Working Paper Series

Classical inequality indices, welfare functions, and the dual decomposition

Oihana Aristondo
José Luis García-Lapresta
Casilda Lasso de la Vegaz
Ricardo Alberto Marques Pereira

ECINEQ WP 2012 – 253

ECINEQ 2012 – 253
April 2012

www.ecineq.org

Classical inequality indices, welfare functions, and the dual decomposition

Oihana Aristondo
BRIDGE Research Group, Universidad del País Vasco

José Luis García-Lapresta
PRESAD Research Group, IMUVA, Universidad Valladolid

Casilda Lasso de la Vega
BRIDGE Research Group, Universidad del País Vasco

Ricardo Alberto Marques Pereira *
Dipartimento di Informatica e Studi Aziendali, Universitµa degli Studi di Trento

Abstract

We consider the classical inequality measures due to Gini, Bonferroni, and De Vergottini and we present a brief review of the three inequality indices and the associated welfare functions, in the correspondence scheme introduced by Blackorby and Donaldson, and Weymark. The three classical inequality indices incorporate different value judgments in the measurement of inequality, leading to different behavior under income transfers between individuals in the population. The welfare functions associated with the Gini, Bonferroni, and (normalized) De Vergottini indices are Schur-concave OWA functions, with larger weights for lower incomes. We examine the dual decomposition and the orness degree of the three welfare functions in the standard framework of aggregation functions on the [0; 1]n domain, and show that it offers interesting insight on the distinct and complementary nature of the classical inequality indices.

Keywords: income inequality and social welfare, classical Gini, Bonferroni, and De Vergottini inequality indices, welfare functions, aggregation functions, WA and OWA functions, dual decomposition, orness

JEL Classification: D63, I32.

* **Contact details**: lapresta@eco.uva.es, casilda.lassodelavega@ehu.es, ricalb.marper@unitn.it

1 Introduction

Income inequality plays a crucial role in Economics and Social Welfare. It has been proved that income inequality has important impacts in terms of development, poverty, and public finance.

Typical issues that arise in these contexts are the evolution of inequality over time in some particular region, the differences of the inequality levels across different countries, and the effect of different policies in the evolution of the inequality. In order to address these and related questions the choice of inequality measure is a central issue.

Basically, an inequality measure is a summary statistic of the income dispersion. Several inequality indices have been proposed in the literature, for comprehensive surveys on inequality measures, see Silber [40] and Chakravarty [12]. One of the most widely used is the Gini index (Gini [24]), based on the absolute values of all pairwise income differences. This index has a very intuitive appeal for its geometrical interpretation in terms of the Lorenz curve and, unlike other inequality measures, it easily accommodates negative incomes. One drawback of the Gini index is that it is insensitive to the position of income transfers within the ordered income profile. In order to overcome this difficulty, a single-parameter class of inequality measures that generalizes the Gini index referred to as the S-Gini family, has been introduced and characterized (Donaldson and Weymark [15, 16], Weymark [43] and Bossert [9]). In this family, different value judgments can be considered by means of a weighting function of incomes.

The Bonferroni and De Vergottini indices are two other classical inequality indices that are recently receiving growing attention, see for instance Nygard and Sandstrom [36], Giorgi [25, 26], Tarsitano [42], Giorgi and Mondani [28], Giorgi and Crescenzi [27], Chakravarty and Muliere [13], Piesch [37], Chakravarty [11] and Bárcena and Imedio [3]. Similarly to the Gini index, they also permit negative incomes.

The Bonferroni index ([8]) measures inequality comparing the overall income mean with the income means of the poorest individuals in the population. The De Vergottini index ([14]) complements the information provided by the Bonferroni index since inequality is captured by comparing the overall income mean with the income means of the richest individuals in the population. The three classical inequality indices –Gini, Bonferroni, and De Vergottini– are formally similar but introduce distinct and complementary information in the study of income inequality. Moreover, in contrast with the Gini index, the Bonferroni and De Vergottini indices are sensitive to the specific position of income transfers within the ordered income profile.

An inequality index is relative if it is invariant when an additional amount of income is proportionally distributed among the whole population. This corresponds to the rightist viewpoint, according to Kolm's designation [32]. In turn, the leftist view requires that inequality remains unchanged when each individual in the population receives the same amount of the extra income. This invariance condition is fulfilled by the absolute inequality indices, which are obtained by multiplying the corresponding relative indices by the mean income.

Choosing a particular index to measure inequality involves a value judgment, because different choices can lead to different results. One criterion is to select ethical indices, that is indices that have a normative interpretation. This means that there is an explicit relationship between the inequality measure and a social welfare ordering defined on incomes. In other words, for these indices it is possible to construct a social welfare function whose contours specify the tradeoffs between inequality and efficiency, as measured by the total income.

An interesting feature of the inequality indices considered in this paper is that the associated welfare functions are of the OWA type. Accordingly, they can be studied in the framework of the dual decomposition of aggregation functions proposed by García-Lapresta and Marques Pereira [22], where each aggregation operator is additively decomposed into a self-dual core and an associated anti-self-dual remainder[1].

The dual decomposition offers interesting insight on the distinct and complementary nature of the three classical inequality indices. In the Gini index case, the dual decomposition reproduces in a natural way the construction of the associated welfare function. As for the Bonferroni and the De Vergottini indices, the corresponding self-dual cores and anti-self-dual remainders express the underlying relationship between the two indices.

The paper is organized as follows. In Section 2, we introduce the basic notation and properties of aggregation functions and we describe the general framework of the dual decomposition of an aggregation function into a self-dual core and an associated anti-self-dual remainder. Moreover, we briefly review the dual decomposition of OWA functions. Section 3 is devoted to inequality indices and the associated welfare functions, focusing on the classical Gini, Bonferroni, and De Vergottini indices. In Section 4 we examine the dual decomposition of the welfare functions associated to the Gini, Bonferroni, and De Vergottini indices. Finally, Section 5 contains some concluding remarks.

2 Aggregation functions

In this section we present notation and basic definitions regarding aggregation functions on $[0,1]^n$ and functions on $[0,\infty)^n$, with $n \in \mathbb{N}$ and $n \geq 2$ throughout the text.

Notation. Points in $[0,1]^n$ are denoted as $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{0} = (0, \dots, 0)$, $\boldsymbol{1} = (1, \dots, 1)$. Accordingly, for every $x \in [0,1]$, we have $x \cdot \boldsymbol{1} = (x, \dots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i \in \{1, \dots, n\}$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in [0,1]^n$, the increasing and decreasing reorderings of the coordinates of $\boldsymbol{x}$ are indicated as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \dots, x_n\} = x_{[n]}$

[1] Other applications of the dual decomposition to the field of Welfare Economics can be found in García-Lapresta *et al.* [20] and Aristondo *et al.* [1].

and $x_{(n)} = \max\{x_1, \dots, x_n\} = x_{[1]}$. In general, given a permutation σ on $\{1, \dots, n\}$, we denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Finally, the arithmetic mean is denoted $\mu(\boldsymbol{x}) = (x_1 + \cdots + x_n)/n$.

We begin by defining standard properties of real functions on $\mathbb{R}^n$. For further details the interested reader is referred to Fodor and Roubens [19], Calvo *et al.* [10], Beliakov *et al.* [4], García-Lapresta and Marques Pereira [22] and Grabisch *et al.* [29].

