f(y)dy = \int_0^\infty f(y)dy = F(x)$.

The influence function for an estimator *T(·)* is defined as

(2) $$IF(y;T;F) = \nabla T_{F \to \Delta y}$$

It describes the effect of an infinitesimal 'contamination' at the point *y* on the estimator: in the mixed distribution $tH + (1-t)F$, it is as if an observation is randomly sampled from distribution *F* with probability (1-*t*) and from $\Delta_y$ with probability *t*. The influence function is also known as the *Gâteaux derivative*, following Gâteaux (1913). It has become a key tool in robust statistics.

An important property of the influence function is that, in all cases in which the frequencies and range of the *y*-values are bounded,

(3) $$\int_0^\infty IF(y;T;F) f(y)dy = 0$$

See part II of Appendix A for the proof of this result, whose significance will become apparent.

The re-centered influence function is defined by adding the influence function to the functional itself:

(4) $$RIF(y;T;F) = T(F) + IF(y;T;F)$$

Because of (3) we have

(5) $$\int_0^\infty RIF(y;T;F) f(y)dy = T(F).$$

The following proposition determines the influence functions and recentered influence functions for the distributional statistics whose directional derivatives are given in Proposition 1. Again, some comments follow the statement of this Proposition.

### Proposition 2

For the following distributional statistics *T(·)*, the influence functions $IF(y;T;F)$ and recentered influence functions $RIF(y;T;F)$ are as shown:

*(a)* The mean: $IF(y;T_\mu;F) = y - \mu_F$ and $RIF(y;T_\mu;F) = y$

*(b)* The $p^{th}$ quantile point:

$$IF(y;T_{vp};F)=\begin{cases}\dfrac{p}{f(v_p)} & y>v_p\\[2ex] \dfrac{-(1-p)}{f(v_p)} & y<v_p\end{cases}\qquad \text{and}$$

$$RIF(y;T_{vp};F)=\begin{cases}v_p+p\big/f(v_p) & y>v_p\\ v_p-(1-p)\big/f(v_p) & y<v_p\end{cases}$$

*(c)* The variance: $IF(y;T_{\sigma^2};F)=-\sigma_F^2+(y-\mu_F)^2$ and $RIF(y;T_{\sigma^2};F)=(y-\mu_F)^2$

*(d)* The Gini coefficient: $IF(y;T_G;F)=-\dfrac{\mu_F+y}{\mu_F}G_F+1-\dfrac{y}{\mu_F}+\dfrac{2}{\mu_F}\displaystyle\int_0^y F(x)dx$ and

$$RIF(y;T_G;F)=-\frac{y}{\mu_F}G_F+1-\frac{y}{\mu_F}+\frac{2}{\mu_F}\int_0^y F(x)dx$$

*(e)* The Atkinson index:

$$IF(y;T_{ATK};F)=\begin{cases}\left\{\dfrac{e}{1-e}+\dfrac{y}{\mu_F}\right\}\left[1-I_F\right]-\dfrac{\left[1-I_F\right]^e y^{1-e}}{(1-e)\mu_F^{1-e}} & 0<e\neq 1\\[2ex] (1-I_F)\left\{\ln\left\{\mu_F(1-I_F)\right\}-\ln(y)+\left(\dfrac{y}{\mu_F}-1\right)\right\} & e=1\end{cases}\qquad\text{and}$$

$$RIF(y;T_{ATK};F)=I_F+\begin{cases}\left\{\dfrac{e}{1-e}+\dfrac{y}{\mu_F}\right\}\left[1-I_F\right]-\dfrac{\left[1-I_F\right]^e y^{1-e}}{(1-e)\mu_F^{1-e}} & 0<e\neq 1\\[2ex] (1-I_F)\left\{\ln\left\{\mu_F(1-I_F)\right\}-\ln(y)+\left(\dfrac{y}{\mu_F}-1\right)\right\} & e=1\end{cases}$$

*(f)* The generalized Lorenz ordinate:

$$IF(y;T_{GLp};F)=\begin{cases}y-(1-p)v_p-T_{GLp}(F) & y<v_p\\ pv_p-T_{GLp}(F) & y\geq v_p\end{cases}\qquad\text{and}$$

$$RIF(y;T_{GLp};F)=\begin{cases}y-(1-p)v_p & y<v_p\\ pv_p & y\geq v_p\end{cases}$$

*(g)* The Lorenz ordinate: $IF(y;T_{Lp};F)=\begin{cases}\dfrac{y-(1-p)v_p}{\mu_F}-T_{Lp}(F).\dfrac{y}{\mu_F} & y<v_p\\[2ex] \dfrac{pv_p}{\mu_F}-T_{Lp}(F).\dfrac{y}{\mu_F} & y\geq v_p\end{cases}$ and

$$RIF(y;T_{Lp};F)=\begin{cases}\dfrac{y-(1-p)v_p}{\mu_F}+T_{Lp}(F).\left(1-\dfrac{y}{\mu_F}\right) & y<v_p\\[2ex] \dfrac{pv_p}{\mu_F}+T_{Lp}(F).\left(1-\dfrac{y}{\mu_F}\right) & y\geq v_p\end{cases}$$

