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Abstract

This paper introduces and analyses, both normatively and statistically, a new class of inequality measures. This class generalizes and comprises different well-known families of inequality measures as particular cases. The elements of this new class are obtained by weighting local inequality evaluated through the Bonferroni curve. The weights are the density functions of the beta distributions over [0,1]. Therefore, the weights are not necessarily monotonic. This allows us to choose the inequality measures that are more or less sensitive to changes that could take place in any part of the distribution. As a consequence of the different weighting schemes attached to the indexes, the elements of the class introduce very dissimilar value judgements in the measurement of inequality and welfare. The possibility of choosing the index that focuses on a specific percentile, and not necessarily on the extremes of the distribution, is one of the advantages of our proposal.

Keywords: Lorenz curve, Bonferroni curve, preference distributions, inequality aversion, beta distribution.

JEL Classification: C10, D31, I38.

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1. Introduction

If inequality is assessed using a single inequality measure, a number of important dimensions of the change in inequality due to a certain policy will not be picked up. Each inequality measure incorporates assumptions about the way in which income differences in different parts of the distribution are summarized. It is therefore desirable to calculate a wide range of inequality indexes incorporating different assumptions, but at the same time having a common theoretical foundation in order to thoroughly evaluate a redistribution policy. Although indexes with distinct theoretical foundations are occasionally employed together[1], this procedure can make it difficult to evaluate their capacity as complementary measures of inequality. It is therefore convenient to have homogeneous families of indexes that provide sufficient information about the distribution and at the same time differ from and complement one another in terms of their normative aspects.

This paper introduces a class of inequality measures whose elements can be expressed as a weighted average of the local inequality in each income percentile. The weights are the density functions of the beta distributions over [0,1], which depend on two positive real parameters. When both parameters change, a set of weights is obtained. These weights represent different attitudes in terms of how inequality is evaluated throughout the income distribution. Consequently, we obtain a family of indexes that have common properties and a clear formal analogy, but at the same time differ and complement each other in the ethical characteristics.

The Gini (1914) and Bonferroni (1930) indexes belong to this class of inequality measures, as well as families of inequality measures previously described in the literature such as the generalized Gini measures of inequality (Kakwani, 1980; Yitzhaki, 1983) and the most recent proposals of Aaberge (2000, 2007) and Imedio et al. (2011). In the indexes of these families, the weights attached to local inequality behave monotonically throughout the income

[1] It is common practice to combine the Gini index (1914) with the Atkinson indexes and/or entropy indexes. Newbery (1970) highlights the differences between the theoretical foundations of the Gini index and the Atkinson indexes. The entropy indexes derive from information theory.

distribution. The greatest weight is assigned to one of the extremes. In the class of inequality measures that we propose, the weights are not necessarily monotonic. They can achieve the maximum or minimum value in any percentile. One of the advantages of our proposal is the possibility of choosing the index that focuses on a specific percentile, and not necessarily on the extremes of the distribution. This flexibility allows us to notice, through a proper selection of indexes, that the impact on inequality of a certain policy depends not only on the interval of incomes more affected by this policy, but also on the index used. The empirical illustration deals with this issue.

Despite the variety of inequality measures in the literature, the previous reasons justify the interest of our proposal. From the theoretical point of view, an additional advantage is the broader and overall treatment of different families of indexes that have been proposed in the literature in a scattered way, showing their common foundations.

Normative aspects are addressed following Yaari's approach (1987, 1988) based on social preference distributions. This approach for relating inequality and welfare is more general than the classic AKS (Atkinson, 1970; Kolm, 1966; Sen, 1973), making it an easy task to compare the level of inequality aversion[2], or preference for equality, that the indexes introduce. In some cases, it also allows indexes to be ordered in terms of this criterion. Families of inequality measures that show increasing inequality aversion approaching the maximum aversion, or Rawlsian leximin[3] belong to this class of inequality measures, as well as those that show decreasing inequality aversion. Social preference distributions also make it easy to analyse the behaviour of the indexes with respect to principles that are more demanding than the Pigou-Dalton Principle of Transfers (PT), namely the Principle of Positional Transfer Sensitivity (PPTS) and the Principle of Diminishing Transfers (PDT). Both the PPTS and the PDT analyse

[2] An index shows inequality aversion if it satisfies the Pigou-Dalton Principle of Transfers. This principle states that an income transfer from a richer to a poorer individual, whose ranks in the income distribution remain unchanged (progressive transfer), reduces income inequality. When the level of inequality aversion increases, the effect of this type of transfer on the inequality index is greater.

[3] The Rawlsian leximin focuses on the poorest individual of the population. Between two distributions, the distribution with the greater minimum income is preferred or, in the event of equality, the distribution in which the minimum income is less frequent. This approach is derived from the theory of social justice defined by Rawls (1971).

the particular sensitivity of the indexes to progressive transfers that may occur in different parts of the distribution.

2. Lorenz and Bonferroni curves. Associated indexes.

Let us assume that the income distribution of a population is represented by the random variable X, whose domain is the semi-straight positive real, $R_0^+ = [0, \infty)$, where F is its distribution function[4], and $\mu = E(X) = \int_0^\infty x dF(x) < \infty$ its mean income.

The associated Lorenz curve, L(p), p=F(x), is defined by:

$$L:[0,1] \rightarrow [0,1] \ , \ L(p) = \frac{1}{\mu}\int_0^x s dF(s) = \frac{1}{\mu}\int_0^p F^{-1}(t)dt \ , \ 0 \le p \le 1,$$

where $F^{-1}(t) = \inf\{s : F(s) \ge t\}$, $0 \le t \le 1$, is the inverse to the left of F, $F^{-1}(0)=0$. For each p=F(x), L(p) is the proportion of total income volume accumulated by the set of units with an income lower than or equal to x. It is clear that for $0 \le p \le 1$ it is $L(p) \le p$, and $L(p)=p$ in the case of perfect equality and $L(p)=0$ for $0 \le p < 1$, $L(1)=1$ if the concentration is maximum. For any distribution, X, the Lorenz curve is increasing and convex and given the mean income, the density function of X is obtained from the curvature of L(p). These properties are the result of the following equalities

$$L'(p) = \frac{F^{-1}(p)}{\mu} > 0 \ , \ L''(p) = \frac{1}{\mu f(F^{-1}(p))} > 0 \ , \ 0 < p < 1.$$

By means of the Lorenz curve, the Gini inequality index, G, is defined as

$$G = 2\int_0^1 (p - L(p))dp = 1 - 2\int_0^1 L(p)dp$$

The value of this index is equal to twice the area between the Lorenz distribution curve and the curve corresponding to perfect equality. It is a normalized index, $G \in [0, 1]$.

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[4] Sometimes F is assumed to be continuous in order to obtain theoretical results in a simpler manner. In such a case, $f(x)=F'(x)$ is the density function of the distribution.

A simple transformation of the Lorenz curve allows the information contained in the curve to be interpreted in an alternative manner. Bonferroni (1930) defines his index through the curve[5]

$$B:[0,1]\to[0,1]\ ,\ B(p)=\begin{cases}\dfrac{L(p)}{p}\ ,0<p\leq 1,\\ 0\ ,\ p=0.\end{cases}$$

It satisfies $B(p)\leq 1$, $0\leq p\leq 1$. For an egalitarian distribution the curve is $B(p)=1$, $0<p\leq 1$, whereas if the concentration is maximum, the curve is $B(p)=0$, $0\leq p<1$ and $B(1)=1$. In the literature, B(p) is known as the Bonferroni curve or the scaled conditional mean curve, given that:

$$B(p)=\frac{E(X/X\leq F^{-1}(p))}{\mu}\ ,\ 0<p=F(x)\leq 1\ ,\ B(0)=0.$$

That is, if $p=F(x)$ is the proportion of units whose income is lower than or equal to x, B(p) is the ratio between the mean income of this group and the mean income of the population.

