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The binomial Gini inequality indices and the binomial decomposition of welfare functions*

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Abstract

In the context of Social Welfare and Choquet integration, we briefly review, on the one hand, the generalized Gini welfare functions and inequality indices for populations of $n \geq 2$ individuals, and on the other hand, the Möbius representation framework for Choquet integration, particularly in the case of k-additive symmetric capacities.

We recall the binomial decomposition of OWA functions due to Calvo and De Baets [14] and we examine it in the restricted context of generalized Gini welfare functions, with the addition of appropriate S-concavity conditions. We show that the original expression of the binomial decomposition can be formulated in terms of two equivalent functional bases, the binomial Gini welfare functions and the Atkinson-Kolm-Sen (AKS) associated binomial Gini inequality indices, according to Blackorby and Donaldson's correspondence formula.

The binomial Gini pairs of welfare functions and inequality indices are described by a parameter $j = 1, \ldots, n$, associated with the distributional judgements involved. The j-th generalized Gini pair focuses on the $(n - j + 1)/n$ poorest fraction of the population and is insensitive to income transfers within the complementary richest fraction of the population.

Keywords: Social welfare, Generalized Gini welfare functions and inequality indices, symmetric capacities and Choquet integrals, OWA functions, Binomial decomposition and k-additivity.

JEL Classification: D31, D63, I31.

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1 Introduction

The Gini inequality index [36, 37, 33, 25] plays a central role in Social Welfare Theory and the measurement of economic inequality [6, 65]. In the literature several extensions of the Gini index have been proposed [23, 70, 71, 72, 27, 17, 13], in particular the generalized Gini welfare functions and the associated inequality indices introduced by Weymark [70] on the basis of the Atkinson-Kolm-Sen (AKS) framework and Blackorby and Donaldson's correspondence formula [9, 10],

$$A(\boldsymbol{x}) = \bar{x} - G(\boldsymbol{x})$$

where $A(\boldsymbol{x})$ denotes a generalized Gini welfare function, $G(\boldsymbol{x})$ is the associated absolute inequality index, and $\bar{x}$ denotes the arithmetic mean of the income distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$ of a population of $n \geq 2$ individuals. Recently, the extended interpretation of this formula in terms of the dual decomposition [32] of aggregation functions has been discussed in [4, 5], see also [31].

The generalized Gini welfare functions introduced by Weymark have the form

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)}$$

where $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$ and, as required by the principle of inequality aversion, $w_1 \geq w_2 \geq \ldots \geq w_n \geq 0$ with $\sum_{i=1}^{n} w_i = 1$. These welfare functions correspond to the S-concave subclass of the ordered weighted averaging (OWA) functions introduced by Yager [73], which in turn correspond [28] to the Choquet integrals associated with symmetric capacities.

The use of non-additivity and Choquet integration [21] in Social Welfare and Decision Theory dates back to the seminal work of Schmeidler [63, 64], Ben Porath and Gilboa [8], and Gilboa and Schmeidler [34, 35]. In the discrete case, Choquet integration [60, 19, 22, 38, 39, 53] corresponds to a generalization of both weighted averaging (WA) and ordered weighted averaging (OWA), which remain as special cases. For recent reviews of Choquet integration see Grabisch and Labreuche [44, 45, 46], and Grabisch, Kojadinovich, and Meyer [43].

The complex structure of Choquet capacities can be described in the k-additivity framework introduced by Grabisch [40, 41], see also Calvo and De Baets [14], Cao-Van and De Baets [16], and Miranda, Grabisch, and Gil [59]. The 2-additive case, in particular, has been examined by Miranda, Grabisch, and Gil [59], and Mayag, Grabisch, and Labreuche [55, 56]. Due to its low complexity and versatility it is relevant in a variety of modelling contexts.

The characterization of symmetric Choquet integrals (OWA functions) has been studied by Fodor, Marichal and Roubens [28], Calvo and De Baets [14], Cao-Van and De Baets [16], and Miranda, Grabisch and Gil [59]. It is shown, Gajdos [30], that in the k-additive case the generating function of the OWA weights is polynomial of degree k. In the symmetric 2-additive case, in particular, the generating function is quadratic and thus the weights are equidistant, as in the classical Gini welfare function.

In this paper we review the analysis of symmetric capacities in the Möbius representation framework and, in particular, we recall the binomial decomposition of OWA functions due to Calvo and De Baets [14], with the addition of a uniqueness result.

Considering the binomial decomposition in the restricted context of generalized Gini welfare functions, with the addition of appropriate S-concavity conditions, we show that the original expression of the binomial decomposition can be formulated in terms of two equivalent functional bases, the binomial Gini welfare functions and the Atkinson-Kolm-Sen (AKS) associated binomial Gini inequality indices, according to the Blackorby and Donaldson's correspondence formula.

The binomial Gini welfare functions, denoted C_j with $j = 1, \dots, n$, have null weights associated with the $j-1$ richest individuals in the population and therefore they are progressively focused on the poorest part of the population. Correspondingly, the associated binomial Gini inequality indices, denoted G_j with $j = 1, \dots, n$, have equal weights associated with the $j-1$ richest individuals in the population and therefore they are progressively insensitive to income transfers within the richest part of the population.

The paper is organized as follows. In Section 2 we review the notions of welfare function and inequality index for populations of $n \geq 2$ individuals. In Section 3 we review the classical Gini index and the associated welfare function, including a graphical representation in relation with the Lorenz curve. In Section 4 we present the basic definitions and results on capacities and Choquet integration, with reference to the Möbius representation framework. In Section 5 we consider the context of symmetric capacities and we recall the binomial decomposition of OWA functions due to Calvo and De Baets [14], with the addition of a uniqueness result. In Section 6 we present the main results of the paper. We focus on generalized Gini welfare functions and we examine the binomial decomposition in two equivalent formulations, either in terms of binomial Gini welfare functions, or in terms of the AKS associated binomial Gini inequality indices. In Section 7 we consider the binomial decomposition of generalized Gini welfare functions in the 2-additive and 3-additive cases. Finally, Section 8 contains some conclusive remarks.

2 Welfare functions and inequality indices

In this section we consider populations of $n \geq 2$ individuals and we briefly review the notions of welfare function and inequality index in the standard framework of aggregation functions on the $[0,1]^n$ domain. The income distributions in this framework are represented by points $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$. The correspondence with the traditional income domain $[0, \infty)$ can be obtained by means of a rescaling with respect to some appropriate conventional upper bound for the income values, given that the welfare functions and inequality indices under consideration rescale analogously. In any case, most of our results extend immediately to the traditional income domain $[0, \infty)$, or even $\mathbb{R}$ itself.

We begin by presenting notation and basic definitions regarding aggregation functions on the domain $[0,1]^n$, with $n \geq 2$ throughout the text. Comprehensive reviews of aggregation functions can be found in Fodor and Roubens [29], Calvo et al. [15], Beliakov et al. [7], and Grabisch et al. [47].

Notation. Points in $[0,1]^n$ are denoted $\boldsymbol{x} = (x_1, \dots, x_n)$, with $\mathbf{1} = (1, \dots, 1)$, $\mathbf{0} = (0, \dots, 0)$. Accordingly, for every $x \in [0,1]$, we have $x \cdot \mathbf{1} = (x, \dots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i = 1, \dots, n$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in [0,1]^n$, the increasing and decreasing reorderings of the coordinates of $\boldsymbol{x}$ are indicated as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \ldots, x_n\} = x_{[n]}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\} = x_{[1]}$. In general, given a permutation σ on $\{1, \ldots, n\}$, we denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Finally, the arithmetic mean is denoted $\bar{x} = (x_1 + \cdots + x_n)/n$.

We define some standard properties of real functions on $[0,1]^n$.

Definition 1 *Let $A : [0,1]^n \longrightarrow [0,1]$ be a function.*

1. *A is* monotonic *if $\mathbf{x} \geq \mathbf{y} \Rightarrow A(\mathbf{x}) \geq A(\mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$. Moreover, A is* strictly monotonic *if $\mathbf{x} > \mathbf{y} \Rightarrow A(\mathbf{x}) > A(\mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$.*

2. *A is* idempotent *if $A(x \cdot \boldsymbol{1}) = x$, for all $x \in [0,1]$. On the other hand, A is* nilpotent *if $A(x \cdot \boldsymbol{1}) = 0$, for all $x \in [0,1]$.*

3. *A is* symmetric *if $A(\mathbf{x}_\sigma) = A(\mathbf{x})$, for any permutation σ on $\{1, \ldots, n\}$ and all $\mathbf{x} \in [0,1]^n$.*

4. *A is* invariant for translations *if $A(\mathbf{x} + t \cdot \boldsymbol{1}) = A(\boldsymbol{x})$, for all $t \in \mathbb{R}$ and $\mathbf{x} \in [0,1]^n$ such that $\mathbf{x} + t \cdot \boldsymbol{1} \in [0,1]^n$. On the other hand, A is* stable for translations *if $A(\mathbf{x} + t \cdot \boldsymbol{1}) = A(\mathbf{x}) + t$, for all $t \in \mathbb{R}$ and $\mathbf{x} \in [0,1]^n$ such that $\mathbf{x} + t \cdot \boldsymbol{1} \in [0,1]^n$.*

5. *A is* invariant for dilations *if $A(\lambda \cdot \mathbf{x}) = A(\mathbf{x})$, for all $\lambda > 0$ and $\mathbf{x} \in [0,1]^n$ such that $\lambda \cdot \mathbf{x} \in [0,1]^n$. On the other hand, A is* stable for dilations *if $A(\lambda \cdot \mathbf{x}) = \lambda \, A(\mathbf{x})$, for all $\lambda > 0$ and $\mathbf{x} \in [0,1]^n$ such that $\lambda \cdot \mathbf{x} \in [0,1]^n$.*

We introduce the majorization relation on $[0,1]^n$ and we discuss the concept of income transfer following the approach in Marshall and Olkin [54], focusing on the classical results relating majorization, income transfers, and bistochastic transformations, see Marshall and Olkin [54, Ch. 4, Prop. A.1].

Definition 2 *The* majorization relation $\preceq$ *on $[0,1]^n$ is defined as follows: given $\mathbf{x}, \mathbf{y} \in [0,1]^n$ with $\bar{x} = \bar{y}$, we say that*

$$\mathbf{x} \preceq \mathbf{y} \quad \text{if} \quad \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \qquad k = 1, \ldots, n \tag{1}$$

where the case $k = n$ is an equality due to $\bar{x} = \bar{y}$. As usual, we write $\mathbf{x} \prec \mathbf{y}$ if $\mathbf{x} \preceq \mathbf{y}$ and not $\mathbf{y} \preceq \mathbf{x}$, and we write $\mathbf{x} \sim \mathbf{y}$ if $\mathbf{x} \preceq \mathbf{y}$ and $\mathbf{y} \preceq \mathbf{x}$. We say that $\mathbf{y}$ majorizes $\mathbf{x}$ if $\mathbf{x} \prec \mathbf{y}$, and we say that $\mathbf{x}$ and $\mathbf{y}$ are indifferent if $\mathbf{x} \sim \mathbf{y}$.

