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Gaston Yalonetzky

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Gaston Yalonetzky*
University of Leeds and OPHI

Abstract

Bennett and Hatzimasoura (2011) derive a new class of poverty measures suitable for ordinal variables. These indices are weighted sums of the population probabilities of attaining each state of the ordinal variable which is below the poverty line. The weights are uniquely determined by the choice of the classisingle parameter and by the number of ordinal states below the poverty line. However, as I show in this note, such restrictive propery is not necessary for the derivation of poverty measures for ordinal variables that satisfy a broad array of desirable properties, including those advocated by Bennett and Hatzimasoura. The class of measures proposed in this note include those of the authors, as a specific subclass. Since the class of admissible poverty measures for ordinal variables is unbounded, the note provides two dominance conditions whose fulfillment ensure the agreement of ordinal poverty comparisons among measures belonging to subfamilies within the class.

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* **Contact details**: G.Yalonetzky@leeds.ac.uk .

Introduction

The issues arising in the measurement of wellbeing using ordinal variables have received renewed attention (e.g. Allison and Foster (2004), Zheng (2011), Yalonetzky (2013)). One such concern is the lack of meaning for the distances between ordinal categories, or states. In a recent contribution, Bennett and Hatzimasoura (2011) derive a new class of poverty measures suitable for ordinal variables. The authors also restate the well-known properties of monotonicity and transfers, from the literature of poverty measurement with continuous variables (e.g. see Foster et al. (2010)), in order to render them sensible in applications to ordinal variables. The proposed indices are weighted sums of the population probabilities of attaining each state of the ordinal variable which is below the poverty line. In the indices of Bennett and Hatzimasoura, the weights are uniquely determined by the choice of the class' single parameter (α) and by the number of ordinal states below the poverty line.

The first task of this note is to show that such restrictive propery (on the weights) is not necessary for the derivation of poverty measures for ordinal variables that satisfy a broad array of desirable properties, including those advocated by Bennett and Hatzimasoura (2011). The note proposes a more general class of measures that includes those of the authors, as a specific subclass. The note identifies subclasses within the general class in relation to the sets of desirable properties fulfilled by their members.

The unboundedness of the general class of admissible poverty measures for ordinal variables, which encompasses all the members of Bennett and Hatzimasoura (2011), justifies looking for dominance conditions whose fulfillment ensure the agreement of ordinal poverty comparisons among measures belonging to broad classes. This note also provides two dominance conditions for subfamilies of measures within the general class of poverty measures with ordinal variables.

The rest of the note proceeds as follows: First, the class of Bennett and Hatzimasoura (2011) is introduced along with the relevant notation. Then, in a separate section, the general class of poverty measures for ordinal variables is introduced. A subsection checks the fulfillment of desirable properties by subclasses of the general class. Then the fourth section is dedicated to the partial orderings determined by two stochastic dominance conditions relevant for ordinal variables. The note concludes with remarks on the interpretation of the poverty measures for ordinal variables.

The proposal by Bennett and Hatzimasoura

Let Y be an ordinal measure of wellbeing with S categories or states (e.g. self-reported health, education measured in levels, etc.), such that: $Y := (y_1, y_2, \ldots, y_S)$ and $0 \leq y_1 < y_2 < \ldots < y_S < \infty$. If a person in a state not better than k ($1 \leq k \leq S$) is deemed poor, then Bennett and Hatzimasoura (2011) propose the following single-parameter class of poverty measures:

$$\pi_\alpha(Y) = \sum_{j=1}^{k} p_j \left(\frac{k-j+1}{k} \right)^\alpha, \tag{1}$$

where $p_j \equiv \Pr[Y = y_j]$ and $\alpha \geq 0$ is the single parameter. The authors carefully explain the rationale and the different meanings of (1), particularly in relation to the different choices of α. The latter also affect the sets of desirable properties fulfilled by members of the class, as shown by the authors. The reader is referred to their paper for further details. Here it is worth pointing out the resemblance between (1) and the Foster-Greer-Thorbecke class for continuous variables; and also that (1) is a weighted sum of the probabilities of being in each state below the poverty line. *The weights are jointly, and uniquely, determined by the number of states below the poverty line (k) and the choice of the single parameter, α.* It is also worth noting that, as long as $\alpha > 0$, people in the poorest states contribute more to overall social poverty, $\pi_\alpha(Y)$, since the weights decrease as j increases. The latter is the dual side of the traditional, monotonicity property, which demands the index to increase, thereby reflecting higher poverty, whenever a poor person becomes poorer. $\pi_\alpha(Y)$ fulfills this property if and only if $\alpha > 0$ (Bennett and Hatzimasoura (2011, p. 12)).

