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Relative bipolarization quasi-ordering based on Between-Group Gini (BGG) curves*

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Abstract

This note proposes a relative bipolarization quasi-ordering based on Between-Group Gini (BGG) curves, which is consistent with the quasi-ordering generated by relative bipolarization indices satisfying key axioms. Therefore the quasi-ordering induced by BGG curves is identical to the one induced by relative bipolarization curves (the secondorder curves in Foster and Wolfson (2010)). An appealing trait of BGG curves is their intuitive and straightforward representation of relative bipolarization situations, including minimum and maximum bipolarization, as well as any intermediate situation of perfect bimodality.

Keywords: Bipolarization, Gini decomposition.

JEL Classification: D30, D31.

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1 Introduction

The last couple of decades has witnessed substantial theoretical developments in the measurement of different polarization concepts. One such concept is that of bipolarization which was motivated, *inter alia*, by concerns over the disappearance of the middle class in developed countries. The seminal contribution for the measurement of bipolarization was an early version of Foster and Wolfson (2010), which introduced the first and second-order bipolarization curves. As Foster and Wolfson (2010) explain, bipolarization captures the degree to which societies may develop bimodal distributions above and below the median. Any progressive transfer that brings two people *on different sides of the median* closer together is deemed to reduce bipolarization; whereas any progressive transfer that brings two people *on the same side of the median* closer together is meant to increase bipolarization. Further significant contributions based on these two aspects of bipolarization were provided by Wang and Tsui (2000), Esteban, Gradin, and Ray (2007), Deutsch, Silber, and Hanoka (2007), Bossert and Schworm (2008), Chakravarty (2009), Lasso de la Vega, Urrutia, and Diez (2010), and Chakravarty and D'Ambrosio (2010).

Now, as Chakravarty (2009) explained, the plethora of bipolarization indices fulfilling desirable properties does not have to produce consistent rankings of distributions normally. Hence Chakravarty (2009) proved that bipolarization indices fulfilling the key axioms rank distributions consistently if and only if the relative bipolarization curves do not cross. That is, he provided the bipolarization-equivalent of the Lorenz consistency conditions. This paper proposes an alternative relative bipolarization quasi-ordering, based on Between-Group Gini (BGG) curves. The latter are based on recursive estimations of Between-Group Gini indices for two non-overlapping income groups divided by the median. The first index is computed with all the observations, and then, at every iteration, the poorest and the richest person are tossed out of the distribution before another Between-Group Gini index is computed. The process is continued until only the median is left (in the case of an odd number of people), and all the computed indices make up the curve (which is then plotted against the number of ditched pairs as a proportion of the population).

The note shows that the quasi-ordering produced by the BGG curves is identical to that of the relative bipolarization curves, and it also illustrates the appeal of the BGG curves in representing relative bipolarization situations including minimum and maximum bipolarization, as well as any intermediate situation of perfect bimodality. Likewise, the note shows that the non-positive slope of the BGG curve is directly related to the degree of dispersion (or clustering) found in each of the two non-overlapping groups divided by the median.

The rest of the note proceeds as follows. The next section provides the notation and a recapitulation of desirable properties for a bipolarization index. Special attention is given to normalization properties. Then the BGG curves are introduced and some bipolarization situations are illustrated with it. Thereafter the relative bipolarization quasi-ordering is shown to be necessary and sufficient for the consistency of relative bipolarization indices satisfying different sets of key desirable properties. The note ends with some concluding remarks.

