Working Paper Series

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ECINEQ WP 2012 – 275

ECINEQ 2012 – 275
November 2012

www.ecineq.org

Multidimensional Gini Indices, Weak Pigou-Dalton Bundle Dominance and Comonotonizing Majorization

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Abstract

This paper considers the problem of constructing a normatively significant multidimensional Gini index of relative inequality. The social evaluation relation (SER) from which the index is derived is required to satisfy a weak version of the Pigou-Dalton Bundle Principle (WPDBP) (rather than Uniform Majorization or similar conditions). It is also desired to satisfy a weak form of the condition of Correlation Increasing Majorization called Comonotonizing Majorization (CM).

The problem of measuring multidimensional inequality is here interpreted to be essentially a problem of setting weights on the different attributes. It is argued that determination of these weights is linked to the problem of determining the weights of the individuals. A number of conditions on the two sets of weights and on their interrelationships are proposed. By combining these conditions with a social evaluation function which is decomposable between equality and efficiency components we obtain a specific SER. The Kolm index derived from this relation is then suggested as the multidimensional inequality index.

It is shown that the proposed index is a multidimensional Gini index satisfying the (inequality index versions of the) properties of WPDBP and CM. The index does not seem to have appeared in the literature before. Moreover, the literature does not seem to contain any other normatively significant multidimensional Gini index that would satisfy both of these properties if the allocation matrices are not restricted to be strictly positive. In this paper this restriction has been relaxed on grounds of potential empirical applicability of the index.

Keywords: multidimensional inequality, Gini index, social evaluation.

JEL Classification: D63.

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1. Introduction

It is now generally recognised that the well-being of an individual depends not only on the individual's income but also on such non-income attributes as health, education etc. and that, therefore, the methods for measuring inequality in the distribution of standard of living in a society need to be extended from the unidimensional to the multidimensional context. This paper is concerned with the extension of the most widely used undimensional inequality index (viz. the Gini) in this direction. We shall be concerned with *relative* inequality.

The normative theory of unidimensional inequality measurement owes its existence to the works of Atkinson (1970), Kolm (1969) and Sen (1973). The pioneering attempts to extend the theory to the multidimensional context were the papers by Kolm (1977) and Atkinson and Bourguignon (1982). For an insightful survey of subsequent research in the field see Weymark (2006). In the specific area of multidimensional Gini indices Gajdos and Weymark (2005) and Decancq and Lugo (2009) are two of the recent important contributions.

Since multiplicity of attributes is the central issue here, one possible approach to the problem is to interpret it as one of determining the relative importances (i.e. the weights) that are to be attached to the different attributes. In this connection it is frequently assumed that the society somehow reaches a consensus regarding the weights on the basis of broad-based discourses. Since this assumption is often insufficient to give practical hints regarding specific values of the weights, one trend in the empirical literature has been to use weights which are computationally simple (for instance, equal weights). However, various other weights have also been used. For a comprehensive survey see Decancq and Lugo (2012).

The theoretical literature has taken the logically more satisfactory route of deriving inequality indices from systems of axioms. In many cases, however, this approach leads to a class of indices rather than to a single index. Typically, the different members of a class correspond to different values of the weights on the attributes and on the individuals. This would not be a major problem in practical applications if it were the case that the different members of a given class were at least ordinally equivalent (i.e. that a given pair of societies were ordered in the same way in terms of inequality by different indices in the class). For many (if not most) of the multidimensional indices axiomatised in the literature, however, this is not the case. Yet the axioms provide no clue as to which specific member of the implied class (i.e, which specific weighting scheme) is to be used. The empirical researcher in search of a numerical specification of the weights is, therefore, again assumed to be guided by a social consensus.

This paper seeks narrower restrictions on the weights by means of conditions imposed on them. The work reported here can be interpreted to be located within the social consensus approach. However, we take a closer look at the informational basis of the consensus on the weights and investigate the consequences of some simple assumptions in this regard.

We shall proceed by assuming that that there is a given list of relevant attributes. Formally, the preparation of the list may be looked upon as a part of the weight setting exercise since attributes which are *excluded* may be interpreted to be assigned zero weights. However, we shall not go into the question how the 'relevance' of an attribute is decided. We assume that the included attributes are basic determinants of the 'capabilities' of the individuals, to use the now standard terminology due to Sen. (See, for instance, Sen (1985)). In a given society there is likely to be a broad unanimity of opinion as to which attributes have these characteristics.

This implies that all attributes included in the list should be given *positive* weights. The determination of these weights, however, remains a complex problem. We shall assume that in setting the weights on the attributes the society is guided by the relative contributions of the attributes to aggregate well-being. But the problem of determining these relative contributions is inseparably linked to that of determining the relative importances of the *individuals*. In conformity with unidimensional theory, the contribution of a given attribute to aggregate well-being can be looked upon as a function of the weighted sum of the levels of well-being of the different individuals w.r.t. the attribute, the weights here being the indicators of the relative importances of the individuals. However, since all individuals are to be treated symmetrically, the weight assigned to an individual in this context should depend (exclusively) on the individual's well-being rather than on his or her personal identity or other characteristics. The problem is that in the multidimensional framework each individual's well-being should depend on the weighted sum of his or her levels of well-being w.r.t. the different attributes and cannot be determined without knowledge about the weights on the attributes.

In this paper the weights of the attributes as well as those of the individuals are functions of the individuals' allocations of the attributes. We shall make simplifying assumptions on the nature of these functions without diluting on the core issue of *interdependence* between the two sets of weights.

The weights will then be used to define a social evaluation relation (SER) on the set of alternative patterns of allocation of the attributes. We shall assume that the SER is representable by a social evaluation function (SEF) and that the SEF is separable into an equity and an efficiency component. A relative inequality index will then be obtained from the SER by standard methods. As stated above, we shall be interested in obtaining multidimensional Gini indices of relative inequality.

We shall, however, seek an SER satisfying two conditions (apart from other standard requirements) which in our opinion seem to have intuitive appeal. One of these conditions is related to the need to make the SER distribution-sensitive. In the context of inequality measurement, the need to ensure such sensitivity by formulating a multidimensional version of the well-known Pigou-Dalton transfer (PDT) condition of unidimensional theory is obvious. However, the literature contains different suggestions regarding the ways of doing so. Until recently the standard procedure has been to use either the Uniform Majorization (UM) condition formulated by Kolm (1977) or some of its variants. Recently, however, attention has been drawn to some limitations of UM and similar conditions. For discussion see, for instance, Lasso de la Vega et. al. (2010). These authors have replaced UM by the Pigou-Dalton Bundle Principle (PDBP) introduced by Fleurbaey and Trannoy (2003) for the purpose of deriving inequality indices. In this paper we shall impose on the SER a weaker form of PDBP to be called the Weak Pigou-Dalton Bundle Principle (WPDBP).

The other condition, to be called Comonotonizing Majorization (CM), is a weaker variant of the so called condition of Correlation Increasing Majorisation (CIM). The idea behind CIM is that the SER and, therefore, the inequality index derived from it, should take into account the pattern of dependence between the different attribute distributions. In particular, a greater degree of dependence should be socially undesirable. (See Atkinson and Bourguignon (1982).) CIM, introduced by Tsui (1999) as a condition on the inequality index, is one way of ensuring this. CM, a condition on the SER, is based on similar intuition but is technically weaker.

Properties of an SER carry over naturally into analogous properties of a derived inequality index and are often given the same names and acronyms. Thus we shall also talk about inequality indices satisfying WPDBP and CM. It will be clear from the context whether the conditions refer to an inequality index or to the underlying SER.

Among the multidimensional inequality indices appearing in the literature, the class of Gini indices derived in Gajdos and Weymark (2005) can be shown to satisfy WPDBP. However, it does not satisfy CM. While List (1999) and Banerjee (2010) propose Gini indices satisfying CIM (and, therefore, CM), these papers share the common feature that these Gini indices are introduced in an ad hoc fashion without deriving them from an SER. Moreover, all members of the suggested classes do not satisfy WPDBP.

Decancq and Lugo (forthcoming) derive two different classes of Gini indices from SER's and also implement them empirically. One of these classes is a subclass of the Gajdos-Weymark class referred to above and, therefore, would violate CM while satisfying WPDBP. The other class derived there can be shown to satisfy both WPDBP and CM if it is assumed that all individuals are allocated strictly positive amounts of all the attributes. However, this assumption is restrictive; and if it is relaxed, some members of this class would violate both WPDBP and CM.