A general class of poverty measures for ordinal variables

Let $k = 3$ and $\alpha = 1$. In that case the three weights of (1) are: $1, \frac{2}{3}, \frac{1}{3}$, for p_1, p_2, p_3, respecitvely. Is this restrictive feature of the weight setting necessary for a well-behaved ordinal poverty measure, i.e. an index that satisfies a set of desirable properties in the context of ordinal variables? Is it possible to have well-behaved measures with alternative sets of weights for one given context (e.g. $k = 3$)? In this section, the note answers the first question negatively and the second question positively, by proposing the following general class of poverty measures for ordinal variables:

$$\pi_w(Y) = \sum_{j=1}^{k} p_j w_j, \tag{2}$$

where w_j is the weight attached to the probability of being in (poor) state j, p_j. Unlike (1), (2) does not depend on any parameter that uniquely determines the set of weights. Not all choices of weights produce well-behaved poverty measures. The next subsection shows the restrictions in the choices of w_j that are required in order to derive subclasses of (2) that satisfy different sets of desirable properties in the context of ordinal variables. As it turns out, these subclasses include, but are not confined to, the measures proposed by Bennett and Hatzimasoura (2011), as specific subfamilies. Hence the restrictions on the weights presented in this note are less stringent than the ones present in (1).

The following subsection lists a minimum set of desirable properties that poverty measures for ordinal variables should fulfill, together with a derivation of the restrictions on w_j that are required to render (2) in fulfillment of the respective properties.

Fulfillment of desirable properties

The first desirable axiom, presented by Bennett and Hatzimasoura (2011), is that of ordinal invariance. Since ordinal variables are only meaningful as rankers, the axiom of ordinal invariance stipulates that poverty measures should be insensitive to order-preserving transformations of the ordinal variables and the poverty lines:

Axiom 1 *Ordinal invariance: Let* Y *have a distribution:* $(p_1, y_1; p_2, y_2; \ldots; p_S, y_S)$. *If* $g : \mathbb{R} \rightarrow \mathbb{R}$ *is a monotone increasing function, then:* $\pi_w(Y) = \pi_w(g[Y])$.

Bennett and Hatzimasoura (2011) show that (1) fulfills axiom 1. The reasoning to show that (2) also fulfills the same axiom is analogous. Intuitively, $\pi_w(Y)$ maps from the probabilities, p_j, which are invariant to any order-preserving transformation of the ordinal variable. Likewise the weights, w_j, are not functions of any scaling of the variable. The next axiom, population replication invariance, is standard in the poverty literature. It demands that the poverty measure should be insensitive to replications of the population:

Axiom 2 *Population replication invariance: If every individual in the population is replicated by an equal amount, then* $\pi_w(Y)$ *should not change.*

$\pi_w(Y)$ fulfills axiom 2 because homogeneous replications do not change the probability distribution and the weights do not depend on those replications either. I now introduce a variation of another standard axiom, which establishes the first restriction on the set of weights, w_j:

Axiom 3 *Normalization: 1)* $\pi_w(Y) = 0$, *if and only if nobody is poor, i.e.* $p_j = 0 \forall j \leq k$; *and 2)* $\pi_w(Y) \leq 1$.

The second part of the normalization axiom puts an upper limit to the poverty measure. This upper limit has different meanings depending on the choice of weights. For instance, if $w_j = 1 \forall j \in [1, 2, \ldots, k]$, then $\pi_w(Y) = 1$ implies that all the population is poor. The following proposition provides the first restriction of weights, with which axiom 3 is fulfilled:

Proposition 1 *:* $\pi_w(Y)$ *satisfies axiom 3 if and only if* $\forall j \in [1, 2, \ldots, k] : \min(w_j) > 0 \wedge \max(w_j) = 1$.

