ECINEQ
Society for the Study of Economic Inequality

Working Paper Series

On a family of achievement and shortfall inequality indices

Satya R. Chakravarty
Nachiketa Chattopadhyay
Conchita D'Ambrosio

ECINEQ WP 2013 - 300

ECINEQ 2013 - 300
August 2013

www.ecineq.org

On a family of achievement and shortfall inequality indices

Satya R. Chakravarty†
Nachiketa Chattopadhyay
Indian Statistical Institute, Kolkata , India

Conchita D'Ambrosio
Université du Luxembourg

Abstract

This paper identifies a family of absolute consistent inequality indices using a weakly decomposable postulate suggested by Ebert (2010). Since one member employs an Atkinson (1970) type aggregation we refer to it as the Atkinson index of consistent inequality. A second member of this family parallels the Kolm (1976) index of inequality while a third member of the family can be regarded as the normalized Theil (1972) consistent mean logarithmic deviation index. Two innovative features of these indices are that no specific structure is imposed on the form of the index at the outset and no transformation of any existing index is considered to ensure consistency. Each of them regards an achievement distribution as equally unequal as the corresponding shortfall distribution. We apply these indices to study inequality in mental health in Britain between 1991 and 2008.

Keywords: Achievement inequality, shortfall inequality, consistency, BHPS.

JEL Classification: D63, I31.

†Contacts of S. Chakravarty: satya@isical.ac.in, N. Chattopadhyay: nachiketa@isical.ac.in and C. D'Ambrosio: conchita.dambrosio@uni.lu.

1. Introduction

A recent concern in the literature on the measurement of inequality is to look at both achievement and shortfall inequalities and establish relation between them. In measuring inequality in some dimension of human well-being represented by a bounded variable, e.g., nutritional intake or health status, researchers often focus on attainments or shortfalls (Sen, 1992). Achievement inequality concentrates on the attainments of the individuals in the dimension, whereas shortfall inequality focuses on the shortfalls of the attainments from the maximum possible level of attainment. Going back to health status, which is the focus of our empirical exercise, shortfall inequality measures differences in bad health while achievement inequality captures inequality in good health.

When achievement and shortfall inequalities are measured identically, we say that there is consistency between the two notions of inequality and the underlying inequality index is called a consistent index. In particular, under consistency they must always move along the same direction. Using data from Australia and Sweden, Clarke et al. (2002) demonstrated that for the commonly used concentration index, the inequality rankings of achievements and shortfalls are not the same. Working within the generalized Gini and generalized coefficient of variation frameworks, Erreygers (2009) characterized respectively the absolute Gini index and the variance as two consistent indicators of inequality. Both were found to be inversely related to the difference between the bounds of the distributions.

Lambert and Zheng (2011) considered a weaker condition within the Zoli (1999) inequality partial ordering framework and showed that for no documented intermediate inequality orderings and the relative ordering for achievements will coincide with that for shortfalls. In contrast, the absolute inequality partial ordering fulfills this condition unambiguously. They also identified two classes of absolute inequality indices which measure achievement and shortfall inequalities identically and showed that the variance is the only subgroup decomposable consistent absolute inequality index. Lasso de la Vega and Aristondo (2012) devised a procedure that enables conversion of any inequality index into an indicator that measure achievement and shortfall inequalities equally. In particular, they considered relative and absolute indices of inequality. They have also analyzed the Theil (1972) mean logarithmic deviation index, the only subgroup decomposable relative inequality index which uses subgroup population proportions as coefficients of subgroup inequality levels, to determine the within-group component of the total inequality.

In this paper we identify a family of absolute consistent inequality indices using a weakly decomposable postulate suggested by Ebert (2010). The properties of the family are investigated in details. Since one member of this family employs an Atkinson (1970) type aggregation we refer to it as the Atkinson (1970) index of consistent inequality. This parametric index contains positive multiples of the standard deviation and the absolute Gini index as special cases. A second member of this family parallels the Kolm (1976) index of inequality. The maximax index drops out as a special case of both the indices. A third member of the family can be regarded as the normalized Theil (1972) consistent mean logarithmic deviation index. Since the Atkinson index contains two well-known indices as special cases, we also develop an axiomatic characterization of this index.

Two innovative features of these indices are that no specific structure is imposed on the form of the index at the outset and no transformation of any existing index is considered to ensure consistency. Each of them regards an achievement distribution as equally unequal as the corresponding shortfall distribution.

The paper is organized as follows. The next section builds the formal framework and presents the analysis. An empirical application to the study of inequality in mental health in Britain is presented in section 3. Finally, Section 4 makes some concluding remarks.