2 Preliminaries

2.1 Notation

Let y be a non-negative continuous variable from distribution Y, observed for N individuals. We can divide the population into two halves. The bottom 50%, with $y \leqslant m$, where m, is the median, and the the top 50%, with $y \geqslant m$. The bottom half belongs to the subset N_L and the top half belongs to the subset N_U. Each set has H individuals, where $H = \frac{N}{2}$ if N is even, and $H = \frac{N}{2} + 1$ if N is odd (both include the median). For reasons of handiness that become apparent below, individuals can be ranked in ascending order according to their value of y the following way:

$$y^L(1) \leqslant y^L(\frac{H-1}{H}) \leqslant ... \leqslant y^L(\frac{2}{H}) \leqslant y^L(\frac{1}{H}) \leqslant m \leqslant y^U(\frac{1}{H}) \leqslant y^U(\frac{2}{H}) \leqslant ... \leqslant y^U(\frac{H-1}{H}) \leqslant y^U(1), \tag{1}$$

where the superscript L means that the individual belongs to the bottom half, and the superscript U means that he belongs to the top half. The value $y^L(\frac{1}{H})$ corresponds to the richest person in the bottom half, whereas $y^L(1)$ is the value of y for the poorest person in the whole population. Likewise $y^U(\frac{1}{H})$ accrues to the poorest person in the top half, whereas $y^U(1)$ is the value of y enjoyed by the wealthiest person in the whole population.

The following definitions are also useful:

$$\mu_U(\frac{H-i}{H}) \equiv \frac{1}{H-i}\sum_{j=1}^{H-i} y^U(\frac{j}{H}) \tag{2}$$

$$\mu_L(\frac{H-i}{H}) \equiv \frac{1}{H-i}\sum_{j=1}^{H-i} y^L(\frac{j}{H}) \tag{3}$$

$$\mu(\frac{H-i}{H}) \equiv \frac{1}{2}[\mu_U(\frac{H-i}{H}) + \mu_L(\frac{H-i}{H})] \tag{4}$$

$\mu_U(\frac{H-i}{H})$ is the mean value of y for the poorest $H-i$ people who belong to the top 50% (i.e. for whom $y \geqslant m$). For instance, if $i = 0$, then μ_U considers everybody above the median, whereas if $i = H-1$, $\mu_U(\frac{1}{H}) = y^U(\frac{1}{H})$. Likewise, $\mu_L(\frac{H-i}{H})$ is the mean value of y for the richest $H-i$ people who belong to the bottom 50% (i.e. for whom $y \leqslant m$). Finally, $\mu(\frac{H-i}{H})$ is the mean value of y that excludes the i poorest and the i richest people in the population.

2.2 Desirable properties for a bipolarization index

The following definitions are helpful for the later introduction of desirable properties for a relative bipolarization index:

Definition 1. *Spread-decreasing Pigou-Dalton transfer (SDPD transfer): Distribution X is obtained from Y by an SDPD transfer if there is a pair of individuals (i,j), such that*

$i \in N_U \wedge j \in N_L$, $x_i^U = y_i^U - \delta$, $x_j^L = y_j^L + \delta$, for $\delta > 0$, $x_i^U \geqslant m \geqslant x_j^L$ and $x_k = y_k \forall k \neq (i,j)$.

Definition 2. *Clustering-increasing Pigou-Dalton transfer (CIPD transfer): Distribution X is obtained from Y by an CIPD transfer if there is a pair of individuals (i,j), such that $i,j \in N_U \vee i,j \in N_L$, $y_i > y_j$, $x_i = y_i - \delta$, $x_j = y_j + \delta$, for $\delta > 0$, $x_i \geqslant x_j$ and $x_k = y_k \forall k \neq (i,j)$.*

Definition 3. *Minimum bipolarization (MIN): Distribution Y exhibits minimum bipolarization if and only if: $y_i = m \forall i \in N$.*

Definition 4. *Maximum bipolarization (MAX): Distribution Y exhibits maximum bipolarization if and only if: $y_i = 0 \forall i \in N_L \wedge y_j = 2m \forall j \in N_U$.*