Hence, according to proposition 1, a well-behaved poverty measure for ordinal variables of the form (2), must have positive weights, but none of them should be higher than 1. The next axiom of monotonicity is the adaptation of the literature's monotonicity axiom (e.g. see Foster et al. (1984)) to the ordinal context, by Bennett and Hatzimasoura (2011):

Axiom 4 *Monotonicity: Let* Y *have a distribution:* $(p_1, y_1; p_2, y_2; \ldots; p_S, y_S)$ *and the poverty line be set at* $k \in [1, 2, \ldots, S]$ *.If the distribution of* $\widetilde{Y}$ *is generated from that of* Y *by moving probability mass* δ *from state* j *to state* i*, such that* $i < j$, $\widetilde{p_j} = p_j - \delta$ *and* $\widetilde{p_i} = p_i + \delta$, *then* $\pi_w\left(\widetilde{Y}\right) > \pi_w(Y)$.

The monotonicity axiom requires a poverty measure to increase when someone becomes poorer. The following proposition provides the second restriction of weights, with which axiom 4 is fulfilled:

Proposition 2 *:* $\pi_w(Y)$ *satisfies axiom 4 if and only if* $\forall i, j \in [1, 2, \ldots, k] \mid i < j : w_i > w_j$.

Proof: Define $\Delta\pi_w \equiv \pi_w\left(\widetilde{Y}\right) - \pi_w(Y)$. Then: $\Delta\pi_w = [w_i - w_j]\,\delta$. Hence: $\Delta\pi_w > 0 \leftrightarrow w_i > w_j$. The same result has to hold for any other combination of i and j. QED.

According to propositions 1 and 2, a poverty measure for ordinal variables of the form (2) must have positive weights which decrease as the state or category increases. Moreover the weight of the poorest state should be equal to 1: $w_1 = 1, w_2 < w_1, w_3 < w_2, \ldots, w_k < w_{k-1}, w_k > 0$. Clearly, the class (1) fulfills this condition when $\alpha > 0$. For instance, as mentioned, with $k = 3$ and $\alpha = 1$: $w_1 = 1, w_2 = \frac{2}{3}, w_3 = \frac{1}{3}$. But many other members of the more general class (2) are also suitable. For instance, with $k = 3$, the following weights are also admissible: $w_1 = 1, w_2 = 0.5, w_3 = 0.2$.

In the poverty literature a transfers axiom is also considered, whereby a poverty measure should increase whenever a regressive transfer takes place among the poor. Bennett and Hatzimasoura (2011) adapt it for ordinal variables by focusing on offseting transfers of probability mass in two different parts of the distribution:

Axiom 5 *Transfers: Let Y have a distribution: $(p_1, y_1; p_2, y_2; \ldots; p_S, y_S)$ and the poverty line be set at $k \in [1, 2, \ldots, S]$ .If the distribution of $\widetilde{Y}$ is generated from that of Y by moving probability mass δ from state i to state $i - m$, such that $\widetilde{p_i} = p_i - \delta$ and $\widetilde{p_{i-m}} = p_{i-m} + \delta$, and by moving probability mass δ from state j to state $j - m$, such that $\widetilde{p_j} = p_j - \delta$ and $\widetilde{p_{j+m}} = p_{j+m} + \delta$, then, for all $i \leq j : \pi_w\left(\widetilde{Y}\right) > \pi_w(Y)$.*

The following proposition provides the third restriction of weights, with which axiom 5 is fulfilled:

Proposition 3 *: $\pi_w(Y)$ satisfies axiom 5 if and only if $\forall i, j, m \in [1, 2, \ldots, k] \mid i \leq j : [w_{i-m} - w_i] > [w_j - w_{j+m}]$.*

Proof: Define $\Delta\pi_w \equiv \pi_w\left(\widetilde{Y}\right) - \pi_w(Y)$. When the transfer depicted in axiom 5 occurs: $\Delta\pi_w = [(w_{i-m} - w_i) - (w_j - w_{j+m})]\,\delta$. Hence: $\Delta\pi_w > 0 \leftrightarrow [w_{i-m} - w_i] > [w_j - w_{j+m}]$. The same result has to hold for any other combination of i and j. QED.