(*h*) The FGT index: $IF(y;T_{FGT\alpha};F)=\begin{cases}\left(1-\dfrac{y}{z}\right)^{\alpha}-T_{FGT\alpha}(F) & y<z\\ -T_{FGT\alpha}(F) & y>z\end{cases}$ and

$$RIF(y;T_{FGT\alpha};F)=\begin{cases}\left(1-\dfrac{y}{z}\right)^{\alpha} & y<z\\ 0 & y>z\end{cases}$$

*(i)* The Watts index: $IF(y;T_W;F)=\begin{cases}\ln\left(\dfrac{z}{y}\right)-T_W(F) & y<z\\ -T_W(F) & y>z\end{cases}$ and

$$RIF(y;T_W;F)=\begin{cases}\ln\left(\dfrac{z}{y}\right) & y<z\\ 0 & y>z\end{cases}$$

*(j)* The TIP curve ordinate at $p\in[0,1]$:

$$IF(y;T_{TIPp};F)=-T_{TIPp}(F)+\begin{cases}\begin{cases}z-y & y<z\\ 0 & y>z\end{cases} & z<v_p\\[3ex] \begin{cases}pz+(1-p)v_p-y & y<v_p\\ p(z-v_p) & y>v_p\end{cases} & z>v_p\end{cases}$$ and

$$RIF(y;T_{TIPp};F)=\begin{cases}\begin{cases}z-y & y<z\\ 0 & y>z\end{cases} & z<v_p\\[3ex] \begin{cases}pz+(1-p)v_p-y & y<v_p\\ p(z-v_p) & y>v_p\end{cases} & z>v_p\end{cases}$$

*(k)* The growth incidence curve ordinate at p:

$$IF(y;T_{GICp};F)=\frac{y}{\mu_F}T_{GICp}(F)+\begin{cases}\dfrac{\gamma pq'(v_p)}{f(v_p)} & y>v_p\\[2ex] \dfrac{-\gamma(1-p)q'(v_p)}{f(v_p)} & y<v_p\end{cases}$$ and

$$RIF(y;T_{GICp};F)=\begin{cases}\gamma\left\{\left[\frac{y}{\mu_F}+1\right]q(v_p)+\frac{pq'(v_p)}{f(v_p)}\right\} & y>v_p\\ \gamma\left\{\left[\frac{y}{\mu_F}+1\right]q(v_p)-\frac{(1-p)q'(v_p)}{f(v_p)}\right\} & y<v_p\end{cases}$$

(*l*) Pro-poorness for the FGT index:

$$IF(y;T_{ppFGT\alpha},F)=\begin{cases}\frac{\alpha y}{z}\left(1-\frac{y}{z}\right)^{\alpha-1}\{\gamma q(y)[q(y)+yq'(y)]-1\}-\left[1+\frac{y}{\mu_F}\right]T_{ppFGT\alpha}(F) & y<z\\ -\left[1+\frac{y}{\mu_F}\right]T_{ppFGT\alpha}(F) & y>z\end{cases}$$ and

$$RIF(y;T_{ppFGT\alpha},F)=\begin{cases}\frac{\alpha y}{z}\left(1-\frac{y}{z}\right)^{\alpha-1}\{\gamma q(y)[q(y)+yq'(y)]-1\}-\left[\frac{y}{\mu_F}\right]T_{ppFGT\alpha}(F) & y<z\\ -\left[\frac{y}{\mu_F}\right]T_{ppFGT\alpha}(F) & y>z\end{cases}$$

*(m)* Pro-poorness for the Watts index:

$$IF(y;T_{ppW},F)=\begin{cases}\{\gamma q(y)[q(y)+yq'(y)]-1\}-\left[1+\frac{y}{\mu_F}\right]T_{ppW}(F) & y<z\\ -\left[1+\frac{y}{\mu_F}\right]T_{ppW}(F) & y>z\end{cases}$$ and

$$RIF(y;T_{ppW},F)=\begin{cases}\{\gamma q(y)[q(y)+yq'(y)]-1\}-\left[\frac{y}{\mu_F}\right]T_{ppW}(F) & y<z\\ -\left[\frac{y}{\mu_F}\right]T_{ppW}(F) & y>z\end{cases}$$

*(n)* Pro-poorness for the headcount ratio:

$$IF(y;T_{ppHC};F)=-\left[1+\frac{y}{\mu_F}\right]T_{ppHC}(F)-zf(z)\frac{y}{\mu_F},\ \ (y\neq z)$$ and

$$RIF\left(y;T_{ppHC};F\right)=-\left[\frac{y}{\mu_F}\right]T_{ppHC}(F)-zf(z)\frac{y}{\mu_F},\ \ (y\neq z)$$