Another traditional reading, which reverses that of majorization, refers to the concept of Lorenz dominance: we say that $\boldsymbol{x}$ is Lorenz superior to $\boldsymbol{y}$ if $\boldsymbol{x} \prec \boldsymbol{y}$, and we say that $\boldsymbol{x}$ is Lorenz indifferent to $\boldsymbol{y}$ if $\boldsymbol{x} \sim \boldsymbol{y}$.

Given an income distribution $\boldsymbol{x} \in [0,1]^n$, with mean income $\bar{x}$, it holds that $\bar{x} \cdot \mathbf{1} \preceq \boldsymbol{x}$ since $k\,\bar{x} \geq \sum_{i=1}^{k} x_{(i)}$ for $k = 1, \ldots, n$. The majorization is strict, $\bar{x} \cdot \mathbf{1} \prec \boldsymbol{x}$, when $\boldsymbol{x}$ is not a uniform income distribution. In such case, $\bar{x} \cdot \mathbf{1}$ is Lorenz superior to $\boldsymbol{x}$. Moreover, for any income distribution $\boldsymbol{x} \in [0,1]^n$ with mean income $\bar{x}$ it holds that $\boldsymbol{x} \preceq (0, \ldots, 0, n\bar{x})$, and the majorization is strict when $\boldsymbol{x} \neq \mathbf{0}$.

The majorization relation is a preorder, in which $\boldsymbol{x} \sim \boldsymbol{y}$ if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ differ by a permutation. In general, $\boldsymbol{x} \preceq \boldsymbol{y}$ if and only if there exists a bistochastic matrix $\mathbf{C}$ (non-negative square matrix of order n where each row and column sums to one) such that $\boldsymbol{x} = \mathbf{C}\boldsymbol{y}$. Moreover, $\boldsymbol{x} \prec \boldsymbol{y}$ if the bistochastic matrix $\mathbf{C}$ is not a permutation matrix.

A particular case of bistochastic transformation is the so-called transfer, also called T-transformation.

Definition 3 *Given* $\mathbf{x}, \mathbf{y} \in [0,1]^n$*, we say that* $\mathbf{x}$ *is derived from* $\mathbf{y}$ *by means of a* transfer *if, for some pair* $i, j = 1, \ldots, n$ *with* $y_i \leq y_j$*, we have*

$$x_i = (1-\varepsilon)\, y_i + \varepsilon y_j \qquad x_j = \varepsilon y_i + (1-\varepsilon)\, y_j \qquad \varepsilon \in [0,1] \tag{2}$$

and $x_k = y_k$ *for* $k \neq i, j$*. These formulas express an income transfer, from a richer to a poorer individual, of an income amount* $\varepsilon(y_j - y_i)$*. The transfer obtains* $\mathbf{x} = \mathbf{y}$ *if* $\varepsilon = 0$*, and exchanges the relative positions of donor and recipient in the income distribution if* $\varepsilon = 1$*, in which case* $\mathbf{x} \sim \mathbf{y}$*. In the intermediate cases* $\varepsilon \in (0,1)$ *the transfer produces an income distribution* $\mathbf{x}$ *which is Lorenz superior to the original* $\mathbf{y}$*, that is* $\mathbf{x} \prec \mathbf{y}$*.*

In general, for income distributions $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ and the majorization relation $\preceq$, it holds that $\boldsymbol{x} \preceq \boldsymbol{y}$ if and only if $\boldsymbol{x}$ can be derived from $\boldsymbol{y}$ by means of a finite sequence of transfers. Moreover, $\boldsymbol{x} \prec \boldsymbol{y}$ if any of the transfers is not a permutation.

Definition 4 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be a function. In relation with the majorization relation* $\preceq$*, the notions of* Schur-convexity *(S-convexity) and* Schur-concavity *(S-concavity) of the function* A *are defined as follows:*

1. *A is* S-convex *if* $\mathbf{x} \preceq \mathbf{y} \;\Rightarrow\; A(\mathbf{x}) \leq A(\mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in [0,1]^n$
2. *A is* S-concave *if* $\mathbf{x} \preceq \mathbf{y} \;\Rightarrow\; A(\mathbf{x}) \geq A(\mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in [0,1]^n$*.*

Moreover, the S-convexity *(resp.* S-concavity*) of a function* A *is said to be* strict *if* $\mathbf{x} \prec \mathbf{y}$ *implies* $A(\mathbf{x}) < A(\mathbf{y})$ *(resp.* $A(\mathbf{x}) > A(\mathbf{y})$*).*

Notice that S-convexity (S-concavity) implies symmetry, since $\boldsymbol{x} \sim \boldsymbol{x}_\sigma$ and thus $A(\boldsymbol{x}) = A(\boldsymbol{x}_\sigma)$.

Definition 5 *A function* $A : [0,1]^n \longrightarrow [0,1]$ *is an* n-ary aggregation function *if it is monotonic and* $A(\boldsymbol{0}) = 0$*,* $A(\boldsymbol{1}) = 1$*. An aggregation function is said to be* strict *if it is strictly monotonic.*

For simplicity, the n-arity is omitted whenever it is clear from the context. Particular cases of aggregation functions are weighted averaging (WA) functions, ordered weighted averaging (OWA) functions, and Choquet integrals, which contain the former as special cases.

Definition 6 *Given a weighting vector* $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$*, with* $\sum_{i=1}^n w_i = 1$*, the* Weighted Averaging (WA) function *associated with* $\mathbf{w}$ *is the aggregation function* $A : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$A(\mathbf{x}) = \sum_{i=1}^{n} w_i\, x_i. \tag{3}$$

Definition 7 *Given a weighting vector* $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$*, with* $\sum_{i=1}^{n} w_i = 1$*, the* Ordered Weighted Averaging (OWA) function *associated with* $\mathbf{w}$ *is the aggregation function* $A : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$A(\mathbf{x}) = \sum_{i=1}^{n} w_i \, x_{(i)}. \tag{4}$$

The traditional form of OWA functions as introduced by Yager [73] (OWA operators) is as follows, $A(\boldsymbol{x}) = \sum_{i=1}^{n} \tilde{w}_i \, x_{[i]}$ where $\tilde{w}_i = w_{n-i+1}$. In [74, 75] the theory and applications of OWA functions are discussed in detail.

The following are two classical results particulary relevant in our framework. The proofs, given here for convenience, are analogous. The first result, see in particular Skala [68], regards a form of dominance relation between weighting structures and OWA functions.

Proposition 1 *Consider two OWA functions* $A, B : [0,1]^n \longrightarrow [0,1]$ *associated with weighting vectors* $\boldsymbol{u} = (u_1, \ldots, u_n) \in [0,1]^n$ *and* $\boldsymbol{v} = (v_1, \ldots, v_n) \in [0,1]^n$*, respectively. It holds that* $A(\boldsymbol{x}) \leq B(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in [0,1]^n$ *if and only if*

$$\sum_{i=1}^{k} u_i \geq \sum_{i=1}^{k} v_i \qquad \textit{for} \quad k = 1, \ldots, n \tag{5}$$

where the case $k = n$ *is an equality due to weight normalization.*

Proof: Suppose first that the weights satisfy (5). Then, it holds that

$$\begin{aligned}
& A(\boldsymbol{x}) - B(\boldsymbol{x}) = \\
& = (u_1 - v_1)\, x_{(1)} + (u_2 - v_2)\, x_{(2)} + \ldots + (u_n - v_n)\, x_{(n)} \\
& \leq (u_1 - v_1)\, x_{(2)} + (u_2 - v_2)\, x_{(2)} + \ldots + (u_n - v_n)\, x_{(n)} \\
& = (u_1 + u_2 - v_1 - v_2)\, x_{(2)} + \ldots + (u_n - v_n)\, x_{(n)} \\
& \leq \ldots = (u_1 + u_2 + \ldots + u_n - v_1 - v_2 - \ldots - v_n)\, x_{(n)} = 0 \,.
\end{aligned}$$

Conversely, suppose that $A(\boldsymbol{x}) \leq B(\boldsymbol{x})$ for all $\boldsymbol{x} \in [0,1]^n$. Consider a point $\boldsymbol{x} \in [0,1]^n$ whose coordinates are zero except for $x_k = \ldots = x_n = 1$, for some $k = 1, \ldots, n$. Then

$$A(\boldsymbol{x}) = \sum_{i=k}^{n} u_i \leq \sum_{i=k}^{n} v_i = B(\boldsymbol{x})$$

which means that $\sum_{i=k}^{n} (u_i - v_i) \leq 0$ where the case $k = 1$ is an equality due to weight normalization. Equivalently, $\sum_{i=1}^{k} (u_i - v_i) \geq 0$, where the equality is now for $k = n$. □

The next result, see for instance Chakravarty [18, p. 28], regards the relation between the weighting structure and the S-convexity or S-concavity of the OWA function.

Proposition 2 *Consider an OWA function* $A : [0,1]^n \longrightarrow [0,1]$ *associated with a weighting vector* $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$*. The OWA function A is S-convex if and only if the weights are non decreasing,* $w_1 \leq \ldots \leq w_n$*, and A is strictly*

S-convex if and only if the weights are increasing, $w_1 < \ldots < w_n$. Analogously, the OWA function A is S-concave if and only if the weights are non increasing, $w_1 \geq \ldots \geq w_n$, and A is strictly S-concave if and only if the weights are decreasing, $w_1 > \ldots > w_n$.

Proof: We prove the statements regarding S-concavity, those regarding S-convexity have analogous proofs. We begin by proving that an OWA function A is S-concave if and only if the weights are non increasing, $w_1 \geq \ldots \geq w_n$. Suppose first that the weights are non increasing. Consider two points $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ with $\boldsymbol{x} \preceq \boldsymbol{y}$, that is,

$$\sum_{i=1}^{k} (x_{(i)} - y_{(i)}) \geq 0 \qquad k = 1, \ldots, n \tag{6}$$

where the case $k = n$ is an equality due to $\bar{x} = \bar{y}$. Then, it holds that

$$\begin{aligned}
& A(\boldsymbol{x}) - A(\boldsymbol{y}) = \\
& = w_1(x_{(1)} - y_{(1)}) + w_2(x_{(2)} - y_{(2)}) + \ldots + w_n(x_{(n)} - y_{(n)}) \\
& \geq w_2(x_{(1)} - y_{(1)}) + w_2(x_{(2)} - y_{(2)}) + \ldots + w_n(x_{(n)} - y_{(n)}) \\
& = w_2(x_{(1)} + x_{(2)} - y_{(1)} - y_{(2)}) + \ldots + w_n(x_{(n)} - y_{(n)}) \\
& \geq \ldots = w_n(x_{(1)} + x_{(2)} + \ldots + x_{(n)} - y_{(1)} - y_{(2)} - \ldots - y_{(n)}) = 0\,.
\end{aligned}$$

Conversely, suppose A is S-concave. Given a point $\boldsymbol{y} \in [0,1]^n$ with $y_1 < y_2 < \ldots < y_n$, consider $\boldsymbol{x} \in [0,1]^n$ obtained from $\boldsymbol{y}$ by a transfer in the following way: for some $i = 1, \ldots, n-1$ and $0 \leq \delta \leq (y_{i+1} - y_i)/2$ we define $x_i = y_i + \delta \leq y_{i+1} - \delta = x_{i+1}$ and $x_j = y_j$ for all $j \neq i, i+1$. Clearly, $\boldsymbol{x} \preceq \boldsymbol{y}$ and thus $A(\boldsymbol{x}) \geq A(\boldsymbol{y})$ due to the S-concavity of A. Since

$$A(\boldsymbol{x}) = A(\boldsymbol{y}) + \delta(w_i - w_{i+1})\,,$$

we obtain $w_i \geq w_{i+1}$ for $i = 1, \ldots, n-1$, which means that weights are non increasing.