On the other hand, the contributions by Lasso de la Vega et. al. (2010) and Tsui (1999) are concerned with derivation of 'Generalized Entropy' (rather than Gini) classes of inequality indices.

Thus, the existing literature does not seem to contain a normatively significant multidimensional Gini index satisfying WPDBP and CM. In this paper we seek such an index.

Section 2 below is devoted to introducing the notations and developing the formal definitions of a multidimensional Gini SER and of a multidimensional Gini index of relative inequality.

It is in Section 3 that we come to the matter of weights. We derive a specific SER from conditions imposed on the weights and from the decomposability assumption. We then derive the proposed Gini index. Section 4 concludes the discussion.

2. Notations, Definitions etc.

Consider a society consisting of n individuals. The standard of living of each individual is considered to depend on his or her levels of m attributes. n and m are positive integers. Since we shall be concerned with distributional issues, assume that $n \geq 2$. The set of individuals $\{1,2,.....,n\}$ and the set of attributes $\{1,2,......,m\}$ will be denoted by N and M respectively.

Let $X = ((x_p^j))$ be the $n \times m$ matrix in which x_p^j denotes the amount of the j-th attribute ($j = 1,2, \ldots\ldots,m$) allocated to the p-th individual ($p = 1,2, \ldots..n$). X will be called an *allocation matrix*. Some contributions to the literature assume that X is positive. This simple assumption may turn out to be restrictive from the empirical point of view. For instance, in any society, especially a less developed one, there may be individuals who are without education. Therefore, if education is one of the attributes, X will fail to be positive. Similarly, if access to housing facility is an attribute, existence of the homeless would cause problems.

We shall assume that X is non-negative. However, each column of X is assumed to have at least one positive entry since inclusion of an attribute in the analysis is meaningful only if there is a positive total amount of the attribute. Let **X** denote the class of all allocation matrices with these characteristics. $\mathbf{x}_p$ will denote the p-th row of X , $p = 1,2,\ldots.., n$, and $\mathbf{x}^j$ will denote its j-th column, $j = 1,2,\ldots..,m$.

We shall be concerned with an inequality index derived from social evaluation of allocation matrices. A *social evaluation relation* is a binary relation R on the set **X**. For any X and Y in **X**, X R Y is interpreted to mean that X is *weakly socially preferred* to Y. The asymmetric and symmetric factors of R will be denoted by P and I respectively.

We shall impose a number of conditions on R. The first few of these are standard assumptions in the existing literature.

Continuity (CONT): The relation R is continuous.

(ORD): R is an ordering.

Monotonicity under Equality (ME): For all X in **X**, let X_μ denote the matrix obtained by replacing each entry in X by the arithmetic mean of the column containing it. If X and Y in **X** are such that $X_\mu = X$ and $Y_\mu = Y$, then X P Y if [$X \geq Y$ and $X \neq Y$].

Kolm Monotonicity (KMON): There exists a mapping f_R: $\mathbf{X} \rightarrow \Re_{++}$ such that, for all X in **X**, $(f_R(X)X_\mu)$ I X.

Anonymity (ANON): For all X and Y in **X** such that Y is obtained by a permutation of the rows of X, X *I* Y.

Population Replication Invariance (PRI): For all X and Y in **X** such that Y is obtained by a *k*-fold replication of the population in X for some positive integer *k* i.e., for all *p* in *N*,

$$\mathbf{x}_p = \mathbf{y}_p = \mathbf{y}_{n+p} = \ldots\ldots = \mathbf{y}_{n(k-1)+p},$$

X *I* Y.

CONT is a "no jump" condition. It requires that, for all X, Y and Z in **X**, if X *P* Y and Z is close to Y, then X *P* Z. Similarly, if X *P* Y and Z is close to X, then Z *P* Y. **ORD** requires that *R* is reflexive, complete and transitive. We shall not assume that *R* is necessarily monotonic. (In this respect we follow Weymark (1981) where the unidimensional case (m = 1) is considered and the discussion is couched in terms of a social evaluation function representing *R*.) The Monotonocity (MON) condition which has been used in the literature states that, for all X and Y in **X** such that X ≥ Y and X ≠ Y, X *P* Y. In the present context this condition appears to be too strong. A *ceteris paribus* increase in an entry in X may increase inequality though it increases the total allocation of an attribute. Requiring that there is a net increase in the society's well-being in all such cases may be considered to be overly demanding, especially in the context of a discussion of inequality. In this paper we have replaced MON by the two conditions (**ME** and **KMON**) which are (jointly) weaker than MON in the presence of **CONT** and **ORD**. MON obviously implies **ME** while the converse is not true (irrespective of the presence of **CONT** or **ORD**). The fact that, in the presence of **CONT** and **ORD**, MON implies **KMON** is easily checked by standard arguments. Kolm Monotocity has been so named here since the statement in the condition appeared and played an important role in the pioneering paper by Kolm (1977) on multidimensional inequality indices. It may be noted, however, that Kolm (1977) did not propose it as a condition but derived it from **CONT**, **ORD** and MON. For any *R*, the function f_R (.) will be called the Kolm function of *R*. **ANON** says that the labelling of the individuals (i.e. which individual is called individual no. 1, which is called no. 2 etc.) is immaterial for social evaluation. **PRI** implies that social evaluation depends on the relative frequencies of the allocations. It is the proportion of the population (rather than the absolute number of individuals) getting a particular allocation of an attribute that is important.

In a discussion of inequality social evaluation is desired to be distribution-sensitive. In unidimensional theory a widely used notion of distribution sensitivity is the Pigou-Dalton (PD) transfer principle. If the attribute in question is income, a PD transfer is an income transfer from a richer to a poorer person by an amount less than their initial income difference. If **x** and **y** are allocation vectors, the following three statements are equivalent (Hardy, Littlewood and Polya (1934) and Marshall and Olkin (1979, Ch.1)): (1) **x** Lorenz dominates **y**; (2) **x** *Pigou-Dalton majorizes* **y** i.e. **x** is obtained from **y** by a *finite* sequence of PD transfers; and (3) **x** = B**y** for some bistochastic matrix B. (A bistochasic matrix is a non-negative matrix in which each row as well as each column sums to 1.)

The multidimensional literature contains generalizations of the concept of Pigou-Dalton majorization. One of these is the concept of Uniform Majorization. (See Kolm (1977).) For all $n \times m$ matrices X and Y in **X** such that X ≠ Y and X is not a row permutation of Y, X is said to be a uniform majorization of Y if X = BY for some bistochastic matrix B. Since X = BY implies, $x^i = By^i$ for all *i* in *M*, x^i Pigou-Dalton majorizes y^i for each *i*; and since the same matrix B is used to majorize all the columns of Y, the majorization is said to be *uniform* across the attributes.

Kolm (1977) used the notion of uniform majorization (UM) to formulate an axiom regarding an inequality-sensitive social evaluation relation: A relation *R* on **X** is said to satisfy Uniform Majorization **(**UM**)** if, for all X and Y in **X** such that X is a uniform majorization of Y, X *P* Y.

It has recently been pointed out that the axiom of UM has some limitations. [See, for instance, Fleurbaey and Trannoy (2003) and Lasso de la Vega et.al. (2010).] Three difficulties have been highlighted. (1) UM fails to recognize that all attributes may not be transferable in principle. (What, for instance, does one mean by 'transferring' education or health?) (2) On the other hand, UM is restrictive in that it confines attention to the case in which all the attributes are transferred in the same proportion. (3) If there is a transfer of this type between two individuals such that neither is unambiguously richer than the other (i.e. individual 1 has more of some attributes than individual 2 but less of the others), the case for requiring that the transfer would lead to an unambiguous increase in the society's well-being is not convincing.

The first two of these three objections to UM do not seem to make this axiom totally irrelevant. There may be cases in which all the attributes under consideration in a specific context happen to be transferable. (For instance, these may be incomes from different sources or incomes at different points of time.) Moreover, we may *choose* to transfer all the attributes in the same proportion.