*(o)* Poverty elasticity of the headcount ratio:

$$IF(y;T_{PEHC};F)=\begin{cases}-T_{EHC}(F)\left[\frac{1}{F(z)}+\frac{y}{\mu_F}\right] & y<z\\ -T_{EHC}(F)\left[\frac{y}{\mu_F}\right] & y>z\end{cases}$$ and

$$RIF(y;T_{PEHC};F)=\begin{cases} -T_{EHC}(F)\left[\dfrac{1-F(z)}{F(z)}+\dfrac{y}{\mu_F}\right] & y<z \\ T_{EHC}(F)\left[1-\dfrac{y}{\mu_F}\right] & y>z \end{cases}$$

The proofs of these result are immediate from those in Proposition 1, simply substituting the general distribution *H(x)* by $\Delta_y(x)=\begin{cases}0 & x<y\\ 1 & x\geq y\end{cases}$ throughout. Other notations are sometimes used for some of these expressions. For example, Firpo et al. (2009) write the influence function for the $p^{th}$ quantile point $v_p = T_{vp}(F) = F^{-1}(p)$ as $IF(y;T_{vp};F)=\dfrac{\mathbf{I}(y>v_p)-(1-p)}{f(v_p)}$ where **I** is an indicator function. Their expression for the Gini influence function (given in an unpublished companion paper) is $IF(y;T_G;F)=A_2(F)+B_2(F)y+C_2(y,F)$ where $A_2(F)=\dfrac{2}{\mu}R(F)$, $B_2(F)=\dfrac{2}{\mu^2}R(F)$, $C_2(y,F)=-\dfrac{2}{\mu}\{y[1-p(y)]+GL(p(y),F)\}$ in which, in our terms, $GL(p(y),F)=\int_{-\infty}^{y}xdF(x)=-\int_{-\infty}^{y}F(x)dx+yF(y)$ and $R(F)=\frac{1}{2}\mu_F(1-G_F)$: in fact this expression is equivalent to ours. The reader will notice that for the poverty-related measures, we do not define influence functions at the poverty line value $y = z$. This is because these influence functions are infinite at $y = z$. Properties (3) and (5) can be verified in all cases, although we must interpret $IF(z;T;F)f(z)dz$ and $RIF(z;T;F)f(z)dz$ as each equal to *T(F)* in the poverty-related cases. Note that in case *(a)*, (3) confirms that the average deviation of $IF(y;\mu;F)=y-\mu$ from the mean is zero. This influence function does not depend on *F*, and furthermore is unbounded in *y*; therefore it does not have infinitesimal robustness.

*Recentered influence function regression* offers a simple way of establishing a direct link between a social evaluation function and individual (or household) characteristics *x*, because of (5), which says that the expected value of the recentered influence function is equal to the corresponding distributional statistic, $T(F)=E_F\left[RIF(y;T,F)\right]$. By the law of iterated expectations, the distributional statistic can thus be written as the conditional expectation of $RIF(y;T,F)$, given observable covariates *x*, and is determined in a recentered influence function regression, as shown in Firpo et al. (2009).

## 4. Conclusion

In this paper, we have laid out a catalog of influence functions and associated recentered influence functions for a range of social evaluation functions widely used in assessing the distributional and poverty impact of public policy. These evaluation functions are distributional statistics that can be viewed as real-valued and continuous functionals of the underlying outcome distribution functions. Given this and the fact that an outcome distribution is fully characterized by its mean and the degree of inequality, several authors have proposed counterfactual decomposition methods to identify the contribution of changes in the mean and relative inequality to variations in social outcomes as measure by a suitable evaluation function. In particular, Datt and Ravallion (1992) decompose change in poverty into a distribution-neutral growth effect, a redistribution effect and a residual interpreted as an interaction term. The Shapley method proposed by Chantreuil and Trannoy (1999) and Shorrocks (1999) is analogous to that of Datt and Ravallion, but does not involve a residual. A limitation of the usefulness of this approach for policy analysis is that it explains variations in social outcomes in terms of changes in summary statistics which may be hard to target with policy instruments.

Ultimately, individual outcomes depend on policy and individual characteristics. These are the fundamental factors driving the observed change in social outcomes. The influence function (i.e. the first-order directional derivative of a distributional statistic) is the cornerstone of the infinitesimal approach to robustness which allows us to link social outcomes to individual characteristics, and hence to assess the heterogeneity of impacts underlying these social outcomes. This link is established in a straightforward manner through the recentered or rescaled influence function (RIF), which is defined as the first two leading terms of the von Mises (1947) linear approximation ( analogous to a one-step Taylor expansion that replaces the distributional statistic by a locally linear function (Hampel et al. 1986)). Since the expected value of the influence function is zero, the law of iterated expectations can be used to link a social evaluation function to individual characteristics. This leads to the conditional expectation of the RIF given the covariates, known as RIF regression.