Definition 1 *Let $A : D^n \longrightarrow \mathbb{R}$ be a function with $D = [0, 1]$ or $D = [0, \infty)$.*

1. *A is* idempotent *if for every $x \in D$:*
$$A(x \cdot \mathbf{1}) = x.$$
2. *A is* symmetric *if for every permutation σ on $\{1, \dots, n\}$ and every $\boldsymbol{x} \in D^n$:*
$$A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x}).$$
3. *A is* monotonic *if for all $\boldsymbol{x}, \boldsymbol{y} \in D^n$:*
$$\boldsymbol{x} \geq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \geq A(\boldsymbol{y}).$$
4. *A is* strictly monotonic *if for all $\boldsymbol{x}, \boldsymbol{y} \in D^n$:*
$$\boldsymbol{x} > \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) > A(\boldsymbol{y}).$$
5. *A is* compensative *if for every $\boldsymbol{x} \in D^n$:*
$$x_{(1)} \leq A(\boldsymbol{x}) \leq x_{(n)}.$$
6. *A is* self-dual *if $D = [0, 1]$ and for every $\boldsymbol{x} \in [0, 1]^n$:*
$$A(\mathbf{1} - \boldsymbol{x}) = 1 - A(\boldsymbol{x}).$$
7. *A is* anti-self-dual *if $D = [0, 1]$ and for every $\boldsymbol{x} \in [0, 1]^n$:*
$$A(\mathbf{1} - \boldsymbol{x}) = A(\boldsymbol{x}).$$
8. *A is* invariant for translations *if for all $t \in \mathbb{R}$ and $\boldsymbol{x} \in D^n$:*
$$A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x})$$
whenever $\boldsymbol{x} + t \cdot \mathbf{1} \in D^n$.

9. A *is* stable for translations *if for all* $t \in \mathbb{R}$ *and* $\boldsymbol{x} \in D^n$*:*

$$A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x}) + t$$

whenever $\boldsymbol{x} + t \cdot \mathbf{1} \in D^n$*.*

10. A *is* scale invariant *(or* homothetic*) if for all* $\lambda > 0$ *and* $\boldsymbol{x} \in D^n$*:*

$$A(\lambda \cdot \boldsymbol{x}) = \lambda \cdot A(\boldsymbol{x})$$

whenever $\lambda \cdot \boldsymbol{x} \in D^n$*.*

Definition 2 *Let* $\{A^{(k)}\}_{k \in \mathbb{N}}$ *be a sequence of functions, with* $A^{(k)} : D^k \longrightarrow \mathbb{R}$ *and* $A^{(1)}(x) = x$ *for every* $x \in D$*, where* $D = [0,1]$ *or* $D = [0,\infty)$*.* $\{A^{(k)}\}_{k \in \mathbb{N}}$ *is* invariant for replications *if for all* $\boldsymbol{x} \in D^n$ *and any number of replications* $m \in \mathbb{N}$ *of* $\boldsymbol{x}$*:*

$$A^{(mn)}(\overbrace{\boldsymbol{x}, \ldots, \boldsymbol{x}}^{m}) = A^{(n)}(\boldsymbol{x}).$$

Definition 3 *Consider the binary relation* $\succcurlyeq$ *on* D^n*, with* $D = [0,1]$ *or* $D = [0,\infty)$*, defined as*

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \;\Leftrightarrow\; \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \;\; \text{and} \;\; \sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)},$$

for every $k \in \{1, \ldots, n-1\}$*. With respect to the binary relation* $\succcurlyeq$*, the notions of* S-convexity *and* S-concavity *of a function* A *are defined as follows.*

1. $A : D^n \longrightarrow D$ *is* S-convex *if for all* $\boldsymbol{x}, \boldsymbol{y} \in D^n$*:*

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \geq A(\boldsymbol{y}).$$

2. $A : D^n \longrightarrow D$ *is* S-concave *if for all* $\boldsymbol{x}, \boldsymbol{y} \in D^n$*:*

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \leq A(\boldsymbol{y}).$$

Moreover, in each case, the S-convexity *(resp.* S-concavity*) of a function* A *is said to be strict if* $A(\boldsymbol{x}) > A(\boldsymbol{y})$ *(resp.* $A(\boldsymbol{x}) < A(\boldsymbol{y})$*) whenever* $\boldsymbol{x} \neq \boldsymbol{y}$*.*

Definition 4 *Given* $\boldsymbol{x}, \boldsymbol{y} \in D^n$*, with* $D = [0,1]$ *or* $D = [0,\infty)$*, we say that* $\boldsymbol{y}$ *is obtained from* $\boldsymbol{x}$ *by a* progressive transfer *if there exist two individuals* $i, j \in \{1, \ldots, n\}$ *and* $h > 0$ *such that* $x_i < x_j$*,* $y_i = x_i + h \leq x_j - h = y_j$ *and* $y_k = x_k$ *for every* $k \in \{1, \ldots, n\} \setminus \{i, j\}$*.*

A classical result (see Marshall and Olkin [34, Ch. 4, Prop. A.1]) establishes that $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ if and only if $\boldsymbol{y}$ can be derived from $\boldsymbol{x}$ by means of a finite sequence of permutations and/or progressive transfers.

Definition 5 *A function $A : [0,1]^n \longrightarrow [0,1]$ is called an n-ary* aggregation function *if it is monotonic and satisfies $A(\mathbf{1}) = 1$ and $A(\mathbf{0}) = 0$. An aggregation function is said to be* strict *if it is strictly monotonic.*

For the sake of simplicity, the n-arity is omitted whenever it is clear from the context.

It is easy to see that every idempotent aggregation function is compensative, and viceversa. Self-duality and stability for translations are important properties of aggregation functions. In turn, anti-self-duality and invariance for translations are incompatible with idempotency. Nevertheless, anti-self-duality and invariance for translations play an important role in this paper as far as they are properties of important functions associated with aggregation functions, such as we shall discuss later.

2.1 Dual decomposition of aggregation functions

In this section we briefly recall the so-called *dual decomposition* of an aggregation function into its self-dual core and associated anti-self-dual remainder, due to García-Lapresta and Marques Pereira [22]. First we introduce the concepts of self-dual core and anti-self-dual remainder of an aggregation function, establishing which properties are inherited in each case from the original aggregation function. Particular emphasis is given to the properties of stability for translations (self-dual core) and invariance for translations (anti-self-dual remainder).

Definition 6 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. The aggregation function $A^* : [0,1]^n \longrightarrow [0,1]$ defined as*

$$A^*(\boldsymbol{x}) = 1 - A(\mathbf{1} - \boldsymbol{x})$$

is known as the dual *of the aggregation function A.*

Clearly, $(A^*)^* = A$, which means that dualization is an *involution*. An aggregation function A is self-dual if and only if $A^* = A$.

2.1.1 The self-dual core of an aggregation function

Aggregation functions are not in general self-dual. However, a self-dual aggregation function can be associated to any aggregation function in a simple manner. The construction of the so-called *self-dual core* of an aggregation function A is as follows.

Definition 7 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function. The function* $\widehat{A} : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$\widehat{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) + A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) - A(\mathbf{1} - \boldsymbol{x}) + 1}{2}$$

is called the core *of the aggregation function* A.

Since $\widehat{A}$ is self-dual, we say that $\widehat{A}$ is the *self-dual core* of the aggregation function A. Notice that $\widehat{A}$ is clearly an aggregation function.

It is interesting to note that the self-dual core reduces to the arithmetic mean in the simple case of $n = 2$, but not in higher dimensions.

The following results[2] can be found in García-Lapresta and Marques Pereira [22].

Proposition 1 *An aggregation function* $A : [0,1]^n \longrightarrow [0,1]$ *is self-dual if and only if* $\widehat{A}(\boldsymbol{x}) = A(\boldsymbol{x})$ *for every* $\boldsymbol{x} \in [0,1]^n$.

Proposition 2 *The self-dual core* $\widehat{A}$ *inherits from the aggregation function* A *the properties of continuity, idempotency (hence, compensativeness), symmetry, strict monotonicity, stability for translations, and invariance for replications, whenever* A *has these properties.*

2.1.2 The anti-self-dual remainder of an aggregation function

We now introduce the *anti-self-dual remainder* $\widetilde{A}$, which is simply the difference between the original aggregation function A and its self-dual core $\widehat{A}$.

Definition 8 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function. The function* $\widetilde{A} : [0,1]^n \longrightarrow \mathbb{R}$ *defined as* $\widetilde{A}(\boldsymbol{x}) = A(\boldsymbol{x}) - \widehat{A}(\boldsymbol{x})$, *that is*

$$\widetilde{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) - A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) + A(\mathbf{1} - \boldsymbol{x}) - 1}{2},$$

is called the remainder *of the aggregation function* A.