2. The Formal Framework

Assume that for any person i the level of achievement x_i takes on values in the non-degenerate interval $D^1 = [0, a]$ and for any $n \in N/\{1\}$ the achievement distribution is denoted by $x = (x_1, x_2, \cdots, x_n) \in D^n$, where $a > 0$ is finite, $D^n = [0, a]^n$, the $n-$ fold Cartesian product of $[0, a]$ and N is the set of positive integers. Let $D = \bigcup_{n \in N/\{1\}} D^n$. For all $n \in N/\{1\}$ and $x \in D^n$, we write $\lambda(x)$ for the mean achievement. The $n-$coordinated vector of ones is denoted by 1^n, where $n \in N/\{1\}$ is arbitrary. By assumption a is the maximum level of achievement and the shortfall experienced by person i in the distribution x is $s_i = a - x_i$. For any $n \in N/\{1\}$ and $x \in D^n$, the associated shortfall distribution $s(x) = (s_1, s_2, \cdots, s_n)$ is as well an element of D^n . For $x, y \in D$, x is obtained from y by a progressive transfer, if there is a pair (i, j) such that $x_i - y_i = y_j - x_j = \eta > 0$, $y_j - \eta \geq y_i + \eta$ and $x_l = y_l$ for all $l \neq i, j$. That is, there is a transfer of a positive amount of achievement η from y_j to a lower level y_i so that the donor j does not become poorer than the recipient i .

We assume the following postulates for an index of inequality $I : D \to \Re^1$, where $\Re^1$ is the real line.

Symmetry (SYM): For all $x \in D$, $I(x) = I(y)$, where y is any permutation of x .

Pigou-Dalton Transfers Principle (PDT): For all $y \in D$, if x is obtained from y by a progressive transfer, then $I(x) \leq I(y)$.

Dalton Population Principle (DPP): For all $n \in N/\{1\}$, $x \in D^n$, $I(x) = I(y)$, where y is the $l-$fold replication of x, that is, each x_i appears l times in y , $l \geq 2$ being any integer.

Continuity (CON): For al $n \in N/\{1\}$, I is a continuous function.

Symmetry demands that inequality should not be sensitive to reordering of the achievements. Thus, for a symmetric index the individuals should not be distinguished by anything other than their levels of achievement in the considered dimension. Symmetry enables us to define the inequality index directly on ordered distributions. The Pigou-Dalton Transfers Principle demands that a progressive transfer of achievement, that is, a transfer from a person to anyone who achieved less so that the donor does not become poorer than the recipient, should not increase inequality. Under symmetry only rank preserving transfers are allowed. Non-increasingness of an inequality index under a rank preserving progressive transfer is equivalent to S-convexity of the index (Dasgupta et al. 1973).[1] According to the Dalton Population Principle, inequality remains invariant under

[1]A real valued function f defined on D^n is called S-convex if $f(xB) \leq f(x)$ for all $x \in D^n$ and for all $n \times n$ bistochastic matrices. An $n \times n$ matrix with non-negative entries is called a bistochastic matrix if each of its columns and rows sums to one. If a function f is S-convex , then - f is S-concave. All S-convex functions are symmetric.

replications of the population. This postulate, which enables us to view inequality as an average concept, becomes helpful for cross-population comparisons of inequality. Continuity is a condition to guarantee that there will be no abrupt changes because of minor observational errors in incomes.

We assume that the inequality index I is translation invariant, that is, of absolute type. Formally, for all $n \in N/\{1\}$, $x \in D^n$, $I(x) = I(x + c1^n)$, where c is a scalar such that $x + c1^n \in D^n$. This means that inequality does not alter under equal absolute changes in all achievements. This notion on inequality invariance contrasts with relative concept which requires inequality to remain unaltered when all achievements are scaled equi-proportionally. However, in view of Lambert and Zheng's (2011) finding that orderings involving intermediate and relative inequality concepts cannot produce identical rankings of achievement and shortfall distributions, we rule out this at the outset.

In order to characterize the family of consistent indices, following Ebert (2010), we consider the following axiom.

Decomposability (DEC): For every $\underline{n} = (n^1, n^2)$, where $n^1 \geq 1$ and $n^2 \geq 1$ are integers, there exist positive weight functions $w^1(\underline{n})$, $w^2(\underline{n})$ and $u(\underline{n})$ such that

$$I(x^1, x^2) = w^1(\underline{n})I(x^1) + w^2(\underline{n})\ I(x^2) + u(\underline{n})\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} I(x_i^1, x_j^2), \qquad (1)$$

where x^j is any arbitrary income distribution over the population with size n^j, $j = 1,2$. This axiom enables us to decompose overall inequality of the achievement distribution (x^1, x^2) into a within-group component $w^1(\underline{n})I(x^1) + w^2(\underline{n})I(x^2)$ and a between-group component $u(\underline{n})\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} I(x_i^1, x_j^2)$. While the former corresponds to the usual within-group term used in the literature (e.g., Bourguignon, 1979, Shorrocks, 1980), the latter depends on pairwise comparisons of incomes. The usual between-group term $I(\lambda_1 1^{n_1}, \lambda_2 1^{n_2})$ is level of inequality that would arise if each achievement in a subgroup were replaced by the mean value of the subgroup, where λ_j is the mean of the distribution x^j, $j = 1,2$. In the usual case the decomposability postulate is stated for any arbitrary number of subgroups. As Ebert (2010) stated, **DEC** can be extended to more than two subgroups. An inequality index satisfying **DEC** is called weakly decomposable.

Ebert (2010) also assumed that the inequality index is normalized, that is, it takes on the value zero if and only if all the incomes are equal. Formally,

Normalization (NOM): For al $n \in N/\{1\}$, $I(x)=0$, if and only if $x \in D^n$ is of the form $x = c1^n$, $c \in D^1$ being arbitrary.