The third issue, however, seems to be more fundamental. Indeed, it seems possible to construct examples of situations where it would be natural to conclude that the society would be indifferent between the pre- and post-transfer allocations (rather than preferring the latter over the former) under uniform majorization. Consider, for instance, the case where $n = 2 = m$, X = $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$. Then Y = BX = $\begin{pmatrix} 10/3 & 8/3 \\ 8/3 & 10/3 \end{pmatrix}$. In the pre-transfer matrix X individual 1 is favourably placed (relative to individual 2) w.r.t. attribute 1 and is in an adverse situation w.r.t. attribute 2. Assume that the society attaches equal importance ("weights") to the two attributes. In this case individual 1's advantage in attribute 1 is exactly balanced by her disadvantage w.r.t. attribute 2 while individual 2's situation is an exact mirror image. It is possible to argue that in this case there is no "net" relative inequality in society. Note now that the same is also true of the post-transfer matrix Y. In other words, the over-all degree of relative inequality can be said to be zero in both X and Y. (Note also that the argument generalizes to the case of unequal weights on the attributes. In this case there would be some "net" over-all inequality. But if the same set of attribute weights apply to both X and Y, the over-all degree of relative inequality can again be arguably the same in the two matrices.) Since the total available amount or arithmetic mean of each attribute is also the same in the two matrices, there seems to be a case for requiring the society to be indifferent between X and Y. At least, the case for requiring that Y is *strictly preferred* to X does not seem to be convincing.

This objection to UM, however, would not apply if the individual who gets the transfer is unambiguously poorer than the individual from whom the transfer is made. Consider, for instance, the case where X = $\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix}$. With the same B as in the preceding paragraph, Y is now $\begin{pmatrix} 10/3 & 10/3 \\ 8/3 & 8/3 \end{pmatrix}$. In X individual 1 gets twice the allocation of individual 2 for both the attributes. Individual 1's relative advantage w.r.t. attribute 1 is now compounded by relative advantage w.r.t. attribute 2. In Y for each attribute individual 1 gets 10/8 times the allocation of individual 2 i.e. in Y there is, again, a compounding of relative advantages across attributes for individual 1. However since 10/8 < 2, there is now a case for believing that over-all *relative* inequality is less in Y than in X. Since, as before, the total amounts of each attribute are the same in the two allocation matrices, there may now be a case for requiring that the society prefers Y to X.

To give a formal statement of the case in which UM would be a reasonable condition and also for later reference for other purposes, at this point we introduce the concept of comonotonic matrices.

For all X in **X** and for all *j* in *M*, $\mathbf{x}^j$ is *non-increasing monotonic* if $x_1^j \geq x_2^j \geq .... \geq x_n^j$. It is *non-decreasing monotonic* if $x_1^j \leq x_2^j \leq ,......... \leq x_n^j$. $\mathbf{x}^j$ is *monotonic* if it is either non-increasing monotonic or non-decreasing monotonic; it is *equally distributed* if it is both non-increasing and non-decreasing comonotonic. For all X in **X** and for all *i* and *j* in *M*, $\mathbf{x}^i$ and $\mathbf{x}^j$ are *comonotonic* if either both $\mathbf{x}^i$ and $\mathbf{x}^j$ are non-increasing monotonic or both of them are non-decreasing monotonic; they are called *countermonotonic* if one of them is non-increasing monotonic, the other is non-decreasing monotonic and neither is equally distributed.

Definition 2.1: For all X in **X**, X is called *non-increasing comonotonic* if $\mathbf{x}^j$ is non-increasing monotonic for all *j* in *M*. It is *non-decreasing comonotonic* if $\mathbf{x}^j$ is non-decreasing monotonic for all *j* in *M*. X is *comonotonic* if it is either non-increasing comonotonic or non-decreasing comonotonic. It is called *mixed monotonic* if there is a non-trivial partition $\{M_1, M_2\}$ of *M* such that $\mathbf{x}^i$ is non-increasing monotonic for all *i* in M_1, $\mathbf{x}^j$ is non-decreasing monotonic for all *j* in M_2 and, for at least one *i* in M_1 and one *j* in M_2, $\mathbf{x}^i$ and $\mathbf{x}^j$ are countermonotonic..

For any X in **X**, there is a unique non-increasing comonotonic matrix Y in **X** such that Y is obtained by rearranging each column of X, if necessary, in non-increasing order. Such a matrix Y will be called the *non-increasing comonotonization* of X. The *non-decreasing comonotonization* of X is similarly defined. Y is a *comonotonization* of X if it is either the non-increasing or the non-decreasing comonotonization of X. □

For example, if n = 3, m = 2, $X = \begin{pmatrix} 4 & 6 \\ 2 & 2 \\ 0 & 1 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 4 & 6 \end{pmatrix}$, $Z = \begin{pmatrix} 4 & 1 \\ 2 & 2 \\ 0 & 6 \end{pmatrix}$ and $W = \begin{pmatrix} 4 & 1 \\ 2 & 2 \\ 3 & 6 \end{pmatrix}$, then X is non-increasing comonotonic, Y non-decreasing comonotonic, Z is mixed monotonic and W does not fall into any of these categories. Further, both X and Y are comonotonizations of Z.

In view of our discussion above, the following condition on the social evaluation relation would seem to be intuitively more transparent than UM.

Weak Uniform Majorization **(**WUM**):** *R* is said to satisfy Weak Uniform Majorization if, for all X and Y in **X** such that (i) Y ≠ X and Y is not a permutation of the rows of X, (ii) Y = BX for some bistochastic matrix B and (iii) both X and Y are non-increasing (or non-decreasing) comonotonic, Y *P* X.

If *R* satisfies **ANON**, part (iii) in the antecedent of the above condition can be amended to require only that X is comonotonic since for any non-increasing comonotonic X and any bisochastic matrix B, Y = BX is either non-increasing comonotonic or a row permutation of a matrix with this characteristic; and a similar statement can be made if X non-decreasing comonotonic.

If m = 1, WUM coincides with unidimensional Pigou-Dalton majorization. In this sense WUM also is a candidate multidimensional generalization of this unidimensional notion.

However, although WUM seems to be more intuitively acceptable than the stronger condition of UM, the focus on comnotonicity makes it applicable only to a special class of matrices. Besides, it continues to share with UM the assumptions that all attributes are transferable and that the transfers are uniform across attributes. It is reasonable to look for other ways of making the social evaluation distribution-sensitive while trying to avoid these special requirements.

With this objective we consider below a condition on the social evaluation relation which is similar to but technically weaker than the Pigou-Dalton Bundle Principle (PDBP) introduced by Fleurbaey and Trannoy (2003). (See Lasso de la Vega et. al. (2010) for an innovative use of PDBP for the purpose of deriving a multidimensional inequality index.)

Consider the case where the proportions of the attributes that are transferred are allowed to differ between attributes and are not restricted to be non-zero for all attributes. We shall, however, assume (i) that transfers from an individual q to an individual p are allowed only if

q is unambiguously richer than p (i.e. q has more of every attribute than p) and (ii) that transfers preserve the relative ranks of the individuals in all the dimensions. These requirements are stated formally in the following definition of a Weak Pigou-Dalton Bundle Transfer.

Definition 2.2: For all X and Y in **X**, Y is said to be derived from X by a **Weak Pigou-Dalton Bundle Transfer (WPDBT)** if there exist p and q in N such that

(i) $\mathbf{x}_q > \mathbf{x}_p$;

(ii) $\mathbf{y}_q = \mathbf{x}_q - \mathbf{d}$ and $\mathbf{y}_p + \mathbf{d}$ for some $\mathbf{d}$ in $\Re^m_+$ such that $\mathbf{d} \neq 0$.

(iii) $\mathbf{y}_r = \mathbf{x}_r$ for all r in $N - \{p, q\}$;

(iv) for all j in M for all r and s in N [$y^j_r \geq y^j_s$ if and only if $x^j_r \geq x^j_s$]

Part (i) of Definition 2.1 states that individual q is unambiguously richer than individual p in the initial allocation matrix X. Part (ii) requires that non-negative amounts of the different attributes are transferred from individual q to individual p. The amounts or the proportions of the transfers need not be the same for all attributes. Neither is it required that some amounts of *all* attributes must be transferred i.e. it is recognized that some attributes may, by their nature, be non-transferable. It is required, however, that the transfer is non-trivial i.e. some amount of at least one attribute is transferred. Part (iii) states that all individuals other than p and q are unaffected by the transfer. Part (iv) states that, for every attribute, the rank of any individual in an ordered re-arrangement (in, say, non-increasing order) of the relevant column of X is unaffected by the transfer.

It is part (iv) of the Definition 2.1which distinguishes WPDBT from the notion of Pigou-Dalton Bundle Transfer in which the condition of invariance of the ranks of the individuals in the columns of X is required to apply only to the two individuals involved in the transfer (i.e. p and q).

Definition 2.3: A social evaluation relation R on **X** is said to satisfy the **Weak Pigou-Dalton Bundle Principle (WPDBP) i**f, for all X and Y in **X** such that Y is obtained from X by a finite sequence of WPDBT's, Y P X.