Since $\widetilde{A}$ is anti-self-dual, we say that $\widetilde{A}$ is the *anti-self-dual remainder* of the aggregation function A. Clearly, $\widetilde{A}$ is not an aggregation function. In particular, $\widetilde{A}(\mathbf{0}) = \widetilde{A}(\mathbf{1}) = 0$ violates idempotency and implies that $\widetilde{A}$ is either non monotonic or everywhere null. Moreover, $-0.5 \le \widetilde{A}(\boldsymbol{x}) \le 0.5$ for every $\boldsymbol{x} \in [0,1]^n$.

The following results[3] can be found in García-Lapresta and Marques Pereira [22].

[2] Excepting that invariance for replications is inherited by the core (the proof is immediate).

[3] Excepting that invariance for replications is inherited by the remainder (the proof is immediate) and that strict S-convexity and S-concavity are also inherited by the remainder.

Proposition 3 *An aggregation function* $A : [0,1]^n \longrightarrow [0,1]$ *is self-dual if and only if* $\widetilde{A}(\boldsymbol{x}) = 0$ *for every* $\boldsymbol{x} \in [0,1]^n$.

Proposition 4 *The anti-self-dual remainder* $\widetilde{A}$ *inherits from the aggregation function* A *the properties of continuity, symmetry, invariance for replications, plus also strict S-convexity and S-concavity, whenever* A *has these properties.*

Summarizing, every aggregation function A decomposes additively $A = \widehat{A} + \widetilde{A}$ in two components: the self-dual core $\widehat{A}$ and the anti-self-dual remainder $\widetilde{A}$, where only $\widehat{A}$ is an aggregation function. The so-called *dual decomposition* $A = \widehat{A} + \widetilde{A}$ clearly shows some analogy with other algebraic decompositions, such as that of square matrices and bilinear tensors into their symmetric and skew-symmetric components.

The following result concerns two more properties of the anti-self-dual remainder based directly on the definition $\widetilde{A} = A - \widehat{A}$ and the corresponding properties of the self-dual core (see García-Lapresta and Marques Pereira [22]).

Proposition 5 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function.*

1. $\widetilde{A}(x \cdot \mathbf{1}) = 0$ *for every* $x \in [0,1]$.
2. *If* A *is stable for translations, then* $\widetilde{A}$ *is invariant for translations.*

These properties of the anti-self-dual remainder are suggestive. The first statement establish that anti-self-dual remainders are null on the main diagonal. The second statement applies to the subclass of stable aggregation functions. In such case, self-dual cores are stable and therefore anti-self-dual remainders are invariant for translations. In other words, if the aggregation function A is stable for translations, the value $\widetilde{A}(\boldsymbol{x})$ does not depend on the average value of the $\boldsymbol{x}$ coordinates, but only on their numerical deviations from that average value. These properties of the anti-self-dual remainder $\widetilde{A}$ suggest that it may give some indication on the dispersion of the $\boldsymbol{x}$ coordinates.

In Maes *et al.* [33], the authors propose a generalization of the dual decomposition framework introduced in García-Lapresta and Marques Pereira [22], based on a family of binary aggregation functions satisfying a form of twisted self-duality condition. Each binary aggregation function in that family corresponds to a particular way of combining an aggregation function A with its dual A^* for the construction of the self-dual core $\widehat{A}$. As particular cases of the general framework proposed in Maes *et al.* [33], one obtains García-Lapresta and Marques Pereira's construction, based on the arithmetic mean, and Silvert's construction, based on the symmetric sums formula (see Silvert [41]). However, the dual decomposition framework introduced in García-Lapresta and Marques Pereira [22] remains the only one which preserves stability under translations, a crucial requirement in the present construction of poverty measures.

2.2 OWA operators

In 1988 Yager [46] introduced OWA operators as a tool for aggregating numerical values in multi-criteria decision making. An OWA operator is similar to a weighted mean, but with the values of the variables previously ordered in a decreasing way. Thus, contrary to the weighted means, the weights are not associated with concrete variables and, therefore, they are symmetric. Because of these properties, OWA operators have been widely used in the literature (see, for instance, Yager and Kacprzyk [47] and Yager *et al.* [48]).

Definition 9 *Given a weighting vector* $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$ *satisfying* $\sum_{i=1}^n w_i = 1$*, the* OWA operator associated with $\boldsymbol{w}$ *is the aggregation function* $A_{\boldsymbol{w}} : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$A_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{[i]}.$$

Simple examples of OWA operators are

$$A_{\boldsymbol{w}}(\boldsymbol{x}) = \begin{cases} \max\{x_1, \ldots, x_n\}\,, & \text{when } \boldsymbol{w} = (1, 0, \ldots, 0)\,, \\ \min\{x_1, \ldots, x_n\}\,, & \text{when } \boldsymbol{w} = (0, \ldots, 0, 1\,)\,, \\ \dfrac{x_1 + \cdots + x_n}{n}\,, & \text{when } \boldsymbol{w} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})\,. \end{cases}$$

OWA operators are continuous, idempotent (hence, compensative), symmetric, and stable for translations. Moreover, an OWA operator $A_{\boldsymbol{w}}$ is self-dual if and only if $w_{n+1-i} = w_i$ for every $i \in \{1, \ldots, n\}$ (see García-Lapresta and Llamazares [21, Proposition 5]).

In general, the self-dual core $\widehat{A}_{\boldsymbol{w}}$ and the anti-self-dual remainder $\widetilde{A}_{\boldsymbol{w}}$ of an OWA operator A_w can be written as

$$\widehat{A}_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{w_i + w_{n-i+1}}{2} \, x_{[i]} \quad \text{and} \quad \widetilde{A}_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{w_i - w_{n-i+1}}{2} \, x_{[i]}\,.$$

As we know, the self-dual core $\widehat{A}_{\boldsymbol{w}}$ is an aggregation function. Moreover, since

$$\sum_{i=1}^{n} \frac{w_i + w_{n-i+1}}{2} = 1,$$

the self-dual core $\widehat{A}_{\boldsymbol{w}}$ is again an OWA operator, that is $\widehat{A}_{\boldsymbol{w}} = A_{\widehat{\boldsymbol{w}}}$ with

$$\widehat{w}_i = \frac{w_i + w_{n-i+1}}{2}$$

for every $i \in \{1, \ldots, n\}$. Notice that $\widehat{A}_{\boldsymbol{w}}$ reduces to the arithmetic mean in the simple case $n = 2$, but not in higher dimensions.

The self-dual core and the anti-self-dual remainder can be equivalently written as follows

$$\widehat{A}_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, \frac{x_{[i]} + x_{[n-i+1]}}{2} \quad \text{and} \quad \widetilde{A}_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, \frac{x_{[i]} - x_{[n-i+1]}}{2} .$$

These expressions show clearly that the self-dual core is a weighted average of pairwise averages of $\boldsymbol{x}$ coordinates (*quasi-midranges*), whereas the anti-self-dual remainder is a weighted average of pairwise differences of $\boldsymbol{x}$ coordinates (*quasi-ranges*). The anti-self-dual remainder is therefore independent of the overall average of the coordinates of $\boldsymbol{x}$ and constitutes a form of dispersion measure. Moreover, it is straightforward to prove that $w_1 \geq \cdots \geq w_n$ implies $\widetilde{A}_{\boldsymbol{w}}(\boldsymbol{x}) \geq 0$ and $w_1 \leq \cdots \leq w_n$ implies $\widetilde{A}_{\boldsymbol{w}}(\boldsymbol{x}) \leq 0$.

3 Inequality indices and welfare functions

In this paper we assume the following definitions of inequality index and welfare function.

Definition 10 *An* inequality index *is a function* $I : [0, \infty)^n \longrightarrow [0, \infty)$ *that is continuous and strictly S-convex. The inequality index is* relative *if* I *is scale invariant and* absolute *whenever* I *is invariant for translations.*

Certain properties which can be considered to be inherent to the concept of inequality have come to be accepted as basic properties for an inequality measure. The crucial axiom in this field is the *Pigou-Dalton transfer principle.* This axiom establishes that a progressive transfer, that is, a transfer from a richer person to a poorer one that does not change the relative positions of the donor and the recipient, should decrease inequality. There are alternative ways that guarantee that this property is fulfilled. Marshall and Olkin [34] show that strict S-convexity implies symmetry and inequality reduction under progressive transfers. Conversely, symmetry and inequality reduction under progressive transfer implies strict S-convexity.