While MIN is universally accepted as the benchmark of minimum bipolarization in the bipolarization literature , MAX does not carry consensus (e.g. note that the corresponding normalization axiom in Chakravarty (2009, p. 108) only relates to MIN). For instance, some of the bipolarization indices that can be derived from one of the families of Wang and Tsui (2000) do not exhibit a maximum at all. By contrast, the classic index by Foster and Wolfson (2010) reaches its maximum value of 1 if and only if MAX is present.[1] Having a maximum value that is well defined and easy to interpret can be advantageous in terms of comparability across distributions, but it comes at the cost of being unable to differentiate among all the different distributions characterized by the maximum definition (e.g. MAX in the case of bipolarization). Moreover, in the case of relative bipolarization measurement, MAX seems to be the only reasonable maximum benchmark since, once attained, bipolarization cannot be further increased either by progressive transfers on one side of the mean or by regressive transfers across the mean (e.g. reversing the SDPD). This situation occurs as long as everybody in the bottom half has zero income and everybody in the top half has the same positive income whichever its value. [2]

Now a bipolarization index, $\mathcal{I}(X) : \mathbb{R}_+^N \to \mathbb{R}_+$, is a continuous function that maps from an N-dimensional vector of real, non-negative numbers to the non-negative segment of the real line. Some bipolarization indices, e.g. FW, actually map onto the real interval $[0,1]$. The desirable properties for a bipolarization index are listed below. The first two properties are widely used in the bipolarization literature.

Axiom 1. *SYM (Symmetry): If X is obtained from Y by multiplying the latter with a permutation matrix[3] then $\mathcal{I}(X) = \mathcal{I}(Y)$.*

Axiom 2. *POP (Population replication): If X is obtained from Y by replicating the latter's population by a constant factor of $\lambda \in \mathbb{N}_{++}$, then $\mathcal{I}(X) = \mathcal{I}(Y)$.*

[1]They propose the following index: $FW = (G_B - G_W)\frac{\mu}{m}$ where G_B and G_W are the between-group and within-group Gini indices, respectively. The two groups are top and bottom halves of the distribution. μ is the population mean.

[2]Interestingly, the alternatives of indices with, or without, a maximum value also exist in the inequality literature. For instance, the Gini coefficient takes its maximum value of $1 - 1/N$ if and only if $y = 0$ for everybody except for one single person for whom $y > 0$, *irrespective of how wealthy this person is*. By contrast, the mean log deviation, for instance, does not have a maximum value.

[3]The permutation matrix is a square N-dimensional matrix with entries of 0 and 1 such that all rows and columns add up to one.

Axiom 3. *SC (Scale invariance): If* $X := (x_1, x_2, ..., x_N)$ *and* $Y := (kx_1, kx_2, ..., kx_N)$ *where* $k \in \mathbb{R}_{++}$*, then* $\mathcal{I}(X) = \mathcal{I}(Y)$.

Relative bipolarization indices usually satisfy scale invariance (e.g. FW, the index of Deutsch et al. (2007), or the P_4^N of Wang and Tsui (2000)). However others have explored constructing bipolarization indices satisfying a less stringent property of unit consistency (e.g. Lasso de la Vega et al. (2010)) whereby only the ordinal comparisons, but not the indices' values, are unaffected by changes in the variable's unit of measurement. A third alternative is a property of translation invariance, which states that: if $X := (x_1, x_2, ..., x_N)$ and $Y := (x_1 + k, x_2 + k, ..., x_N + k)$ where $k \in \mathbb{R}$, then $\mathcal{I}(X) = \mathcal{I}(Y)$. The index P_3^N of Wang and Tsui (2000) is translation invariant.

The next two axioms restate the expected effects of SDPD and CIPD transfers on a bipolarization index:

Axiom 4. *SPREAD: If* X *is obtained from* Y *by an SDPD transfer, then* $\mathcal{I}(X) < \mathcal{I}(Y)$.

Axiom 5. *CLU: If* X *is obtained from* Y *by a CIPD transfer, then* $\mathcal{I}(X) > \mathcal{I}(Y)$.