The following theorem can now be stated.

Theorem 1: Assume that ***NOM*** holds for any two-person society. Then the *inequality index I satisfies **CON**, **DPP**, **DEC**, **SYM** and translation invariance if and only it is of the form*

$$I_\psi(x) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(|x_i - x_j|), \tag{2}$$

where $\psi : D^1 \to R^1$ *is continuous and* $\psi(0)=0$.

Proof. See Appendix A. □

In the pairwise comparison between persons i and j with achievements x_i and x_j respectively, the person who has achieved less, say j, will feel deprived from the comparison with the better off individual. The difference $(x_i - x_j)$ can be taken as an indicator of his deprivation. Likewise, $(x_j - x_i)$ represents person $i's$ deprivation if $x_j > x_i$. The index I in (2) aggregates all deprivations that may arise in all pairwise comparisons in the society using the transformation ψ. Since deprivation is not likely to decrease if the gap $|x_i - x_j|$ increases, which may result if, given $x_j > x_i$, there is a transfer from x_i to x_j, we assume that ψ is non-decreasing. Also $\psi(0)=0$ ensures that in any pairwise comparison if there is no feeling of deprivation by none of the two persons, then the overall deprivation is zero. Note that $\psi(|x_i - x_j|) = \psi(|s_i - s_j|)$. Thus, the index in (2) measure inequality consistently.

In the following theorem we show that non-decreasingness of ψ is necessary and sufficient for I_ψ to satisfy **PDT**.

Theorem 2: *The inequality index* I_ψ *given by equation (2) satisfies **PDT** if and only if the function* ψ *identified in Theorem 1 above is non-decreasing.*

Proof. See Appendix A. □

As an illustrative example, let $\psi(t) = t^r$, where $r \geq 0$ is a constant. (Since $r = 0$ makes the index a constant, we omit this case in the discussion below.) Then the corresponding index becomes

$$I_\varepsilon(x) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|x_i - x_j\right|^r . \quad (3)$$

This family was suggested by Ebert (2010) as a class of weakly decomposable indices. However, his assertion that I_r fulfills **PDT** if and only if $r \geq 1$ is not true (see Proposition 3 of Ebert, 2010). To see this, consider the distribution $y = (2,4,6,8,10)$. The distribution $x = (3,4,5,8,10)$ is obtained from y by a transfer of 1 unit of achievement from the third richest person to the poorest person. Then for $r = 0.5$, $I_{0.5}(x) = 1.45 < I_{0.5}(y) = 1.55$, which is a counter example to Proposition 3 of Ebert (2010).

One major problem with I_ψ is that it may not be bounded above. For instance, let $\psi(x_i, x_j) = e^{\theta|x_i - x_j|}, \theta > 0$. Then for any unequal distribution x the resulting index $I_\theta(x) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta\left|x_i - x_j\right|}$ increases as θ increases. As $\theta \to \infty$, for any unequal x, $I_\theta(x) \to \infty$. In an empirical exercise we may often be interested in comparing two different inequality indices. If one of them is bounded and the other is unbounded then the comparison does not look feasible. Therefore, it is desirable that all indices under comparison should be bounded. In order to avoid this problem, we consider a particular cardinalization of I_ψ that becomes bounded. However, since the definition of this particular cardinalization of I_ψ relies on the inverse of the function ψ, we assume that ψ is increasing.

Let $\Psi = \left\{\psi : D^1 \to \Re \,\middle|\, \psi \text{ is increasing, continuous and } \psi(0) = 0\right\}$. Given any achievement distribution $x \in D^n$ and $\psi \in \Psi$, we define the representative summary deprivation $I_\psi^R(x)$ as that level of deprivation which when arises in all pairwise comparisons will make the existing distribution inequality equivalent. Thus, in a two-person achievement distribution (x_1, x_2), $I_\psi^R(x)$ is simply the distance between the origin and the foot of the perpendicular drawn on the horizontal axis from the point of intersection of the iso-inequality contour and the 45^0 line passing through the origin. Formally, for any $x \in D^n$ and $\psi \in \Psi$,

$$\frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi\left(I_\psi^R(x)\right) = \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi\left(\left|x_i - x_j\right|\right). \quad (4)$$

From (4) it follows that

$$I_\psi^R(x) = \psi^{-1}\left[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi\left(\left|x_i - x_j\right|\right)\right]. \quad (5)$$

The representative summary deprivation is a particular numerical representation of the inequality index identified in (2). That is, for all $n \in N/\{1\}$, $x, y \in D^n$,

$$I_\psi^R(x) \geq I_\psi^R(y) \leftrightarrow I_\psi(x) \geq I_\psi(y). \quad (6)$$

Given that I_ψ^R is a particular cardinalization of the index in (2), we can as well use I_ψ^R as an index of inequality.

The following theorem establishes some important properties of I_ψ^R.