Moreover, A is strictly S-concave if and only if the weights are decreasing, $w_1 > \ldots > w_n$. Suppose first that the weights are decreasing. Consider two points $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ with $\boldsymbol{x} \prec \boldsymbol{y}$ as in (6), but now with $\sum_{i=1}^{k}(x_{(i)} - y_{(j)}) > 0$ for at least one $k = 1, \ldots, n-1$. We can then repeat the analysis of $A(\boldsymbol{x}) - A(\boldsymbol{y})$ as above but now, using the fact that the weights are decreasing, at least one of the inequalities will be strict.

Conversely, suppose A is strictly S-concave. We can then repeat the argument above but now, with $\varepsilon > 0$, the strict S-concavity of A leads to the decreasingness of the weights. □

We will now review the basic concepts and definitions regarding welfare functions and inequality indices. In line with the standard framework of aggregation functions, we refer consistently throughout the paper to the individual income domain $[0,1]$, instead of the traditional income domain $[0,\infty)$. As long as the functions involved are stable (or invariant) for dilations, which is the case of generalized Gini welfare functions and the associated inequality indices, a rescaling of the income domain corresponds to an identical rescaling (if any) of the function values, thereby obtaining equivalent constructions. In fact, given a conventional upper bound λ for the income within the traditional domain $[0, \infty)$ and an income profile $\boldsymbol{x} \in [0,1]^n$, the equivalence of the two domain frameworks is directly expressed by the law of stability for dilations, $\lambda A(\boldsymbol{x}) = A(\lambda \cdot \boldsymbol{x})$.

Certain properties which are generally considered to be inherent to the concepts of welfare and inequality have come to be accepted as basic axioms for welfare and inequality measures, see for instance Kolm [50, 51]. The crucial axiom in this field is the *Pigou-Dalton transfer principle*, which states that welfare (inequality) measures should be non-decreasing (non-increasing) under transfers. This axiom translates directly into the properties of S-concavity and S-convexity in the context of symmetric functions on $[0,1]^n$. In fact, a function is S-concave (S-convex) if and only if it is symmetric and non-decreasing (non-increasing) under transfers, see for instance Marshall and Olkin [54].

Definition 8 *An aggregation function $A : [0,1]^n \longrightarrow [0,1]$ is a* welfare function *if it is continuous, idempotent, and S-concave. The welfare function is said to be* strict *if it is a strict aggregation function which is strictly S-concave.*

Due to monotonicity and idempotency, a welfare function is non decreasing over $[0,1]^n$ but increasing along the diagonal $\boldsymbol{x} = x \cdot \mathbf{1}$ with $x \in [0,1]$. Moreover, notice that S-concavity implies symmetry. Due to S-concavity, a welfare function ranks any Lorenz superior income distribution with the same mean as $\boldsymbol{x}$ as no worse than $\boldsymbol{x}$, whereas a strict welfare function ranks it as better.

Give a welfare function A, the *uniform equivalent income* $\tilde{x}$ associated with an income distribution $\boldsymbol{x}$ is defined as the income level which, if equally distributed among the population, would generate the same welfare value, $A(\tilde{x} \cdot \mathbf{1}) = A(\boldsymbol{x})$. Due to the idempotency of A, we obtain $\tilde{x} = A(\boldsymbol{x})$. Since $\bar{x} \cdot \mathbf{1} \preceq \boldsymbol{x}$ for any income distribution $\boldsymbol{x} \in [0,1]^n$, S-concavity implies $A(\bar{x} \cdot \mathbf{1}) \geq A(\boldsymbol{x})$ and therefore $A(\boldsymbol{x}) \leq \bar{x}$ due to the idempotency of the welfare function. In other words, the mean income $\bar{x}$ and the uniform equivalent income $\tilde{x}$ are related by $0 \leq \tilde{x} \leq \bar{x} \leq 1$.

In the literature, our welfare function A corresponds to the idempotent representation of the social evaluation function W, which is assumed to be continuous, non decreasing, increasing along the diagonal of $[0,1]^n$, and S-concave (thus symmetric). Under these assumptions there exists a unique uniform equivalent income $\tilde{x}$ such that $W(\tilde{x} \cdot \mathbf{1}) = W(\boldsymbol{x})$. Defining $A(\boldsymbol{x}) = \tilde{x}$ we obtain our welfare function A, which is an increasing transform of the social evaluation function W. Moreover, A is stable for translations (dilations) if and only if W is translatable (homothetic), see Blackorby and Donaldson [9, 10], and Blackorby, Donaldson, and Auersperg [11].

We now define the notions of absolute and relative inequality indices. The former were introduced by Kolm [50, 51] and developed by Blackorby and Donaldson [10], Blackorby, Donaldson, and Auersperg [11], and Weymark [70]. Following Kolm (1976), inequality measures are described as relative when they are invariant for multiplicative transformations (dilation invariance), and absolute when they are invariant for additive transformations (translation invariance).

Definition 9 *A function $G : [0,1]^n \longrightarrow [0,1]$ is an* absolute inequality index *if it is continuous, nilpotent, S-convex, and invariant for translations. The absolute inequality index is said to be* strict *if it is strictly S-convex.*

Definition 10 *A function* $G_R : [0,1]^n \longrightarrow [0,1]$ *is a* relative inequality index *if it is continuous, nilpotent, S-convex, and invariant for dilations. The relative inequality index is said to be* strict *if it is strictly S-convex.*

In relation with the properties of the majorization relation discussed earlier, it holds that: over all income distributions $\boldsymbol{x} \in [0,1]^n$ with the same mean income $\bar{x}$, a welfare function has minimum value $A(0,\ldots,0,n\bar{x})$, and an absolute inequality index has maximum value $G(0,\ldots,0,n\bar{x})$.

In the AKS framework introduced by Atkinson [6], Kolm [49], and Sen [65], a welfare function which is stable for translations induces an associated absolute inequality index by means of the correspondence formula $A(\boldsymbol{x}) = \bar{x} - G(\boldsymbol{x})$, see Blackorby and Donaldson [10]. Analogously, a welfare function which is stable for dilations induces an associated relative inequality index by means of the correspondence formula $A(\boldsymbol{x}) = \bar{x}\,(1 - G_R(\boldsymbol{x}))$, see Blackorby and Donaldson [9]. In both cases the welfare functions and the associated inequality indices are said to be *ethical*, see also Sen [67], Blackorby, Donaldson, and Auersperg [11], Weymark [70], Blackorby and Donaldson [12], and Ebert [26].

Definition 11 *Given a welfare function* $A : [0,1]^n \longrightarrow [0,1]$ *which is stable for translations, the associated* Atkinson-Kolm-Sen (AKS) absolute inequality index $G : [0,1]^n \longrightarrow [0,1]$ *is defined as*

$$G(\mathbf{x}) = \bar{x} - A(\mathbf{x}) \tag{7}$$

The fact that A *is stable for translations ensures the translational invariance of* G*. The absolute inequality index can be written as* $G(\mathbf{x}) = \bar{x} - \tilde{x}$ *and represents the per capita income that could be saved if society distributed incomes equally without any loss of welfare.*

Definition 12 *Given a welfare function* $A : [0,1]^n \longrightarrow [0,1]$ *which is stable for dilations, the associated* Atkinson-Kolm-Sen (AKS) relative inequality index $G_R : [0,1]^n \longrightarrow [0,1]$ *is defined as*

$$G_R(\mathbf{x}) = 1 - \frac{A(\mathbf{x})}{\bar{x}} \tag{8}$$

for $\mathbf{x} \neq \boldsymbol{0}$*, and* $G_R(\boldsymbol{0}) = 0$*. The fact that* A *is stable for dilations ensures the dilational invariance of* G*. The relative inequality index can be written as* $G_R(\mathbf{x}) = (\bar{x} - \tilde{x})/\bar{x}$*, with* $\tilde{x} \leq \bar{x}$*, and represents the fraction of total income that could be saved if society distributed the remaining amount equally without any welfare loss. In other words, it can be interpreted as the proportion of welfare loss due to inequality.*

In the AKS framework, a welfare function A which is stable for both translations and dilations is associated with both absolute and relative inequality indices G and G_R, respectively, with $G(\boldsymbol{x}) = \bar{x}\,G_R(\boldsymbol{x})$ for all $\boldsymbol{x} \in [0,1]^n$.

A class of welfare functions that play an important role in this paper is that of the generalized Gini welfare functions introduced by Weymark [70], see also Mehran [57], Donaldson and Weymark [23, 24], Yaari [71, 72], Ebert [27], Quiggin [61], Ben-Porath and Gilboa [8].

Definition 13 *Given a weighting vector* $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$*, with* $w_1 \geq \cdots \geq w_n \geq 0$ *and* $\sum_{i=1}^n w_i = 1$*, the* generalized Gini welfare function *associated with* $\mathbf{w}$ *is the function* $A : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$A(\mathbf{x}) = \sum_{i=1}^{n} w_i x_{(i)} \tag{9}$$

and the associated generalized Gini inequality index *is defined as*

$$G(\mathbf{x}) = \bar{x} - A(\mathbf{x}) = -\sum_{i=1}^{n} (w_i - \frac{1}{n})\, x_{(i)}\,. \tag{10}$$

The generalized Gini welfare functions, which are strict if and only if $w_1 > \ldots > w_n > 0$, are clearly stable for both translations and dilations. For this reason they have a natural central role within the AKS framework and Blackorby and Donaldson's correspondence formula.

3 The Gini inequality index and the associated welfare function

Consider a population of $n \geq 2$ individuals whose income distribution is represented by $\boldsymbol{x} = (x_1, \ldots, x_n) \in [0,1]^n$. At the individual level, the standard $[0,1]$ domain of the aggregation functions framework can be obtained by rescaling the traditional income domain $[0,\infty)$ with respect to some appropriate conventional upper bound for the income values. The welfare functions and inequality indices under consideration in this paper rescale analogously. In fact, most of our results extend immediately to the traditional income domain $[0,\infty)$, or even $\mathbb{R}$ itself.