Although from a formal standpoint the Bonferroni curve represents inequality in an equivalent manner to the Lorenz curve and both curves are determined mutually, the information they yield is different. The values of L(p) are fractions of the total income, while the values of B(p) refer to relative income levels.

The shape of B(p) depends on the shape of the underlying distribution, F. It is verified:

$$B'(p)=\frac{1}{\mu p^2}\int_0^p\frac{tdt}{f(F^{-1}(t))}>0\ ,\ B''(p)=-\frac{1}{\mu p^3}\int_0^p\frac{t^2 f'(F^{-1}(t))dt}{(f(F^{-1}(t)))^3}\ ,$$

provided that $\lim_{p\to 0^+}(p^2/f(F^{-1}(p)))=0$. Therefore, B(p) is increasing but the concavity/convexity of the Bonferroni curve depends on the concavity/convexity of the associated distribution. If F is convex (concave), in which case f is increasing (decreasing) and the majority of the population has high (low) incomes, B(p) is concave (convex). If f is bell shaped and asymmetrical to the right, F is convex/concave and B(p) is concave/convex. When the distribution function is concave/convex, the lowest and the highest incomes are the most

[5] In the following equality, if the minimum income is $x_0>0$, then $B(0)=\lim_{p\to 0^+}(L(p)/p)=L'(0^+)=x_0/\mu$

frequent and there is a tendency for polarization, in which case B(p) is convex/concave. That is, unlike what occurs with the Lorenz curve, the shape of the Bonferroni curve yields information on the associated distribution.

In the graphic analysis, the differences in inequality between two distributions can be more clearly observed with their corresponding Bonferroni curves than with the Lorenz curves. Figure 1 represents these curves for the distributions of annual disposable income for Spanish households before and after benefits in 2007 using EU-SILC data.

Figure 1. Lorenz curve and Bonferroni curve

There is practically no difference between the Lorenz curves, while the separation between the Bonferroni curves is greater, especially in the lower tail of the distribution. Although this type of interpretation is not statistically significant, it facilitates comparisons.

From curve B(p), the Bonferroni index[6], B, is defined as

$$B = 1 - \int_0^1 B(p)dp = \int_0^1 (1 - B(p))dp = \int_0^1 \frac{p - L(p)}{p} dp .$$

Its value coincides with the area between the Bonferroni curve of the existing distribution and the curve corresponding to the case of perfect equality. It is evident that $B \in [0, 1]$.

When each of these curves is compared in a certain percentile $x=F^{-1}(p)$, $0 \leq p \leq 1$ with its corresponding curve in case of equidistribution, we obtain the inequality accumulated up to this percentile. If we use the Lorenz curve, then:

$$D_L(p) = p - L(p) , \ 0 \leq p = F(x) \leq 1 \qquad [1]$$

is the difference between the share in the total amount of income of individuals whose income is lower than or equal to x in the case of equidistribution and the real share in total income in the distribution under consideration.

If we consider the Bonferroni curve, the function

[6] Very little attention has been given to this index in the literature on economics up to relatively recent years. Nygard and Sandström (1981), Tarsitano (1990), Giorgi and Crescenzi (2001) and Giorgi and Nadarajah (2010) refer to the properties of this index. In Chakravarty (2007) and Bárcena and Imedio (2008), the B index is interpreted as a deprivation measure.

$$D_B(p) = 1 - B(p) = \frac{\mu - E(X|X \leq x)}{\mu}, \quad 0 \leq p = F(x) \leq 1 \qquad [2]$$

measures the relative difference between the mean income of the population and the mean income of individuals whose income is lower than or equal to x.

Although both curves are determined mutually, we introduce value judgements when choosing one of them to measure local inequality because we attach greater or lesser importance to inequality located in certain parts of the distribution. Local inequality measured through $D_L(p)$ is greater in the middle of the distribution and $D_B(p) = D_L(p)/p$ is greater on the left-hand side of the distribution. Effectively, $D_B(p)$ is strictly increasing in $(0,1)$, $D_B(0^+) = 1$, $D_B(1) = 0$, while $D_L(p)$ reaches it maximum value in $p = F(\mu)$, with $D_L(0) = D_L(1) = 0$.

In the next section we introduce a new class of inequality measures that is the main contribution of this paper. In this class, local inequality is measured by means of the function $D_B(p)$ and the density functions of the beta distribution in [0,1] are used as weights.

3. The β class of inequality measures

Let us assume $D:[0,1] \to R$ is a function such that for each $p \in [0,1]$, $D(p)$ measures inequality accumulated up to percentile p and $\omega:[0,1] \to R$ is a non-negative weight function such that $\int_0^1 \omega(p)dp = \int_0^\infty \omega(F(x))dF(x) = 1$. It is clear that the real number

$$I_{D,\omega} = \int_0^1 D(p)\omega(p)dp$$

measures inequality in the distribution F. Its value depends on the functions D and ω, which respectively introduce a way to evaluate cumulative local inequality and a criterion to weight this inequality along the income distribution. This procedure to generate inequality indexes underlies (sometimes in an implicit manner) the papers of Amato (1948), Giacardi (1950a, 1950b), Mehran (1976), Benedetti (1980), Yitzhaki (1983) and Piccolo (1991), among others.

In the class of indexes that we propose, local inequality is measured by means of the function $D_B(p) = D_L(p)/p$ defined in [2], and the density functions of the beta distribution in [0, 1] are used as weights. That is:

$$\omega_{(s,t)} : [0,1] \to R_0^+,\ \omega_{(s,t)}(p) = (b(s,t))^{-1} p^{s-1}(1-p)^{t-1}, s > 0, t > 0, \quad [3]$$

where b(s,t) is the Euler beta function.

The above is set out in the following definitions.

Definition. For each $(s,t) \in R^+ \times R^+$, the index I(s,t) is given by:

$$I(s,t) = \int_0^1 D_B(p)\omega_{(s,t)}(p)dp = (b(s,t))^{-1} \int_0^1 (1 - B(p))p^{s-1}(1-p)^{t-1} dp. \quad [4]$$

We denote the biparametric set $\boldsymbol{\beta}=\{I(s,t)\}_{s,t>0}$ as the beta class of inequality measures.

It is immediate that $I(s,t)$ (respectively $\mu I(s,t)$) is a relative (respectively absolute) measure of inequality, with I(s,t)=0 in case of equidistribution and I(s,t)=1 in case of maximum concentration. That is, $I(s,t)$ is a normalized compromise index.

The elements of $\boldsymbol{\beta}$ are consistent with the ordering of the distribution induced by the Bonferroni curve, and for s ≥ 2 with those induced by the Lorenz[7] curve. Therefore, the elements of $\boldsymbol{\beta}$ satisfy the Pigou-Dalton Transfer Principle: progressive transfers decrease income inequality.

The $\boldsymbol{\beta}$ class adds a broad set of judgements relative to the weight that the social evaluator attaches to the local inequality accumulated in different parts of the distribution. These judgements are derived from the shape of the function $\omega_{(s,t)}(\cdot)$. So, we have:

(i) If $0 < s < 1$, $0 < t < 1$, then $\omega_{(s,t)}(p)$ is U shaped, is symmetric for s=t, and reaches its minimum value for $p = (s-1)/(s+t-2)$.

(ii) If $0 < s < 1$, $t \geq 1$, $\omega_{(s,t)}(p)$ is decreasing and convex.

(iii) If $s \geq 1$, $0 < t < 1$, $\omega_{(s,t)}(p)$ is increasing and convex.

[7] If X and Y are two income distribution functions ($B_X(p) \geq B_Y(p)$, $0 \leq p \leq 1$)$\Rightarrow$ ($I_X(s,t) \leq I_Y(s,t)$). Analogously, the Lorenz consistency: ($L_X(p) \geq L_Y(p)$) $\Rightarrow$ ($I_X(s,t) \leq I_Y(s,t)$).

(iv) If s = 1 (respectively t = 1), t ≥ 1 (respectively s ≥ 1), $\omega_{(s,t)}(p)$ is decreasing (respectively increasing), and $\omega_{(1,1)}(p) = 1$.