As an illustration consider the case in which $n = 3$, $m = 2$, $X = \begin{pmatrix} 10 & 9 \\ 2 & 8 \\ 7 & 6 \end{pmatrix}$ and $Y = \begin{pmatrix} 8 & 9 \\ 4 & 8 \\ 7 & 6 \end{pmatrix}$.

In X individual 1 is unambiguously richer than individual 2. Y is obtained from X by transferring 2 units of the first attribute from individual 1 to individual 2. The third individual's allocations are left unchanged. This is a WPDBT since, as is easily checked, all parts of Definition 2.2 are satisfied. By a simple extension of the notion behind the unidimensional Pigou-Dalton transfer principle, it seems reasonable to require that Y is socially preferred to Y. By repeated application of this intuition it would be reasonable to require that an allocation matrix Z is socially preferred to X if Z is obtained from X by a *finite* sequence of WPDBT's rather than by a single WPDBT.

A comment about the requirement of rank preservation is in order. It is well-known that in the case of a unidimensional PD transfer the condition regarding preservation of ranks of all the individuals would be superfluous in the presence of ANON. In that case it would suffice to require that the amount transferred is less than the difference between the two individuals' allocations. For instance, if, in the above example, attribute 1 is the only attribute under consideration and 7 units are transferred from individual 1 to individual 2, the allocations will change from $\mathbf{x} = \begin{pmatrix} 10 \\ 2 \\ 7 \end{pmatrix}$ to $\mathbf{y} = \begin{pmatrix} 3 \\ 9 \\ 7 \end{pmatrix}$. The rankings of the third individual (whose allocation is not changed) with respect to individuals 1 and 2 are affected by the transfer. However, this is of no consequence since ANON implies that the society is indifferent between $\mathbf{y} = \begin{pmatrix} 3 \\ 9 \\ 7 \end{pmatrix}$ and $\mathbf{z} = \begin{pmatrix} 9 \\ 3 \\ 7 \end{pmatrix}$. The move from **x** to **z** is, however, rank-preserving. Since the society would strictly prefer **x** to **z** even if rank preservation is required, it would (by **ORD**) strictly prefer **x** to **y** although **y** does not preserve ranks.

It can be checked that essentially the same argument will apply in the multidimensional framework if the majorization of the allocation matrix is *uniform*. Again, **ANON** would make a rank-preservation requirement superfluous.

The case of *non-uniform* transfers is, however, different. It is easily seen that the presence of the second columns in the matrices X and Y in the above example would prevent **ANON** from making rank preservation superfluous in the way that it did in the two preceding paragraphs. In particular, the matrix $\begin{pmatrix} 3 & 9 \\ 9 & 8 \\ 7 & 6 \end{pmatrix}$ is not a row permutation of $\begin{pmatrix} 9 & 9 \\ 3 & 8 \\ 7 & 6 \end{pmatrix}$.

It seems that the intuitive plausibility of the condition that Y is preferred to X if Y is obtained from X by a finite sequence of WPDBT's (as defined in Definition 2.3) is understood more easily than that of the similar condition in which the requirement in part (iv) of the Definition is relaxed to require only the preservation of the rank between the individuals involved in the transfer. However, we do not pursue the matter further here. In any case from the formal point of view the *stronger* requirement in part (iv) of Definition 2.4 makes WPDBP a *weaker* (rather than a stronger) condition than PDBP. Hence, if PDBP is considered to be a plausible condition, the judgement about WPDBP can hardly be different.

It may be noted that **WPDBP** does not rule out the case where all attributes are transferable, transfers are uniform and allocation matrices are comonotonic. In fact, it can be checked that if X and Y in **X** are such that (i) X is a uniform majorization of Y and (ii) Y and, hence, X (or a row permutation of X) are comonotonic, then it is possible to find a finite sequence of WPDBT's by which X can be obtained from Y. Thus, **WPDBP** implies WUM.

To summarise, in most cases in the existing literature the objective of making the social evaluation relation distribution-sensitive is sought to be fulfilled by imposing on it the condition of UM (or some similar condition). In view of its limitations, in this paper we replace UM by the intuitively more transparent condition of **WPDBP**. In doing so we also ensure that WUM is satisfied.

We are now in a position to define a social evaluation relation.

Definition 2.4: A Multidimensional Social Evaluation Relation (MSER), *R*, is **a** binary relation on **X** satisfying **CONT**, **ORD**, **ME**, **KMON**, **ANON**, **PRI** and **WPDBP**.

A scalar-valued mapping E_R on **X** that represents *R* is called the social evaluation function of *R*. It may be note that since we do not require *R* to satisfy monotonicity, E_R is also not constrained to have this property.

Distribution sensitivity is, by definition, concerned with equity considerations. One aspect of such considerations is captured by generalizations of the Pigou-Dalton transfer principle such as **WPDBP**. In multi-attribute theory, however, there are other aspects of the matter. One such aspect relates to the pattern of inter-relation among the attributes. One of the axioms to be stated now deals with this aspect. The proposed axiom is weaker than the well-known axiom of Correlation Increasing Majorization.

To introduce the weaker axiom we use, again, the concept of comonotonic matrices. The following condition is proposed.

Definition 2.5 Comonotonizing Majorization (CM): A binary relation *R*, on **X** satisfies **CM** if, for X and Y in **X** such that (i) X is mixed monotonic and (ii) Y is a comonotonization of X, X *P* Y.

CM is a weaker condition than the axiom of Correlation Increasing Majorization (CIM). While the literature contains different formulations of CIM, the essential idea is that if Y is obtained from X by rearranging the entries in each column of X in such a way that the correlations between the different columns of the matrix increase, then X strictly dominates Y. It is easily seen that that this is a stronger requirement than **CM**. Dardanoni (1996) used a variant of CIM which, in terms of the present framework, would read as follows: for any pair of distribution matrices X and Y, if X is not comonotonic and Y is a comonotonization of X, then X is strictly preferred to Y. This variant also is a stronger condition than **CM** since it implies **CM** but the converse is not true.

CM has a simple intuitive meaning. Consider, for instance, the case in which $n = 2 = m$, Y = $\begin{pmatrix} 9 & 7 \\ 6 & 3 \end{pmatrix}$ and X = $\begin{pmatrix} 9 & 3 \\ 6 & 7 \end{pmatrix}$. X is mixed monotonic and Y is a comonotonization of X. Consider now the question whether society should strictly prefer X to Y. In Y individual 1 has a greater allocation of both the attributes than individual 2. In X individual 1 has more of attribute 1 than individual 2 but has less of attribute 2. Therefore, in Y the adversity faced by individual 2 w.r.t. attribute 1 is compounded by the adversity faced by her w.r.t. attribute 2. In X, however, individual 1 is favourably treated w.r.t. attrute1 while w.r.t. attribute 2 it is individual 2 who receives favourable treatment. So far as over-all equity is concerned, X seems to be better than Y. On the other hand, the total available amounts of both the attributes are the same in X as in Y. Thus, whatever be the relative weights ("prices") of the two attributes, the aggregate value of the attributes is also the same in X and Y. If, intuitively, over-all well-being in the society is considered to be determined by aggregate amounts of the attributes (or the aggregate value of all the attributes) on the one hand and distributive equity on the other, there seems to be reasonable grounds for requiring that society strictly prefers X to Y.

The reader will notice that while **WPDBP** has been imposed as part of the definition of an **MSER**, **CM** has only been given the status of an additional desirable property. This is in deference to the fact that that there are indices in the literature which have come to be accepted as multidimensional inequality indices but which do not satisfy **CM**. The results of this paper can easily be suitably restated if **CM** is a definitional requirement of an **MSER**.

In this paper we are interested in obtaining ordinal indices of relative inequality derived from social evaluation relations. An inequality index is a scalar-valued mapping, D, on **X** with the interpretation that if X and Y in **X** are such that D(X) $\geq$ D(Y), the degree of inequality in the society is at least as great under X as that under Y. The properties that are intuitively expected of an inequality index are similar to the corresponding properties imposed on a social evaluation relation and are given the same names and acronyms. In the subsequent discussion it will be clear from the context whether the conditions refer to an inequality index or to a social evaluation relation. We introduce the following definitions.

Definition 2.6: A Multidimensional Inequality Index (MII) is a scalar-valued mapping D on **X** satisfying the properties of (i) Continuity (**CONT**), (ii) Anonymity (**ANON**), (iii) Population Replication Invariance (**PRI**) and (iv) Weak Pigou-Dalton Bundle Principle (**WPDBP**) [i.e. for all X and Y in **X** such that X is obtained from Y by a finite sequence of Weak Pigou-Dalton Bundle Transfers (WPDBT's), D(X) < D(Y)].