The classical absolute Gini inequality index G^c is traditionally defined as

$$G^c(\boldsymbol{x}) = \frac{1}{2n^2} \sum_{i,j=1}^{n} |x_i - x_j|\,. \tag{11}$$

However, the double summation expression for $n^2 G^c(\boldsymbol{x})$ in (11) corresponds to

$$\begin{array}{rrrrr}
(x_{(n)} - x_{(n-1)}) + & (x_{(n)} - x_{(n-2)}) & + \ldots + & (x_{(n)} - x_{(2)}) & + \; (x_{(n)} - x_{(1)}) \\
& + \; (x_{(n-1)} - x_{(n-2)}) & + \ldots + & (x_{(n-1)} - x_{(2)}) & + \; (x_{(n-1)} - x_{(1)}) \\
& & \vdots & & \\
& & + & (x_{(3)} - x_{(2)}) & + \; (x_{(3)} - x_{(1)}) \\
& & & & + \; (x_{(2)} - x_{(1)})
\end{array}$$

which can be rewritten as

$$(n-1)x_{(n)} + ((n-2)-1)x_{(n-1)} + \ldots + (1-(n-2))x_{(2)} + (-(n-1))x_{(1)}\,. \tag{12}$$

The classical *absolute Gini inequality index* G^c can thus be written in the form

$$G^c(\boldsymbol{x}) = -\sum_{i=1}^{n} \frac{n-2i+1}{n^2}\, x_{(i)} \tag{13}$$

where $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$. Notice that the coefficients of G^c have zero sum, $\sum_{i=1}^{n}(n-2i+1) = 0$. This expression shows explicitly the coefficients associated with the ordered income variables and is thereby the most convenient in our presentation.

Given an income distribution $\boldsymbol{x} \in [0,1]^n$, the so called Lorenz area measures the deviation from the uniform income distribution and is related with the classical relative Gini inequality index. In a population of $n \geq 2$ individuals, the graphical representation of the classical relative Gini can be derived as follows. Consider the auxiliary functions

$$V(\boldsymbol{x}) = \sum_{i=1}^{n}(x_{(1)} + \ldots + x_{(i)}) = nx_{(1)} + (n-1)x_{(2)} + \ldots + x_{(n)} \tag{14}$$

$$U(\boldsymbol{x}) = \sum_{i=1}^{n}(x_{(i)} + \ldots + x_{(n)}) = x_{(1)} + 2x_{(2)} + \ldots + nx_{(n)}\,. \tag{15}$$

We can easily express $U(\boldsymbol{x})$ in terms of $V(\boldsymbol{x})$,

$$\begin{aligned} U(\boldsymbol{x}) &= \sum_{i=1}^{n}[(x_{(1)} + \ldots + x_{(n)}) - (x_{(1)} + \ldots + x_{(i)}) + x_{(i)}] \\ &= n^2\bar{x} - V(\boldsymbol{x}) + n\bar{x} = n(n+1)\bar{x} - V(\boldsymbol{x}) \end{aligned} \tag{16}$$

where $\bar{x} = (x_{(1)} + \ldots + x_{(n)})/n$. Since

$$\begin{aligned} n^2 G^c(\boldsymbol{x}) &= -\sum_{i=1}^{n}(n-2i+1)x_{(i)} \\ &= -((n-1)x_{(1)} + (n-3)x_{(2)} + \ldots + (-n+1)x_{(n)}) \end{aligned} \tag{17}$$

we can write $G^c(\boldsymbol{x})$ in terms of $\bar{x}$ and $V(\boldsymbol{x})$,

$$n^2 G^c(\boldsymbol{x}) = -(V(\boldsymbol{x}) - U(\boldsymbol{x})) = n(n+1)\bar{x} - 2V(\boldsymbol{x})\,. \tag{18}$$

Consider now the area illustrated in Fig. 1, i.e., the sum of the grey rectangles. We are interested in the vertical differences between the diagonal i/n values, associated with uniform cumulative income distribution, and the actual cumulative income distribution expressed by the $h_i(\boldsymbol{x})$ values,

$$h_i(\boldsymbol{x}) = \frac{x_{(1)} + \ldots + x_{(i)}}{x_{(1)} + \ldots + x_{(n)}} \tag{19}$$

where we assume $\boldsymbol{x} \neq \mathbf{0}$. The total area $H(\boldsymbol{x})$ indicated in Fig. 1 is then

$$\begin{aligned} H(\boldsymbol{x}) &= \sum_{i=1}^{n}\Big(\frac{i}{n} - h_i(\boldsymbol{x})\Big) = \sum_{i=1}^{n}\Big(\frac{i}{n} - \frac{x_{(1)} + \ldots + x_{(i)}}{x_{(1)} + \ldots + x_{(n)}}\Big) \\ &= \frac{1}{n\bar{x}}\Big[\sum_{i=1}^{n}\big(i\,\bar{x} - (x_{(1)} + \ldots + x_{(i)})\big)\Big] \\ &= \frac{1}{n\bar{x}}\Big[\frac{n(n+1)}{2}\,\bar{x} - V(\boldsymbol{x})\Big] = \frac{n}{2\bar{x}}\,G^c(\boldsymbol{x}) \end{aligned} \tag{20}$$

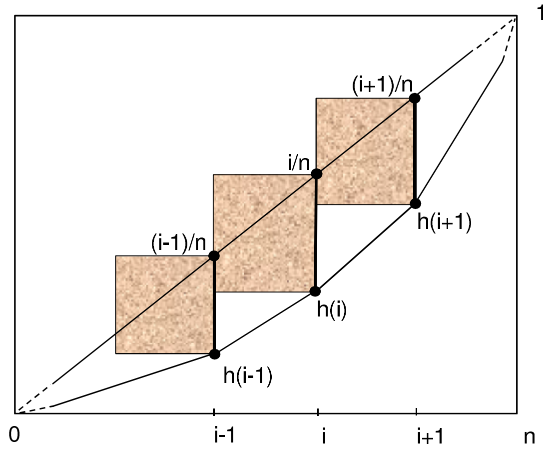

Figure 1: The classical relative Gini in the discrete case.

using (18). Finally, we obtain $G^c(\boldsymbol{x}) = \bar{x}\, G^c_R(\boldsymbol{x})$, where G^c_R stands for the *classical relative Gini inequality index*,

$$G^c_R(\boldsymbol{x}) = \frac{H(\boldsymbol{x})}{n/2}\,. \tag{21}$$

In the AKS framework, the *welfare function* associated with the classical absolute Gini inequality index is

$$A^c(\boldsymbol{x}) = \bar{x} - G^c(\boldsymbol{x}) \tag{22}$$

and it can be written as

$$A^c(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{2(n-i)+1}{n^2}\, x_{(i)} = \sum_{i=1}^{n} \frac{1}{n}\, x_{(i)} + \sum_{i=1}^{n} \frac{n-2i+1}{n^2}\, x_{(i)} \tag{23}$$

where the coefficients of $A^c(\boldsymbol{x})$ have unit sum, $\sum_{i=1}^{n}(2(n-i)+1) = n^2$.

The pair A^c, G^c is the classical instance of Blackorby and Donaldson's correspondence formula for generalized Gini welfare functions and inequality indices. In what follows we will describe how to obtain a general expansion of Blackorby and Donaldson's correspondence formula with respect to analogous pairs of generalized Gini welfare functions and inequality indices.

4 Capacities and Choquet integrals

In this section we present a brief review of the basic facts on Choquet integration, focusing on the Möbius representation framework. For recent reviews of Choquet integration see [44, 43, 45, 46] for the general case, and [59, 55, 56] for the 2-additive case in particular.

Consider a finite set of interacting individuals $N = \{1, 2, \ldots, n\}$. Any subsets $S, T \subseteq N$ with cardinalities $0 \le s,\ t \le n$ are usually called coalitions. The concepts of capacity and Choquet integral in the definitions below are due to [21, 69, 22, 38, 39].

Definition 14 *A* capacity *on the set N is a set function $\mu : 2^N \longrightarrow [0,1]$ satisfying*

(i) $\mu(\emptyset) = 0$, $\mu(N) = 1$ (boundary conditions)

(ii) $S \subseteq T \subseteq N \quad \Rightarrow \quad \mu(S) \leq \mu(T)$ (monotonicity conditions).

Capacities are also known as *fuzzy measures* [69] or *non-additive measures* [22]. A capacity μ is said to be additive over N if $\mu(S \cup T) = \mu(S) + \mu(T)$ for all coalitions S, $T \subseteq N$, with $S \cap T = \emptyset$. Alternatively, the capacity μ is subadditive over N if $\mu(S \cup T) \leq \mu(S) + \mu(T)$ for all coalitions S, $T \subseteq N$ with $S \cap T = \emptyset$, with at least two such coalitions for which μ is subadditive in the strict sense. Analogously, the capacity μ is superadditive over N if $\mu(S \cup T) \geq \mu(S) + \mu(T)$ for all coalitions S, $T \subseteq N$ with $S \cap T = \emptyset$, with at least two such coalitions for which μ is superadditive in the strict sense. In the additive case, $\sum_{i=1}^{n} \mu(\{(i)\}) = 1$.

Definition 15 *Let μ be a capacity on N. The* Choquet integral $\mathcal{C}_\mu : [0,1]^n \longrightarrow [0,1]$ *with respect to μ is defined as*

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)} \qquad \mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n \tag{24}$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$. Moreover, $A_{(i)} = \{(i), \ldots, (n)\}$ and $A_{(n+1)} = \emptyset$.

The Choquet integral is a continuous and idempotent aggregation function. Within each comonotonicity cone of the domain $[0,1]^n$, the Choquet integral reduces to a weighted mean, whose weights depend on the comonotonicity cone. In fact, given $\boldsymbol{x} \in [0,1]^n$, we can write $\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, x_{(i)}$ where $w_i = \mu(\{(i), (i+1), \ldots, (n)\}) - \mu(\{(i+1), \ldots, (n)\})$, with $w_i \geq 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$ due to the boundary and monotonicity conditions of the capacity.

In the additive case, since

$$\mu(A_{(i)}) = \mu(\{(i)\}) + \mu(\{(i+1)\}) + \ldots + \mu(\{(n)\}) = \mu(\{(i)\}) + \mu(A_{(i+1)}) \tag{25}$$

the Choquet integral reduces to a weighted averaging function (WA),

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)} = \sum_{i=1}^{n} \mu(\{(i)\}) x_{(i)} = \sum_{i=1}^{n} \mu(\{i\}) x_i \tag{26}$$

where the weights are given by $w_i = \mu(\{i\})$, $i = 1, \ldots, n$.

A capacity μ can be equivalently represented by its Möbius transform m_μ [62, 20, 41, 53, 58] in the following way.

Definition 16 *Let μ be a capacity on the set N. The* Möbius transform $m_\mu : 2^N \longrightarrow \mathbb{R}$ *associated with the capacity μ is defined as*

$$m_\mu(T) = \sum_{S \subseteq T} (-1)^{t-s} \mu(S) \qquad T \subseteq N \tag{27}$$

where s and t denote the cardinality of the coalitions S and T, respectively.

Conversely, given the Möbius transform m_μ, the associated capacity μ is obtained as

$$\mu(T) = \sum_{S \subseteq T} m_\mu(S) \qquad\qquad T \subseteq N\,. \tag{28}$$

In the Möbius representation, the boundary conditions take the form

$$m_\mu(\emptyset) = 0 \qquad\qquad \sum_{T \subseteq N} m_\mu(T) = 1 \tag{29}$$

and the monotonicity conditions can be expressed as follows: for each $i = 1, \ldots, n$ and each coalition $T \subseteq N \setminus \{i\}$, the monotonicity condition is written as

$$\sum_{S \subseteq T} m_\mu(S \cup \{i\}) \geq 0 \qquad\qquad T \subseteq N \setminus \{i\} \qquad i = 1, \ldots, n\,. \tag{30}$$

This form of the monotonicity conditions derives from the original monotonicity conditions in Definition 14, expressed as $\mu(T \cup \{i\}) - \mu(T) \geq 0$ for each $i \in N$ and $T \subseteq N \setminus \{i\}$.