Definition 11 *A* welfare function *is a function* $W : [0, \infty)^n \longrightarrow [0, \infty)$ *that is continuous, strictly S-concave and monotonic.*

Similarly to the inequality field, strictly S-concavity is equivalent to symmetry and the increment of the welfare level under progressive transfers. Any welfare function allows the definition of the "equally distributed equivalent income", as the income level that if equally distributed among the population would generate the same value of the W function.

An inequality index is called *ethical* if it implies, and is implied, by a welfare function. If the welfare function W is homothetic, there is a one-to-one relationship between W and a relative inequality index (see Blackorby and Donaldson [6]). Following the Kolm [31], Atkinson [2] and Sen [39] approaches, every relative inequality index, I, may be associated to a homothetic welfare function, $W : [0,\infty)^n \longrightarrow [0,\infty)$, according to the following expression

$$W(\boldsymbol{x}) = \mu(\boldsymbol{x})\,(1 - I(\boldsymbol{x}))\,. \tag{1}$$

Conversely, given a homothetic welfare function we can recover the relative index associated using the above relation. The index $I(\boldsymbol{x})$ gives the fraction of total income that could be saved if society distributed the remaining amount equally without any welfare loss. In other words, it can be interpreted as the proportion of welfare loss due of inequality.

In turn, Kolm [32] and Blackorby and Donaldson [7] approaches allow the derivation of a translatable welfare function from an absolute inequality index according to the following expression

$$W(\boldsymbol{x}) = \mu(\boldsymbol{x}) - I_A(\boldsymbol{x}). \tag{2}$$

This absolute index, I_A, represents the per capita income that could be saved if society distributed incomes equally without any loss of welfare.

Remark 1 If W is a homothetic welfare function (particularly, if W is the welfare function associated with a relative inequality index), it is possible to work in $[0,1]^n$ instead of $[0,\infty)^n$. Given an income distribution $\boldsymbol{x} = (x_1,\ldots,x_n) \in [0,\infty)^n$ such that $\boldsymbol{x} \neq \boldsymbol{0}$, since $\dfrac{1}{x_{[1]}} \cdot \boldsymbol{x} \in [0,1]^n$ and W is homothetic, we have

$$W(\boldsymbol{x}) = W\left(x_{[1]} \cdot \frac{1}{x_{[1]}} \cdot \boldsymbol{x}\right) = x_{[1]} \cdot W\left(\frac{1}{x_{[1]}} \cdot \boldsymbol{x}\right).$$

Obviously, $W(\boldsymbol{0}) = 0$.

A class of welfare functions that will play an important role in this paper is what is referred to as Generalize Gini welfare functions (see Mehran [35], Donaldson and Weymark [15, 16], Weymark [43], Yaari [44, 45], Ebert [18], Quiggin [38] and Ben-Porath and Gilboa [5]).

Definition 12 *Given a weighting vector* $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]$, *with* $0 < w_1 < \cdots < w_n$ *and* $\sum_{i=1}^{n} w_i = 1$, *the* generalized Gini welfare function *(or* rank dependent general welfare function*) associated with* $\boldsymbol{w}$ *is the function* $W_{\boldsymbol{w}} : [0,\infty)^n \longrightarrow [0,\infty)$ *defined as*

$$W_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{[i]}.$$

Positivity of w_i guarantees that $W_{\boldsymbol{w}}$ satisfies the Pareto Principle, that is, it is increasing in x_i. Increasingness of the sequence of coefficients is necessary and sufficient for S-concavity of $W_{\boldsymbol{w}}$. On the other hand, all the functions $W_{\boldsymbol{w}}$ are both, stable for translations and homothetic. Thus, by Remark 1, the generalized Gini welfare functions can be considered as OWA operators, when they are restricted to $[0,1]^n$.

3.1 The Gini index

Corrado Gini introduced in 1912 the now called Gini index ([24]), the most popular measure of inequality. It is based on the average of the absolute differences between all possible pairs of observations. The Gini index is defined as half of the ratio of that average to the mean of the distribution (hence proposing a relative measure of variability). Specifically, for any unordered income distribution the formula given by Gini [24] was

$$G(\boldsymbol{x}) = \frac{1}{2n^2\mu(\boldsymbol{x})} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|. \tag{3}$$

This index varies between 0, which reflects complete equality, and 1. It is relative and invariant under replications of the population, which allows inequality comparisons between societies with different incomes and different populations. Moreover, inequality as measured by this index depends on the significance of the income gaps in society.

Graphically, the Gini index can be computed as twice the area between the line of equality and the Lorenz curve (Gastwirth [23], Kendall and Stuart [30], Dorfman [17]). This curve plots the cumulative income share, ranked in increasing order, on the vertical axis against the distribution of the population on the horizontal axis.

Mehran [35] highlights the linear structure of the index and the implicit weighting scheme involved in (3), that assigns a particular weight to an individual according to his ranking in the income distribution (Sen [39]). In particular, it can be shown than an alternative formula for $G(\boldsymbol{x})$ is

$$G(\boldsymbol{x}) = 1 - \frac{1}{n^2\mu(\boldsymbol{x})} \sum_{i=1}^{n} (2i-1)\, x_{[i]}.$$

See Yitzhaki [49] for alternative formulations of the Gini index.

The decrease of $G(\boldsymbol{x})$ under a progressive transfer does not depend where the transfer takes place as long as it occurs between two persons with a fixed rank difference. In other words, this index is insensitive to the incomes of the individuals involved in the transfers.

When the Gini coefficient is multiplied by the mean income an absolute index is obtained.

Definition 13 *The* absolute Gini inequality index *is defined as*

$$G_A(\boldsymbol{x}) = \mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n} \frac{2i-1}{n}\, x_{[i]}.$$

Remark 2 From (1) and (2), the *Gini welfare function* is simultaneously obtained as

$$W_G(\boldsymbol{x}) = \mu(\boldsymbol{x})(1 - G(\boldsymbol{x})) = \mu(\boldsymbol{x}) - G_A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} \frac{2i-1}{n}\, x_{[i]}.$$

3.2 The Bonferroni index

The Bonferroni index is another example of relative index that has a natural upper bound 1. It is based on the comparison of the partial means and the general mean of an income distribution.

Let us denote by $m_i(\boldsymbol{x})$ the mean income of the $n-i+1$ persons with lowest income, that is

$$m_i(\boldsymbol{x}) = \frac{1}{n-i+1}\sum_{j=i}^{n} x_{[j]}.$$

The Bonferroni index is computed according to the following

$$B(\boldsymbol{x}) = \frac{1}{n\mu(\boldsymbol{x})}\sum_{i=1}^{n} (\mu(\boldsymbol{x}) - m_i(\boldsymbol{x})).$$

Then, $B(\boldsymbol{x})$ represents the amount by which the mean of ratios $m_i(\boldsymbol{x})/\mu(\boldsymbol{x})$ falls short of unity.

B is not invariant for replications. However it fulfils a stronger redistributive criterion than the Pigou-Dalton condition. The decrement in the B index due to a progressive transfer is larger the poorer are the two participants. This property is referred to as the *principle of positional transfer sensitivity* (Mehran [35] and Zoli [50]).

When multiplied by the mean income it becomes an absolute index.

Definition 14 *The* absolute Bonferroni inequality index *is defined as*

$$B_A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n}(\mu(\boldsymbol{x}) - m_i(\boldsymbol{x})) = \mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n} m_i(\boldsymbol{x}).$$

Remark 3 From (1) and (2), the *Bonferroni welfare function* is simultaneously obtained as

$$W_B(\boldsymbol{x}) = \mu(\boldsymbol{x})(1 - B(\boldsymbol{x})) = \mu(\boldsymbol{x}) - B_A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} m_i(\boldsymbol{x}).$$

Proposition 6 *The Bonferroni welfare function is expressed by*

$$W_B(\boldsymbol{x}) = \sum_{i=1}^{n} u_i\, x_{[i]},$$

where $u_i = \displaystyle\sum_{j=n-i+1}^{n} \frac{1}{jn}$, *for* $i = 1, \ldots, n$.