Theorem 3: *For any $\psi \in \Psi$, I_ψ^R is continuous and bounded between 0 and a, where the lower bound is achieved when the achievements are equally distributed. The index I_ψ^R also satisfies **SYM**, **DPP**, **PDT** and **NOM**.*

Proof. See Appendix A. □

In order to illustrate I_ψ^R, let $\psi(t) = t^r$, where $r > 0$, so that the resulting index becomes

$$I_r^R(x) = \left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|x_i - x_j\right|^r\right)^{1/r}. \quad (7)$$

The index I_r^R is the symmetric mean of order r of the deprivation gaps $\left|x_i - x_j\right|$. Since it employs the Atkinson (1970)-type aggregation, we refer to it as the Atkinson consistent inequality index. For a given distribution x, the index increases as r increases. As the value of r increases, more weight is assigned to higher gaps in the aggregation if

$r > 1$, whereas the opposite holds if $r < 1$. For $r = 1$, I_{ψ}^{R} becomes twice the absolute Gini index of inequality, whereas for $r = 2$, it is $\sqrt{2}$ times the standard deviation. As $r \to \infty$, $I_{r}^{R} \to \max_{i,j}\left\{\left|x_i - x_j\right|\right\}$, the maximax index.

It may be worthwhile to mention here that there have been several generalizations of Gini's mean difference in the literature. Donaldson and Weymark (1981) used a rank-ordered income vector to investigate an ethical equality measure called the S-Gini (see also Yitzhaki, 1983). Chakravarty (1988) has suggested a family of Gini coefficient called E-Gini that satisfies the diminishing transfers principle. Ebert (1988) has also characterized two families of ethical inequality measures which are generalizations of the Gini coefficient.

Now, as a second example, let us consider $\psi(x_i, x_j) = e^{\theta|x_i - x_j|}, \theta > 0$, which generates the Kolm (1976) consistent inequality index given by

$$I_{\theta}^{R}(x) = \frac{1}{\theta}\log\left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta\left|x_i - x_j\right|}\right). \tag{8}$$

A transfer of achievement from a person i to a richer person j will increase I_{θ}^{R} by a larger amount as θ increases. As $\theta \to 0$, I_{θ}^{R} approaches zero, whereas I_{θ}^{R} becomes the maximax index as $\theta \to \infty$.

Finally, if we assume that $\psi(t) = \log(1 + t)$, the corresponding index turns out to be

$$I_{\log}^{R}(x) = e^{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\log\left(1 + \left|x_i - x_j\right|\right)} - 1. \tag{9}$$

The Theil (1972) mean logarithmic deviation index of inequality aggregates logarithmic transforms of gaps between the mean achievement and the actual achievements. By considering the transformation $\psi(t) = \log(1 + t)$, in equation (9) we simply normalize the average of logarithmic transform of deprivation gaps. Consequently, this index may be called the normalized Theil consistent mean logarithmic deviation index.

Given any $\psi \in \Psi$ there exists a corresponding consistent inequality index I_{ψ}^{R}. These indices will differ in the way how we specify ψ. However, we can uniquely identify a particular member of the family $I_{\psi\in\Psi}^{R}$, using the following axiom.

LIH (Linear Homogeneity): For all $c > 0$, for all $x \in D^n$, such that $cx \in D^n$, $I_{\psi}^{R}(cx) = cI_{\psi}^{R}(x)$.

LIH implies that equal proportional changes in all achievements changes the index by the same proportion. (See Ebert, 1988, Chakravarty, 2009, for a discussion.)

The following theorem shows that the Atkinson index is the only member of the family $I_{\psi\in\Psi}^{R}$ that satisfies **LIH**.

Theorem 4: *A consistent inequality index $I_{\psi\in\Psi}^{R}$ satisfies **LIH** if and only if $I_{\psi}^{R}(x)$ is given by (6).*

Proof. See Appendix A. □

Theorem 4 clearly demonstrates that if for empirical purpose one wishes to use a member of the family $I_{\psi\in\Psi}^{R}$ that satisfies linear homogeneity, then the only choice is the Atkinson index.

3. An Empirical Application

We use data from eighteen waves of the British Household Panel Survey (BHPS) covering the period 1991-2008. BHPS includes at every wave a psychological measure of mental stress (the 12-item General Health Questionnaire, or GHQ-12, henceforth GHQ, see Goldberg, 1972), based on responses to the General Health Questionnaire of the adult population. This measure consists of twelve questions covering feelings of strain, depression, inability to cope, anxiety-based insomnia, and lack of confidence (see Table 1 for details). Responses are made on a four-point scale of frequency of a feeling in relation to a person's usual state: ‘Not at all’, ‘No more than usual’, ‘Rather more than usual’, and ‘Much more than usual’. The GHQ is widely used in medical, psychological and sociological research, and is considered to be a robust indicator of the individual's psychological state. This paper uses the Caseness version of the GHQ score, which counts the number of questions for which the response is in one of the two ‘low well-being' categories. This count is reversed so that higher scores indicate higher levels of well-being, running from 0 (all twelve responses indicating poor psychological health) to 12 (no responses indicating poor psychological health). Our sample, obtained by dropping missing values for GHQ, is composed of approximately 8000 individuals, depending on the year of analysis. All indices are estimated using sample weights. The mean value of the GHQ score is approximately 10 while the standard deviation is 3.