(v) If s >1, t >1, $\omega_{(s,t)}(p)$ is bell shaped, if s = t it is symmetric and reaches its maximum for $p = (s-1)/(s+t-2)$.

Therefore, in (i) (respectively (v)) less (more) weight is attached to local inequality in the middle incomes and more (less) weight to the tails. These weights are more focused in the middle incomes as s and t are greater and closer to each other. In the rest of the cases, except for s = t = 1, greater weight is attached to the local inequality in one of the tails of the distribution. Figure 2 shows the functions $\omega_{(s,t)}$ for different values of its parameters.

Figure 2. $\omega_{(s,t)}$ functions.

When local inequality is measured through the Lorenz differences, $D_L(p)=p-L(p)=pD_B(p)$, an equivalent expression for the indexes is:

$$I(s,t) = (b(s,t))^{-1}\int_0^1 (p - L(p))p^{s-2}(1-p)^{t-1}dp\,. \qquad [5]$$

Nevertheless, I(s,t) is not really a weighted mean of $D_L(p)$ because the weights used are the functions:

$$\pi_{(s,t)} : [0,1] \to R^+,\ \pi_{(s,t)}(p) = \frac{\omega_{(s,t)}(p)}{p} = (b(s,t))^{-1}p^{s-2}(1-p)^{t-1}, s>0, t>0\,, \qquad [6]$$

and $\int_0^1 \pi_{(s,t)}(p)dp = (s+t-1)/(s-1) \neq 1$.

Expression [5] proves that the elements of $\boldsymbol{\beta}$ are linear measures of Mehran (1976). They can also be expressed in terms of the income differences $F^{-1}(p) - \mu$, $0 \le p \le 1$, as

$$I(s,t) = \frac{1}{\mu}\int_0^1 (F^{-1}(p) - \mu)\Pi_{(s,t)}(p)dp\,,$$

where

$$\Pi'_{(s,t)}(p) = \pi_{(s.t)}(p)\,,\ 0<p<1,\ \int_0^1 \Pi_{(s,t)}(p)dp = 0\,.$$

The last condition for normalized indexes is equivalent to $\Pi_{(s,t)}(1)=1$.

The above results are summed up in the following proposition.

Proposition 1. The $\boldsymbol{\beta}$ class is a subset of the linear measures of Mehran. Their elements are compromise indexes and consistent with the ordering of the distribution induced by the Bonferroni curve, and for s ≥ 2 with those induced by the Lorenz curve.

The $\boldsymbol{\beta}$ class comprises not only known indexes, but also families of inequality measures frequently used in the literature. For $(s,t)=(1,1)$ and $(s,t)=(2,1)$ we obtain the Bonferroni, B, and Gini, G, coefficients, respectively:

$$I(1,1)=\int_0^1(1-B(p))dp=B, \qquad [7]$$

$$I(2,1)=2\int_0^1(p-L(p))dp=G. \qquad [8]$$

For some particular values of the parameters of $\boldsymbol{\beta}$, we obtain families that generalize the G and B indexes. For s = 2 we obtain the family of the generalized Gini indexes, $\boldsymbol{\gamma}=\{I(2,t)\}_{t>0}$, where

$$I(2,t)=t(t+1)\int_0^1(1-B(p))p(1-p)^{t-1}dp=t(t+1)\int_0^1(p-L(p))(1-p)^{t-1}dp=$$

$$=1-t(t+1)\int_0^1(1-p)^{t-1}L(p)dp,\ t>0. \qquad [9]$$

If t = 1 and $s\in N=\{1,2,...\}$ is a positive integer, we obtain the countable family $\boldsymbol{\alpha}=\{I(s,1)\}_{s\in N}$ defined in Aaberge (2007).

$$I(s,1)=s\int_0^1(1-B(p))p^{s-1}dp=s\int_0^1(p-L(p))p^{s-2}dp=1-s\int_0^1 p^{s-2}L(p)dp,\ s>0. \qquad [10]$$

B and G belong to this family.

When s =1, another interesting family is obtained. The elements of this family are:

$$I(1,t)=t\int_0^1(1-B(p))(1-p)^{t-1}dp=t\int_0^1(p-L(p))p^{-1}(1-p)^{t-1}dp=1-t\int_0^1(1-p)^{t-1}B(p)dp,\ t>0. \qquad [11]$$

B also belongs to this family. Imedio et al. (2011) introduce and analyse the countable family, $\boldsymbol{\delta}=\{I(1,t)\}_{t\in N}$, and compare it, in normative terms, with $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}_N=\{I(2,t)\}_{t\in N}$ the last one being $\boldsymbol{\gamma}$ restricted to positive integer values of the parameter.

In the indexes belonging to the families $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, the differences $D_B(p)$ and $D_L(p)$ are weighted monotonically along the distribution. In $\boldsymbol{\gamma}$ only the Lorenz differences are weighted monotonically. In these three families the focus is on one of the tails of the distribution. On the other hand, we can choose indexes in class $\boldsymbol{\beta}$ that are more sensitive in certain parts of the distribution, or even in a specific percentile. If policymakers are interested in both tails of the distribution and attach less weight to the inequality accumulated by middle incomes, they must select an index $I(s,t)$ with $(s,t)\in(0,1)\times(0,1)$. In this case, when t increases (s), given s (t), the minimum value of the weight is reached for values of p that approach 1 (0). Particularly, if s = t, we obtain the minimum of the weight function for p = 0.5. On the contrary, if policymakers want to use an index that is more sensitive to changes taking place in the middle of the distribution, they must consider an index in which $(s,t)\in(1,+\infty)\times(1,+\infty)$. Given s (t), the maximum weight is attached to incomes with values of p that approach 0 (1) as t (s) increases. In particular, if s=t, the weight function is bell-shaped and symmetric with respect to p=0.5, where it reaches its maximum. In short, the parameters t and s introduce different value judgements in measuring inequality.

When the parameters s and t are both positive integers, $(s,t)\in N\times N$, we obtain a set of indexes that we will call the subclass $\boldsymbol{\beta}_N$, $\boldsymbol{\beta}_N\subset\boldsymbol{\beta}$. The cardinal of $\boldsymbol{\beta}_N$ is less than the cardinal of $\boldsymbol{\beta}$, but the set of value judgements introduced by the elements of $\boldsymbol{\beta}_N$ in the evaluation of inequality is still wide. Among its elements we find classic indexes such as B and G, families that generalize these indexes and uncommon indexes that are interesting by themselves. $\boldsymbol{\beta}_N$ is represented in the following triangular diagram.

$I(1,1)$

$I(1,2)$ $I(2,1)$

$I(1,3)$ $I(2,2)$ $I(3,1)$

$I(1,4)$ $I(2,3)$ $I(3,2)$ $I(4,1)$

....

$I(1,s+t-1)$ $I(2,s+t-2)$ .... $I(s+t-2,2)$ $I(s+t-1,1)$

....

Focus on low incomes Focus on high incomes

The vertex of the triangle is the Bonferroni index, B=I(1,1). Family $\boldsymbol{\alpha}=\{I(s,1)\}_{s\in N}$ is located on the right-hand side. On the left side of the triangle, whose vertex is the Gini index G=I(2,1), we find the indexes of the family $\boldsymbol{\gamma}_N=\{I(2,t)\}_{t\in N}$. The elements of $\boldsymbol{\delta}=\{I(1,t)\}_{t\in N}$ are on the left side of the triangle.

Other interesting families that belong to $\boldsymbol{\beta}_N$ are those whose elements are the indexes in each row of the triangular diagram. The sum of the parameters of the indexes of each of these families (finite) is a constant. In row n – 1, n ≥ 2, we have n–1 indexes, such that $\{I(s,t)\}_{s+t=n}$:

$$I(1,n-1),\ I(2,n-2),......,\ I(n-2,2),\ I(n-1,1).$$

The weight these indexes attach to the accumulated local inequality, as assessed by $D_B(p)$,

$$\omega_{(s,n-s)}(p)=\frac{(n-1)!}{(s-1)!(n-s-1)!}p^{s-1}(1-p)^{n-s-1},$$

is strictly decreasing for s=1, bell-shaped, and takes it maximum for $p=(s-1)/n-2$, $2\leq s\leq n-2$; and is strictly increasing for s = n–1. Therefore, given s+t, when s increases, a lower weight is attached to inequality on the left-hand side of the distribution, while more weight is attached to inequality in middle and right-hand side incomes. Figure 3 shows the weights of the four indexes $\{I(s,t)\}_{s+t=5}$.