Proof: The derivation is as follows

$$\sum_{i=1}^{n} m_i(\boldsymbol{x}) = \frac{x_{[n]}}{1} + \frac{x_{[n-1]} + x_{[n]}}{2} + \cdots + \frac{x_{[1]} + \cdots + x_{[n]}}{n} =$$

$$= \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) x_{[n]} + \left(\frac{1}{2} + \cdots + \frac{1}{n}\right) x_{[n-1]} + \cdots + \frac{1}{n}\, x_{[1]} =$$

$$= n\sum_{i=1}^{n} u_i\, x_{[i]}. \blacksquare$$

Remark 4 The weights introduced in the previous proposition satisfy the following conditions

1. $0 < u_1 < u_2 < \cdots < u_{n-1} < u_n < 1$.
2. $u_1 = \dfrac{1}{n^2}$ and $u_{i+1} = u_i + \dfrac{1}{(n-i)\, n}$, for $i = 1, \ldots, n-1$.
3. $\displaystyle\sum_{i=1}^{n} u_i = 1$, since

$$\sum_{i=1}^{n}\left(\sum_{j=n-i+1}^{n}\frac{1}{j}\right) = \frac{1}{n} + \left(\frac{1}{n-1} + \frac{1}{n}\right) + \left(\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}\right) +$$

$$+ \cdots + \left(\frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n}\right) + \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n}\right) =$$

$$= 1\,\frac{1}{1} + 2\,\frac{1}{2} + \cdots + (n-2)\,\frac{1}{n-2} + (n-1)\,\frac{1}{n-1} + n\,\frac{1}{n} = n\,.$$

3.3 The De Vergottini index

The De Vergottini index ([14]) captures another aspect of the inequality. It compares the total mean income with the mean of the i-richest person group. If $M_i(\boldsymbol{x})$ denotes the mean income of the i-persons with highest incomes, that is

$$M_i(\boldsymbol{x}) = \frac{1}{i}\sum_{j=1}^{i} x_{[j]},$$

then the De Vergottini index is

$$V(\boldsymbol{x}) = \frac{1}{n\mu(\boldsymbol{x})}\sum_{i=1}^{n}(M_i(\boldsymbol{x}) - \mu(\boldsymbol{x})).$$

With respect to other redistributive criteria, the reduction in the V index due to a progressive transfer is larger the richer are the two participants.

V is also a compromise index in the sense that if multiplied by the mean, then the counterpart absolute index is obtained.

Definition 15 *The* absolute De Vergottini inequality index *is defined as*

$$V_A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n}(M_i(\boldsymbol{x}) - \mu(\boldsymbol{x})) = \frac{1}{n}\sum_{i=1}^{n} M_i(\boldsymbol{x}) - \mu(\boldsymbol{x}).$$

In contrast with the relative Bonferroni index, whose maximum value is

$$B_{max} = \frac{n-1}{n}$$

in correspondence with the income profile in which only one individual accumulates all the income, the De Vergottini index does not have a unit upper bound. The maximum inequality value corresponds to the same income profile as for the Bonferroni index, $x_{[1]} = n\mu(\boldsymbol{x})$, $x_{[2]} = \cdots = x_{[n]} = 0$, but the value is now

$$V_{\max} = \sum_{j=2}^{n}\frac{1}{j}.$$

This value only depends on the population size and may be used to normalize the index. Our proposal is to use the normalization factor

$$c = \frac{n}{n-1}V_{\max},$$

because it ensures that the maximum value of the normalized De Vergottini index, $\overline{V}(\boldsymbol{x}) = V(\boldsymbol{x})/c$, is the same as that of $B(\boldsymbol{x})$, i.e. $(n-1)/n$. Similarly, we denote the absolute normalized De Vergottini index by $\overline{V}_A = V_A/c$.

Remark 5 From (1) and (2), the *normalized De Vergottini welfare function* is simultaneously obtained as

$$W_{\overline{V}}(\boldsymbol{x}) = \mu(\boldsymbol{x})\left(1 - \overline{V}(\boldsymbol{x})\right) = \mu(\boldsymbol{x}) - \overline{V}_A(\boldsymbol{x}) = \frac{c+1}{c}\,\mu(\boldsymbol{x}) - \frac{1}{c\,n}\sum_{i=1}^{n} M_i(\boldsymbol{x}).$$

Remark 6 For $n = 2$, the Gini, Bonferroni and normalized De Vergottini welfare functions coincide:

$$W_G(x_1, x_2) = W_B(x_1, x_2) = W_{\overline{V}}(x_1, x_2) = \frac{x_{[1]} + 3\,x_{[2]}}{4}.$$

However, this fact is not true in higher dimensions. For instance, for $n = 3$ we have

$$W_G(x_1, x_2, x_3) = \frac{10\,x_{[1]} + 30\,x_{[2]} + 50\,x_{[3]}}{90}$$

$$W_B(x_1, x_2, x_3) = \frac{10\,x_{[1]} + 25\,x_{[2]} + 55\,x_{[3]}}{90}$$

$$W_{\overline{V}}(x_1, x_2, x_3) = \frac{10\,x_{[1]} + 34\,x_{[2]} + 46\,x_{[3]}}{90}.$$

Proposition 7 *The weighting scheme implicit in the normalized De Vergottini welfare function* $W_{\overline{V}}$ *is expressed by*

$$\frac{1}{n}\sum_{i=1}^{n} M_i(\boldsymbol{x}) = \sum_{i=1}^{n} v_i\, x_{[i]},$$

where $v_i = \sum_{j=i}^{n} \frac{1}{j\,n}$, *for* $i = 1, \ldots, n$.

Proof: The derivation is as follows

$$\sum_{i=1}^{n} M_i(\boldsymbol{x}) = \frac{x_{[1]}}{1} + \frac{x_{[1]} + x_{[2]}}{2} + \cdots + \frac{x_{[1]} + \cdots + x_{[n]}}{n} =$$

$$= \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) x_{[1]} + \left(\frac{1}{2} + \cdots + \frac{1}{n}\right) x_{[2]} + \cdots + \frac{1}{n}\, x_{[n]} =$$

$$= n \sum_{i=1}^{n} v_i\, x_{[i]}. \blacksquare$$

Remark 7 The weights introduced in the previous proposition satisfy the following conditions

1. $0 < v_n < v_{n-1} < \cdots < v_2 < v_1 < 1$.
2. $v_n = \frac{1}{n^2}$ and $v_{i-1} = v_i + \frac{1}{(i-1)\,n}$, for $i = 2, \ldots, n$.

3. $\sum_{i=1}^{n} v_i = 1$, since

$$\sum_{i=1}^{n}\left(\sum_{j=i}^{n}\frac{1}{j}\right)=\sum_{i=1}^{n}\left(\sum_{j=n-i+1}^{n}\frac{1}{j}\right)=n\,.$$

Remark 8 The normalized De Vergottini welfare function can be written as

$$W_{\overline{V}}(\boldsymbol{x})=\sum_{i=1}^{n} w_i^V x_{[i]} \qquad w_i^V=\frac{c+1-n\,v_i}{c\,n} \qquad i=1,\ldots,n$$

where $\sum_{i=1}^{n} w_i^B = \sum_{i=1}^{n} w_i^V = 1$ and the lowest Bonferroni and De Vergottini weights are $w_1^B = w_1^V = 1/n^2$, since $w_1^B = u_1 = 1/n^2$ and

$$w_1^V=\frac{c+1-n\,v_1}{c\,n}=\frac{n(nv_1-1))+(1-n\,v_1)(n-1)}{n(nv_1-1)n}=\frac{1}{n^2}$$

where we have used that $c=\dfrac{n(nv_1-1)}{n-1}$.