We estimate inequality in mental health using two versions of the Atkinson consistent index introduced in equation (7), two versions of the Kolm consistent inequality index of equation (8) and the Theil consistent index reported in equation (9). We label the two Atkinson consistent measures A(1) and A(2), which correspond respectively to $r=1$ and $r=2$. A(1) is twice the absolute Gini index of inequality and A(2) is $\sqrt{2}$ times the standard deviation which assigns more weight to higher gaps in the aggregation. The two Kolm consistent indices are labeled K(1) and K(2) that correspond to $\theta=1$ and $\theta=2$ respectively .The Theil index is indicated by T.

For comparison purposes of the errors committed with the use of an inappropriate index, we compute the Gini coefficient, the most popular inequality measure, which is also defined on a variable assuming zero values, such as GHQ. The standard Gini coefficient is a measure of relative inequality, hence invariant to multiplication of achievements, and is an inconsistent index measuring achievement and shortfall inequalities differently. We report its values and rankings for both inequality in good mental health (Gini(g)) and in bad mental health (Gini(b)). For an easy inspection of these errors, we plot in Figure 1 the rankings of the three Gini measures, K(1) and T ordered by the results of the standard Gini coefficient for inequality in good mental health. See Table 2 for all index values and rankings. All indices, but Gini(b), agree that the least unequal year is 1991 while the most unequal is 2008. The latter does not hold also for T according to which this position is reached in 1996. The rankings of A(1) and Gini(g) coincide for all the years but for 1994 and 1999 where the position is reversed. The difference between these two indices is the mean of GHQ, which changes only very little over the years, since for the Gini coefficient the absolute version of the index is obtained by multiplication of the relative index by the mean. Other than these similarities, the results differ considerably between the indices, especially for Gini(b). These findings confirm the results of Erreygers (2009) for British infant mortality and survival rates and of Clarke et al. (2002) for income-related inequality in health status and morbidity between Sweden and Australia.

In Figure 2 we plot the values of the indices here introduced that we estimate as a ratio of their 1991 value. According to all indices inequality in mental health had an increasing trend over the years especially when more weight to higher gaps is assigned in the aggregation, A(2) with respect to A(1) and K(1) with respect to K(2). Both T and A(1), which is proportional to the absolute Gini coefficient, have an oscillatory trend over the years with peaks in years 1996, 2000 and 2005 and troughs in 1994, 1999, 2004 and 2007.

4. Conclusion

Achievement inequality in a dimension of human well-being looks at interpersonal differences on the attainment levels in the dimension for different individuals in a society. The shortfall inequality in the dimension is concerned with shortfalls of attainments from the maximum possible value of the attainment. An inequality index is said to be consistent if it measures attainment inequality and shortfall inequality equally. This paper develops a general approach to the measurement of consistent inequality. Because of the underlying aggregation procedures, we refer to three members of the general family as the Atkinson (1970), Kolm (1976) and Theil (1972) consistent inequality indices. Positive multiples of the standard deviation and the absolute Gini index turn out to members of the Atkinson family. Essential to our characterization and investigation of properties of different indices is the weakly decomposable postulate suggested by Ebert (2010). Finally, a numerical application of our indices is provided using data on mental health in Britain. Our empirical findings confirm the results obtained earlier by Erreygers (2009) and Clarke et al. (2002).

References

Aczel, J., 1966. Lectures on functional equations and their applications, Academic Press, New York.

Apostol, T., 1974. Mathematical analysis, 2nd ed., Addison Wesley, Boston.

Atkinson, A. B., 1970. On the measurement of inequality. Journal of Economic Theory 2, 244–263.

Bourguignon, F., 1979. Decomposable income inequality measures. Econometrica 47, 901–920.

Chakravarty, S.R., 1988. Extended Gini indices of inequality. International Economic Review 29, 147–156.

Chakravarty, S.R., 2009. Inequality, polarization and poverty: Advances in distributional analysis, Springer, New York.

Clarke, P.M., Gerdtham, U.-G., Johannesson, M., Kerstin Bingeforsd, Smith, L., 2002. On the measurement of relative and absolute income-related health inequality. Social Science & Medicine 55, 1923–1928.

Dasgupta, P.S., Sen, A.K and D. A. Starrett, D.A., 1973. Notes on the measurement of inequality. Journal of Economic Theory 6, 180–187.

Donaldson, D., Weymark, J.A., 1980. A single parameter generalization of the Gini indices of inequality. Journal of Economic Theory 22, 67–68.

Ebert, U., 1988. A family of aggregative compromise inequality measure. International Economic Review 29, 363–376.

Ebert, U., 2010. The decomposition of inequality reconsidered: weakly decomposable measures. Mathematical Social Sciences 60, 94–103.

Erreygers, G., 2009. Can a single indicator measure both attainment and shortfall inequality? Journal of Health Economics 28, 885–893.

Goldberg, D.P., 1972. The Detection of psychiatric illness by questionnaire. Oxford University Press, Oxford.

Kolm, S., 1976. Unequal inequalities. Journal of Economic Theory 12, 416–442.

Lambert, P., Zheng, B., 2011. On the consistent measurement of attainment and shortfall Inequality. Journal of Health Economics 30, 214–219

Lasso de la Vega, C., Aristondo, O., 2012. Proposing indicators to measure achievement and shortfall inequality consistently. Journal of Health Economics 31, 578–583.