Figure 3. Weights for $\{I(s,t)\}_{s+t=5}$

The meaning of the Bonferroni index within the family $\boldsymbol{\beta}_N$ is reinforced by the following property.

Proposition 2. (a) For each $n \in N$, n≥2, B is the arithmetic mean of the indexes $\{I(s,t)\}_{s+t=n}$.

That is, B is the arithmetic mean of each row of the triangular diagram: $\sum_{s=1}^{n-1} I(s,n-s) = (n-1)B$.

(b) The Bonferroni index, B, is a weighted mean of all the generalized Gini indexes with a positive integer parameter: $B = I(1,1) = \sum_{i=1}^{\infty} I(2,i)/i(i+1)$, $\sum_{i=t}^{\infty} 1/i(i+1) = 1$.

Proof. (a) From the values of the beta function when its parameters are positive integer values and applying the Newton expansion of the binomial we get:

$$\sum_{s=1}^{n-1} (b(s,n-s))^{-1} p^{s-1}(1-p)^{n-s-1} = (n-1)\sum_{s=1}^{n-1} \binom{n-2}{s-1} p^{s-1}(1-p)^{n-s-1} =$$

$$= (n-1)\sum_{s=0}^{n-2} \binom{n-2}{s} p^{s}(1-p)^{n-s-2} = (n-1)[p+(1-p)]^{n-2} = n-1.$$

From the previous equality, [4] and [7], we get:

$$\sum_{s=1}^{n-1} I(s,n-s) = \sum_{s=1}^{n-1} (b(s,n-s))^{-1} \int_0^1 (1-B(p))p^{s-1}(1-p)^{n-s-1}dp = (n-1)\int_0^1 (1-B(p))dp = (n-1)B.$$

(b) For $|1-p| < 1$ and, in particular, if 0<p≤1, then $(1/p) = \sum_{i=1}^{\infty} (1-p)^{i-1}$, where convergence is uniform. From this, [7] and [9], we obtain:

$$B = I(1,1) = \sum_{i=1}^{\infty} \int_0^1 (p-L(p))(1-p)^{i-1} = \sum_{i=1}^{\infty} I(2,i)/i(i+1), \sum_{i=1}^{\infty} 1/i(i+1) = 1.$$

The Gini index, G=I(2,1), satisfies a property that is similar to the previous one, 2(a), when the index corresponding to s=1 is eliminated in each row of the triangular diagram.

Proposition 3. For each $n \in N$, n≥3, G is the weighted mean of the indexes $\{I(s,t)\}_{s\geq 2, s+t=n}$: I(2,n–2), I(3, n–3), ..., I(n–1,1). It is satisfied:

$$\frac{2}{(n-1)(n-2)} \sum_{s=1}^{n-1} (s-1)I(s,n-s) = G, \quad \frac{2}{(n-1)(n-2)} \sum_{s=1}^{n-1} (s-1) = 1.$$

Proof. It is analogous to the one in proposition 1 (a). In this case we have:

$$\frac{1}{(n-1)(n-2)}\sum_{s=2}^{n-1}(s-1)(b(s,n-s))^{-1}p^{s-2}(1-p)^{n-s-1}=\sum_{s=2}^{n-1}\binom{n-3}{s-1}p^{s-2}(1-p)^{n-s-1}=$$

$$=\sum_{s=0}^{n-3}\binom{n-3}{s}p^{s}(1-p)^{n-s-3}=[p+(1-p)]^{n-3}=1.$$

Taking into account expression [5], we obtain:

$$\frac{2}{(n-1)(n-2)}\sum_{s=1}^{n-1}(s-1)I(s,n-s)=2\int_0^1(p-L(p))dp=G,\ \text{being}\ \ \frac{2}{(n-1)(n-2)}\sum_{s=1}^{n-1}(s-1)=1.$$

The previous propositions show the algebraic relationship among the elements of $\boldsymbol{\beta}_{\mathbf{N}}$.

In the following section we analyse some normative aspects of the elements of $\boldsymbol{\beta}$.

4. Normative characterizations.

In order to establish the relationship between inequality and social welfare we follow the Yaari approach (1987, 1988). If F is the income distribution and ϕ:[0, 1]$\rightarrow$R is a distribution function[8] that represents social preferences, the Yaari social welfare function (YSWF) is given by

$$W_\phi(F)=\int_{R^+}xd\phi(F(x))=\int_0^1F^{-1}(p)d\phi(p)=\int_0^1\phi'(p)F^{-1}(p)dp\,.$$

Thus, W_ϕ is additive and linear in the incomes and weights them according to the rankings attached to the individuals in the distribution[9]. The weight attached to the income of an individual with rank p, 0<p<1, is $\phi'(p)\geq 0$. Yaari (1988) shows that $W_\phi(F)$ presents an aversion to inequality if, and only if, $\phi'(p)$ is decreasing, which is equivalent to the concavity of ϕ.

If μ is the mean of F and L(p) is its Lorenz curve, YSWF can be expressed as a social welfare function associated to a linear measure of inequality of the type defined in Mehran (1976). Then,

[8] We assume the distribution function to be a class C^2 function, which is twice continuously derivable. When necessary, we will admit the existence of higher order derivatives in later results.

$$W_\phi(F) = \mu\left[1 - I_\phi(F)\right],$$

Where

$$I_\phi(F) = 1 - \frac{W_\phi(F)}{\mu} = 1 - \frac{1}{\mu}\int_{R_0^+} x\phi'(F(x))dF(x), \quad [12]$$

or

$$I_\phi(F) = \int_0^1 (p - L(p))\pi_\phi(p)dp, \;\; \pi_\phi(p) = -\phi''(p), \quad [13]$$

which yields an explicit relationship between the preference distribution and the weighting scheme of the Lorenz differences.

According to the Blackorby and Donaldson approach (1978), the expression $\mu\left[1 - I_\phi(F)\right]$ is the equally distributed equivalent income[10], in which case $\mu I_\phi(F)$ measures the loss of social welfare due to inequality.

If $\Im_\beta = \{\phi_{(s,t)}\}_{s,t>0}$ is the family of preference distributions associated to the elements of **β,** from [6] and [13] we obtain:

$$\phi''_{(s,t)}(p) = -\pi_{(s,t)}(p) = -(B(s,t))^{-1} p^{s-2}(1-p)^{t-1}, s > 0, t > 0. \quad [14]$$

The functions of $\Im_\beta$ are strictly concave. Therefore, in all the indexes of the **β** class and in their corresponding YSWFs, there is an underlying preference (aversion) for equality (inequality). This common attitude, however, presents different degrees of intensity depending on the index.

Although it is not always necessary to know the preference distributions in order to prove some normative properties of the indexes, it is convenient to know the preference distributions to study certain aspects. For example, the possible ranking of the elements of a family of inequality measures according to their inequality aversion is equivalent to rank their respective preference functions depending on their concavity.

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[9] Ben Porath and Gilboa (1994) axiomatize the YSWF for discrete distributions. Zoli (1999) shows that the YSWF dominance is equivalent to the inverse stochastic dominance of Muliere and Scarsini (1989).
[10] This refers to a level of income such that if it is equally attached to all the individuals of the population, it will provide an identical level of social welfare, according to the specified SWF, to that of the existing distribution. This concept is the basis of the AKS approach (Atkinson, 1970; Kolm, 1966; Sen, 1973) for relating social welfare and inequality.