3.4 Orness of the Gini, Bonferroni and normalized De Vergottini welfare funcions

The notion of *orness* (or *attitudinal character*) of OWA operators was introduced by Yager [46] for reflecting the andlike or orlike aggregation behavior of OWA operators.

Definition 16 *Let $A_{\boldsymbol{w}}$ the OWA operator associated with the weighting vector $\boldsymbol{w}=(w_1,\ldots,w_n)\in[0,1]^n$. The* orness *of $A_{\boldsymbol{w}}$ is defined by*

$$A^o_{\boldsymbol{w}}=\frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i.$$

Remark 9 The orness of $A_{\boldsymbol{w}}$ coincides with the value $A_{\boldsymbol{w}}(\boldsymbol{x}^o)$, where $\boldsymbol{x}^o_i=\dfrac{n-i}{n-1}$, i.e.,

$$A^o_{\boldsymbol{w}}=w_1+w_2\,\frac{n-2}{n-1}+\cdots+w_{n-1}\,\frac{1}{n-1}.$$

The orness of the extreme OWA operators maximum, arithmetic mean and minimum are 1, 0.5 and 0, respectively:

1. $A_{\boldsymbol{w}}(\boldsymbol{x})=\max\{x_1,\ldots,x_n\}$, where $\boldsymbol{w}=(1,0,\ldots,0)$: $A^o_w=1$.

2. $A_{\boldsymbol{w}}(\boldsymbol{x})=\dfrac{x_1+\cdots+x_n}{n}$, where $\boldsymbol{w}=\left(\dfrac{1}{n},\dfrac{1}{n},\ldots,\dfrac{1}{n}\right)$: $A^o_w=\dfrac{1}{2}$.

3. $A_{\boldsymbol{w}}(\boldsymbol{x}) = \min\{x_1, \ldots, x_n\}$, where $\boldsymbol{w} = (0, \ldots, 0, 1)$: $A^o_{\boldsymbol{w}} = 0$.

Proposition 8 *The orness of the Gini welfare function is* $W^o_G = \dfrac{1}{3} - \dfrac{1}{6n}$.

Proof: From the definition

$$W_G(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} \frac{2i-1}{n}\, x_{[i]}$$

and since $W^o_G = W_G(\boldsymbol{x}^o)$ with $x^o_{[i]} = \dfrac{n-i}{n-1}$, we obtain

$$W^o_G = W_G(\boldsymbol{x}^o) = \sum_{i=1}^{n} w^G_i x^o_{[i]} = \frac{1}{n} \sum_{i=1}^{n} \frac{2i-1}{n}\, \frac{n-i}{n-1} =$$

$$= \frac{1}{(n-1)\,n^2} \left(-n^2 \;+\; (2n+1) \sum_{i=1}^{n} i \;-\; 2 \sum_{i=1}^{n} i^2 \right) =$$

$$= \frac{1}{(n-1)\,n^2} \left(-n^2 \;+\; (2n+1)\, \frac{n(n+1)}{2} \;-\; 2\, \frac{n(n+1)(2n+1)}{6} \right) =$$

$$= \frac{2n-1}{6n} = \frac{1}{3} - \frac{1}{6n},$$

where we have used $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. ■

Proposition 9 *The orness of the Bonferroni welfare function is* $W^o_B = \dfrac{1}{4}$.

Proof: From the definition

$$W_B(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} m_i(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n-i+1} \sum_{j=i}^{n} x_{[j]}$$

and since $W_B^o = W_B(\boldsymbol{x}^o)$ with $x_{[i]}^o = \dfrac{n-i}{n-1}$, we obtain

$$
\begin{aligned}
W_B^o &= W_B(\boldsymbol{x}^o) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n-i+1}\sum_{j=i}^{n}\frac{n-j}{n-1} = \\
&= \frac{1}{n(n-1)}\sum_{i=1}^{n}\frac{1}{n-i+1}\left(n(n-i+1)-\sum_{j=i}^{n} j\right) = \\
&= \frac{1}{n(n-1)}\sum_{i=1}^{n}\frac{1}{n-i+1}\left(n(n-i+1)-\frac{(n-i+1)(n+i)}{2}\right) = \\
&= \frac{1}{n(n-1)}\sum_{i=1}^{n}\left(\frac{n}{2}-\frac{i}{2}\right) = \frac{1}{n(n-1)}\left(\frac{n^2}{2}-\frac{n(n+1)}{4}\right) = \\
&= \frac{1}{n(n-1)}\,\frac{n(n-1)}{4} = \frac{1}{4},
\end{aligned}
$$

where we have used that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proposition 10 *The orness of the normalized De Vergottini welfare function is* $W_{\overline{V}}^o = \dfrac{1}{2} - \dfrac{1}{4\,c}$.

Proof: From the definition

$$
W_{\overline{V}}(\boldsymbol{x}) = \frac{c+1}{c}\,\mu(\boldsymbol{x}) - \frac{1}{c\,n}\sum_{i=1}^{n} M_i(\boldsymbol{x}) = \frac{c+1}{c}\,\mu(\boldsymbol{x}) - \frac{1}{c\,n}\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i} x_{[j]}\right)
$$

and since $W^o_{\overline{V}} = W_{\overline{V}}(\boldsymbol{x}^o)$ with $x^o_{[i]} = \dfrac{n-i}{n-1}$, we obtain

$$
\begin{aligned}
W^o_{\overline{V}} = W_{\overline{V}}(\boldsymbol{x}^o) &= \frac{c+1}{c}\,\mu(\boldsymbol{x}^o) - \frac{1}{c\,n}\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i}\frac{n-j}{n-1}\right) = \\
&= \frac{c+1}{c}\,\frac{1}{2} - \frac{1}{c\,n(n-1)}\sum_{i=1}^{n}\left(n - \frac{1}{i}\sum_{j=1}^{i} j\right) = \\
&= \frac{c+1}{2\,c} - \frac{1}{c\,n(n-1)}\sum_{i=1}^{n}\left(n - \frac{i+1}{2}\right) = \\
&= \frac{c+1}{2\,c} - \frac{1}{c\,n(n-1)}\sum_{i=1}^{n}\left(\frac{2n-1}{2} - \frac{i}{2}\right) = \\
&= \frac{c+1}{2\,c} - \frac{1}{c\,n(n-1)}\left(\frac{n(2n-1)}{2} - \frac{n(n+1)}{4}\right) = \\
&= \frac{c+1}{2\,c} - \frac{1}{c\,n(n-1)}\,\frac{3n(n-1)}{4} = \frac{c+1}{2\,c} - \frac{3}{4\,c} = \frac{2c-1}{4\,c} = \\
&= \frac{1}{2} - \frac{1}{4\,c},
\end{aligned}
$$

where we have used that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\mu(\boldsymbol{x}^o) = \dfrac{1}{2}$.

4 Dual decomposition of Gini, Bonferroni and normalized De Vergottini welfare functions

This section identifies the self-dual core and the anti-self-dual remainder of the Gini, Bonferroni and normalized De Vergottini welfare functions and highlights the relationships among them.

4.1 The Gini welfare function

Proposition 11 *The absolute Gini inequality index is anti-self-dual, i.e., it satisfies* $G_A(\mathbf{1} - \boldsymbol{x}) = G_A(\boldsymbol{x})$*, for every* $\boldsymbol{x} \in [0,1]^n$.