The Choquet integral in Definition 15 can be expressed in terms of the Möbius transform in the following way [53, 41],

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{T \subseteq N} m_\mu(T) \min_{i \in T}(x_i)\,. \tag{31}$$

Defining a capacity μ on a set N of n elements requires $2^n - 2$ real coefficients, corresponding to the capacity values $\mu(T)$ for $T \subseteq N$. In order to control exponential complexity, Grabisch [40] introduced the concept of k-additive capacities.

Definition 17 *A capacity μ on the set N is said to be* k-additive *if its Möbius transform satisfies $m_\mu(T) = 0$ for all $T \subseteq N$ with $t > k$, and there exists at least one coalition $T \subseteq N$ with $t = k$ such that $m_\mu(T) \neq 0$.*

In the k-additive case, with $k = 1, \ldots, n$, the capacity μ is expressed as follows in terms of the Möbius transform m_μ,

$$\mu(T) = \sum_{S \subseteq T,\, s \leq k} m_\mu(S) \qquad\qquad T \subseteq N \tag{32}$$

and the boundary and monotonicity conditions (29) and (30) take the form

$$m_\mu(\emptyset) = 0 \qquad\qquad \sum_{T \subseteq N,\, t \leq k} m_\mu(T) = 1 \tag{33}$$

$$\sum_{S \subseteq T,\, s \leq k-1} m_\mu(S \cup i) \geq 0 \qquad\qquad T \subseteq N \setminus i \qquad i = 1, \ldots, n\,. \tag{34}$$

Finally, we examine the particular case of symmetric capacities and Choquet integrals, which play a crucial role in this paper.

Definition 18 *A capacity μ is said to be* symmetric *if it depends only on the cardinality of the coalition considered, in which case we use the simplified notation*

$$\mu(T) = \mu(t) \qquad \textit{where} \qquad t = |T|\,. \tag{35}$$

Accordingly, for the Möbius transform m_μ associated with a symmetric capacity μ we use the notation

$$m_\mu(T) = m_\mu(t) \qquad \text{where} \qquad t = |T|\,. \tag{36}$$

In the symmetric case, the expression (28) for the capacity μ in terms of the Möbius transform m_μ reduces to

$$\mu(t) = \sum_{s=1}^{t} \binom{t}{s} m_\mu(s) \qquad\qquad t = 1, \ldots, n \tag{37}$$

and the boundary and monotonicity conditions (29) and (30) take the form

$$m_\mu(0) = 0 \qquad\qquad \sum_{s=1}^{n} \binom{n}{s} m_\mu(s) = 1 \tag{38}$$

$$\sum_{s=1}^{t} \binom{t-1}{s-1} m_\mu(s) \geq 0 \qquad\qquad t = 1, \ldots, n\,. \tag{39}$$

The monotonicity conditions correspond to $\mu(t) - \mu(t-1) \geq 0$ for $t = 1, \ldots, n$.

The Choquet integral (24) with respect to a symmetric capacity μ reduces to an Ordered Weighted Averaging (OWA) function [28, 73],

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} [\mu(n-i+1) - \mu(n-i)]x_{(i)} = \sum_{i=1}^{n} w_i\, x_{(i)} = A(\boldsymbol{x}) \tag{40}$$

where the weights $w_i = \mu(n-i+1) - \mu(n-i)$ satisfy $w_i \geq 0$ for $i = 1, \ldots, n$ due to the monotonicity of the capacity μ, and $\sum_{i=1}^{n} w_i = 1$ due to the boundary conditions $\mu(0) = 0$ and $\mu(n) = 1$. Comprehensive reviews of OWA functions can be found in [74] and [75].

The weighting structure of the OWA function (40) is of the general form $w_i = f(\frac{n-i+1}{n}) - f(\frac{n-i}{n})$ where f is a continuous and increasing function on the unit interval, with $f(0) = 0$ and $f(1) = 1$. Gajdos [30] shows that the OWA function A is associated with a k-additive capacity μ, with $k = 1, \ldots, n$, if and only if f is polynomial of order k. In fact, in (37), the k-additive case is obtained simply by taking $m_\mu(k+1) = \ldots = m_\mu(n) = 0$, and the binomial coefficient of the Möbius value $m_\mu(k)$ corresponds to $t(t-1)\ldots(t-k+1)/k!$, which is polynomial of order k in the coalition cardinality t.

5 The binomial decomposition

In this section we consider OWA functions $A : [0,1]^n \longrightarrow [0,1]$ and we review the analysis of the associated symmetric capacities in the Möbius representation framework. In particular, we recall the binomial decomposition of OWA functions due to Calvo and De Baets [14], with the addition of a uniqueness result.

We begin by introducing the convenient notation

$$\alpha_j = \binom{n}{j} m_\mu(j) \qquad j = 1, \ldots, n\,. \tag{41}$$

In this notation the upper boundary condition (38) reduces to

$$\sum_{j=1}^{n} \alpha_j = 1 \tag{42}$$

and the monotonicity conditions (39) take the form

$$\sum_{j=1}^{i} \frac{\binom{i-1}{j-1}}{\binom{n}{j}} \alpha_j \geq 0 \qquad\qquad i = 1, \ldots, n\,. \tag{43}$$

Considering the OWA function $A : [0,1]^n \longrightarrow [0,1]$ associated with a symmetric capacity μ as in (40),

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, x_{(i)} \qquad w_i = \mu(n-i+1) - \mu(n-i) \tag{44}$$

the capacity values as in (37) expressed in the new notation take the form

$$\begin{aligned} \mu(n-i+1) &= \frac{\binom{n-i+1}{1}}{\binom{n}{1}} \alpha_1 + \frac{\binom{n-i+1}{2}}{\binom{n}{2}} \alpha_2 + \ldots + \frac{\binom{n-i+1}{n-i}}{\binom{n}{n-i}} \alpha_{n-i} + \frac{\binom{n-i+1}{n-i+1}}{\binom{n}{n-i+1}} \alpha_{n-i+1} \\ \mu(n-i) &= \frac{\binom{n-i}{1}}{\binom{n}{1}} \alpha_1 + \frac{\binom{n-i}{2}}{\binom{n}{2}} \alpha_2 + \ldots + \frac{\binom{n-i}{n-i}}{\binom{n}{n-i}} \alpha_{n-i}\,. \end{aligned} \tag{45}$$

Accordingly, the weights w_i, $i = 1, \ldots, n$ as in (44) are given by

$$\begin{aligned} w_i &= \frac{\binom{n-i}{0}}{\binom{n}{1}} \alpha_1 + \frac{\binom{n-i}{1}}{\binom{n}{2}} \alpha_2 + \ldots + \frac{\binom{n-i}{n-i-1}}{\binom{n}{n-i}} \alpha_{n-i} + \frac{\binom{n-i}{n-i}}{\binom{n}{n-i+1}} \alpha_{n-i+1} \\ &= \sum_{j=1}^{n-i+1} \frac{\binom{n-i}{j-1}}{\binom{n}{j}} \alpha_j \end{aligned} \tag{46}$$

where the coefficients α_j, $j = 1, \ldots, n$ are subject to conditions (42) and (43). We have used the standard formula $\binom{p}{q} = \binom{p+1}{q+1} - \binom{p}{q+1}$, with $p, q = 0, 1, \ldots$

Notice that the boundary and monotonicity conditions (42) and (43) correspond to the standard conditions $w_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$.

Finally, the OWA function $A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, x_{(i)}$ is written as

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, x_{(i)} = \sum_{i=1}^{n} \sum_{j=1}^{n-i+1} \alpha_j \frac{\binom{n-i}{j-1}}{\binom{n}{j}} x_{(i)} \tag{47}$$

where the coefficients α_j, $j = 1, \ldots, n$ are subject to conditions (42) and (43). In order to illustrate the weighting structure of the OWA function A, in terms of the coefficients α_j, $j = 1, \ldots, n$, we can write (47) explicitly as follows,

$$\begin{aligned} A(\boldsymbol{x}) &= \Big[\alpha_1 \frac{\binom{n-1}{0}}{\binom{n}{1}} + \alpha_2 \frac{\binom{n-1}{1}}{\binom{n}{2}} + \ldots + \alpha_{n-1} \frac{\binom{n-1}{n-2}}{\binom{n}{n-1}} + \alpha_n \frac{\binom{n-1}{n-1}}{\binom{n}{n}}\Big]\, x_{(1)} \\ &\quad + \Big[\alpha_1 \frac{\binom{n-2}{0}}{\binom{n}{1}} + \alpha_2 \frac{\binom{n-2}{1}}{\binom{n}{2}} + \ldots + \alpha_{n-2} \frac{\binom{n-2}{n-3}}{\binom{n}{n-2}} + \alpha_{n-1} \frac{\binom{n-2}{n-2}}{\binom{n}{n-1}}\Big]\, x_{(2)} \\ &\quad + \ldots + \Big[\alpha_1 \frac{\binom{1}{0}}{\binom{n}{1}} + \alpha_2 \frac{\binom{1}{1}}{\binom{n}{2}}\Big]\, x_{(n-1)} + \Big[\alpha_1 \frac{\binom{0}{0}}{\binom{n}{1}}\Big]\, x_{(n)}\,. \end{aligned} \tag{48}$$

A fact that emerges clearly from expression (48) is that the values of the coefficients α_j, $j = 1, \ldots, n$ are uniquely determined by the weighting structure of the OWA function A: w_n determines α_1, then w_{n-1} determines α_2, and so on.

Interchanging the two summations in (47),

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} \Big[\sum_{j=1}^{n-i+1} \alpha_j \frac{\binom{n-i}{j-1}}{\binom{n}{j}} \Big] x_{(i)} = \sum_{j=1}^{n} \alpha_j \Big[\sum_{i=1}^{n-j+1} \frac{\binom{n-i}{j-1}}{\binom{n}{j}} x_{(i)} \Big] \qquad (49)$$

we obtain the binomial decomposition of OWA functions due to Calvo and De Baets [14]. Following the notation introduced by the authors, we begin with the following definition.