The expressions of the functions $\phi_{(s,t)}$ when $(s,t)\in N\times N$, correspondents to the indexes of $\boldsymbol{\beta}_N$**,** are obtained from [13] and [14] integrating twice. The constant of integration are determined when we impose the conditions $\phi_{(s,t)}(0)=0$ and , $\phi_{(s,t)}(1)=1$. We get this result:

$$\phi_{(1,t)}(p)=\begin{cases} p-p\ln(p)\ ,\ t=1 \\ t\left[p-p\ln(p)+\sum_{i=1}^{t-1}\dfrac{(-1)^i}{i}\dbinom{t-1}{i}\left(p-\dfrac{p^{i+1}}{i+1}\right)\right]\ ,\ t\geq 2. \end{cases}$$

$$\phi_{(s,t)}(p)=(B(s,t))^{-1}\sum_{i=0}^{t-1}\frac{(-1)^i}{s+i-1}\binom{t-1}{i}\left(p-\frac{p^{s+i}}{s+i}\right),\ s\geq 2,\ t\geq 1.$$

Particularly, the preference distributions associated to the Bonferroni and Gini indexes are:

$$\phi_B(p)=\phi_{(1,1)}(p)=p-p\ln(p),\ 0<p\leq 1,\ \phi_{(1,1)}(0)=0.$$

$$\phi_G(p)=\phi_{(2,1)}(p)=2p-p^2,\ 0\leq p\leq 1.$$

Both functions are strictly increasing and strictly concave in the interval [0,1], but ϕ_B is more concave than ϕ_G (Figure 4). Thus, B shows more inequality aversion than G, which influences the normative properties of both indexes. For example, the different behaviour when considering principles of transfers that are more demanding than the Pigou-Dalton Principle.

The relationship between B and G in terms of preference for equality can be extended to the families $\boldsymbol{\gamma}_N$, $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$**.** Imedio and Bárcena (2007) and in Imedio et alt. (2010) show that the functions of families $\Im_{\gamma N}=\{\phi_{(2,t)}\}_{t\in N}$, $\Im_\alpha=\{\phi_{(s,1)}\}_{s\in N}$ and $\Im_\delta=\{\phi_{(1,t)}\}_{t\in N}$ are ranked according to their degree of concavity. As the corresponding parameter increases, the concavity of the functions of $\Im_{\gamma N}$ and $\Im_\delta$ increases, while the opposite occurs for the functions of the family $\Im_\alpha$. In fact, the functions of $\Im_\delta$ and $\Im_{\gamma N}$ converge to the function of maximum concavity in the interval [0,1], which is constant and equal to the unit, except for p=0. That is:

$$\lim_{t\to+\infty}\phi_{(1,t)}(p)=\lim_{t\to+\infty}\phi_{(2,t)}(p)=\begin{cases}0\ ,p=0\ , \\ 1\ ,0<p\leq 1\ .\end{cases}$$

However, the functions of $\Im_\alpha$ converge to the identity in the interval $[0,1]$:

$$\lim_{s \to +\infty} \phi_{(s,1)}(p) = p, \ 0 \le p \le 1,$$

with null concavity.

Consequently, the elements of $\boldsymbol{\beta}$ cover the total range of inequality aversion from maximum aversion (rawlsian leximin) to indifference.

The following figures show the preference functions of the indexes I(1,2), I(1,1)=B, I(2,1)=G and I(3,1), and the preference functions of I(2,3), I(2,2), I(2,1)=G and I(3,1). The extreme case functions, maximum concavity and linear function are included in both figures.

Figure 4. Preference functions.

In the triangular diagram that represents $\boldsymbol{\beta}_N$, the vertex is the Bonferroni index, I(1,1). When we move from I(1,1) over the right-hand side of the triangle, family $\boldsymbol{\alpha}$, the indexes show a decreasing aversion to inequality and attach less weight to low incomes. The opposite occurs when we move from the vertex (from I(2,1)) over the left-hand side of the triangle, family $\boldsymbol{\delta}$ (family $\boldsymbol{\gamma}_N$). In this case the indexes show an increasing aversion to inequality and pay more attention to the incomes in the right-hand side of the distribution.

The sum of the parameters of the indexes in the same row of the triangle is a constant, s+t=n, n≥2. If n≥3, when we move over any row from left to right, we start with an element of $\boldsymbol{\delta}$ and finish with an element of $\boldsymbol{\alpha}$. The degree of aversion to inequality decreases in each row when we move from left to right. As an example, Figure 5 shows the four preference functions of the indexes of the fourth row of the triangle, $\{I(s,t)\}_{s+t=5}$.

Figure 5. Preference distribution of the indexes $\{I(s,t)\}_{s+t=5}$.

As shown in Figure 4, the preference distribution of I(1,4) has the greatest degree of concavity. The degree of concavity, together with the preference for equality, decrease successively as we go from I(2,3) to I(3,2) and to I(4,1). The same behaviour can be observed in the indexes of any row of the triangle, but as s+t increases, the number of indexes also increases. Hence the degree of inequality aversion (or concavity of their corresponding preference functions) covers a broader spectrum and there is a smoother change in the concavity between consecutive indexes.

5. Transfer Principles

The indexes of β satisfy the Pigou-Dalton Principle of Transfers (PDPT) given the concavity of their respective preference distributions. However, when studying this type of measures it is common to analyse if they satisfy more demanding redistributive criteria. An obvious step is to consider principles by which the effect of a transfer is greater when it takes place in the lower part of the distribution. Kolm (1966) and Mehran (1976) propose two alternative versions of a principle of this type. According to the Principle of Diminishing Transfers (PDT), a progressive transfer between two individuals with a given difference in income implies that the lower the income of these individuals, the greater the reduction (increase) in the index (social welfare). A different version of the PDT is given by the Principle of Positional Transfer Sensitivity (PPTS). According to the PPTS, when there is a given difference in ranks among the individuals for whom the transfer takes place, the effect of the transfer is greater when it occurs among individuals in the lower part of the distribution. Although both principles are analogous with regard to the transfers, the income difference between the donor and the recipient is relevant for the PDT, while the proportion of individuals located between both is relevant for the PPTS. The following result shows how both principles are satisfied.

Proposition 4. Let F be an income distribution with mean μ and $I_\phi(F)$ an inequality index whose preference distribution, ϕ, is concave. Then

(i) (Mehran, 1976; Zoli, 1999) Index $I_\phi(F)$ satisfies the PPTS if, and only if, $\phi'''(p) > 0$.

(ii) (Aaberge, 2000) Index $I_\phi(F)$ satisfies the PDT if, and only if, $\phi''(F(x))F'(x)$ is strictly increasing for x>0. This is equivalent to the condition

$$-\frac{\phi'''(F(x))}{\phi''(F(x))} > \frac{F''(x)}{(F'(x))^2}\ ,\ x > 0\,. \qquad [15]$$

Proof. (i) Let δ be a positive income transfer from a small portion, dp, of the population in the p-th percentile to the individuals in a lower percentile p–s, s>0. From [12] we obtain that for a sufficiently small ε>0 the reduction of the index is

$$\nabla I_{\phi}(\delta)=\frac{1}{\mu}\int_{p-\varepsilon/2}^{p+\varepsilon/2}\delta\phi'(t)dt-\frac{1}{\mu}\int_{p-s-\varepsilon/2}^{p-s+\varepsilon/2}\delta\phi'(t)dt\ .$$

By applying the mean value theorem and taking the limit when ε→0, we obtain

$$dI_{\phi}(\delta)=\frac{\delta}{\mu}\left[\phi'(p)-\phi'(p-s)\right]dp\ .$$

It is clear that I_{ϕ} satisfies the PDPT, $dI_{\phi}(\delta)<0$, if, and only if, $\phi'(p)$ is strictly decreasing, which is equivalent to condition $\phi''(p)<0$, which characterizes the concavity of ϕ. Moreover, I_{ϕ} will satisfy the PPTS if, and only if

$$\frac{\delta}{\mu}\left[\phi'(q)-\phi'(q-s)\right]<\frac{\delta}{\mu}\left[\phi'(p)-\phi'(p-s)\right],$$

for all q<p, which is equivalent to the strict increase of function ϕ'', that is, $\phi'''(p)>0$.