Definition 2.7: A Multidimensional Inequality Index, D, satisfies the property of Comonotonizing Majorization (**CM**) if, for all X and Y in **X** such that X is mixed monotonic and Y is a comonotonization of X, D(X) < D(Y).

Definition 2.8: A Multidimensional Index of Relative Inequality (MIRI) is an MII, D, such that, for all X and Y in **X** and for all positive scalar *k*, D(X) ≥ D(Y) if and only if D(*k*X) ≥ D(*k*Y).

Given an MSER, *R*, on **X** an MII, D_R (say), is said to be *normatively significant* or *generated by R* if, for all X and Y in **X** such that $X_\mu = Y_\mu$, $D_R(X) \geq D_R(Y)$ if and only if Y *R* X.

Let $\mathbf{X}^1$ denote the set of all allocation matrices (vectors) when $m = 1$. In this unidimensional case the standard method of constructing a normatively significant relative inequality index D_R from a social evaluation relation *R* on $\mathbf{X}^1$ is to consider the Equally Distributed Equivalent Income (EDEI) function i.e. a mapping, E_R from $\mathbf{X}^1$ into $\Re$ such that, for any **x** in $\mathbf{X}^1$**,** $E_R(\mathbf{x})$ is the scalar for which $(E_R(\mathbf{x}).1_n)$ *I* **x**. **KMON** ensures that the function is well-defined. (Put $E_R(\mathbf{x}) = f_R(\mathbf{x})\mu(\mathbf{x})$ where f_R is as in the statement of **KMON** and $\mu(\mathbf{x})$ is the mean of **x**.) **ORD**, **ME** and **KMON** imply that the function is a representation of *R* (i.e. it is a social evaluation function).
D_R is then defined to be such that, for all **x** in $\mathbf{X}^1$,

$$D_R(\mathbf{x}) = 1-(E_R(\mathbf{x}) / \mu(\mathbf{x})) = 1 - f_R(\mathbf{x})$$

In the multidimensional case ($m > 1$), however, μ(X) would be the vector of the means of the *m* columns of X. The above procedure will, therefore, be inapplicable. A general strategy in this case is to construct a mapping F_R from **X** into $\Re$ such that, for all X and Y in **X** such that $X_\mu = Y_\mu$, $F_R(X) \geq F_R(Y)$ if and only if X *R* Y. If such a mapping exists, then the mapping $D_R : \mathbf{X} \rightarrow \Re$, defined to be such that, for all X in **X,** $D_R(X) = 1 - F_R(X)$, D_R is taken to be an inequality index generated by *R*.

In view of our definition of a social evaluation relation *R*, it is natural here to take the Kolm function f_R as the mapping F_R of the preceding paragraph since **KMON** would guarantee that the mapping is well-defined and, together with **ORD** and **ME**, it would imply that, for all X and Y in **X**, if $X_\mu = Y_\mu$, then $f_R(X) \geq f_R(Y)$ if and only if X *R* Y. Let K_R: $\mathbf{X} \rightarrow \Re$ now be the mapping such that, for all X in **X**, $K_R(X) = 1 - f_R(X)$. It follows easily that K_R is an MII as per Definition 2.6 and that it is normatively significant. It is called the *Kolm inequality index* generated by *R*. However, it is an MIRI if and only if *R* satisfies an additional property viz. Homotheticity.

Definition 2.9: An MSER, *R*, on **X** satisfies Homotheticity (**HOM**) if, for all X and Y in **X** and for all positive scalar λ, X *R* Y if and only if λX *R* λY.

If an MGSER is assumed to satisfy **HOM**, it implies that social evaluation is invariant w. r. t. a common proportional change in the units in which all the attributes are measured. A stronger variant of this property is Strong Homotheticity (SHOM) which requires social evaluation to be invariant w. r. t. *independent* changes in the units in which the different attributes are measured i.e., for all X and Y in **X** and for all $m \times m$ diagonal matrices Q with positive entries along the main diagonal, X *R* Y if and only if (XQ) *R* (YQ). (See Tsui (1995).) However, in the context of measurement of inequality, the acceptability of this stronger requirement has been questioned. (See Bourguignon (1999).) Therefore, we shall desire *R* to satisfy the weaker requirement, **HOM**.

We shall be concerned with the multidimensional versions of the well-known unidimensioanal Gini social evaluation relation and of the Gini index of relative inequality.

Definition 2.10: The unidimensional Gini social evaluation relation (UGSER) is the relation *R* on $\mathbf{X}^1$ whose EDEI function, $E^G{}_R$: $\mathbf{X}^1 \to \Re$, is defined as follows: For all $\mathbf{x}= (x_1,x_2,........,x_n)$ in $\mathbf{X}^1$, $E^G{}_R(\mathbf{x})=\sum_{p=1}^{n} r_p x_p$ where, for all p in N, $r_p = (2s_p - 1) / n^2$, s_p being the rank of the p-th individual in a rearrangement of X in decreasing order (with ties broken arbitrarily).

The relative inequality index, g (say), generated by the UGSER is the classical unidimensional Gini index (UGI) of relative inequality: For all **x** in $\mathbf{X}^1$, $g(\mathbf{x}) = 1 - (E^G{}_R(\mathbf{x}) / \mu(\mathbf{x}))$

Weymark (1981, 2006)) extended the notion of UGSER by defining a generalised Gini social evaluation relation by allowing the weights $(r_1.r_{2,......,}r_n)$ to be *n* arbitrary real numbers such that (i) $r_p > 0$ for all p in N, (ii) $r_p > r_q$ if and only if $x_p < x_q$ and (iii) $\sum_{p=1}^{n} r_p = 1$. In this paper we confine ourselves to multidimensional extensions of UGSER as defined above. The corresponding exercise for generalised Gini social evaluation would be similar in nature.

Definition 2.11: A **Multidimensional Gini Social Evaluation Relation** (**MGSER**) is a relation *R* on **X** such that (i) *R* is an MSER as per **Definition 2.4** and (ii) if $m = 1$, then *R* coincides with UGSER as defined in **Definition 2.10**.

Definition 2.12: A **Multidimensional Gini Index of Relative Inequality** (**MGIRI**) is a scalar-valued mapping D on **X** such that (i) D is an MIRI as per **Definition 2.8** and (ii) if $m = 1$, then D coincides with the unidimensional Gini index of relative inequality, g.

In this paper we are interested in normatively significant MGIRI's satisfying the property of **CM**. As noted in the Introduction, the existing literature does not seem to contain such an MGIRI if the admissible allocation matrices are not restricted to be strictly positive.

3. A Normatively Significant Multidimensional Gini Index

For all X in **X** let X* denote the matrix obtained by dividing each entry in X by the arithmetic mean of the column containing it. It may be called the *scaled* version of X. Let **X*** = {X*: X ε **X**} i.e. **X*** is the set of all scaled allocation matrices.

In the following *r* and *v* are mappings from **X*** into $\Re^n$ and $\Re^m$ respectively; *R* is a relation on **X** and, for all X in **X**, W(X*) is the matrix of Gini weights i.e. the matrix in which, for all *j* in *M*, the *j*-th column, $\mathbf{w}^j(X^*)$, is the vector of classical Gini weights on the individuals obtained from the allocation vector $\mathbf{x}^j$ [i.e., for all *p* in *N* and for all *j* in *M*, the (*p*-th row, *j*-th column) entry w^j_p in W(X*) is $(2s^j_p - 1) / n^2$ where s^j_p is the rank of individual *p* in a rearrangement of $\mathbf{x}^{j*}$ in non-increasing order]. For all *p* in *N*, $\mathbf{w}_p(X^*)$ will denote the *p*-th row of W(X*).

A prime on a vector will denote its transpose.

r and *v* are interpreted to be the (column) vectors of weights assigned to the individuals and to the attributes respectively; and *R* is interpreted to be the social evaluation relation.

It may be noted that the EDEI function of the UGSER, *R*, stated in Definition 2.10 can be written as follows: For all x in $\mathbf{X}^1$, $E^G_R(x) = [r'(x^*)x^*]\mu_x$ where μ_x denotes the mean of x, x* denotes the vector x scaled by μ_x and the vector *r´* of the rank-based weights of the individuals is written as a function of x* since it can be determined from this scaled version of x. Thus, $E^G_R(x)$ is a product of two parts: an "equality" part that depends on the scaled version of x and an "efficiency" part consisting of the per capita amount of the attribute. Our search for an MGSER will be motivated by this feature of the UGSER. It may also be noted that this type of decomposability is true not only of the UGSER but also of the SER's underlying many other unidimensional indices of relative inequality satisfying the Pigou-Dalton transfer principle.