One example of fulfillment of axiom 5 is provided by members of (1) for which $\alpha > 1$, as shown by Bennett and Hatzimasoura (2011). However, many other members of the more general class (2) also fulfill the axiom of transfers, e.g. the one with weights: $w_1 = 1, w_2 = 0.5, w_3 = 0.2$.

It is also easy to show that the class (2) fulfills a traditional focus axiom, whereby $\pi_w(Y)$ is insensitive to changes in the wellbeing of non-poor people (as long as they do not change their poverty status). Likewise all members of class (2) are additively decomposable, and hence subgroup consistent. For more details on these axioms the reader is referred to Foster et al. (2010).

Partial orderings

As the previous section shows, entire subclasses from (2) are admissible as well-behaved measures of poverty with ordinal variables, in terms of their fulfillment of several desirable properties/axioms. These include, but are not restricted to, measures from (1); i.e. measures characterized by $w_j = \left(\frac{k-j+1}{k}\right)^{\alpha}$, with certain restrictions on the values of α (e.g. $\alpha > 1$ is required for the fulfillment of the transfers axiom). Since so many choices of weights, w_j, generate well-behaved poverty measures, it is natural to inquiry into the existence of dominance conditions whose fulfillment guarantees the robustness of poverty orderings, in the context of ordinal variables, to different choices of weights. In this section I derive two dominance conditions that are relevant for the general class (2).

A first-order dominance condition

Let $\Delta\pi_w \equiv \pi_w(A) - \pi_w(B)$, where A and B are two distributions and let $W := (w_1, w_2, \ldots, w_S)$ be the vector of weights for all the states of the variable.[1] Define also $F_j \equiv \Pr[Y \leq y_j]$ and $\Delta F_j \equiv F_j(A) - F_j(B)$. The following theorem provides the first-order dominance condition:

Theorem 1 *: $\Delta\pi_w \leq 0 \forall W \mid w_j < w_{j-1} \; \forall j \leq k, w_i = 0 \forall i > k \wedge \forall k \in [1, 2, \ldots, S]$ if and only if $\Delta F_{j-1} \leq 0 \forall j \in [1, 2, \ldots, S]$.*

Theorem 1 states that poverty in A is not higher than that in B for all poverty lines and for all members of (2) which fulfill axiom 4 (monotonicity) if and only if the cumulative distribution of A is never above that of B.

Proof: Note that $\pi_w(Y) = \sum_{j=1}^{S} p_j w_j \leftrightarrow w_j = 0 \forall j > k$ (i.e. the focus axiom holds). Hence using summation by parts, and noting that $\Delta F_S W_s = \Delta F_0 W_0 = 0$,[2] it is possible to show that:

$$\Delta\pi_w = -\sum_{j=1}^{S} [w_j - w_{j-1}] \Delta F_{j-1} \tag{3}$$

From (3) it is then straightforward to derive theorem 1. QED. Note that this condition applies to all poverty measures satisfying monotonicity, irrespective of whether they fulfill the transfers axiom. These include all the members of (1) for which $\alpha > 0$. By contrast, the second-order dominance condition in the next subsection is less stringent in terms of the comparison of cumulative distributions, but only applies to poverty measures satisfying the transfers axiom.

A second-order dominance condition

The following theorem provides the second-order dominance condition:

Theorem 2 *: $\Delta\pi_w \leq 0 \forall W \mid w_j < w_{j-1} \wedge [(w_j - w_{j-1}) - (w_{j-1} - w_{j-2}) > 0] \; \forall j \leq k, w_i = 0 \forall i > k \wedge \forall k \in [1, 2, \ldots, S]$ if and only if $\sum_{t=1}^{j} \Delta F_t \leq 0 \forall t \in [1, 2, \ldots, S-1]$.*

[1] For the focus axiom to hold it is necessary that: $w_j = 0 \forall j > k$.