Taking the RIF regression model to be linear means that one can apply standard OLS to the estimation of both aggregate and partial distributional policy effects. Furthermore, this fact makes the extension of the standard Oaxaca-Blinder decomposition of changes in the mean of outcome distribution to variations in generic social evaluation functions both simple and meaningful. Thus, in the case of variation in poverty outcomes, the analyst can now move beyond Datt-Ravallion and Shapley to decompose such variation to reflect changes in the distribution of individual or household characteristics and returns to these characteristics. A major stumbling block on the way may be the computation of the relevant influence function. It is our hope that this paper will make that difficulty irrelevant. While the limitations of this approach must be firmly kept in mind, nonetheless we remain confident about its usefulness in an operational environment as a first-order approximation of policy impact, much the same as the use of the envelope (theorem) approach in fiscal incidence analysis.

# Appendix A

*Part I: proof of results stated in Proposition 1.*

*(a)* $\nabla T_{\mu,\, F\to H} = \frac{d}{dt}\Big[\int x\big(th(x)+(1-t)f(x)\big)dx\Big]\Big|_{t=0} = \frac{d}{dt}\big[t\mu_H + (1-t)\mu_F\big]\Big|_{t=0} = \mu_H - \mu_F$

*(b)* Let $w(t)$ be the $p^{th}$ quantile point of $tH+(1-t)F$, so that $tH(w(t))+(1-t)F(w(t)) = p$ and $w(0) = v_p$. Then $\frac{\partial p}{dt} = 0 = H(w(t)) - F(w(t))$ + $w'(t)\big[th(w(t))+(1-t)f(w(t))\big]$ , i.e. $w'(t) = \frac{F(w(t))-H(w(t))}{th(w(t)+(1-t)f(w(t))}$. Now $\nabla T_{vp,\, F\to H}$ = $w'(0) = \frac{F(w(0))-H(w(0))}{f(w(0))} = \frac{F(v_p)-H(v_p)}{f(v_p)} = \frac{p-H(v_p)}{f(v_p)}$ as claimed.

*(c)* $\nabla T_{\sigma^2,\, F\to H} = \frac{d}{dt}\int\big(x - t\mu_H - (1-t)\mu_F\big)^2\big(th(x)+(1-t)f(x)\big)\Big|_{t=0} = \int\big(x-\mu_F\big)^2\big(dH - dF\big)$
$+\int 2\big(x-\mu_F\big)\big(\mu_F - \mu_H\big)dF = -\sigma_F^2 + \int\big(x-\mu_F\big)^2 dH = \sigma_H^2 - \sigma_F^2 + \big(\mu_H - \mu_F\big)^2$ as claimed.

*(d)* $\mu_F G_F = \int F(x)\big[1-F(x)\big]dx \Rightarrow \mu_{tH+(1-t)F}G_{tH+(1-t)F} = \int\big[tH(x)+(1-t)F(x)\big]\big[1-tH(x)-(1-t)F(x)\big]dx$
$\Rightarrow \frac{d}{dt}\Big[\mu_{tH+(1-t)F}G_{tH+(1-t)F}\Big]\Big|_{t=0} = \int\big[H(x)-F(x)\big]\big[1-2F(x)\big]dx$. Since $\frac{d}{dt}\Big[\mu_{tH+(1-t)F}\Big] =$ $\mu_H - \mu_F$, $\int\big[H(x)-F(x)\big]\big[1-2F(x)\big]dx = \big(\mu_H - \mu_F\big)G_F + \mu_F\nabla T_{G,\, F\to H}$ implying the result.

*(e)* Let $\xi_0$ be the equally distributed equivalent (EDE) income for F, and let $\xi(t)$ be the EDE income for $tH+(1-t)F$. Then $I_F = 1-\frac{\xi_0}{\mu_F}$ and $I_{tH+(1-t)F} = 1-\frac{\xi(t)}{\mu_{tH+(1-t)F}}$, where $U(\xi_0) = \int U(x)f(x)dx$ and $U(\xi(t)) = t\int U(x)h(x)dx + (1-t)U(\xi_0)$. Differentiating and setting *t = 0,* $U'(\xi_0)\xi'(0) = \int U(x)h(x)dx - U(\xi_0)$ & $\nabla T_{ATK,F\to H}\big(F\big) = \frac{d}{dt}I_{tH+(1-t)F}\Big|_{t=0}$
$= \frac{-\xi'(0)\mu_F + \xi_0\big(\mu_H - \mu_F\big)}{\mu_F^2} = \left\{\frac{U(\xi_0) - \int U(x)h(x)dx}{\mu_F U'(\xi_0)}\right\} + \frac{\xi_0\big(\mu_H-\mu_F\big)}{\mu_F^2}$. Setting $U(x) = x^{1-e}$ for the case $0 < e \neq 1$ and then $U(x) = \ln(x)$ for the case $e = 1$, and substituting, the cited values for $\nabla T_{ATK,F\to H}\big(F\big)$ follow after some manipulation.