Definition 19 *The* binomial OWA functions $C_j : [0,1]^n \longrightarrow [0,1]$*, with* $j = 1, \ldots, n$*, are defined as*

$$C_j(\mathbf{x}) = \sum_{i=1}^{n} w_{ji}\, x_{(i)} = \sum_{i=1}^{n} \frac{\binom{n-i}{j-1}}{\binom{n}{j}} x_{(i)} \qquad j = 1, \ldots, n \qquad (50)$$

where the binomial weights w_{ji}*,* $i, j = 1, \ldots, n$ *are null when* $i + j > n + 1$ *according to the usual convention that* $\binom{p}{q} = 0$ *when* $p < q$*, with* $p, q = 0, 1, \ldots$ *In fact, with the exception of* $C_1(\mathbf{x}) = \bar{x}$*, the binomial OWA functions* C_j*,* $j = 2, \ldots, n$ *have an increasing number of null weights, in correspondence with* $x_{(n-j+2)}, \ldots, x_{(n)}$*.*

The weight normalization of the binomial OWA functions, $\sum_{i=1}^{n} w_{ji} = 1$ for $j = 1, \ldots, n$, is due to the column-sum property of binomial coefficients,

$$\sum_{i=1}^{n} \binom{n-i}{j-1} = \sum_{i=0}^{n-1} \binom{i}{j-1} = \binom{n}{j} \qquad j = 1, \ldots, n\,. \qquad (51)$$

Proposition 3 *Any OWA function* $A : [0,1]^n \longrightarrow [0,1]$ *can be written uniquely as*

$$A(\boldsymbol{x}) = \alpha_1 C_1(\boldsymbol{x}) + \alpha_2 C_2(\boldsymbol{x}) + \ldots + \alpha_n C_n(\boldsymbol{x}) \qquad (52)$$

where the coefficients α_j*,* $j = 1, \ldots, n$ *are subject to conditions (42) and (43).*

Proof: The expression of the binomial decomposition (52) is due to Calvo and De Baets [14, Theorem 2], on the basis of considerations which we have presented in (42) - (49). On the other hand, uniqueness is due to the fact that the weights w_i, $i = 1, \ldots, n$ of the OWA function A are in one-to-one correspondence with the coefficients α_j, $j = 1, \ldots, n$ satisfying (42) and (43), as indicated after (48). □

In the binomial decomposition (52), the k-additive case, with $k = 1, \ldots, n$, is obtained simply by taking $\alpha_{k+1} = \ldots = \alpha_n = 0$, in which case (52) reduces to $A(\boldsymbol{x}) = \alpha_1 C_1(\boldsymbol{x}) + \ldots + \alpha_k C_k(\boldsymbol{x})$.

An interesting result concerning the cumulative properties of binomial weights is due to Calvo and De Baets [14].

Proposition 4 *The binomial weights* $w_{ji} \in [0,1]$*, with* $i, j = 1, \ldots, n$*, have the following cumulative property,*

$$\sum_{k=1}^{i} w_{j-1,k} \leq \sum_{k=1}^{i} w_{jk} \qquad i = 1, \ldots, n \quad j = 2, \ldots, n\,. \qquad (53)$$

Proof: Given the binomial weight definition in (50), we wish to prove that

$$\sum_{k=1}^{i} \frac{\binom{n-k}{j-2}}{\binom{n}{j-1}} \leq \sum_{k=1}^{i} \frac{\binom{n-k}{j-1}}{\binom{n}{j}} \qquad i = 1, \dots, n \quad j = 2, \dots, n \tag{54}$$

which can be written as

$$\binom{n}{j} \sum_{k=1}^{i} \binom{n-k}{j-2} \leq \binom{n}{j-1} \sum_{k=1}^{i} \binom{n-k}{j-1}. \tag{55}$$

Using the column-sum property of binomial coefficients, inequality (55) takes the form

$$\binom{n}{j} \left[\binom{n}{j-1} - \binom{n-i}{j-1} \right] \leq \binom{n}{j-1} \left[\binom{n}{j} - \binom{n-i}{j} \right] \tag{56}$$

or, equivalently,

$$\binom{n}{j} \binom{n-i}{j-1} \geq \binom{n}{j-1} \binom{n-i}{j} \tag{57}$$

which holds trivially for $n < i + j$, and for $n \geq i + j$ reduces to $n - j \geq n - i - j$, always true for $i = 1, \dots, n$ and $j = 2, \dots, n$. $\square$

Given that binomial weights have the cumulative property (53), Proposition 1 implies that the binomial OWA functions C_j, $j = 1, \dots, n$ satisfy the relations $1 \geq \bar{x} = C_1(\boldsymbol{x}) \geq C_2(\boldsymbol{x}) \geq \dots \geq C_n(\boldsymbol{x}) \geq 0$, for any $\boldsymbol{x} \in [0,1]^n$.

Summarizing, the binomial decomposition (52) holds for any OWA function A in terms of the binomial OWA functions C_j, $j = 1, \dots, n$ and the corresponding coefficients α_j, $j = 1, \dots, n$ subject to conditions (42) and (43).

6 The binomial Gini inequality indices

In this section we focus on generalized Gini welfare functions $A : [0,1]^n \longrightarrow [0,1]$, corresponding to S-concave OWA functions. We begin by showing that the functions C_j in the binomial decomposition (52) are themselves generalized Gini welfare functions. We then reexamine the binomial decomposition in the restricted context of generalized Gini welfare functions A, adding appropriate S-concavity conditions to the original monotonicity conditions.

The binomial weights w_{ji}, $i, j = 1, \dots, n$ as in (50) have regularity properties which have interesting implications at the level of the functions C_j, $j = 1, \dots, n$.

Proposition 5 *The binomial weights $w_{ji} \in [0,1]$, with $i, j = 1, \dots, n$, have the following properties,*

i. *for* $j = 1$ $\quad 1/n = w_{11} = w_{12} = \dots = w_{1,n-1} = w_{1n}$

ii. *for* $j = 2$ $\quad 2/n = w_{21} > w_{22} > \dots > w_{2,n-1} > w_{2n} = 0$

iii. *for* $j = 3, \dots, n$ $\quad j/n = w_{j1} > w_{j2} > \dots > w_{j,n-j+2} = \dots = w_{jn} = 0$

Proof: Concerning (i), we obtain $w_{1i} = 1/n$ for $i = 1, \ldots, n$ directly from the binomial weight definition in (50). Moreover, given $i = 2, \ldots, n$ and $j = 2, \ldots, n$, we have

$$w_{j,i-1} - w_{ji} = \frac{1}{\binom{n}{j}} \left[\binom{n-i+1}{j-1} - \binom{n-i}{j-1} \right] = \frac{1}{\binom{n}{j}} \binom{n-i}{j-2} \tag{58}$$

which means, for $i = 2, \ldots, n$ and $j = 2$, that $w_{2,i-1} - w_{2i} = 1/\binom{n}{2}$ constant, as in (ii). Finally, for $j = 3, \ldots, n$, we obtain from (58) that

$$w_{j,i-1} - w_{ji} > 0 \qquad \text{for} \quad i = 2, \ldots, n-j+2 \tag{59}$$

$$w_{j,i-1} - w_{ji} = 0 \qquad \text{for} \quad i = n-j+3, \ldots, n \tag{60}$$

which proves (iii). □

The functions C_j, $j = 1, \ldots, n$ are continuous, idempotent, and stable for translations, where the latter two properties follow immediately from $\sum_{i=1}^{n} w_{ji} = 1$ for $j = 1, \ldots, n$. Moreover, given that binomial weights are non increasing, $w_{j1} \geq w_{j2} \geq \ldots \geq w_{jn}$ for $j = 1, \ldots, n$, Proposition 2 implies that the functions C_j, $j = 1, \ldots, n$ are S-concave, with strict S-concavity applying only to C_2.

In relation with these properties, we conclude that the functions C_j, $j = 1, \ldots, n$, which we hereafter call *binomial Gini welfare functions*, are generalized Gini welfare functions on the income domain $\boldsymbol{x} \in [0,1]^n$. It is then natural to express the binomial decomposition (52) entirely in the context of generalized Gini welfare functions, restricting the domain of the coefficients α_j, $j = 1, \ldots, n$ by means of appropriate S-concavity conditions.

Proposition 6 *Any generalized Gini welfare function* $A : [0,1]^n \longrightarrow [0,1]$ *can be written uniquely as*

$$A(\boldsymbol{x}) = \alpha_1 C_1(\boldsymbol{x}) + \alpha_2 C_2(\boldsymbol{x}) + \ldots + \alpha_n C_n(\boldsymbol{x}) \tag{61}$$

where the coefficients α_j, $j = 1, \ldots, n$ *are subject to conditions (42), (43), and*

$$\sum_{j=2}^{n} \frac{\binom{n-i}{j-2}}{\binom{n}{j}} \alpha_j \geq 0 \qquad i = 2, \ldots, n\,. \tag{62}$$

Proof: In order to ensure the S-concavity of an OWA function A, thereby obtaining a generalized Gini welfare function, the weights (46) must satisfy the conditions $w_{i-1} - w_i \geq 0$ for $i = 2, \ldots, n$, as in Proposition 2,

$$w_{i-1} - w_i = \sum_{j=1}^{n-i+2} \frac{\binom{n-i+1}{j-1} - \binom{n-i}{j-1}}{\binom{n}{j}} \alpha_j \geq 0 \qquad i = 2, \ldots, n \tag{63}$$

since $\binom{n-i}{j-1} = 0$ for $j = n-i+2$. Therefore, given that the overall coefficient of α_1 is null, and that both binomial coefficients in the numerator are null for $j > n-i+2$, we obtain the S-concavity conditions (62). □

Summarizing, the binomial decomposition (61) holds for any generalized Gini welfare function A in terms of the binomial Gini welfare functions C_j, $j = 1, \ldots, n$ and the corresponding coefficients α_j, $j = 1, \ldots, n$ subject to conditions (42), (43), and (62).

Definition 20 *Consider the binomial Gini welfare functions* $C_j : [0,1]^n \longrightarrow [0,1]$*, with* $C_j(\mathbf{x}) = \sum_{i=1}^n w_{ji} x_{(i)}$ *for* $j = 1, \dots, n$*. The* binomial Gini inequality indices $G_j : [0,1]^n \longrightarrow [0,1]$*, with* $j = 1, \dots, n$*, are defined as*

$$G_j(\mathbf{x}) = \bar{x} - C_j(\mathbf{x}) \qquad j = 1, \dots, n \tag{64}$$

which means that

$$G_j(\mathbf{x}) = -\sum_{i=1}^n v_{ji} x_{(i)} = -\sum_{i=1}^n \left[w_{ji} - \frac{1}{n} \right] x_{(i)} \qquad j = 1, \dots, n \tag{65}$$

where the coefficients v_{ji}*,* $i, j = 1, \dots, n$ *are equal to* $-1/n$ *when* $i + j > n + 1$*, since in such case the binomial weights* w_{ji} *are null. The weight normalization of the binomial Gini welfare functions,* $\sum_{i=1}^n w_{ji} = 1$ *for* $j = 1, \dots, n$*, implies that* $\sum_{i=1}^n v_{ji} = 0$ *for* $j = 1, \dots, n$*.*

The binomial Gini inequality indices G_j, $j = 1, \dots, n$ are continuous, nilpotent, and invariant for translations, where the latter two properties follow immediately from $\sum_{i=1}^n v_{ji} = 0$ for $j = 1, \dots, n$. Moreover, the G_j are S-convex: given $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ with $\bar{x} = \bar{y}$, we have that $\boldsymbol{x} \preceq \boldsymbol{y} \Rightarrow C_j(\boldsymbol{x}) \geq C_j(\boldsymbol{y}) \Rightarrow G_j(\boldsymbol{x}) \leq G_j(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, due to the S-concavity of the C_j, $j = 1, \dots, n$.