Marshall, A.W., Olkin I., Arnold, B.C., 2011. Inequalities: theory of majorization and its applications, 2nd ed., Springer, New York.

Rudin, W., 1976. Principles of mathematical analysis, 3rd ed., McGraw Hill, New York.

Sen, A.K., 1992. Inequality reexamined. Harvard University Press, Harvard.

Shorrocks, A.F., 1980. The class of additively decomposable inequality measures. Econometrica 48, 613–625.

Shorrocks, A.F., 1983. Ranking income distributions. Economica 50, 3-17.

Theil, H., 1972. Statistical decomposition analysis. North Holland, Amsterdam.

Yitzhaki, S., 1983. On an extension of the Gini inequality index. International Economic Review 24, 617–628.

Zoli, C., 1999. Intersecting generalized Lorenz curves and the Gini index. Social Choice and Welfare 16, 183–196.

TABLE 1: QUESTIONS AT THE BASIS OF GHQ

The following 12 questions from the BHPS questionnaire are at the basis of GHQ:

1. *Here are some questions regarding the way you have been feeling over the last few weeks. For each question please ring the number next to the answer that best suits the way you have felt.*

Have you recently . . .
a) been able to concentrate on whatever you're doing?

Better than usual 1
Same as usual 2
Less than usual 3
Much less than usual . . . 4

then
b) lost much sleep over worry?
e) felt constantly under strain?
f) felt you couldn't overcome your difficulties?
i) been feeling unhappy or depressed?
j) been losing confidence in yourself?
k) been thinking of yourself as a worthless person?

with the responses:
Not at all 1
No more than usual 2
Rather more than usual 3
Much more than usual 4

then
c) felt that you were playing a useful part in things?
d) felt capable of making decisions about things?
g) been able to enjoy your normal day-to-day activities?
h) been able to face up to problems?
l) been feeling reasonably happy, all things considered?

with the responses:
More so than usual 1
About same as usual 2
Less so than usual 3
Much less than usual 4

TABLE 2: INEQUALITY IN GHQ: INDEX VALUES AND YEARLY RANKINGS

years	A(1)	oA(1)	A(2)	oA(2)	K(1)	oK(1)	K(2)	oK(2)	T	oT	Gini(g)	oGini(g)	Gini(b)	oGini(b)
1991	2.455	1	3.702	1	7.702	1	9.552	1	1.564	1	0.119	1	0.713	4
1992	2.633	2	3.918	2	7.988	2	9.754	2	1.677	8	0.130	2	0.707	1
1993	2.686	5	4.007	4	8.072	3	9.797	3	1.700	13	0.133	5	0.708	2
1994	2.669	3	3.997	3	8.135	4	9.853	4	1.684	11	0.132	4	0.711	3
1995	2.762	12	4.150	5	8.344	5	9.973	5	1.719	17	0.137	12	0.716	6
1996	2.782	15	4.163	7	8.376	7	10.004	7	1.734	18	0.138	15	0.714	5
1997	2.751	11	4.175	8	8.448	9	10.040	9	1.690	12	0.136	11	0.723	7
1998	2.743	10	4.176	9	8.437	8	10.036	8	1.675	6	0.136	10	0.728	10
1999	2.684	4	4.180	11	8.526	14	10.091	13	1.590	2	0.131	3	0.746	18
2000	2.811	16	4.266	16	8.552	16	10.100	15	1.717	16	0.140	16	0.725	8
2001	2.739	9	4.150	6	8.371	6	9.988	6	1.681	9	0.136	9	0.725	9
2002	2.725	8	4.177	10	8.461	10	10.051	10	1.646	5	0.134	8	0.734	14
2003	2.713	6	4.186	12	8.519	11	10.102	16	1.619	4	0.133	6	0.741	16
2004	2.722	7	4.216	13	8.522	12	10.088	12	1.613	3	0.134	7	0.744	17
2005	2.829	17	4.291	17	8.524	13	10.081	11	1.710	15	0.141	17	0.730	12
2006	2.774	13	4.228	14	8.531	15	10.097	14	1.684	10	0.137	13	0.730	11
2007	2.781	14	4.259	15	8.579	17	10.122	17	1.676	7	0.138	14	0.734	13
2008	2.857	18	4.364	18	8.661	18	10.167	18	1.704	14	0.142	18	0.736	15