*(f)* Let $p = tH(w(t)) + (1-t)F(w(t))$ as in *(b),* so that $GL_{tH+(1-t)F}(p) =$ $\int_0^{w(t)} x\big[th(x)+(1-t)f(x)\big]dx$, $\mathrm{w}(0) = v_p$ and $\nabla T_{GLp,F\to H} = \frac{d}{dt}\left[\int_0^{w(t)} x\big[th(x)+(1-t)f(x)\big]dx\right]\Bigg|_{t=0}$
$= \int_0^{v_p} x\big[h(x) - f(x)\big]dx + w'(0)v_p f(v_p)$ and $w'(0) = \frac{p-H(v_p)}{f(v_p)}$. So $\nabla T_{GLp,F\to H} = \int_0^{v_p} xh(x)dx$
$+v_p\big[p - H(v_p)\big] - T_{GLp}(F)$ as claimed.

*(g)* $\frac{d}{dt}\left[\mu_{tH+(1-t)F}L_{tH+(1-t)F}(p)\right]\Big|_{t=0} = \nabla_{GLp,F\to H}$ and $\frac{d}{dt}\left[\mu_{tH+(1-t)F}\right]\Big|_{t=0} = \mu_H - \mu_F$, whence $(\mu_H - \mu_F)L_F(p) + \mu_F \nabla_{Lp,F\to H} = \nabla_{GLp,F\to H} \Rightarrow \nabla_{Lp,F\to H} = \frac{1}{\mu_F}\nabla_{GLp,F\to H} + \left(1 - \frac{\mu_H}{\mu_F}\right)L_F(p)$, which is as claimed.

*(h), (i)* Immediate when the operator $\frac{d}{dt}\Big|_{t=0}$ is applied to the expressions $T_{FGT\alpha}(tH+(1-t)F) = \int_0^z \left(1-\frac{x}{z}\right)^{\alpha}\left[th(x)+(1-t)f(x)\right]dx$ and $T_W(tH+(1-t)F) = \int_0^z \ln\left(\frac{z}{x}\right)\left[th(x)+(1-t)f(x)\right]dx$.

*(j)* As for *(b)* and *(f)*, let $p = tH(w(t)+(1-t)F(w(t))$ so that $w'(0) = \frac{p - H(v_p)}{f(v_p)}$. Now

$$TIP_{tH+(1-t)F}(p) = \begin{cases} \int_0^{w(t)} (z-x)\left[th(x)+(1-t)f(x)\right]dx & w(t) < z \\ \int_0^{z} (z-x)\left[th(x)+(1-t)f(x)\right]dx & w(t) > z \end{cases}$$

and $\nabla T_{TIPp,F\to H}(F) = \frac{d}{dt}TIP_{tH+(1-t)F}(p)\Big|_{t=0}$. Differentiating, we have

$$\frac{d}{dt}TIP_{tH+(1-t)F}(p) = \begin{cases} \int_0^{w(t)} (z-x)\left[h(x)-f(x)\right]dx + w'(t)(z-w(t))\left[th(w(t))+(1-t)f(w(t))\right] & w(t) < z \\ \int_0^{z} (z-x)\left[h(x)-f(x)\right]dx & w(t) > z \end{cases}$$

and setting t = 0,

$$\nabla T_{TIPp,F\to H}(F) = -T_{TIPp}(F) + \begin{cases} \int_0^{v_p} (z-x)\left[h(x)-f(x)\right]dx + \left[p - H(v_p)\right](z - v_p) & v_p < z \\ \int_0^{z} (z-x)\left[h(x)-f(x)\right]dx & v_p > z \end{cases}$$

which is as claimed since $T_{TIPp}(F) = \begin{cases} \int_0^{v} (z-x)f(x)dx & v \le z \\ \int_0^{z} (z-x)f(x)dx & v \ge z \end{cases}$

*(k)-(o)* For these results, we need to allow for changing income growth patterns in computing the directional derivatives. Let the income distributions at times 0 and 1 be $F$ and $\tilde{F}$ respectively, where $\mu_{\tilde{F}} = (1+\gamma)\mu_F$. An income $x$ at time 0 increases to $x\left[1+\gamma q(x)\right]$ at time 1. For $t \in [0,1]$, consider distributions $tH+(1-t)F$ at time 0 and

$t\tilde{H}+(1-t)\tilde{F}$ at time 1, i.e. *H* stays unchanged as *F* experiences growth. The mean is $t\mu_H+(1-t)\mu_F$ at time 0 and $t\mu_H+(1-t)(1+\gamma)\mu_F$ at time 1, i.e. the aggregate growth rate is $\gamma(t)=\dfrac{(1-t)\gamma\mu_F}{t\mu_H+(1-t)\mu_F}$, whence

(A1) $\quad \gamma(0)=\gamma\,,\quad \gamma'(0)=\gamma\dfrac{\mu_H}{\mu_F}$

Let $q_t(x)$ be the growth elasticity of *x* in the distribution $tH+(1-t)F$ (so that $q_0(x)\equiv q(x)$). An income of *x* in period 0 grows to $x\left[1+\gamma(t)q_t(x)\right]$ in period 1. If there are no rank changes from one period to the next, as we shall assume, then

(A2) $\quad tH(x)+(1-t)F(x)=t\tilde{H}\left(x\left[1+\gamma(t)q_t(x)\right]\right)+(1-t)\tilde{F}\left(x\left[1+\gamma(t)q_t(x)\right]\right)\ \forall x$.