In fact, the binomial Gini inequality indices G_j, $j = 1, \dots, n$ in (64) correspond to the Atkinson-Kolm-Sen (AKS) absolute inequality indices associated with the binomial welfare functions C_j, $j = 1, \dots, n$, in the spirit of Blackorby and Donaldson's correspondence formula. Together, as we discuss below, the binomial Gini welfare functions C_j and the binomial Gini inequality indices G_j, $j = 1, \dots, n$ can be regarded as two equivalent functional bases for the class of generalized Gini welfare functions and inequality indices.

In analogy with the binomial weights w_{ji}, $i, j = 1, \dots, n$, their inequality counterparts v_{ji}, $i, j = 1, \dots, n$ have interesting regularity properties, which follow directly from Proposition 5.

Proposition 7 *The coefficients* $v_{ji} \in [-1/n, (n-1)/n]$*, with* $i, j = 1, \dots, n$*, have the following properties,*

i. *for* $j = 1$ $\qquad 0 = v_{11} = v_{12} = \ldots = v_{1,n-1} = v_{1n}$

ii. *for* $j = 2$ $\qquad 1/n = v_{21} > v_{22} > \ldots > v_{2,n-1} > v_{2n} = -1/n$

iii. *for* $j = 3, \dots, n$ $\qquad \frac{j-1}{n} = v_{j1} > v_{j2} > \ldots > v_{j,n-j+2} = \ldots = v_{jn} = -1/n$

Notice that $C_1(\boldsymbol{x}) = \bar{x}$ and $G_1(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in [0,1]^n$. On the other hand, $C_2(\boldsymbol{x})$ has $n-1$ positive linearly decreasing weights and one null last weight, and the associated $G_2(\boldsymbol{x})$ has linearly increasing coefficients and is in fact proportional to the classical Gini index, $G_2(\boldsymbol{x}) = \frac{n}{n-1} G^c(\boldsymbol{x})$. The remaining $C_j(\boldsymbol{x})$, $j = 3, \dots, n$, have $n - j + 1$ positive non-linear decreasing weights and $j - 1$ null last weights, and the associated $G_j(\boldsymbol{x})$, $j = 3, \dots, n$ have $n - j + 2$ non-linear increasing weights and $j - 1$ equal last weights.

Therefore, the only strict binomial welfare function is $C_1(\boldsymbol{x}) = \bar{x}$ and the only strict binomial inequality index is $G_2(\boldsymbol{x}) = \frac{n}{n-1} G^c(\boldsymbol{x})$. In the remaining $G_j(\boldsymbol{x})$, $j = 3, \dots, n$ the last $j - 1$ coefficients coincide and thus they are non strict absolute inequality indices, in the sense that they are insensitive to income transfers involving only the $j-1$ richest individuals of the population.

As an immediate consequence of Proposition 6, substituting for the binomial Gini welfare functions in terms of the binomial Gini inequality indices, $C_j(\boldsymbol{x}) = \bar{x} - G_j(\boldsymbol{x})$ for $j = 1, \ldots, n$, we obtain the following result.

Proposition 8 *Any generalized Gini welfare function $A : [0,1]^n \longrightarrow [0,1]$ can be written uniquely as*

$$A(\boldsymbol{x}) = \bar{x} - \alpha_2 G_2(\boldsymbol{x}) - \ldots - \alpha_n G_n(\boldsymbol{x}) \tag{66}$$

where the coefficients α_j, $j = 2, \ldots, n$ are subject to the conditions

$$\sum_{j=2}^{n} \Big[1 - n\frac{\binom{i-1}{j-1}}{\binom{n}{j}}\Big]\alpha_j \leq 1 \qquad\qquad i = 1, \ldots, n \tag{67}$$

and

$$\sum_{j=2}^{n} \frac{\binom{n-i}{j-2}}{\binom{n}{j}}\, \alpha_j \geq 0 \qquad\qquad i = 2, \ldots, n \tag{68}$$

Notice that $G_1(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in [0,1]^n$ and thus its absence in (66) is in any case immaterial.

Proof: The expression of the binomial decomposition (66) is obtained directly from (61) in Proposition 6 by substituting for $\alpha_1 = 1 - \alpha_2 - \alpha_3 - \ldots - \alpha_n$, as in the boundary condition (42).

The S-concavity conditions (68) are the same as in Proposition 6, which do not involve the coefficient α_1. Consider now the monotonicity conditions (43). Substituting for $\alpha_1 = 1 - \alpha_2 - \alpha_3 - \ldots - \alpha_n$, we obtain

$$\frac{1}{n} + \left[\frac{\binom{i-1}{1}}{\binom{n}{2}} - \frac{1}{n}\right] \alpha_2 + \left[\frac{\binom{i-1}{2}}{\binom{n}{3}} - \frac{1}{n}\right] \alpha_3 + \ldots + \left[\frac{\binom{i-1}{i-1}}{\binom{n}{i}} - \frac{1}{n}\right] \alpha_i$$

$$-\frac{1}{n}(\alpha_{i+1} + \ldots + \alpha_n) \geq 0 \qquad\qquad i = 1, \ldots, n \tag{69}$$

which correspond to the following n monotonicity conditions in terms of the $n-1$ coefficients α_j, $j = 2, \ldots, n$,

$$\sum_{j=2}^{n} \Big[1 - n\frac{\binom{i-1}{j-1}}{\binom{n}{j}}\Big]\alpha_j \leq 1 \qquad\qquad i = 1, \ldots, n\,. \tag{70}$$

The first and the last of these monotonicity conditions are always of the form $\alpha_2 + \alpha_3 + \ldots + \alpha_n \leq 1$ and $\alpha_2 + 2\,\alpha_3 + \ldots + (n-1)\,\alpha_n \geq -1$, respectively. □

In the binomial inequality decomposition (66) the level of k-additivity of the generalized Gini welfare function A is controlled by the coefficients α_2, ..., α_n subject to the conditions (67). As k-additivity increases, the binomial decomposition of A includes an increasing number of binomial Gini inequality indices which are progressively insensitive to income transfers within the richest part of the population. Moreover, the binomial Gini inequality indices are increasingly stronger, $0 = G_1(\boldsymbol{x}) \leq G_2(\boldsymbol{x}) \leq \ldots \leq G_n(\boldsymbol{x}) \leq 1$ for any $\boldsymbol{x} \in [0,1]^n$, in correspondence with the analogous but inverse ordering of binomial Gini welfare functions obtained after Proposition 4.

7 The 2-additive and 3-additive cases

In this section we examine the binomial decomposition of generalized Gini welfare functions (66) in the 2-additive and 3-additive cases, focusing on the particular form of the monotonicity and S-concavity conditions (67) and (68). The 3-additive case is illustrated for a population of size $n = 6$.

Consider the 2-additive case. The monotonicity conditions (67) take the form

$$\left[1 - \frac{n(i-1)}{\binom{n}{2}}\right]\alpha_2 \leq 1 \qquad\qquad i = 1, \ldots, n \tag{71}$$

which are equivalent to

$$-1 \leq \alpha_2 \leq 1 \tag{72}$$

corresponding to the first and last of the n conditions (71), the others been dominated by these two. In turn, the S-concavity conditions (68) take the common form

$$\frac{1}{\binom{n}{2}}\,\alpha_2 \geq 0 \tag{73}$$

which is equivalent to

$$\alpha_2 \geq 0\,. \tag{74}$$

Notice that in the 2-additive case both the monotonicity and the S-concavity conditions are independent of n.

As an immediate consequence of Proposition 8, we have the following result.

Proposition 9 *Any 2-additive generalized Gini welfare function $A : [0,1]^n \longrightarrow [0,1]$ can be written uniquely as*

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{(i)} = \bar{x} - \alpha_2 G_2(\boldsymbol{x}) \tag{75}$$

where $G_2(\boldsymbol{x})$ is the binomial Gini inequality index

$$G_2(\boldsymbol{x}) = -\sum_{i=1}^{n} v_{2i}\, x_{(i)} = -\sum_{i=1}^{n} \frac{n-2i+1}{n(n-1)}\, x_{(i)} \tag{76}$$

and the coefficient α_2 is subject to the conditions (72) and (74).

Given that G_2 is proportional to the classical absolute Gini inequality index

$$G_2(\boldsymbol{x}) = \frac{n}{n-1}\, G^c(\boldsymbol{x}) \tag{77}$$

any 2-additive Gini welfare function can be written as

$$A(\boldsymbol{x}) = \bar{x} - \frac{n}{n-1}\, \alpha_2 G^c(\boldsymbol{x}) \tag{78}$$

where α_2 is a free parameter subject to the conditions (72) and (74).

The strict case $\alpha_2 > 0$ in (78) corresponds to the well-known Ben Porath and Gilboa's formula [8] for Weymark's generalized Gini welfare functions with linearly decreasing (inequality averse) weight distributions, see also [42].

In particular, with $\alpha_2 = (n-1)/n$, we obtain the classical Gini welfare function

$$A(\boldsymbol{x}) = A^c_G(\boldsymbol{x}) \qquad \alpha_2 = \frac{n-1}{n}\,. \tag{79}$$

Other interesting parametric choices for α_2 could be $\alpha_2 = (n-l)/n$ with $l = 0, 1, \ldots, n$. In the case $l = 0$ all the Choquet capacity structure lies in the non-additive Möbius values $m_\mu(2)$, the case $l = 1$ corresponds to the classical absolute Gini inequality index, and the remaining cases correspond to increasingly weak structure being associated with the values $m_\mu(2)$, towards the additive case $l = n$. In other words, the parametric choices associated with $l = 0, 1, \ldots, n$ correspond to an interpolation between $A(\boldsymbol{x}) = \bar{x} = C_1(\boldsymbol{x})$ (with $l = n$) and $A(\boldsymbol{x}) = C_2(\boldsymbol{x})$ (with $l = 0$) through the intermediate (with $l = 1$) case $A(\boldsymbol{x}) = A^c(\boldsymbol{x})$, the classical Gini welfare function.

Consider now the 3-additive case. The monotonicity conditions (67) take the form

$$\left[1 - n\frac{\binom{i-1}{1}}{\binom{n}{2}}\right]\alpha_2 + \left[1 - n\frac{\binom{i-1}{2}}{\binom{n}{3}}\right]\alpha_3 \leq 1 \qquad\qquad i = 1, \ldots, n\,. \tag{80}$$

In turn, the S-concavity conditions (63) take the form

$$\frac{1}{\binom{n}{2}}\,\alpha_2 + \frac{n-i}{\binom{n}{3}}\,\alpha_3 \geq 0 \qquad i = 2, \ldots, n\,. \tag{81}$$

The S-concavity conditions reduce to the $i = 2$ and $i = n$ cases,

$$\alpha_3 \geq -\frac{1}{3}\,\alpha_2 \qquad \alpha_2 \geq 0 \tag{82}$$

since the intermediate conditions

$$\alpha_3 \geq -\frac{(n-2)}{3(n-i)}\,\alpha_2 \qquad i = 3, \ldots, n-1 \tag{83}$$

are dominated by the first, given the last constraint $\alpha_2 \geq 0$. Notice that in the 3-additive case the monotonicity conditions depend on n, but the S-concavity conditions are independent of n.

As an immediate consequence of Proposition 8, we have the following result.