The following conditions are proposed on the mappings *r* and *v* and the relation *R*.

Condition 1: For all X* in **X***, (i) $r(X^*) > 0_n$ and $v(X^*) > 0_m$; and (ii) if $m = 1$, then *r*(X*) is the classical Gini weight vector for individuals for the allocation vector X.

Condition 2: For all X* in **X*** and for all *p* and *q* in *N*,

$$(r_p(X^*) / r_q(X^*)) = (\mathbf{w}_p(X^*)v(X^*) / \mathbf{w}_q(X^*)v(X^*)).$$

Condition 3: For all X* in **X*** and for all *i* and *j* in *M*,

$$(v_i(X^*) / v_j(X^*)) = (r'(X^*)\mathbf{x}^{i*} / r'(X^*)\mathbf{x}^{j*}).$$

Condition 4: Let E: $\mathbf{X} \rightarrow \Re$ be such that, for all X in **X,** E(X) = $[r'(X^*)X^*v(X^*)]h(\mu(X))$ where, for all X in **X**, $\mu(X)$ denotes the vector of the column means of X and h is a scalar-valued continuous function which is increasing in each of its arguments and is homogeneous of degree 1.

For all X and Y in **X**, X *R* Y if and only if $E(X) \geq E(Y)$.

Part (i) of **Condition 1** requires that each individual receives a positive weight, whatever may be the pattern of allocations. Similarly, each attribute is given a positive weight. As explained in the Introduction, it is a difficult task to start with a comprehensive list of all conceivable attributes and to suggest a procedure for determining their weights which would *not* be constrained, *a priori*, to be non-zero. We assume instead that that there is a social consensus on which attributes are to be included in the analysis. It is, therefore, natural to require that all the *included* attributes are to be given positive weights. Part (ii) of the Condition requires that in the unidimensional case $r(X^*)$ coincides with the classical Gini weight vector.

Condition 2 states that, for any pair of individuals, the ratio between their weights equals the ratio between the weighted sums of their Gini weights for the different attributes where, for each individual, the attribute weights, v_j's, are used for the purpose of aggregating over the attributes. In other words an individual's weight is proportional to the weighted sum of her Gini weights w.r.t. to the different attributes. Since $r > 0_n$, $v > 0_m$ and the matrix W of Gini weights is positive by definition, the ratios on both l.h.s. and r.h.s. of Condition 2 are well-defined.

Condition 3 states that, for any pair of attributes, the ratio between their weights is given by the ratio between the weighted sums of the individual allocations of the attributes where, for each attribute, the individual weights, r_p's, are used for the purpose of aggregating over the individuals. The ratio on the l.h.s. of the equation in the statement of Condition 3 is well-defined since $v > 0_m$; that on the r.h.s. is well-defined since $r > 0_n$ and since each column in an allocation matrix (and, therefore, each column in its scaled version) contain at least one positive entry by virtue of the fact that the matrix has a positive row. Intuitively, for any X, $r'(X^*)X^*v(X^*)$ may be considered to be proportional to the degree of *equality* prevailing in the society. As will be seen later, if the weight vectors satisfy Conditions 1, 2, and 3, then this expression attains its maximal value when each attribute is equally distributed while the minimal value is attained when a particular individual is allocated the total available amounts of all the attributes. Since $r'(X^*)X^*v(X^*) = \sum_{j=1}^{m} v_j\, [r'(X^*)\mathbf{x}^{j*}]$ and since the expression in square brackets on the r.h.s. can be considered to be an indicator of the degree of equality in the distribution of the *j*-th attribute, Condition 3 can be interpreted to mean that, for each *j*, v_j is proportional to the marginal contribution of attribute *j* to the over-all degree of equality.

Condition 4 states that the mapping E defined in the condition is an SEF of *R*. For all X, E(X) is decomposable into the components *r´*(X*)X**v*(X*) and h(μ(X)). The latter can be interpreted to be an efficiency component in the sense that it is increasing in the per capita availability of each of the attributes but is independent of the distribution of the attributes among the individuals. Homogeneity of degree one implies that an equiproportional increase in the per capita amounts of all the attributes increases the value of this component in the same proportion. On the other hand, the component *r´*(X*)X**v*(X*) is a weighted sum of the *scaled* allocations of the attributes to the individuals. The scaling makes it independent of the per capita amounts of the attributes and, as remarked in the preceding paragraph, relates it to the degree of equality.

The "equality-efficiency" decomposition of the social evaluation proposed in Condition 4 is similar in spirit to an assumption made in Magadalou and Nock (2011) under which the divergence of a (unidimensional) distribution from a reference distribution which is perfectly egalitarian is separable into an efficiency loss and a loss on account of inequality. They, however, considered additive separability.

It may be noted that the function h(μ(X)) in the efficiency component has been described in general terms. This component also can possibly take the form of a weighted sum viz. a weighted sum of the column means of X. However, by definition, the weights here will depend exclusively on the column means i.e. they will be independent of the distribution of the attributes. None of the results to be proved below will be affected by such a specification of h. The more general form in which h has been stated in Condition 4 avoids the unnecessary notational inconvenience of incorporating, for each X, two different sets of attribute weights in the analysis.

We now derive a particular relation on **X** as an implication of the Conditions stated above. In course of this derivation we shall use the Perron-Frobenius theory of the eigen values and eigen vectors of non-negative square matrices. For a square matrix A if there exists a scalar λ and a non-zero (column) vector *x* such that Ax = λx, then λ is called an eigen value (or characteristic root) of B and x is called the associated (column) eigen (or characteristic) vector. A real square matrix of order *n* has *n* (not necessarily distinct or non-zero or real) eigen values and *n* associated eigen vectors. The eigen value which is maximal (in the sense that its modulus value is at least as great as that of any other eigen value) is called the *maximal* (or the *first*) eigen value of A; the associated eigen vector is called its *first eigen vector*.)

It is easily seen that if x is an eigen vector of A, so is *k*x for any non-zero scalar *k*. Thus, eigen vectors are determined only upto multiplication by non-zero scalars. Therefore, when we speak of *the* first eigen vector of a square matrix, the implicit assumption is that it is made unique by choosing a mormalization rule. The most widely used normalization rule is to require that the squares of the components of any eigen vector sum to one. In this paper we follow this normalization rule. .

For our purposes we shall use the facts stated in the following Lemmas.

Lemma 1: Let A be square matrix in which all entries are positive real numbers.
(i) The maximal eigen value of A, $\lambda^*(A)$ (say), is real, positive and unique.
(ii) λ^* is continuously increasing in all entries in A.
(iii) The first eigen vector of A is real and positive.
(iv) No other eigen vctor of A is positive.

Lemma 2: For any real square matrix A and any positive scalar k, the maximal eigen value of the matrix kA is k times that of A and the first eigen vector of kA is the same as that of A.

Lemma 3: If A and B are $n \times m$ and $m \times n$ real matrices respectively, then AB and BA have the same spectrum (i.e. set of distinct eigen values) with the possible exception of zero eigen values.

Lemma 1 is a part of Perron's Theorem. A more general version of the Theorem in which A is only required to be square, non-negative and indecomposable is called the Perron-Frobenius Theorem. The Lemma, as stated, will suffice for our purpose. For proof and discussion see Debreu and Herstein (1953) and Horn and Johnson (1985). Lemma 2 is easily checked. For Lemma 3 see Meyer (2000, Ch. 7, Exercise 7.1.19).

We now prove the following Proposition.

Proposition 3.1: The mappings r and v satisfy **Conditions 1**, **2 and 3** if and only if, for all X in **X**, $r(X^*)$ and $v(X^*)$ are the first eigen vectors of $W(X^*)X^{*\prime}$ and $X^{*\prime}W(X^*)$ respectively.

Proof:

I. The 'Only If' part:

Let the mappings r and v satisfy Conditions 1, 2 and 3.

Condition 2 implies that, for all X* in **X*** and for all p in N, $r_p(X^*) = \alpha(X^*)\mathbf{w}_p(X^*)v(X^*)$ for some scalar $\alpha(X^*)$. Hence,

$$r(X^*) = \alpha(X^*)W(X^*)v(X^*) = W(X^*)v(X^*) \quad ....(1)$$

since part (ii) of Condition 1 implies that $\alpha(X^*) = 1$.