Figure 1: Inequality in GHQ: Yearly Rankings

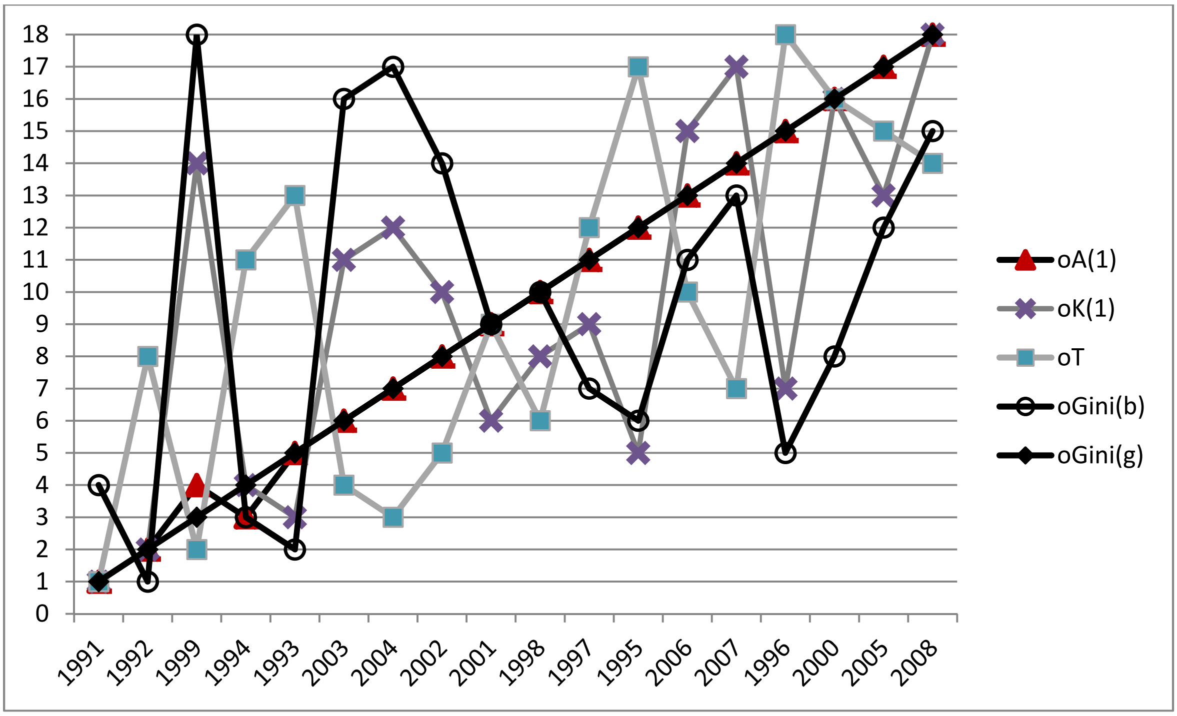
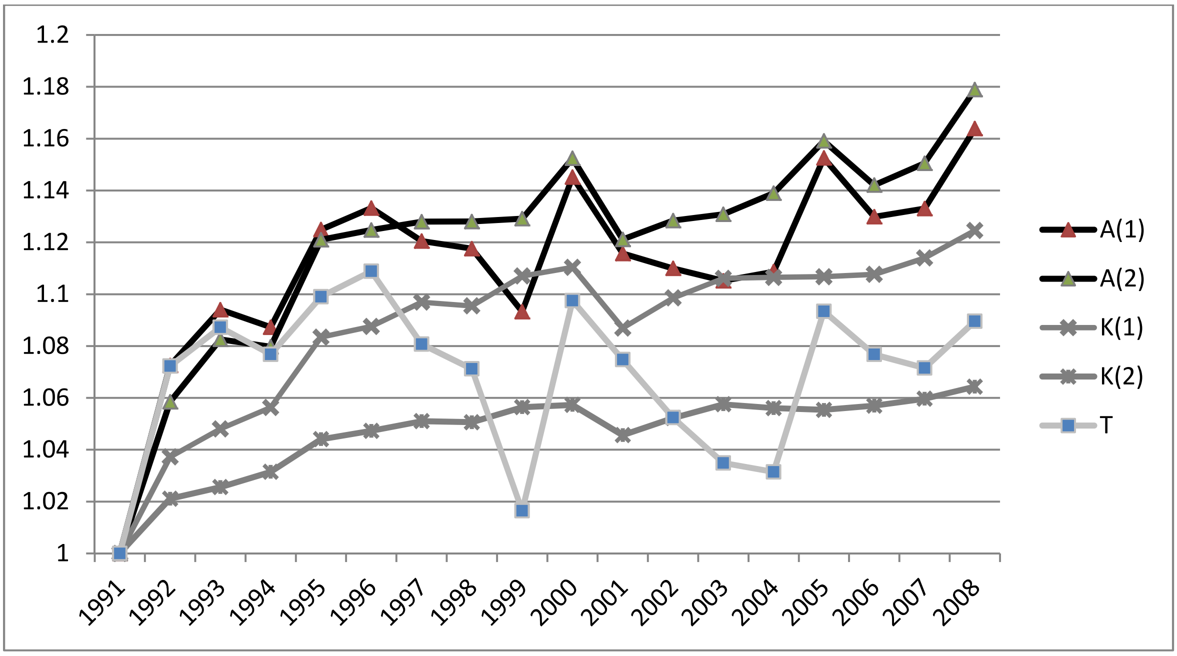


Figure 2: Inequality in GHQ: Index Values



Appendix

Proof of Theorem 1.

Ebert (2010) showed that if an inequality index satisfies **NOM** for a two-person society, then it satisfies **DEC** and **DPP** if and only if $I(x)=\frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}I(x_i,x_j)$. Now, let $x_i > x_j$.

Then by translation invariance, $I(x)=\frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}I(x_i-x_j,0)$. Likewise, if $x_j > x_i$,

$I(x)=\frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}I(0,x_j-x_i)$ which in view of **SYM** becomes $I(x)=\frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}I(x_j-x_i,0)$.