(ii) Let δ be a positive income transfer from a small fraction, dF(x), of individuals with income level x to individuals with income x–d, 0<d<x. Then, for a sufficiently small ε>0 from [12] it follows that the reduction in the index is

$$\nabla I_{\phi}(\delta)=-\frac{1}{\mu}\left[\int_{x-d-\varepsilon/2}^{x-d+\varepsilon/2}\delta\phi'(F(x))dF(x)-\int_{x-\varepsilon/2}^{x+\varepsilon/2}(-\delta)\phi'(F(x))dF(x)\right].$$

By applying the mean value theorem and making ε→0, we obtain that the relative change in the index is

$$dI_{\phi}(\delta)=\frac{\delta}{\mu}\left[\phi'(F(x))-\phi'(F(x-d))\right]dF(x)\ .$$

Hence, $dI_{\phi}<0$ if, and only if, ϕ' is a strictly decreasing function, which is equivalent to the concavity of ϕ. In order for the reduction of the index to be greater the lower the income level x, for all y<x the following expression must be satisfied

$$\frac{\delta}{\mu}\left[\phi^{'}(F(y))-\phi^{'}(F(y-d))\right]<\frac{\delta}{\mu}\left[\phi^{'}(F(x))-\phi^{'}(F(x-d))\right],$$

for all d>0. By making d→0, and given that ϕ is concave, the above expression is equivalent to $\phi^{''}(F(y))F^{'}(y)<\phi^{''}(F(x))F^{'}(x)$ for y<x. That is, $\phi^{''}(F(x))F^{'}(x)$ must be a strictly increasing function of x, hence $(\phi^{''}(F(x))F^{'}(x))^{'}>0$, x>0, from which we obtain [15]. □

The above proposition proves that an inequality measure satisfies, or does not satisfy, the PPTS depending on the properties of its preference distribution, ϕ, irrespective of the income distribution to which it is applied. It is, therefore, a characteristic of the index. However, the same does not occur with the PDT. That $I_\phi(F)$ satisfies the PDT does not only depend on the properties of its preference distribution, but also on the shape of the income distribution. Expression [15] gives the relationship that must be satisfied by both distributions. That is, given ϕ, index $I_\phi(F)$ verifies the PDT only for a given class of income distributions whose extension depends on the degree of inequality aversion of ϕ.

By applying the above result to the preference distributions associated to the elements of $\boldsymbol{\beta}$, we obtain the behaviour of these indexes with regard to both principles.

Proposition 5.

a) The elements of $\boldsymbol{\beta}=\{I(s,t)\}_{s,t>0}$ satisfy the PPTS if, and only if, the following is satisfied[11]:

$$A(s,t,p)=\left[(s+t-3)p-s+2\right]>0\text{ , }0<p<1. \qquad [16]$$

b) Let F be the distribution function and I(s,t) the inequality index associated to F, the PDT is satisfied if, and only if:

$$\frac{(s+t-3)F(x)-s+2}{F(x)(1-F(x))}>\frac{F^{''}(x)}{(F^{'}(x))^{2}}\text{ , }x>0. \qquad [17]$$

According to the above, the Bonferroni index, B=I(1,1), satisfies the PPTS, because A(1,1,p)=1−p, 0<p<1. However, this is not true for the Gini index, G=I(2,1), because

[11] The sign of the third derivative of the preference distribution is the same as the sign of A(s,t,p). It is obvious that the sign can or cannot be constant in the interval [0,1] depending on the values of s and t.

A(2,1,p)=0, 0<p<1. In the case of a fixed difference in ranks, the Gini coefficient attaches an equal weight to a given transfer irrespective of where it takes place in the income distribution Other indexes show the opposite behaviour with respect to the PPTS. In the case of a fixed difference in ranks, they assign more weight to transfers at the upper than at the central and the lower parts of the distribution. This is the case of I(3,1) because A(3,1,p)=p−1<0, 0<p<1. There are other indexes which behave in a non-uniform manner with respect to the PPTS. For example, for I(3,3) $A(3,3,p) = 3p - 1$, therefore, it satisfies the PPTS if p>1/3, but does not if 0<p<1/3.

In the triangular diagram that represents $\boldsymbol{\beta}_N$, the elements in any of the rows, $\{I(s, n - s)\}_{1 \le s \le n-1}$, n≥2, can show different behaviours with respect to the PPTS. When we move to the right in a row, the indexes go from satisfying the PPTS for I(1,n–1)∈$\boldsymbol{\delta}$, I(2,n–2)∈$\boldsymbol{\gamma}_N$, to only satisfying the principle in a subinterval of [0,1] and, finally, to satisfying the contrary to what the PPTS implies in the case of I(n–1,1)∈$\boldsymbol{\alpha}$.

With respect to the PDT, we should observe that if an index shows inequality aversion ($\phi''(p) < 0$) and the third derivative of the preference function is non-negative ($\phi'''(p) \ge 0$), then it will satisfy the PDT for all the concave income distributions ($F''(p) < 0$) because the condition [15] is then satisfied. Thus, all the elements of $\boldsymbol{\gamma}_N$ and $\boldsymbol{\delta}$ satisfy the PDT when the income distributions are concave. In these cases, the concavity of F is a sufficient condition.

Particularly, the Gini index satisfies the PDT when the income distribution is strictly concave. For the Bonferroni index, B=I(1,1), expression [15], or equivalently [17], is $(1/F(x)) > (F''(x))/(F'(x))^2$, $x > 0$, which is equivalent to the strict concavity of ln(F(x)); a less demanding condition than the concavity of F(x). Therefore, the set of distributions for which B satisfies PDT strictly contains the set of distributions for which G satisfies this principle. It is, let Ω_I be the set of distributions for which the index I satisfies the PDT, the relation of inclusion is $\Omega_G \subset \Omega_B$.

The behaviour of the indexes of γ_N, α and δ with respect to PDT and the relations of inclusion among the set of distributions for which these indexes satisfy the PDT, are studied in depth in Imedio at al. (2010). It is shown that as the degree of inequality aversion of the indexes of a family increases (decreases), the set of distributions for which the indexes satisfy the PDT is larger (smaller).

6. Illustration

This section empirically illustrates the different degree of inequality aversion of the indexes of the $\boldsymbol{\beta}$ family and their different sensitivity to changes in the income distribution depending on the part of the distribution where the change affects most.

We use data from the European Union Statistics on Income and Living Conditions (EU-SILC) for the year 2008 on Spain[12]. We consider three variables:

- Total annual disposable household income (X_1).
- Total annual disposable household income before social transfers other than old-age and survivor's benefits (disposable household income before transfers, X_2).
- Total annual disposable household income plus tax on income and social contributions (income before taxes, X_3).

Social transfers are skewed towards the left tail of the distribution. On the other hand, tax on income and social contributions are skewed towards the right tail of the distribution.

We evaluate the effect of both income components on the indexes $\{I(s,t)\}_{s+t=6}$, $(s,t) \in N \times N$. They are the elements of the fifth row of the triangle that represents $\boldsymbol{\beta}_N$. The five selected indexes have different characteristics[13]. Their degree of inequality aversion diminish

[12] EU-SILC is the reference source for comparative statistics on income distribution, living conditions and social exclusion at the European level. Household incomes are adjusted ('equivalised') to take account of the differences between them in terms of size and composition. We use the modified OECD equivalence. This scale attaches a value of 1 to the first adult member of the household, 0.5 to the remaining adult members and 0.3 to each member under 14 years of age.