Finally, two normalization axioms can be considered. Weak normalization (WNORM) states that the bipolarization index should attain its minimum if and only if X is characterized by MIN, whereas strong normalization (SNORM) states that the bipolarization index should attain its minimum and maximum values if and only if X is characterized by MIN and MAX, respectively:

Axiom 6. *WNORM (Weak Normalization):* $\mathcal{I}(X) = 0$ *if and only if* X *exhibits minimum bipolarization.*

Axiom 7. *SNORM (Strong Normalization): a)* $\mathcal{I}(X) = 0$ *if and only if* X *exhibits minimum bipolarization; and b)* $\mathcal{I}(X) = 1$ *if and only if* X *exhibits maximum bipolarization.*

3 The BGG curves

When there are two non-overlapping groups, the Gini index can be decomposed into a between-group and a within-group component. The between-group component is a function of the difference between the means of the two groups.[4] In the case of two median groups from a population with $2(H - i)$ individuals, the between-group Gini index ($G_B(\frac{H-i}{H})$) is:

$$G_B(\frac{H-i}{H}) = \frac{\mu_U(\frac{H-i}{H}) - \mu_L(\frac{H-i}{H})}{4\mu(\frac{H-i}{H})} \tag{5}$$

So for instance, the overall between-group Gini index for the whole population ($i = 0$ so that $N = 2H$, as above), is: $G_B(1) = \frac{\mu_U(1) - \mu_L(1)}{4\mu(1)}$. Now the between-group Gini (BGG) curves are constructed the following way: Let the horizontal axis run from $\frac{i}{H} = 0$ to 1, and compute $G_B(\frac{H-i}{H})$ for each $i \in [0, H]$, as defined above:

[4] For more details on the decomposition of the Gini index see Lambert and Aronson (1993, especially footnote 1).

$$BGG(\frac{i}{H}) \equiv G_B(\frac{H-i}{H}),\ i \in [0,H] \tag{6}$$

As it is easy to show, 1) the BGG curve never increases with i, and 2) is scale invariant. Likewise, 3) whenever there is maximum bipolarization according to NORM, the BGG curve is a horizontal line intersecting the vertical axis at the value of 0.5. On the other extreme, 4) whenever there is minimum bipolarization according to NORM, the BGG curve is a horizontal line overlapping with the horizontal axis.

Table 1 and figure 1 illustrate the BGG curves, with information for three hypothetical distributions (A,B,C) with the same population size (ten people). Distribution A exhibits some degree of bipolarization in between the two extremes of minimum and maximum bipolarization. As the poorest and richest individuals are tossed out recursively, the between-group Gini indices decrease in value, which is reflected in the downward slope of the BGG curve. Distribution B is obtained from A via an SDPD transfer involving individuals 3 and 9 (see rank column), and its BGG curve lies closer to the horizontal axis than A's. This makes sense since all the between-group Gini indices involving individuals 3 and 9 decrease in value as a consequence of the Pigou-Dalton transfer across the median. By contrast, distribution C is also obtained from A via a CIPD transfer involving individuals 2 and 4 (both below the median). Accordingly, the BGG curve of C is further away from the horizontal axis, compared to the curve of A. The reason is that, at first, the between-group Gini indices do not change as long as individual 2 is not tossed out, because μ_L remains unaffected by the transfer. However, for the between-group Gini indices computed without 2 but with 4, μ_L is lower in C than in A due to the transfer, while μ_U has the same values as before. Hence the BGG curve of C has to lie above that of A across that interval. But then, once individual 4 gets also removed, the BGG curves of both distributions coincide again.