Proposition 10 *Any 3-additive generalized Gini welfare function* $A : [0,1]^n \longrightarrow [0,1]$ *can be written uniquely as*

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{(i)} = \bar{x} - \alpha_2 G_2(\boldsymbol{x}) - \alpha_3 G_3(\boldsymbol{x}) \tag{84}$$

where $G_2(\boldsymbol{x})$ *is as in (76), and* $G_3(\boldsymbol{x})$ *is the binomial Gini inequality index*

$$G_3(\boldsymbol{x}) = -\sum_{i=1}^{n} v_{3i}\, x_{(i)} = -\sum_{i=1}^{n} \frac{2n^2 - 2 + 3i - 6in + 3i^2}{n(n-1)(n-2)}\, x_{(i)} \tag{85}$$

and the coefficients α_2 *and* α_3 *are subject to the conditions (80) and (82).*

Notice that in G_3 the last 2 coefficients coincide ($v_{3,n-1} = v_{3n} = -1/n$) and thus G_3 is a non strict absolute inequality index, in the sense that it is insensitive to income transfers involving the 2 richest individuals in the population.

Example 1 *Consider the case $n = 6$. The weights w_{ji} of the binomial Gini welfare functions C_j, $j = 1, \ldots, 6$ are given by*

$$\begin{aligned}
C_1 :& \quad (\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}) \\
C_2 :& \quad (\tfrac{5}{15}, \tfrac{4}{15}, \tfrac{3}{15}, \tfrac{2}{15}, \tfrac{1}{15}, 0) \\
C_3 :& \quad (\tfrac{10}{20}, \tfrac{6}{20}, \tfrac{3}{20}, \tfrac{1}{20}, 0, 0) \\
C_4 :& \quad (\tfrac{10}{15}, \tfrac{4}{15}, \tfrac{1}{15}, 0, 0, 0) \\
C_5 :& \quad (\tfrac{5}{6}, \tfrac{1}{6}, 0, 0, 0, 0) \\
C_6 :& \quad (1, 0, 0, 0, 0, 0)
\end{aligned}$$

On the other hand, the coefficients v_{ji} of the binomial Gini inequality indices G_j, $j = 1, \ldots, 6$ are given by

$$\begin{aligned}
G_1 :& \quad (0, 0, 0, 0, 0, 0) \\
G_2 :& \quad (-\tfrac{5}{30}, -\tfrac{3}{30}, -\tfrac{1}{30}, \tfrac{1}{30}, \tfrac{3}{30}, \tfrac{5}{30}) \\
G_3 :& \quad (-\tfrac{20}{60}, -\tfrac{8}{60}, \tfrac{1}{60}, \tfrac{7}{60}, \tfrac{10}{60}, \tfrac{10}{60}) \\
G_4 :& \quad (-\tfrac{15}{30}, -\tfrac{3}{30}, \tfrac{3}{30}, \tfrac{5}{30}, \tfrac{5}{30}, \tfrac{5}{30}) \\
G_5 :& \quad (-\tfrac{4}{6}, 0, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}) \\
G_6 :& \quad (-\tfrac{5}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6})
\end{aligned}$$

Consider now the 3-additive case for $n = 6$. We have the following six monotonicity conditions (80) in terms of the two coefficients α_2 and α_3,

$$\begin{cases}
\alpha_2 + \alpha_3 \le 1 \\
3\alpha_2 + 5\alpha_3 \le 5 \\
2\alpha_2 + 7\alpha_3 \le 10 \\
2\alpha_2 - \alpha_3 \ge -10 \\
3\alpha_2 + 4\alpha_3 \ge -5 \\
\alpha_2 + 2\alpha_3 \ge -1
\end{cases} \tag{86}$$

and the corresponding feasible region is illustrated in Fig. 2. The dark subregion is obtained with the two extra conditions (82) associated with S-concavity,

$$\begin{cases}
\alpha_2 + 3\alpha_3 \ge 0 \\
\alpha_2 \ge 0 \, .
\end{cases} \tag{87}$$

The overall region in Fig. 2 refers to the binomial decomposition of OWA functions in Proposition 3, whereas the dark subregion refers to the binomial decomposition of generalized Gini welfare functions in Propositions 6 and 8.

8 Conclusions

We have considered the binomial decomposition of OWA functions due to Calvo and De Baets [14] and we have examined it in the restricted context of gener-

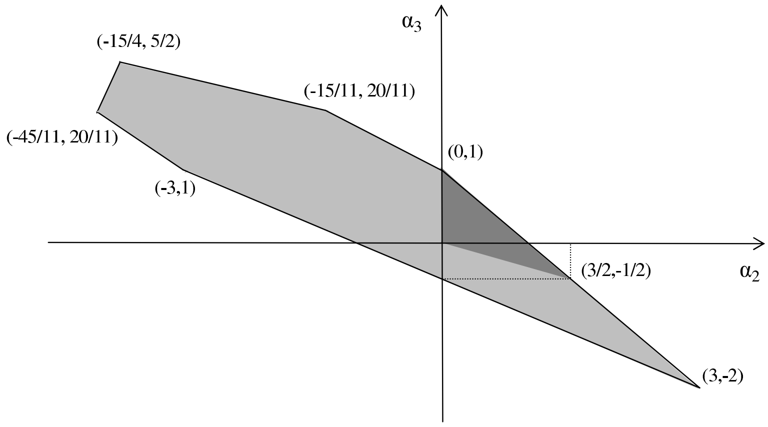

Figure 2: Feasible region associated with conditions (86) and (87).

alized Gini welfare functions, with the addition of appropriate S-concavity conditions. The original expression of the binomial decomposition can be equivalently formulated either in terms of the binomial Gini welfare functions C_j, $j = 1, \ldots, n$, or in terms of the Atkinson-Kolm-Sen (AKS) associated binomial Gini inequality indices G_j, $j = 1, \ldots, n$, according to Blackorby and Donaldson's correspondence formula $G_j(\boldsymbol{x}) = \bar{x} - C_j(\boldsymbol{x})$, with $j = 1, \ldots, n$.

The first pair of binomial Gini welfare function and inequality index is $C_1(\boldsymbol{x}) = \bar{x}$ and $G_1(\boldsymbol{x}) = 0$, for all $\boldsymbol{x} \in [0,1]^n$. In the second pair, $C_2(\boldsymbol{x})$ has $n-1$ positive linearly decreasing weights and one null last weight, and the associated $G_2(\boldsymbol{x})$ has linearly increasing coefficients and is in fact proportional to the classical Gini index, $G_2(\boldsymbol{x}) = \frac{n}{n-1}\, G^c(\boldsymbol{x})$. In the remaining pairs, $C_j(\boldsymbol{x})$, $j = 3, \ldots, n$, have $n-j+1$ positive non-linear decreasing weights and $j-1$ null last weights, and the associated $G_j(\boldsymbol{x})$, $j = 3, \ldots, n$ have $n-j+1$ non-linear increasing weights and $j-1$ equal last weights.

The binomial Gini welfare functions C_j, $j = 1, \ldots, n$ have null weights associated with the $j-1$ richest individuals in the population and therefore, as j increases from 1 to n, they behave in analogy with poverty measures which progressively focus on the poorest part of the population. Correspondingly, the binomial Gini inequality indices G_j, $j = 1, \ldots, n$ have equal weights associated with the $j-1$ richest individuals in the population and therefore, as j increases from 1 to n, they are progressively insensitive to income transfers within the richest part of the population.

The binomial Gini welfare functions and inequality indices bear some analogy with the S-Gini family of welfare functions and absolute inequality indices introduced by Donaldson and Weymark[23], and independently by Kakwani [48] as an extension of a poverty measure proposed by Sen [66], see also Donaldson e Weymark [24], Yitzhaki [76], Bossert [13], Aaberge [1, 2, 3]. The welfare functions of the S-Gini family are of the form

$$A^S_\delta(\boldsymbol{x}) = \sum_{i=1}^{n} \Big[\Big(\frac{n-i+1}{n}\Big)^\delta - \Big(\frac{n-i}{n}\Big)^\delta \Big]\, x_{(i)} \tag{88}$$

where $\delta \in [1, \infty)$ is an inequality aversion parameter. In analogy with the binomial Gini welfare functions, $A_1^S(\boldsymbol{x}) = \bar{x}$ and $A_2^S(\boldsymbol{x}) = \bar{x} - G^c(\boldsymbol{x})$, where G^c is the classical Gini. As the inequality aversion parameter increases, A_δ^S tends to the limit case $A_\infty^S(\boldsymbol{x}) = x_{(1)}$. In other words, the full range of the inequality aversion parameter, from $\delta = 1$ to $\delta = \infty$, corresponds to a continuous interpolation of the index $j = 1, \ldots, n$, with $A_1^S = C_1$ and $A_\infty^S = C_n$.

The welfare functions of the S-Gini family (88) are of the general form

$$A_f(\boldsymbol{x}) = \sum_{i=1}^{n} \Big[f\Big(\frac{n-i+1}{n}\Big) - f\Big(\frac{n-i}{n}\Big) \Big]\, x_{(i)} \tag{89}$$

where f is a continuous and increasing function on the unit interval, with $f(0) = 0$ and $f(1) = 1$. The integer parametric choices $f(t) = t^k$, with $k = 1, \ldots, n$, can be seen in relation with the k-additivity of the welfare function, as discussed in Gajdos [30].

An alternative generalization of the classical Gini which again has some analogy with the binomial Gini welfare functions and inequality indices is that proposed by Lorenzen [52], see Weymark [70],

$$A_j^L(\boldsymbol{x}) = \sum_{i=1}^{j} \frac{j+n-2i+1}{nj}\, x_{(i)} = \sum_{i=1}^{j} \frac{1}{n}\, x_{(i)} + \sum_{i=1}^{j} \frac{n-2i+1}{nj}\, x_{(i)} \tag{90}$$

with $j = 1, \ldots, n$. The extreme cases are $A_1^L(\boldsymbol{x}) = x_{(1)}$ and $A_n^L(\boldsymbol{x}) = \bar{x} - G^c(\boldsymbol{x}) = A^c(\boldsymbol{x})$, where G^c is the classical Gini. As j increases from 1 to n, the Lorenzen welfare function A_j^L involves only the j poorest individuals in the population, to whom it assigns linearly decreasing positive weights. Analogously, the binomial Gini welfare functions C_{n-j+1}, for $j = 1, \ldots, n-1$, also involve only the j poorest individuals but assign them non-linear binomial weights, from $C_n(\boldsymbol{x}) = x_{(1)}$ to $C_2(\boldsymbol{x}) = A^c(\boldsymbol{x}) - \frac{1}{n-1}\, G^c(\boldsymbol{x})$, where G^c is the classical Gini and A^c is the associated welfare function.

Finally, a continuous inequality aversion parameter is also present in the two classical families of decomposable inequality indices, the Kolm family [50] of absolute inequality indices and the Atkinson family [6] of relative inequality indices. The associated welfare functions correspond to the two classical families of decomposable aggregation functions, the exponential quasi-arithmetic means and the power quasi-arithmetic means, respectively. The former are stable for translations and are thus associated with absolute inequality indices, whereas the latter are stable for dilations and are thus associated with relative inequality indices.

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