For *t* = 0, this says that

(A3) $\quad F(x)=\tilde{F}\left(x\left[1+\gamma q(x)\right]\right)\ \forall x$

i.e. that income growth in *F* involves no rank changes; also, differentiating, that

(A4) $\quad f(x)=\tilde{f}\left(x\left[1+\gamma q(x)\right]\right).\left[1+\gamma q(x)+x\gamma q'(x)\right]\ \forall x$

Differentiating with respect to *t* in (A2) and setting $t=0$, we have $H(x)-F(x)=H\left(x\left[1+\gamma q(x)\right]\right)-\tilde{F}\left(x\left[1+\gamma q(x)\right]\right)+x\tilde{f}\left(x\left[1+\gamma q(x)\right]\right).\left[\left\{\frac{d}{dt}\gamma(t)q_t(x)\Big|_{t=0}\right\}\right]$. Using (A1), (A3) and (A4), this reduces to

(A5) $\quad \dfrac{d}{dt}q_t(x)\Big|_{t=0}=-\dfrac{q(x)h(x)\left[1+\gamma q(x)+x\gamma q'(x)\right]}{f(x)}-\left\{q(x)\dfrac{\mu_H}{\mu_F}\right\}$

which will be important for what follows. Finally, among these general results for income growth scenarios, note that if that $q_t(x)$ is continuously differentiable, then $q_t'(x)\to q'(x)\ \forall x$ as $t\to 0$.

*(k)* We have $T_{GICp}(tH+(1-t)F)=\gamma(t)q_t(w(t))$ in this case, where $w(t)$ is the $p^{th}$ quantile point of $tH+(1-t)F$, where $w(0)=v_p$ and $w'(0)=\dfrac{p-H(v_p)}{f(v_p)}$ as in cases (*b*), *(f)* and *(j)*. Now $\dfrac{d}{dt}\left[\gamma(t)q_t(w(t))\right]=\gamma'(t)q_t(w(t))+\gamma(t)q_t'(w(t))w'(t)$ and so, at *t* = 0, we have $\nabla T_{GICp,F\to H}\ =\ \gamma'(0)q(v_p)+\gamma(0)q'(v_p)w'(0)=\gamma q(v_p)\dfrac{\mu_H}{\mu_F}+\gamma q'(v_p)\left[\dfrac{p-H(v_p)}{f(v_p)}\right]$ as claimed.

*(l)-(m)* In each of these cases, the distributional statistic is of the form $T(F)=\int_0^z xu'(x)\left[q(x)-1\right]f(x)dx$, where *z* is the poverty line and *u(x)* is the poverty contribution function. For the FGT index, $u(x)=\left(1-\dfrac{x}{z}\right)^{\alpha}$ and for the Watts index,

$u(x) = \ln\left(\frac{z}{x}\right)$. Generically, $T\left(tH + (1-t)F\right) = \int_0^z xu'(x)\left[q_t(x) - 1\right]\left\{th(x) + (1-t)f(x)\right\}dx$ and $\nabla T_{F\to H} = \frac{d}{dt}\left[T\left(tH + (1-t)F\right)\right]\Big|_{t=0} = \int xu'(x)\left\{\left[q(x) - 1\right]\left\{h(x) - f(x)\right\} + \left[\frac{d}{dt}q_t(x)\Big|_{t=0}\right]f(x)\right\}dx$. Using (A5), we find that $\nabla T_{F\to H} = -\left[1 + \frac{\mu_H}{\mu_F}\right]T(F) - \int xu'(x)\left\{\gamma q(x)\left[q(x) + xq'(x)\right] - 1\right\}h(x)dx$. The cited results for the FGT and Watts poverty indices follow.

*(n)* In this case, we have $T_{ppHC}(tH + (1-t)F) = z\left(q_t(z) - 1\right)\left\{th(z) + (1-t)f(z)\right\}$ and so $\nabla T_{ppHC,F\to H} = \frac{d}{dt}T_{ppHC}(tH + (1-t)F)\Big|_{t=0} = z\left(q(z) - 1\right)\left\{h(z) - f(z)\right\} + zf(z)\frac{d}{dt}q_t(z)\Big|_{t=0}$. Using (A5), this reduces to $\nabla T_{ppHC,F\to H} = -zh(z)\left\{1 + \gamma q(z)\left(q(z) + zq'(z)\right)\right\} - \left[1 + \frac{\mu_H}{\mu_F}\right]T_{ppHC}(F) - zf(z)\frac{\mu_H}{\mu_F}$.