Table 1: Bipolarization quasi-orderings based on BGG curves: an illustration

Rank	A	B	C
1	10	10	10
2	20	20	27
3	30	38	30
4	40	40	33
5	50	50	50
6	60	60	60
7	70	70	70
8	80	80	80
9	90	82	90
10	100	100	100

Figure 1: Bipolarization quasi-orderings based on BGG curves: an illustration

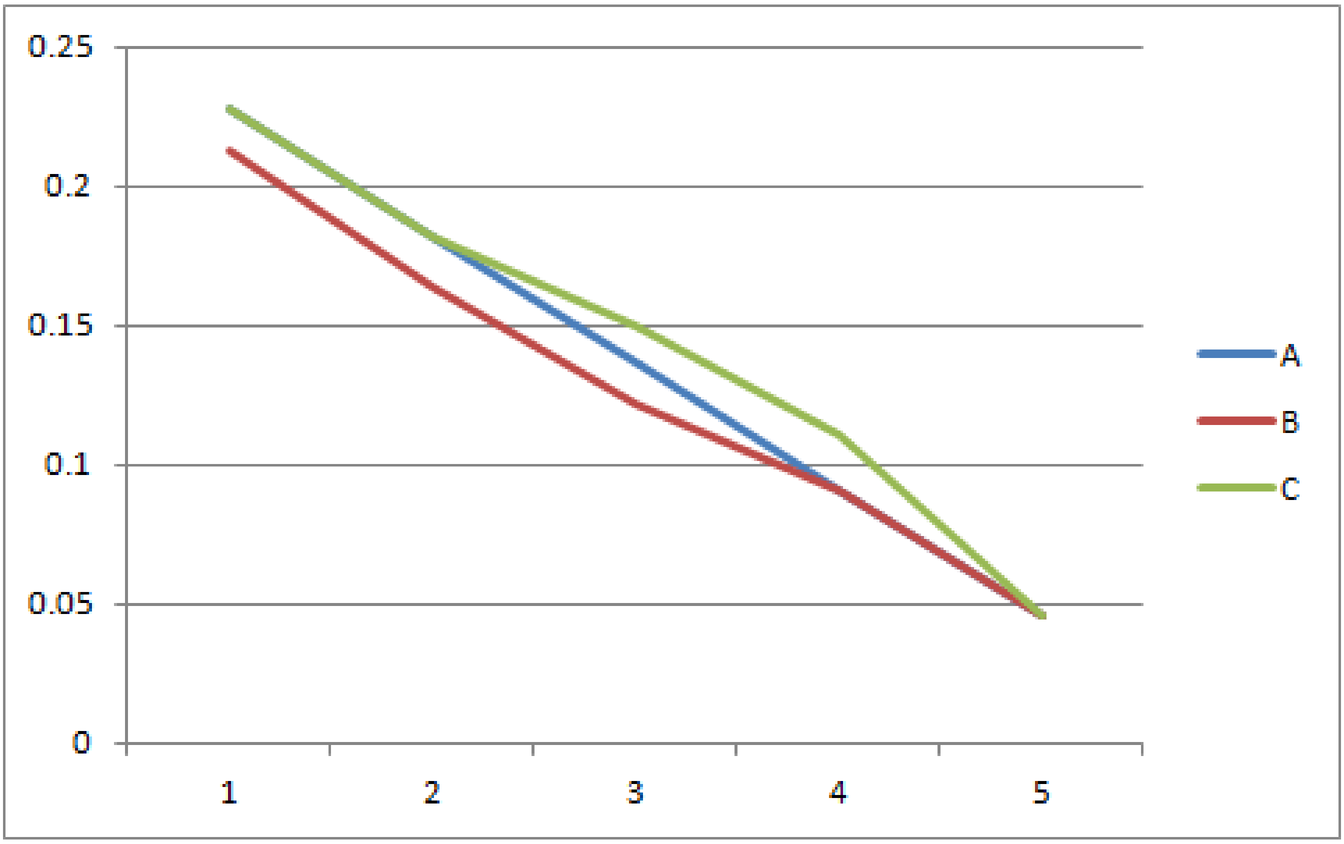
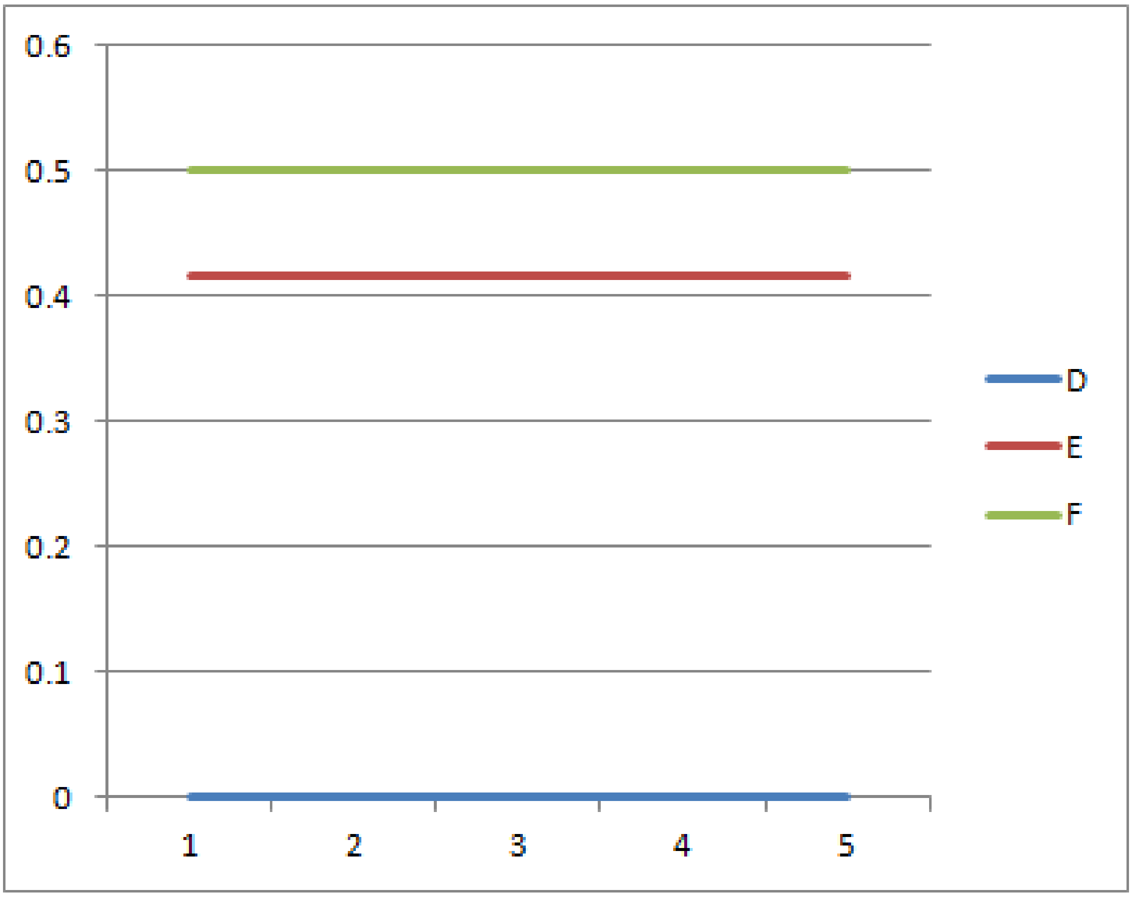