[13] Even though any selection of indexes is arbitrary, the one considered combines a reduced number of different indexes : one from the $\boldsymbol{\delta}$ family, other form $\boldsymbol{\gamma}_N$, other form $\boldsymbol{\alpha}$ and two more that do not belong to known families (I(3,3) and I(2,4)). Any row of the triangle that represents $\boldsymbol{\beta}_N$ have a similar composition of indexes.

from $I(1,5)$ to $I(5,1)$. The weight attached to $I(1,5)$ is strictly decreasing in $(0,1)$, focusing on low incomes. The weights attached to $I(2,4)$, $I(3,3)$ and $I(4,2)$ reach their maximum values in $p = 0,25$, $p = 0,50$ and $p = 0,75$, respectively. Index $I(5,1)$ focuses on high incomes and the corresponding weight[14] function is strictly increasing in $(0,1)$.

Table 1 shows the values of $\{I(s,t)\}_{s+t=6}$ for X_1, X_2 and X_3 respectively. Under these values we show the percentage of variation in inequality when we go from X_2 to X_1 (social transfers) and from X_3 to X_1 (tax on income).

Table 1. Inequality indexes $\{I(s,t)\}_{s+t=6}$, $(s,t) \in N \times N$

Distribution	**I(1,5)**	**I(2,4)**	**I(3,3)**	**I(4,2)**	**I(5,1)**
X_1	0.667	0.520	0.408	0.299	0,177
X_2	0.727	0.563	0.438	0.320	0.189
X_3	0.689	0.547	0.434	0.323	0.194
Variation					
X_2 to X_1	**− 8.2%**	**− 7.6%**	**− 6,8%**	**− 6.6%**	**− 6.3%**
X_3 to X_1	**− 3.2%**	**− 4.9%**	**− 6.0%**	**− 7,4%**	**− 8.8%**

Both income components reduce inequality. The percentage reductions in the indexes differ depending on the effect these components have on the income distribution. Social transfers (mainly received by people with low incomes) produce a greater reduction in the indexes focused on the left tail of the distribution. Tax collection is mainly concentrated in the right tail of the distribution. The redistributive effect of the tax is more pronounced when we use indexes that show more sensitivity to changes in higher incomes.

The above demonstrates that the same transfer can be evaluated as having a greater or lesser impact on inequality depending on the part of the distribution in which the index used pays more attention. If policymakers focus on low incomes and choose I(1,5) to measure inequality, then transfers will reduce inequality more than taxes (8.2% vs. 3.2%). On the other hand, if policymakers focus on higher incomes and evaluate the impact of both components of income with I(5,1), they will point to taxes as being more effective for reducing inequality

[14] We have to note that each inequality measure summarises the distribution of income and that the weights are attached to the cumulative local inequality up to each percentile. Therefore, the value of the index does not only depends on the point where the weighting function reaches its maximum.

(8.8% vs. 6.3%). Using $I(3,3)$ both income components have a similar effect on inequality. In short, when measuring inequality it is important to bear in mind the characteristics of the indexes being used.

7. Conclusions

The indexes of the β class introduce different criteria in the measurement of inequality. At the same time, they share a common set of properties and have a clear formal analogy. Because of this we are able to treat different well-known families of inequality measures uniformly. Other advantage of this class of inequality measures is that it is possible to choose the indexes that focus on a particular percentile of the income distribution.

Each element of β weights cumulative local inequality in a different way. The characteristics of the weighting schemes define the properties of the indexes: i.e. income percentiles where the index attaches more importance, level of inequality aversion and the effect of transfers depending on rank or income differences between the donor and the receiver, and depending on the part of the distribution where the receiver and the donor are located.

The subclass β_N is obtained when the two parameters of the indexes of β are positive integers and the subclass contains the families γ_N, α and δ. These families cover a wide spectrum of inequality aversion that can be introduced by an index. These families introduce indexes with different behaviours with respect to the PPTS or to the PDT and contain infinite and countable indexes. But the indexes in the same row of the triangle that represent β_N are finite families of the kind $\{I(s, n-s)\}_{1 \le s \le n-1}$, n≥2. The degree of inequality aversion in these families changes, but does not reach the extreme, while the responses to the principles of transfers also change. When we move from left to right in each row, the sensitivity of the indexes to changes in different parts of the income distribution also changes.

It is obvious that the real effectiveness of an economic policy to affect inequality or welfare in a certain part of the distribution does not depend on the index used to assess the impact. Even so, it is interesting to be able to choose indexes that are more sensitive to changes in the part of the distribution we are interested in focusing our attention on. All of this allows us to quantitatively assess the policy to be implemented in a better manner.

In practice, the choice of a small set of elements of $\boldsymbol{\beta}$ allows inequality to be measured according to different distributive criteria, the preferences of the social evaluator and the particular nature of each empirical case. If we attempt to rank a set of income distributions with these indexes, it is likely that different rankings will be obtained depending on the index. Bearing in mind the characteristics of each measure, a result of this type would be highly revealing, such as the case of robust rankings.

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9. References

Aaberge, R.: Characterizations of Lorenz curves and income distributions. Social Choice and Welfare **17**, 639-653 (2000)

Aaberge, R.: Gini's nuclear family. Journal of Economic Inequality, **5-**3, 305-322 (2007)

Amato, V.: Sulla misura della concentrazione dei redditi. Rivista Italiana di Demografia e Statistica, **2**, 504-529 (1948)

Atkinson, A. B.: On the measurement of inequality. Journal of Economic Theory **2,** 244-263 (1970)

Bárcena, E., Imedio, L. J.: The Bonferroni, Gini and De Vergottini indices. Inequality, welfare and deprivation in the European Union in 2000, Research on Economic Inequality **16**, 231-257 (2008)

Benedetti, C.: Contro-esempi in statistica e probabilità, Studi in Onore di Paolo Fortunati, Clueb, Bologna vol 1, 23-29 (1980)

Blackorby, C., Donaldson, D.: Measures of relative equality and their meaning in terms of social welfare. Journal of Economic Theory **18**, 59-80 (1978)

Bonferroni, C. E.: Elementi di statistica generale, Libreria Seber, Firenze. (1930)

Chakravarty, S. R.: A deprivation-based axiomatic characterization of the absolute Bonferroni index of inequality, Journal of Economic Inequality **5** (3), 339-351 (2007)

Eurostat, "Cross-sectional EUSILC UDB 2008 microdata", Release 01-03-2011, European Commission, Eurostat (2010).

Giacardi, F.: Un criterio per la costruzione di indici di concentrazione, Rivista Italiana di Demografia e Statistica 4(3-4), 527-538 (1950 a)

Giacardi, F.: Indici di concentrazione, Giornale degli Economisti e Annali di Economia 9(5-6), 282-290 (1950 b)

Gini, C.: Sulla misura della concentrazione e della variabilità dei caratteri. Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti 73, 1203-1248. (1914)

Giorgi, G. M., Crescenzi, M.: A look a the Bonferroni inequality measure in a reliability framework, Statistica **41**, 571-573 (2001)

Giorgi, G. M., Nadarajah, S.: Bonferroni and Gini indices for various parametric families of distributions, Metron, **68** (1), 23-46 (2010)

Imedio-Olmedo, L. J., Bárcena-Martín, E.: Dos familias numerables de medidas de desigualdad. Investigaciones Económicas **XXXI**(1), 191-217 (2007)

Imedio-Olmedo, L. J., Bárcena-Martín, E., Parrado-Gallardo, E. M.: A class of Bonferroni Inequality Indices. Journal of Public Economic Theory, **13** (1), 97-124 (2011)

Kakwani, N. C.: On a class of poverty measures. Econometrica **48**, 437-446 (1980)

Kolm, S.-Ch.: The optimal production of social justice. In: H. Guitton, Margolis J. (eds.), Proceedings of the International Economic association Conference on Public economics, Biarritz. Public Economics. pp. 145-201, Macmillan, London (1969)

Mehran, F.: Linear measures of income inequality. Econometrica **44** (4), 805-809 (1976)

Muliere, P., Scarsini M.: A note on stochastic dominance and inequality measures. Journal of Economic Theory **49**, 314-323 (1989)

Newbery, D. M.: A theorem of the measurement of inequality, Journal of Economic Theory **2**, 264-266 (1970)

Nygard, F., Sandström, A.: Measuring Income Inequality, Stockholm: Almqvist and Wicksell International (1981)