Similarly, Condition 3 would imply

$$v'(X^*) = \beta(X^*)r'(X^*)X^* \quad ....(2)$$

for some scalar $\beta(X^*)$. Condition 1 implies that $\beta(X^*) > 0$.

Hence, $v(X^*) = \beta(X^*)X^{*\prime}r(X^*) = \beta(X^*)X^{*\prime}W(X^*)v(X^*)$

i.e. $[X^{*\prime}W(X^*)]v(X^*) = (1/\beta(X^*))v(X^*) = \lambda(X^*)v(X^*)$

where $\lambda(X^*)$ denotes the positive scalar $1 / \beta(X^*)$.

Since $X^{*\prime}W(X^*)$ is a square matrix of order m, it follows that $\lambda(X^*)$ is an eigen value of this matrix and $v(X^*)$ is the associated characteristic vector. Moreover since X^* is in **X*** and since $W(X^*)$ is positive by construction, all entries in $X^{*\prime}W(X^*)$ are positive. Since Condition 1 requires $v(X^*) > 0$, it follows from part (iv) of Lemma 1 that $\lambda(X^*)$ is the maximal eigen value of $X^{*\prime}W(X^*)$ and $v(X^*)$ is the first eigen vector of this matrix.

Similarly, Eqns. (1) and (2) also imply

$r(X^*) = W(X^*)\beta(X^*)X^{*\prime}r(X^*) = \beta(X^*)W(X^*)X^{*\prime}r(X^*)$ i.e. $[W(X^*)X^{*\prime}]r(X^*) = \lambda(X^*)r(X^*)$.

Hence, by the same argument as above, it follows that $r(X^*)$ is its first eigen vector of the positive $n \times n$ matrix $W(X^*)X^{*\prime}$ since, by Lemma 3, $\lambda(X^*)$ is also the maximal eigen value of this matrix.

II. The 'If' part:

Let r and v be such that, for all X^* in **X***, $r(X^*)$ and $v(X^*)$ are the first eigen vectors of $W(X^*)X^{*\prime}$ and $X^*W'(X^*)$ respectively, made unique by normalization. Condition 1 is then satisfied in view of part (iii) of Lemma 1.

Since $v(X^*)$ is the first eigen vector of $X^{*\prime}W(X^*)$, $[X^{*\prime}W(X^*)]v(X^*) = \lambda(X^*)v(X^*)$ where $\lambda(X^*)$ is the maximal eigen value of $X^{*\prime}W(X^*)$. Pre-multiplication by $W(X^*)$ yields $[W(X^*)X^{*\prime}][W(X^*)v(X^*)] = \lambda(X^*)[W(X^*)v(X^*)]$. Since $\lambda(X^*)$ is also the maximal eigen value of $W(X^*)X^{*\prime}$, it follows that $W(X^*)v(X^*)$ is the first eigen vector of $W(X^*)X^{*\prime}$. Since $r(X^*)$ has been defined to be the first eigen vector of this matrix, it follows that

$$r(X^*) = \theta(X^*)\, W(X^*)v(X^*) \qquad \text{...........(3)}$$

for some positive scalar $\theta(X^*)$. The last equality is easily seen to imply that r and v satisfy Condition 2.

Finally, since $v(X^*)$ is the first eigen vector of $X^{*\prime}W(X^*)$,

$$v(X^*) = (1/\lambda(X^*))(X^{*\prime}W(X^*))v(X^*) = (1/\lambda(X^*))X^{*\prime}(r(X^*) / \theta(X^*) \quad \text{(by Eqn. (3))}$$

$$= \delta(X^*)X^{*\prime}r(X^*)$$

where $\delta(X^*) = [1 / (\lambda(X^*)\theta(X^*))]$ is a positive scalar. The last equality implies Condition 3. □

It can now be shown that Conditions 1 through 4 characterize a specific binary relation on X.

Proposition 3.2: There exist mappings r^* and v^* from **X*** into $\Re^{n}_{++}$ and $\Re^{m}_{++}$ respectively and a relation R^* on **X** satisfying **Conditions 1, 2,3 and 4** if and only if, for all X and Y in **X**, $[(X\ R^*\ Y) \Leftrightarrow [(\lambda(X^*)h(\mu(X))] \geq \lambda(Y^*)h(\mu(Y))]]$ where, for all X in **X**, $\lambda(X^*)$ denotes the maximal eigen value of $X^{*\prime}W(X^*)$.

Proof:

I. The 'Only If' part:

Let r^*, v^* and R^* be as in the statement of the Proposition. Conditions 1, 2 and 3 imply that, for all X in **X**, $r^*(X^*)$ and $v^*(X^*)$ are the first eigen vectors of $W(X^*)X^{*\prime}$ and $X^*W'(X^*)$ respectively (by Proposition 3.1).

Hence, for any X in **X**, $r^{*\prime}(X^*)X^*v^*(X^*) = v^{*\prime}(X^*)X^{*\prime}r^*(X^*)$ (by transposition)
$= v^{*\prime}(X^*)X^{*\prime}W(X^*)v^*(X^*)$ (by Eqn (1))
$= v^{*\prime}(X^*)[X^{*\prime}W(X^*)]v^*(X^*)$
$= \lambda(X^*)v^{*\prime}(X^*)v^*(X^*)$ (where λ is as in the Proposition)
$= \lambda(X^*)$ (by the normalization rule for $v(X)$)

Hence, this part of the Proposition follows from Condition 4.

II. The 'If' part:

Let R^* on **X** be such that, for all X and Y in **X**, $[(X\, R^*\, Y) \Leftrightarrow \lambda(X^*)h(\mu(X)) \geq \lambda(Y^*)h(\mu(Y))]$ where, for all X in **X**, $\lambda(X^*)$ is as stated in the Proposition. To show that there exist mappings r^* and v^* from **X*** into $\Re^{n}_{++}$ and $\Re^{m}_{++}$ respectively such that r^*, v^* and R^* satisfy Conditions 1 through 4, let r^* and v^* be such that, for all X* in **X***, $r^*(X^*)$ and $v^*(X^*)$ are the first eigen vectors of $W(X^*)X^{*\prime}$ and $X^{*\prime}W(X^*)$ respectively. Conditions 1, 2 and 3 are then satisfied in view of the 'if' part of Proposition 3.1. Condition 4 is satisfied since, as shown above, $\lambda(X^*) = r^{*\prime}(X^*)X^*v^*(X^*)$ for all X in **X**. □

We now proceed to show that the relation R^* on **X** characterized in **Proposition 3.2** is a Multidimensiaonal Gini Social Evaluation Relation satisfying **CM**. For this purpose the asymmetric and symmetric factors of R^* will be denoted by P^* and I^* respectively.

Proposition 3.3: R^* is an MGSER satisfying **CM** and **HOM**.

Proof:

I. R^* is an MGSER:

We first show that R^* is an MSER as per Definition 2.4. Note that R^* satisfies **CONT** since h is continuous and since, by part (ii) of Lemma 1, $\lambda(X^*)$, the maximal eigen value of $X^{*\prime}W(X^*)$, is continuous in each entry of $X^{*\prime}W(X^*)$, and, hence, in each entry of X* and of X. Since λ and h are scalar valued, R^* trivially satisfies **ORD**. To see that it satisfies **ME**, let X and Y in **X** be such that $X = X\mu$ and $Y = Y_\mu$. If now $[X \geq Y$ and $X \neq Y]$, then $\mu(X) \geq \mu(Y)$ and $\mu(X) \neq \mu(Y)$. Since h is increasing in each argument, it follows that $h(\mu(X)) > h(\mu(Y))$. Moreover, $X = X_\mu$ implies $X^* = 1_{n \times m}$ and $W(X^*) = (1/n)_{n \times m}$ so that $X^{*\prime}W(X^*) = 1_{m \times m}$. It follows that $\lambda(X^*) = m$. Similarly, since $Y = Y_\mu$, we also have $\lambda(Y^*) = m$. Thus, $\lambda(X^*)h(\mu(X) > \lambda(Y^*)h(\mu(Y))$. Hence, by definition of R^*, X P^* Y.