Table 2 and figure 2 illustrate the BGG curves for some extreme situations. Distribution D is characterized by complete equality, which also coincides with minimum bipolarization. Hence its BGG curve is the flat line overlapping with the horizontal axis. In the other extreme distribution E exhibits maximum bipolarization, as defined above, since half of its population have a zero value whereas the other half enjoy the same positive amount. Hence its BGG curve is also a flat line. Its projection to the left crosses the vertical axis at 0.5. because all the between-group Gini indices remain the same after every pair of individuals is tossed out, and in all cases it reaches its maximum value of 0.5. Finally, distribution F also exhibits a flat BGG curve because it is characterized by perfect bimodality. However, the BGG curve of F lies below that of E since the relative spread between the bottom-half and top-half means is not maximized when the observations in the bottom half of the distribution have strictly positive incomes. The spread can always be increased further by implementing regressive transfers across the median until the bottom half of the distributions has zero incomes. This is indeed the rationale behind the benchmark of maximum relative bipolarization, which is also respected in the quasi-ordering induced by relative bipolarization curves.

Table 2: Bipolarization quasi-orderings based on BGG curves: an illustration

Rank	D	E	F
1	55	10	0
2	55	10	0
3	55	10	0
4	55	10	0
5	55	10	0
6	55	100	110
7	55	100	110
8	55	100	110
9	55	100	110
10	55	100	110

Figure 2: Bipolarization quasi-orderings based on BGG curves: an illustration



Thus far the BGG bipolarization quasi-ordering has been introduced for samples of the same size. However, as in the case of the bipolarization curves of Foster and Wolfson (2010) it is also possible to represent the quasi-ordering in terms of quantiles. This representation is useful to show that the slope of the BGG curve is non-positive and that it depends on the degree of dispersion (or clustering) found in the two non-overlapping groups separated by the median. First, define the BGG curve in terms of quantiles:

$$BGG(q) \equiv \frac{1}{2}\frac{\mu_U(1-q) - \mu_L(1-q)}{\mu_U(1-q) + \mu_L(1-q)},\ q \in [0,1] \tag{7}$$

where q is the centile and:

$$\mu_U(1-q) \equiv \frac{\int_{j=0}^{1-q} y^U(j)dj}{\int_{j=0}^{1-q} dj} \tag{8}$$

$$\mu_L(1-q) \equiv \frac{\int_{j=0}^{1-q} y^L(j)dj}{\int_{j=0}^{1-q} dj} \tag{9}$$

Now the slope of the BGG curve is:

$$S(q) \equiv \frac{dBGG(q)}{dq} = [\mu_U(1-q) + \mu_L(1-q)]^{-2}[\mu_L(1-q)\frac{\partial \mu_U(1-q)}{\partial q} - \mu_U(1-q)\frac{\partial \mu_L(1-q)}{\partial q}] \tag{10}$$

$S(q) \leqslant 0$ since $\frac{\partial \mu_U(1-q)}{\partial q} \leqslant 0$ and $\frac{\partial \mu_L(1-q)}{\partial q} \geqslant 0$. The former partial derivative is non-positive because every time q increases, an individual who is wealthier than the average at the top half (i.e in N_U) is tossed out. Likewise, the latter partial derivative is non-negative because every time q increases, an individual who is poorer than the average at the bottom half (i.e. in N_L) is removed.

Finally, the magnitudes of the two partial derivatives, $\frac{\partial \mu_U(1-q)}{\partial q}$ and $\frac{\partial \mu_L(1-q)}{\partial q}$, affect the steepness of the $S(q)$. The higher the degree of clustering in each of the two groups, the lower the magnitude of the partial derivatives. In the limit, $\frac{\partial \mu_U(1-q)}{\partial q} = 0$ if and only if $y^U(j) = y^U \quad \forall j \in N_U$, and $\frac{\partial \mu_L(1-q)}{\partial q} = 0$ if and only if $y^L(j) = y^L \quad \forall j \in N_L$, where y^U and y^L are constants. That is, when there is no variation whatsoever within both groups *simultaneously*, then $S(q) = 0\ \forall q$ and the BGG curve is a flat horizontal line. The latter is precisely the case of distributions D, E and F shown above, which represent, respectively, cases of minimum bipolarization (in which $y^U = y^L$), perfect bimodality with intermediate bipolarization (in which $y^U > y^L > 0$) and maximum bipolarization (in which $y^U > y^L = 0$).