Proof: The derivation is as follows:

$$G_A(\mathbf{1}-\boldsymbol{x}) = \mu(\mathbf{1}-\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}\,(1-x)_{[i]} =$$

$$= 1-\mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n}\frac{2(n-i+1)-1}{n}\,\left(1-x_{[i]}\right) =$$

$$= 1-\mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n}\left(2-\frac{2i-1}{n}\right)\left(1-x_{[i]}\right) =$$

$$= 1-\mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n}2\left(1-x_{[i]}\right) + \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}\,\left(1-x_{[i]}\right) =$$

$$= 1-\mu(\boldsymbol{x}) - 2 + 2\,\mu(\boldsymbol{x}) + 1 - \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}\,x_{[i]} =$$

$$= \mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}\,x_{[i]} = G_A(\boldsymbol{x}),$$

where we have used that $\displaystyle\sum_{i=1}^{n}\frac{2i-1}{n} = n.$ ■

Remark 10 Since $W_G(\mathbf{1}-\boldsymbol{x}) = \mu(\mathbf{1}-\boldsymbol{x}) - G_A(\mathbf{1}-\boldsymbol{x}) = 1-\mu(\boldsymbol{x}) - G_A(\boldsymbol{x})$, the dual Gini welfare function can be written as $W_G^*(\boldsymbol{x}) = 1 - W_G(\mathbf{1}-\boldsymbol{x}) = \mu(\boldsymbol{x}) + G_A(\boldsymbol{x})$.

Proposition 12 *The self-dual core of the Gini welfare function is the arithmetic mean.*

Proof: Taking into account Remark 10, we have

$$\widehat{W}_G(\boldsymbol{x}) = \frac{W_G(\boldsymbol{x}) + W_G^*(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - G_A(\boldsymbol{x}) + \mu(\boldsymbol{x}) + G_A(\boldsymbol{x})}{2} = \mu(\boldsymbol{x}).\blacksquare$$

Proposition 13 *The anti-self-dual remainder of the Gini welfare function is minus the absolute Gini index.*

Proof: Taking into account Remark 10, we have

$$\widetilde{W}_G(\boldsymbol{x}) = \frac{W_G(\boldsymbol{x}) - W_G^*(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - G_A(\boldsymbol{x}) - \mu(\boldsymbol{x}) - G_A(\boldsymbol{x})}{2} = -G_A(\boldsymbol{x}).\blacksquare$$

4.2 The Bonferroni and De Vergottini welfare functions

Proposition 14 *The duality relation between the absolute Bonferroni and the absolute De Vergottini inequality indices is expressed by*

$$B_A(\mathbf{1}-\boldsymbol{x}) = V_A(\boldsymbol{x}) \quad \textit{and} \quad V_A(\mathbf{1}-\boldsymbol{x}) = B_A(\boldsymbol{x}),$$

for every $\boldsymbol{x} \in [0,1]^n$

PROOF: The derivation is as follows:

$$B_A(\mathbf{1}-\boldsymbol{x}) = \mu(\mathbf{1}-\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n} m_i(\mathbf{1}-\boldsymbol{x}) = (1-\mu(\boldsymbol{x})) - \frac{1}{n}\sum_{i=1}^{n}(1-M_i(\boldsymbol{x})) =$$
$$= 1-\mu(\boldsymbol{x}) - 1 + \frac{1}{n}\sum_{i=1}^{n} M_i(\boldsymbol{x}) = V_A(\boldsymbol{x}).$$

On the other hand, $V_A(\mathbf{1}-\boldsymbol{x}) = B_A(\mathbf{1}-(\mathbf{1}-\boldsymbol{x})) = B_A(\boldsymbol{x})$. ■

Remark 11 Since $W_B(\mathbf{1}-\boldsymbol{x}) = \mu(\mathbf{1}-\boldsymbol{x}) - B_A(\mathbf{1}-\boldsymbol{x}) = 1-\mu(\boldsymbol{x}) - V_A(\boldsymbol{x})$, the dual Bonferroni welfare function can be written as $W_B^*(\boldsymbol{x}) = 1 - W_B(\mathbf{1}-\boldsymbol{x}) = \mu(\boldsymbol{x}) + V_A(\boldsymbol{x})$.

Proposition 15 *The self-dual core and the anti-self-dual remainder of the Bonferroni welfare function are given by*

$$\widehat{W}_B(\boldsymbol{x}) = \mu(\boldsymbol{x}) - \frac{B_A(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2} \quad \textit{and} \quad \widetilde{W}_B(\boldsymbol{x}) = -\frac{B_A(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2}.$$

PROOF: Taking into account Remark 11, the derivations are as follows:

$$\widehat{W}_B(\boldsymbol{x}) = \frac{W_B(\boldsymbol{x}) + W_B^*(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - B_A(\boldsymbol{x}) + \mu(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2} =$$
$$= \mu(\boldsymbol{x}) - \frac{B_A(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2}.$$

$$\widetilde{W}_B(\boldsymbol{x}) = \frac{W_B(\boldsymbol{x}) - W_B^*(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - B_A(\boldsymbol{x}) - \mu(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2} =$$
$$= -\frac{B_A(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2}. ■$$

Remark 12 Since

$$W_{\overline{V}}(\mathbf{1}-\boldsymbol{x}) = \mu(\mathbf{1}-\boldsymbol{x}) - \frac{V_A(\mathbf{1}-\boldsymbol{x})}{c} = 1 - \mu(\boldsymbol{x}) - \frac{B_A(\boldsymbol{x})}{c},$$

the dual normalized De Vergottini welfare function can be written as

$$W^*_{\overline{V}}(\boldsymbol{x}) = 1 - W_{\overline{V}}(\mathbf{1}-\boldsymbol{x}) = \mu(\boldsymbol{x}) + \frac{B_A(\boldsymbol{x})}{c}.$$

Proposition 16 *The self-dual core and the anti-self-dual remainder of the normalized De Vergottini welfare function are given by*

$$\widehat{W}_{\overline{V}}(\boldsymbol{x}) = \mu(\boldsymbol{x}) + \frac{B_A(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2\,c} \quad \text{and} \quad \widetilde{W}_{\overline{V}}(\boldsymbol{x}) = -\frac{B_A(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2\,c}.$$

Proof: Taking into account Remark 12, the derivations are as follows:

$$\widehat{W}_{\overline{V}}(\boldsymbol{x}) = \frac{W_{\overline{V}}(\boldsymbol{x}) + W^*_{\overline{V}}(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - \frac{1}{c}V_A(\boldsymbol{x}) + \mu(\boldsymbol{x}) + \frac{1}{c}B_A(\boldsymbol{x})}{2} =$$

$$= \mu(\boldsymbol{x}) + \frac{B_A(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2\,c}.$$

$$\widetilde{W}_{\overline{V}}(\boldsymbol{x}) = \frac{W_{\overline{V}}(\boldsymbol{x}) - W^*_{\overline{V}}(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - \frac{1}{c}V_A(\boldsymbol{x}) - \mu(\boldsymbol{x}) - \frac{1}{c}B_A(\boldsymbol{x})}{2} =$$

$$= -\frac{B_A(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2\,c}. \blacksquare$$

It may be worth noting the dual behavior of the decomposition components for the Bonferroni and the normalized De Vergottini welfare functions. On the one hand, the anti-self-dual remainders are equal but for the normalization constant. The role played by the absolute Gini index in the anti-self-dual remainder of the Gini welfare function, is replaced now by an average of the respective absolute indices. As regards the self-dual cores, the components are completely symmetric but, once again, the normalization constant.

5 Concluding remarks

We have examined the dual decomposition of the OWA welfare functions associated with the Gini, Bonferroni, and De Vergottini indices in the standard framework of aggregation functions on the $[0,1]^n$ domain. The dual decomposition highlights the distinct and complementary nature of the three classical inequality indices. In the Gini index case, the central result is $G_A(\mathbf{1}-\boldsymbol{x}) = G_A(\boldsymbol{x})$ and the dual decomposition reproduces in a natural way the canonical construction of the associated welfare function. In the Bonferroni and De Vergottini cases, the central result is $B_A(\mathbf{1}-\boldsymbol{x}) = V_A(\boldsymbol{x})$ (and vice-versa) and the natural dual relationship between the two indices emerges very clearly in the way the self-dual cores and anti-self-dual remainders of the two welfare functions combine the two inequality indices. An appropriate normalization of the De Vergottini index is considered. Finally, the orness of the welfare functions associated with three classical inequality indices has been computed, obtaining values in the $(0, 1/2)$ interval due to the common emphasis on poorer incomes. In the large population asymptotic limit, the orness values of the Gini (1/3), Bonferroni (1/4), and De Vergottini (1/2) welfare functions recall the character of the associated classical inequality indices and constitute further evidence of the duality pattern illustrated by the dual decomposition.