Piccolo, D.: Risultati asintotici per misure di variabilità sull'insieme dei numeri interi. Quaderni di Statistica ed Econometria, **XIII**, 91-107 (1991)

Porath, E. B., Gilboa, I.: Linear measures, the Gini index, and the income equality trade-off. Journal of Economic Theory **64** (2), 443-467 (1994)

Rawls, J.: A theory of justice. Harvard University Press, Cambridge (1971)

Sen, A. K.: On economic inequality. Clarendon Press, Oxford (1973)

Tarsitano, A.: The Bonferroni index of income inequality, Income and Wealth distribution, Inequality and Poverty, C. Dagum and M. Zenga, eds., Springer-Verlag, 228-242, (1990)

Yaari, M. E: The dual theory of choice under risk, Econometrica **55** (1), 99-115 (1987)

Yaari, M. E.: A controversial proposal concerning inequality measurement. Journal of Economic Theory **44** (2), 381-397 (1988)

Yitzhaky, S.: On an extension of the Gini index. International Economic Review **24**, 617-628 (1983)

Zoli, C.: Intersecting generalized Lorenz curves and the Gini index. Social Choice and Welfare 16, 183-196 (1999)

Figure 1. Lorenz curve and Bonferroni curve

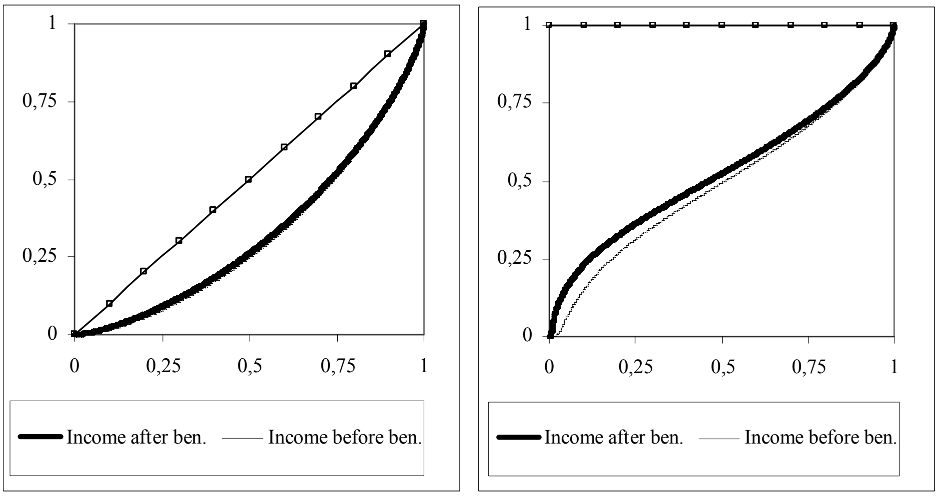

Figure 2. $\omega_{(s,t)}$ functions.

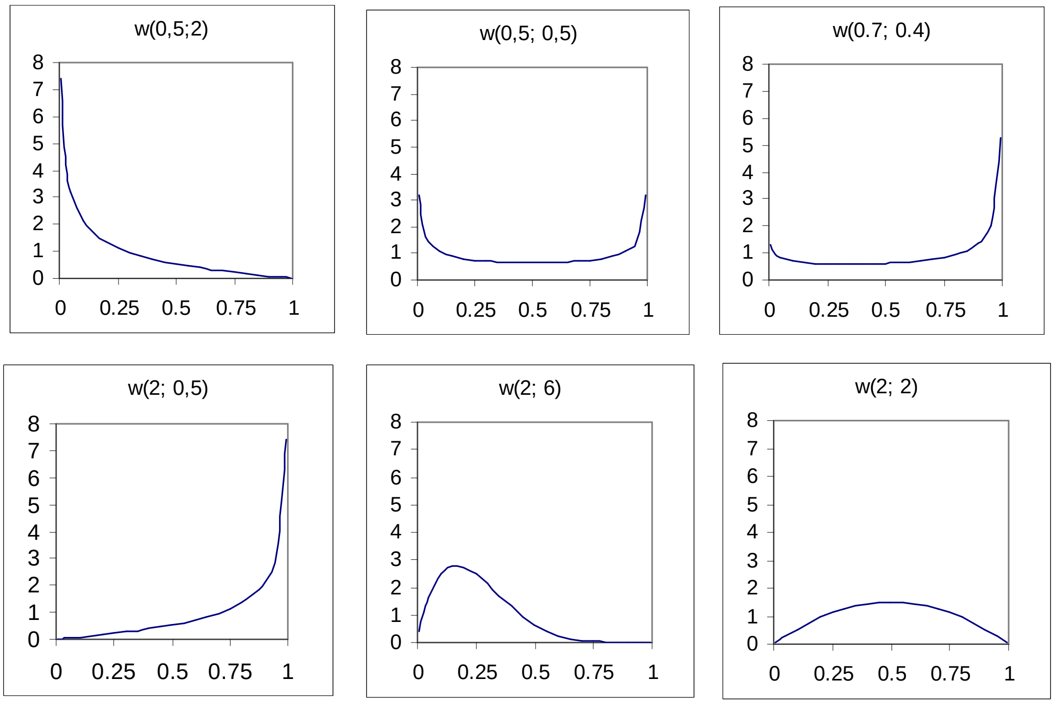

Figure 3. Weights for $\{I(s,t)\}_{s+t=5}$

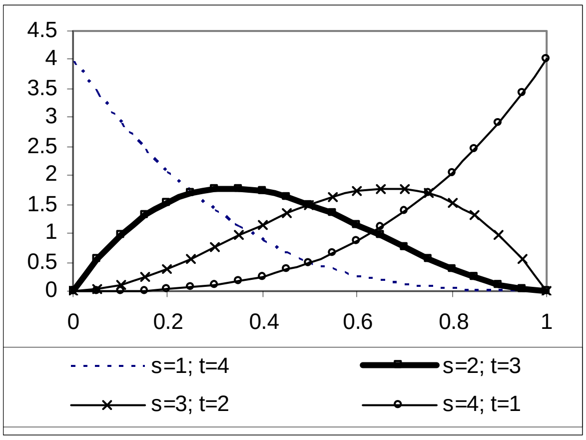

Figure 4. Preference functions.

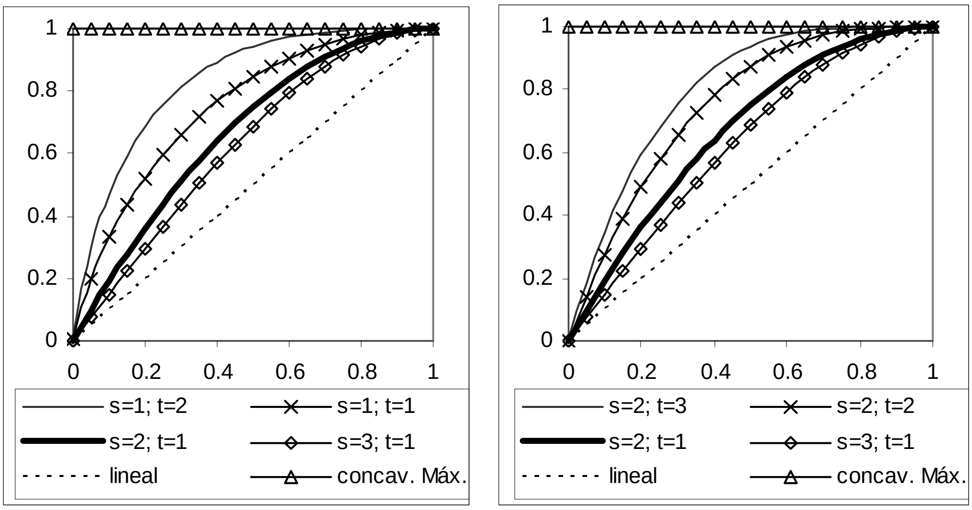

Figure 5. Preference distribution of the indexes $\{I(s,t)\}_{s+t=5}$.