This gives $I(x)=\frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(|x_i-x_j|)$, where $\psi : D^1 \to \Re^1$. Continuity of ψ follows from the fact that I is continuous. $\psi(0)=0$ is an implication of the axiom **NOM** for a two-person society. Conversely, it is easy to check that I satisfies **DPP**, **DEC**, **SYM** and translation invariance. □

Proof of Theorem 2.

$D^2 \subset \Re^2$ is convex. Define the real valued function $f : D^2 \to \Re^1$ by $f(x_i,x_j)=|x_i-x_j|$.

Now, consider $u=(x_i,x_j), v=(x_k,x_l), u,v\in D^2$ such that $f(u)=|x_i-x_j|\geq|x_l-x_k|=f(v)$.

For any $0<c<1$, the convex combination $cu+(1-c)v=c(x_i,x_j)+(1-c)(x_k,x_l)\in D^2$, since D^2 is convex. Then, $f(cu+(1-c)v)=f(cx_i+(1-c)x_k, cx_j+(1-c)x_l)=$

$$|c(x_j-x_i)+(1-c)(x_l-x_k)|\leq c|x_i-x_j|+(1-c)|x_l-x_k|\leq|x_i-x_j|=f(u) .$$

Given that $u,v\in D^2$ and $c\in(0,1)$ are arbitrary, f is quasi-convex. Since a non-decreasing transform of a quasi-convex function is quasi-convex, ψ is quasi-convex. Thus, I_ψ, being a finite sum of quasi-convex functions is also quasi-convex. Note that I_ψ is also symmetric. All symmetric quasi-convex functions are S-convex (Marshall et al. 2011, p. 98). Hence I_ψ is an S-convex function, which we know is equivalent to **PDT** under rank preserving transfers.

To demonstrate the converse, consider the distribution $y = (c, c+\delta, c+\delta \cdots, c+\delta) \in D^n$, where $c \in D^1$ and $\delta > 0$ is arbitrary such that $c+\delta \in D^1$. The distribution $x = \left(c+\frac{\delta}{2}, c+\frac{\delta}{2}, c+\delta \cdots, c+\delta\right) \in D^n$ is obtained from y by a rank preserving transfer of $\frac{\delta}{2}$ units of income from the second person to the first person. PDT demands that $I_\psi(y) \geq I_\psi(x)$ which on simplification becomes $(n-1)\psi(\delta) \geq 2(n-2)\psi\left(\frac{\delta}{2}\right)$ for all $n \in N/\{1\}$. Substituting $n=3$ in this inequality we get $\psi(\delta) \geq \psi\left(\frac{\delta}{2}\right)$. Non-decreasingness of ψ follows from this inequality since $\delta > 0$ is arbitrary. This therefore completes the proof of the necessity part of the theorem.□

Proof of Theorem 3.

Since x_i, x_j are drawn from the compact set D^1, the non-negative deviations will also take values in the compact set $[0, a]$. Now, since ψ defined on the compact set $[0, a]$ is increasing and the continuous image of a compact set is compact (Rudin, 1976, p.89), $\psi(|x_i - x_j|)$ takes values in the compact set $[\psi(0), \psi(a)]$, which, in view of the fact that $\psi(0) = 0$, can be rewritten as $[0, \psi(a)]$. Continuity and increasingness of the function ψ implies that the average function $\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(|x_i - x_j|)$ is continuous and takes values in $[0, \psi(a)]$. Observe that increasingness of ψ ensures the existence of ψ^{-1}. Continuity and increasingness of ψ^{-1} on $[0, \psi(a)]$ now follows from Theorem 4.53 of Apostol (1974, p.95). This in turn demonstrates continuity of I_ψ^R.

For boundedness, note that if the achievement distribution x is perfectly equal, each $|x_i - x_j|$ becomes zero which implies that $\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(|x_i - x_j|) = \psi(0)$. Hence if x is perfectly equal $I_{\psi}^{R}(x) = \psi^{-1}(\psi(0)) = 0$. Likewise, it can be shown that I_{ψ}^{R} is bounded above by a.

Since an increasing function is non-decreasing, by Theorem 2, $\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(|x_i - x_j|)$ satisfies **SYM**, **DPP**, **PDT** and **NOM**. Given that ψ^{-1} is also increasing and $\psi^{-1}(0) = 0$, I_{ψ}^{R} satisfies **SYM**, **DPP**, **PDT** and **NOM**. This completes the proof of the theorem. □

Proof of Theorem 4.

The idea of the proof is taken from Chakravarty (2009).

LIH requires that

$$c\psi^{-1}\left[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(|x_i - x_j|)\right] = \psi^{-1}\left[\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi(|cx_i - cx_j|)\right], \tag{1a}$$

where $c > 0$ is a scalar satisfying the condition laid down in the axiom. The only continuous solution to the functional equation given by (1a) is

$$\psi(t) = A + \frac{Bt^r}{r} \quad , \tag{2a}$$

where A, B are constants. (See Aczel, 1966, p.151). The condition $\psi(0) = 0$ along with continuity of ψ requires that $A = 0$ and $r > 0$. Increasingness of ψ demands that $B > 0$. Substituting the form ψ given by (2a) in (5) we get the desired form of the index. This establishes the necessary part of the theorem. The sufficiency is easy to verify. □