*(o)* Here $T_{EHC}(tH + (1-t)F) = -zq_t(z)\left(\frac{th(z) + (1-t)f(z)}{tH(z) + (1-t)F(z)}\right)$ and therefore $\nabla T_{EHC,F\to H} =$

$$= \frac{d}{dt}T_{ppHC}(tH + (1-t)F)\Big|_{t=0} = -zq(z)\left(\frac{h(z)F(z) - f(z)H(z)}{F(z)^2}\right) - \frac{zf(z)}{F(z)}\frac{d}{dt}q_t(z)\Big|_{t=0}$$, that is,

$$\nabla T_{EHC,F\to H} = -T_{EHC}(F)\left[\frac{H(z)}{F(z)} + \frac{\mu_H}{\mu_F} + \frac{\gamma h(z)}{f(z)}\left[q(z) + zq'(z)\right]\right]$$ using (A5).

## *Part II: proof of property (3)*

From (1), $T\left((1-t)F + tH\right) \approx T\left(F\right) + t\nabla T_{F\to H} + o(t^2)$ which can be extended: if $t = \sum_{i=1}^{n} t_i$, then $T\left((1-t)F + \sum_{i=1}^{n} t_i H_i\right) \approx T\left(F\right) + \sum_{i=1}^{n} t_i \nabla T_{F\to H_i} + o(t^2)$. Let the distribution $F$ comprise values $y_1, y_2, \ldots y_n$ with frequencies $f(y_1), f(y_2), \ldots f(y_n)$ and let $t_i = tf(y_i)$ and $H_i = \Delta y_i,\ 1 \le i \le n$. Then $\sum_{i=1}^{n} t_i \nabla T_{F\to H_i} = t\int_{-\infty}^{\infty} IF(y;T;F)f(y)dy$ and $\sum_{i=1}^{n} t_i H_i = t\int \Delta y f(y)dy = tF$. Thus $T\left((1-t)F + tF\right) \approx T\left(F\right) + t\int_{-\infty}^{\infty} IF(y;T;F)f(y)dy + o(t^2)$ proving the result.

## **Appendix B :** *distributional statistics and their recentered influence functions in tabular form*

| DISTRIBUTIONAL STATISTIC | RECENTERED INFLUENCE FUNCTION |
|---|---|
| mean<br>$\mu_F = \int x f(x) dx$ | $RIF(y;T_\mu;F) = y$ |
| $p^{th}$ quantile point<br>$v_p = F^{-1}(p)$ | $RIF(y;T_{vp};F) = \begin{cases} v_p + p/f(v_p) & y > v_p \\ v_p - (1-p)/f(v_p) & y < v_p \end{cases}$ |
| variance<br>$\sigma_F^2 = \int (x-\mu)^2 f(x) dx$ | $RIF(y;T_{\sigma^2};F) = (y - \mu_F)^2$ |
| Gini coefficient<br>$G_F = \frac{1}{\mu_F} \int F(x)[1 - F(x)] dx$ | $RIF(y;T_G;F) = -\frac{y}{\mu_F} G_F + 1 - \frac{y}{\mu_F} + \frac{2}{\mu_F} \int_0^y F(x) dx$ |

| DISTRIBUTIONAL STATISTIC | RECENTERED INFLUENCE FUNCTION |
|---|---|
| Atkinson inequality index $I_F(e) = 1 - \frac{\xi}{\mu_F},\ e > 0$, (here, $U(x) = \begin{cases} x^{1-e} / 1-e & 0 < e \neq 1 \\ \ln(x) & e = 1 \end{cases}$ and $U(\xi) = \int U(x) f(x) dx$ ) | $RIF(y;T_{ATK};F) = I_F + \begin{cases} \left\{\frac{e}{1-e} + \frac{y}{\mu_F}\right\}\left[1 - I_F\right] - \frac{\left[1-I_F\right]^e y^{1-e}}{(1-e)\mu_F^{1-e}} & 0 < e \neq 1 \\ (1-I_F)\left\{\ln\left\{\mu_F\left(1-I_F\right)\right\} - \ln(y) + \left(\frac{y}{\mu_F} - 1\right)\right\} & e = 1 \end{cases}$ |
| generalized Lorenz ordinate at $p$: $GL_F(p) = \int_0^{v_p} x f(x) dx$ | $RIF(y;T_{GLp};F) = \begin{cases} y - (1-p)v_p & y < v_p \\ p v_p & y \geq v_p \end{cases}$ |
| Lorenz ordinate at $p$: $L_F(p) = \int_0^{v_p} x f(x) dx \Big/ \mu_F$ | $RIF(y;T_{Lp};F) = \begin{cases} \frac{y-(1-p)v_p}{\mu_F} + T_{Lp}(F).\left(1 - \frac{y}{\mu_F}\right) & y < v_p \\ \frac{p v_p}{\mu_F} + T_{Lp}(F).\left(1 - \frac{y}{\mu_F}\right) & y \geq v_p \end{cases}$ |