To verify **KMON** we have to show that R^* has a Kolm function. Thus, we have to show that, for all X in **X**, there exists a positive scalar $f_{R^*}(X)$ such that $[f_{R^*}(X)X_\mu]\ I^*\ X$ i.e. such that

$\lambda(Y^*)$ $h(\mu(Y)) = \lambda(X^*)$ $h(\mu(X)$ where Y denotes the matrix $f_{R^*}(X)X_\mu$. Since $Y^* = 1_{n \times m}$, an argument similar to one used in the preceding paragraph shows that $\lambda(Y^*) = m$. Moreover, $\mu(Y) = f_{R^*}(X)\mu(X)$ so that $h(\mu(Y)) = f_{R^*}(X)h(\mu(X)$ since h is homogeneous of degree 1. These remarks suggest a constructive proof of the fact that *R**satisfies **KMON**: To show that, for any X in **X**, a positive scalar $f_{R^*}(X)$ with the desired property exists, put $f_{R^*}(X) = \lambda(X^*) / m$.

To see that *R** satisfies **ANON** note that if X and Y in **X** are such that Y is a row permutation of X, then $\mu(X) = \mu(Y)$ and $X^{*\prime}W(X^*) = Y^{*\prime}W(Y^*)$ so that $\lambda(X^*) = \lambda(Y^*)$. Hence, X *I** Y.

PRI is verified in a similar way: If X and Y in **X** are such that Y is obtained by a *k*-fold replication of the population in X for some positive integer *k*, then we again have $\mu(X) = \mu(Y)$ and $X^{*\prime}W(X^*) = Y^{*\prime}W(Y^*)$ so that X *I** Y.

To prove that *R** satisfies **WPDBP** i.e. to show that if X and Y in **X** are such that X is obtained from Y by a *finite* sequence of WPDBT's, then X *P** Y, it suffices to show that X *P** Y if X is obtained from Y by a *single* WPDBT. Recall that, according to Definition 2.2 of WPDBT, the notion of such a transfer from individual *q* to individual *p* includes the requirements that $\mathbf{x}_q > \mathbf{x}_p$ and that, for each attribute, the relative ranking between any pair of individuals in the matrix Y is not reversed in the matrix X.

Now noting that the (*i*-th row, *j*-th column) entry in $X^{*\prime}W(X^*)$ is $\mathbf{x^*}^{i\prime}\mathbf{w}^j(X^*)$, it is seen that the requirements mentioned in the preceding paragraph imply that, for all *i* and *j* in *M*, $\mathbf{x^*}^{i\prime}\mathbf{w}(X^*)^j \geq \mathbf{y^*}^{i\prime}\mathbf{w}(Y^*)^j$ with strict inequality holding for all *i* such that d_i, the amount of transfer of attribute *i*, is positive. Thus, $X^{*\prime}W(X) \geq Y^{*\prime}W(Y^*)$ and $X^{*\prime}W(X^*) \neq Y^{*\prime}W(Y^*)$. Hence, by part (ii) of Lemma 1, $\lambda(X^*) > \lambda(Y^*)$. On the other hand, a WPDBT leaves all column means of an allocation matrix unchanged. Hence, $h(\mu(X)) = h(\mu(Y))$. By definition of *R**, therefore, X *P** Y.

This completes the proof of the fact that *R** is an MSER. To see that it is an MGSER according to Definition 2.11, we note that if *m* = 1, *R** reduces to the Unidimensional Gini Social Evaluation Relation as per Definition 2.10.

II. *R** satisfies **CM**:

Let X and Y in **X** be such that X is mixed monotonic and Y is a comonotonization of X. Note that X* is then mixed monotonic and Y* is a comonotonization of X*. Since X* is mixed monotonic, there exists a non-trivial partition $\{M_1, M_2\}$ of *M* such that (i) $\mathbf{x^*}^i$ is non-increasing monotonic for all *i* in M_1, (ii) $\mathbf{x^*}^j$ is non-decreasing monotonic for all *j* in M_2 and (iii) $\mathbf{x^*}^i$ and $\mathbf{x^*}^j$ are countermonotonic for at least one *i* in M_1 and one *j* in M_2. Under these circumstances it can be checked that, for all *i* and *j* in *M*, $\mathbf{x^*}^{i\prime}\mathbf{w}(X^*)^j \geq \mathbf{y^*}^{i\prime}\mathbf{w}(Y^*)^j$ with strict inequality holding whenever *i* is in M_1 and *j* is in M_2 or vice versa. Thus, we again have $X^{*\prime}W(X^*) \geq Y^{*\prime}W(Y^*)$ and $X^{*\prime}W(X^*) \neq Y^{*\prime}W(Y^*)$. Hence, $\lambda(X^*) > \lambda(Y^*)$. Again, since comonotonization of a matrix leaves column means unchanged, $h(\mu(X)) = h(\mu(Y))$. Thus, X *P** Y.

III. R^* satisfies **HOM**:

For all X in **X** and for all positive scalar k, we have: $(k\text{X})^* = \text{X}^*$ and $\mu(k\text{X}) = k\mu(\text{X})$ i.e. $h(\mu(k\text{X})) = k h(\mu(\text{X}))$. Hence, for all X and Y in **X**, X R^* Y if and only if $(k\text{X})$ R^* $(k\text{Y})$. □

As seen above, the social evaluation of an allocation matrix X is considered to given by the product of the equality and the efficiency components, $r^{*\prime}(\text{X}^*)\text{X}^*v^*(\text{X}^*)$ and $h(\mu(\text{X}))$ respectively. It may be noted that in this expression the weight vectors r^* and v^* are functions of the scaled version X* of X. In other words, $r^*(\text{X}^*)$ and $v^*(\text{X}^*)$ are *distributional* weights (i.e. weights that are proposed to be used for the purpose of calculating the value of the inequality index). Thus, for any j in M, $v_j^*(\text{X}^*)$ is the weight on the j-th attribute determined from the purely distributional point of view. It does not reflect the society's ethical judgement regarding the "over-all" importance of the attribute. As noted before, a weighting procedure may be implicit in the efficiency part h(.) of the social evaluation. Thus, if the society desires to assign relatively high importance to an attribute irrespective of the contribution of this attribute to equality, this value judgement can be accomodated (at least partly) through this component of the social evaluation function. The assumption implicit in the weight-setting procedure characterized in the above propositions is that the society sets the *distributional* weights of the attributes on the basis of their observed contributions to the degree of equality.

Consider now the Kolm inequality index, G^* (say), generated by R^*:

For all X in **X**, $G^*(\text{X}) = 1 - f_{R^*}(\text{X})$ where f_{R^*} is the Kolm function of R^* i.e., for all X in **X**, $f_{R^*}(\text{X}) = \lambda(\text{X}^*) / m$.

The following result is now an immediate consequence of the Propositions proved above.

Proposition 3.4: G^* is an MGIRI as per Definition 2.12; it satisfies **CM** and is normatively significant.

Proof is omitted. □

In case of perfect equality (i.e. where $\text{X} = \text{X}_\mu$) $\text{X}^{*\prime}\text{W}(\text{X}^*) = 1_{m \times m}$ so that $\lambda(\text{X}^*) = m$. On the other hand, if there is perfect inequality (in the sense that a particular individual is allocated the total available amounts of all the attributes), it can be checked that X* has one row in which each entry is n while all other entries are zero and that in this case $\lambda(\text{X}^*) = m / n$. Hence, if it is required that the index lies in the closed interval [0, 1] and that the values 0 and 1 are attained in the cases of perfect equality and perfect inequality respectively, the following normalised version, G*, of G obtained:

$$\text{For all } n \times m \text{ matrices X in } \mathbf{X},\ G^*(\text{X}) = (n / (n-1))\,(1 - (\lambda(\text{X}^*) / m)$$

where $\lambda(X^*)$ is the maximal eigen value of $X^{*\prime}W(X^*)$ and $W(X^*)$ is the matrix in which, for all j in M, the j-th column is the vector of the classical Gini weights of the individuals for the j-th attribute.

4. Conclusion

In this paper the problem of measuring multidimensional inequality has been interpreted to be essentially a problem of setting weights on the different attributes. It has been argued that determination of these weights is linked to the problem of determining the weights of the individuals. A number of conditions on the two sets of weights and on their interrelationships have been proposed. By combining these conditions with a social evaluation function which is decomposable between an equality and an efficiency component we obtained a specific social evaluation relation. The Kolm index derived from this relation has been suggested as the multidimensional inequality index. It has been shown that the proposed index is a multidimensional Gini index satisfying the properties of WPDBP and CM. The index does not seem to have appeared in the literature before. Moreover, the literature does not seem to contain any other normatively significant multidimensional Gini index that would satisfy both of these properties in the absence of the restriction that allocation matrices are strictly positive. In this paper this restriction has been relaxed on grounds of empirical applicability.

Acknowledgement

For comments on earlier versions of the paper and on closely related material I am indebted to Koen Decancq and to participants at various Annual Conferences of the Indian Econometric Society.

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