[2] Because $F_S = 1$ and $F_0 = 0$.

Theorem 2 states that poverty in A is not higher than that in B for all poverty lines and for all members of (2) which fulfill axioms 4 and 5 (monotonicity *and* transfers) if and only if *the cumulative of the cumulative* distribution of A is never above that of B.

Proof: Summing by parts (3) yields:

$$\begin{aligned} \Delta\pi_w &= -[w_S - w_{S-1}] \sum_{j=1}^{S-1} \Delta F_j + \\ & \sum_{j=1}^{S} [(w_j - w_{j-1}) - (w_{j-1} - w_{j-2})] \sum_{t=1}^{j-2} \Delta F_t \end{aligned} \quad (4)$$

From (3) it is then straightforward to derive theorem 2. QED. As mentioned, this condition applies only to poverty measures satisfying monotonicity and transfers axioms. These include all the members of (1) for which $\alpha > 1$.

Concluding remarks

The contribution by Bennett and Hatzimasoura (2011), i.e. the class (1), is a major step forward in the measurement of poverty using ordinal variables. However, as this note shows, class (1) is just an example of many other measures that are also suitable for the measurement of poverty with ordinal variables, in the sense that they fulfill several desirable properties. The note proposes a more general class, (2), which encompasses (1). Then the note derives restrictions on the choices of weights attached to the probabilities of attaining states below the poverty line. These restrictions define subclasses of measures that satisfy sets of desirable properties. For instance, the note shows that, in order to satisfy monotonicity, the measures' weights have to be positive and monotonically decreasing, e.g. the highest weight must be attached to the lowest state of wellbeing. The note also shows that the differences between weights need to be greater among pairs of equally-spaced states at lower levels of wellbeing, if the measures are expected to fulfill the transfers axiom.

Compared to class (1), the more general class (2) involves a choice of k parameters, i.e. the weights. By contrast, class (1) requires the choice of just one single paramter (α). Arguably, whether the latter can be deemed an advantage, or not, depends on whether we value, or not, the many more degrees of freedom available in the general class. In any case, the note provides two dominance conditions whose fulfillment ensure the ordinal robustness of poverty comparisons. In such situations, there is no need to make choices among poverty measures; unless, of course, there is also interest in cardinal comparisons.

Finally, it is worth asking whether there is any special meaning for these measures; for its own sake, but also because certain measures' interpretability could be defended as a criterion for choosing among several options. Bennett and Hatzimasoura (2011) discuss at length the meaning of class (1). Clearly, whenever $w_j = 1 \forall j$ (i.e. $\alpha = 0$), the measure is the classic headcount ratio, which violates important properties like monotonicity and transfers. Whenever $\alpha > 0$,Bennett and Hatzimasoura (2011) advance the interpretation of class (1) as a weighted average of the different poverty headcounts that result from setting the poverty lines at all states from 1 until k. It is easy to show that the same interpretation is applicable to the members of class (2), as long as they fulfill the monotonicity axiom. In that case, following Bennett and Hatzimasoura (2011), all members of class (2) can be interpreted as the expected percentile rank of a person who faces a lottery with a survival function determined by the vector W, and with a zero chance of being non-poor.

The most interesting interpretation, among those put forward by Bennett and Hatzimasoura (2011), is that for $\alpha = 1$. That measure is the same expected percentile rank, but for a person who faces equal probabilities of obtaining any wellbeing state below the poverty line. On the grounds of this interpretation, that measure, i.e. the simil from class (1) of the average poverty gap, may have an advantage over other measures satisfying monotonicity. However, the measure does not satisfy the transfers axiom. Among the subclass of poverty measures satisfying the transfers axiom, there is no measure that stands out for having any particulary appealing interpretation vis-a-vis others (unlike the case of $\alpha = 1$ of class (1)). While this lack of clear criteria for choosing among measures bolsters the case for dominance approaches when the main concern is about ordinal comparisons, in the context of cardinal comparisons this note shows that there is a plethora of defendable choices of well-behaved measures, including, but not restricted to, those in class (1). Whether that is a good thing, or not, is a up to further discussion.

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