| **DISTRIBUTIONAL STATISTIC** | **RECENTERED INFLUENCE FUNCTION**, *continued* |
|---|---|
| FGT index for poverty line $z$<br>$P^{FGT}{}_{\alpha} = \int_0^z \left(1-\frac{x}{z}\right)^{\alpha} f(x)dx$ | $RIF(y;T_{FGT\alpha};F) = \begin{cases} \left(1-\frac{y}{z}\right)^{\alpha} & y<z \\ 0 & y>z \end{cases}$ |
| Watts index for poverty line $z$<br>$P^{W} = \int_0^z \ln\left(\frac{z}{x}\right) f(x)dx$ | $RIF(y;T_{W};F) = \begin{cases} \ln\left(\frac{z}{y}\right) & y<z \\ 0 & y>z \end{cases}$ |
| TIP curve ordinate for poverty line $z$ at $p$<br>$TIP_F(p) = T_{TIPp}(F) = \begin{cases} \int_0^{v_p} (z-x) f(x)dx & v_p \le z \\ \int_0^{z} (z-x) f(x)dx & v_p \ge z \end{cases}$ | $RIF(y;T_{TIPp};F) = \begin{cases} \begin{cases} z-y & y<z \\ 0 & y>z \end{cases} & z<v_p \\ \begin{cases} pz+(1-p)v_p-y & y<v_p \\ p(z-v_p) & y>v_p \end{cases} & z>v_p \end{cases}$ |

| **DISTRIBUTIONAL STATISTIC** | **RECENTERED INFLUENCE FUNCTION**, *continued* |
|---|---|
| Growth incidence curve ordinate at p<br><br>$GIC_F(p) = \gamma q(v_p)$<br><br>(here, $\gamma$ is the aggregate growth rate and $q(x)$ is the growth pattern) | $RIF(y;T_{GICp};F) = \begin{cases} \gamma\left\{\left[\frac{y}{\mu_F}+1\right]q(v_p) + \frac{pq'(v_p)}{f(v_p)}\right\} & y > v_p \\ \gamma\left\{\left[\frac{y}{\mu_F}+1\right]q(v_p) - \frac{(1-p)q'(v_p)}{f(v_p)}\right\} & y < v_p \end{cases}$ |
| Additive pro-poorness for the FGT index<br><br>$PP_\alpha^{FGT}(F) = \frac{\alpha}{z}\int_0^z x\left(1-\frac{x}{z}\right)^{\alpha-1}[q(x)-1]f(x)dx$ | $RIF(y;T_{ppFGT\alpha},F) = \begin{cases} \frac{\alpha y}{z}\left(1-\frac{y}{z}\right)^{\alpha-1}\{\gamma q(y)[q(y)+yq'(y)]-1\} - \left[\frac{y}{\mu_F}\right]T_{ppFGT\alpha}(F) & y < z \\ -\left[\frac{y}{\mu_F}\right]T_{ppFGT\alpha}(F) & y > z \end{cases}$ |
| Additive pro-poorness for the Watts index<br><br>$PP^W(F) = \int_0^z [q(x)-1]f(x)dx$ | $RIF(y;T_{ppW},F) = \begin{cases} \{\gamma q(y)[q(y)+yq'(y)]-1\} - \left[\frac{y}{\mu_F}\right]T_{ppW}(F) & y < z \\ -\left[\frac{y}{\mu_F}\right]T_{ppW}(F) & y > z \end{cases}$ |

| **DISTRIBUTIONAL STATISTIC** | **RECENTERED INFLUENCE FUNCTION**, *continued* |
|---|---|
| Additive pro-poorness for the headcount ratio<br>$PP^{HC}(F) = z\left(q(z)-1\right)f(z)$ | $RIF\left(y;T_{ppHC};F\right) = -\left[\dfrac{y}{\mu_F}\right]T_{ppHC}(F) - zf(z)\dfrac{y}{\mu_F}, \quad (y \neq z)$ |
| Poverty elasticity of the headcount ratio,<br>$EHC(F) = -\left( zq(z)f(z) \big/ F(z) \right)$ | $RIF(y;T_{PEHC};F) = \begin{cases} -T_{EHC}(F)\left[\dfrac{1-F(z)}{F(z)} + \dfrac{y}{\mu_F}\right] & y < z \\ T_{EHC}(F)\left[1 - \dfrac{y}{\mu_F}\right] & y > z \end{cases}$ |

For all poverty-related measures, recentered influence functions are infinite at the poverty line value $y = z$.

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