Acknowledgments

O. Aristondo and C. Lasso de la Vega gratefully acknowledge the funding support of the Spanish Ministerio de Ciencia e Innovación (Project ECO2009-11213), and the Basque Departamento de Educación e Investigación (Project GIC07/146-IT-377-07). J.L. García-Lapresta gratefully acknowledges the funding support of the Spanish Ministerio de Ciencia e Innovación (Project ECO2009-07332) and ERDF.

References

[1] O. Aristondo, J.L. García-Lapresta, C. Lasso de la Vega, R.A. Marques Pereira, The Gini index, the dual decomposition of aggregation functions, and the consistent measurement of inequality, International Journal of Intelligent Systems 27 (2012) 132–152.

[2] A.B. Atkinson, On the measurement of inequality, Journal of Economic Theory 2 (1970) 244–263.

[3] E. Bárcena, L. Imedio, The Bonferroni, Gini and De Vergottini indices. Inequality, welfare and deprivation in the European Union in 2000. Research on Economic Inequality 16 (2008) 231–257.

[4] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer, Heidelberg, 2007.

[5] E. Ben Porath, I. Gilboa, Linear measures, the Gini index and the income-equality tradeoff, Journal of Economic Theory 64 (1994) 443–467.

[6] C. Blackorby, D. Donaldson, Measures of relative equality and their meaning in terms of social welfare, Journal of Economic Theory 18 (1978) 59–80.

[7] C. Blackorby, D. Donaldson, A theoretical treatment of indices of absolute inequality, International Economic Review 21 (1980) 107–136.

[8] C. Bonferroni, Elementi di Statistica Generale, Libreria Seber, Firenze, 1930.

[9] W. Bossert, An axiomatization of the single-series Ginis, Journal of Economic Theory 50 (1990) 82–92.

[10] T. Calvo, A. Kolesárova, M. Komorníková, R. Mesiar, Aggregation operators: Properties, classes and construction methods, in: T. Calvo, G. Mayor, R. Mesiar (Eds.), Aggregation Operators: New Trends and Applications, Physica-Verlag, Heidelberg, 2002, pp. 3–104.

[11] S.R. Chakravarty, A deprivation-based axiomatic characterization of the absolute Bonferroni index of inequality, Journal of Economic Inequality 5 (2007) 339–351.

[12] S.R. Chakravarty, Inequality, Polarization and Poverty: Advances in Distributional Analysis, Springer-Verlag, New York, 2009.

[13] S.R. Chakravarty, P. Muliere, Welfare indicators: a review and new perspectives. Measurement of inequality, Metron-International Journal of Statistics 61 (2003) 457–497.

[14] M. De Vergottini, Sugli indici di concentrazione, Statistica 10 (1940) 445–454.

[15] D. Donaldson, J. Weymark, A single-parameter generalization of the Gini indices of inequality, Journal of Economic Theory 22 (1980) 67–86.

[16] D. Donaldson, J. Weymark, Ethically flexible Gini indices for income distributions in the continuum, Journal of Economic Theory 29 (1983) 353–358.

[17] R. Dorfman, A formula for the Gini coefficient, The Review of Economics and Statistics 61 (1979) 146–149.

[18] U. Ebert, Measurement of inequality: an attempt at unification and generalization, Social Choice and Welfare 5 (1988) 147–169.

[19] J. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer Academic Publishers, Dordrecht, 1994.

[20] J.L. García-Lapresta, C. Lasso de la Vega, R.A. Marques Pereira, A.M. Urrutia, A class of poverty measures induced by the dual decomposition of aggregation functions, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 18 (2010) 493–511.

[21] J.L. García-Lapresta, B. Llamazares, Majority decisions based on difference of votes, Journal of Mathematical Economics 45 (2001) 463–481.

[22] J.L. García-Lapresta, R.A. Marques Pereira, The self-dual core and the anti-self-dual remainder of an aggregation operator, Fuzzy Sets and Systems 159 (2008) 47–62.

[23] J.L. Gastwirth, The estimation of the Lorenz curve and Gini index, The Review of Economics and Statistics 54 (1972) 306–316.

[24] C. Gini, Variabilità e Mutabilità, Tipografia di Paolo Cuppini, Bologna, 1912.

[25] G.M. Giorgi, A methodological survey of recent studies for the measurement of inequality of economic welfare carried out by some Italian statisticians, Economic Notes 13 (1984) 145–157.

[26] G.M. Giorgi, Concentration index, Bonferroni, in: Encyclopedia of Statistical Sciences, Wiley, New York, 1998, pp. 141–146.

[27] G.M. Giorgi, M. Crescenzi, A look at the Bonferroni inequality measure in a reliability framework, Statistica 41 (2001) 571–573.

[28] G.M. Giorgi, R. Mondani, Sampling distribution of the Bonferroni inequality index from exponential population, The Indian Journal of Statistics, Series B 57 (1995) 10–18.

[29] M. Grabisch, J.L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, Cambridge, 2009.

[30] L. Kendall, A. Stuart, The Advanced Theory of Statistics (4th edition), volume 1: Distribution Theory, Charles Griffin & Company Limited, London & High Wycombe, 1977.

[31] S.C. Kolm, The optimal production of social justice, in: J. Margolis, H. Guitton (Eds.), Publics Economics, Macmillan, London, 1969, pp. 145–200.

[32] S.C. Kolm, Unequal inequalities I, Journal of Economic Theory, 12 (1976) 416–442.

[33] K. Maes, S. Saminger, B. De Baets, Representation and construction of self-dual aggregation operators, European Journal of Operational Research 177 (2007) 472–487.

[34] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and Its Applications, Mathematics in Science and Engineering, volume 143, Academic Press, New York, 1979.

[35] F. Mehran, Linear measures of income inequality, Econometrica 44 (1976) 805–809.

[36] F. Nygard, A. Sandstrom, Measuring Income Inequality, Almqvist and Wicksell, Stockholm, 1981.

[37] W. Piesch, Bonferroni-index und De Vergottini-index, Diskussionspapiere aus dem Institutfür Volkswirtschaftslehre der Universität Hohenheim, volume 259, Department of Economics, University of Hohenheim, Germany, 2005.

[38] J. Quiggin, Generalized expected utility theory: the rank-dependent model, Mathematical Social Sciences 1 (1993) 409–430.

[39] A.K. Sen, On Economic Inequality, Oxford University Press, Oxford, 1973.

[40] J. Silber, Handbook on Income Inequality, Kluwer Academic Press, Boston, 1999.

[41] W. Silvert, Symmetric summation: A class of operations on fuzzy sets, IEEE Transactions on Systems, Man, and Cybernetics 9 (1979) 657–659.

[42] A. Tarsitano, The Bonferroni index of income inequality, in: C. Dagum, M. Zenga (Eds.), Income and Wealth Distribution, Inequality and Poverty, Springer-Verlag, Berlin, 1990, pp. 228–242.

[43] J. Weymark, Generalized Gini inequality indices, Mathematical Social Sciences 1 (1981) 409–430.

[44] M.E. Yaari, The dual theory of choice under risk, Econometrica 55 (1987) 95–115.

[45] M.E. Yaari, A controversial proposal concerning inequality measurement, Journal of Economic Theory 44 (1988) 381–397.

[46] R.R. Yager, Ordered weighted averaging operators in multicriteria decision making, IEEE Transactions on Systems, Man and Cybernetics 8 (1988) 183–190.

[47] R.R. Yager, J. Kacprzyk (Eds.), The Ordered Weighted Averaging Operators: Theory and Applications, Kluwer Academic Publishers, Norwell, 1997.

[48] R.R. Yager, J. Kacprzyk, G. Beliakov (Eds.), Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice, Springer-Verlag, Berlin, 2011.

[49] S. Yitzhaki, More than a dozen alternative ways of spelling Gini, Research on Economic Inequality 8 (1998) 13–30.

[50] C. Zoli, Intersecting generalized Lorenz curves and the Gini index, Social Choice and Welfare 16 (1999) 183–196.