3.1 Relative bipolarization quasi-orderings based on BGG curves

In the case of populations with the same size, the bipolarization quasi-ordering based on the BGG curve states that if the BGG curve of population C is never above that of D, and at least once below the curve of D, then relative bipolarization in C is strictly lower than in D. That is, if the strict bipolarization quasi-ordering is denoted by $\prec^{bp}$ (meaning "less bipolarized than"), then:

Definition 5. *BGG bipolarization quasi-ordering:* $C \prec^{bp} D$ *if and only if* $BGG^C(\frac{i}{H}) \leqslant BGG^D(\frac{i}{H})\ \ \forall i \in [0,H]$ *and* $\exists j\ \mid\ BGG^C(\frac{j}{H}) < BGG^D(\frac{j}{H})$.

The following theorem states that, in populations of the same size, all relative bipolarization indices satisfying symmetry, spread-decreasing Pigou-Dalton transfers and clustering-increasing Pigou-Dalton transfers, rank distributions consistently if and only if the relationship $\prec^{bp}$ holds between them:

Theorem 1. *Let X and Y have the same population N, then the following statements are equivalent: i) $X \prec^{bp} Y$; ii) $\mathcal{I}(X) < \mathcal{I}(Y)$ for all $\mathcal{I}$ satisfying SYM, SPREAD and CLU.*

Proof. The proof is based on the ideas of Chakravarty (2009). First note that the quasi-ordering $\prec^{bp}$ also satisfies SYM, since it is also based on ordering individuals according to their values of the variable.

Now if $X \prec^{bp} Y$ then one can obtain X from Y through SDPD. If the transfer of δ takes place between individuals i (above the median) and j (below the median) as in the definition of SDPD above, then, assuming that person i is closer to the median in absolute terms (without loss of generality) the BGG curve of X in terms of the BGG curve of Y is the following:

$$BGG^X(\frac{k}{H}) = \frac{\mu_U(\frac{H-k}{H}) - \mu_L(\frac{H-k}{H}) - 2\frac{\delta}{H-k}}{4\mu(\frac{H-k}{H})} \quad \forall k \in [0, j-1] \tag{11}$$

$$= \frac{\mu_U(\frac{H-k}{H}) - \mu_L(\frac{H-k}{H}) - \frac{\delta}{H-k}}{2[\mu_U(\frac{H-k}{H}) - \frac{\delta}{H-k} + \mu_L(\frac{H-k}{H})]} \quad \forall k \in [j, i-1] \tag{12}$$

$$= \frac{\mu_U(\frac{H-k}{H}) - \mu_L(\frac{H-k}{H})}{4\mu(\frac{H-k}{H})} \quad \forall k \in [i, H] \tag{13}$$

More explicitly:

$$BGG^X(\frac{k}{H}) = BGG^Y(\frac{k}{H}) - \frac{\frac{2\delta}{H-k}}{4\mu(\frac{H-k}{H})} \quad \forall k \in [0, j-1] \tag{14}$$

$$= BGG^Y(\frac{k}{H})\frac{4\mu(\frac{H-k}{H})}{2[\mu_U(\frac{H-k}{H}) - \frac{\delta}{H-k} + \mu_L(\frac{H-k}{H})]} - \frac{\frac{\delta}{H-k}}{2[\mu_U(\frac{H-k}{H}) - \frac{\delta}{H-k} + \mu_L(\frac{H-k}{H})]} \quad \forall k \in [j, i-1] \tag{15}$$

$$= BGG^Y(\frac{k}{H}) \quad \forall k \in [i, H] \tag{16}$$

Then it is straightforward to see that $\delta > 0$ reduces the value of 11 (i.e. 14) and leaves 13 (i.e. 16) unchanged. With simple derivation it is also easy to show that $\delta > 0$ reduces the value of 12 (i.e. 15). Now if, instead, person j is closer to the median in absolute terms then the derivation of BGG^X from BGG^Y is very similar but now the previous formulas hold, respectively for the intervals of k: $[0, i-1], [i-1, j-1], [j, H]$. Hence the same results hold. The conclusion is likewise not altered when i and j are equally distant from the median. Therefore, since $X \prec^{bp} Y$ and BGG^X was obtained from BGG^Y using SDPD, it has to be the case that $\mathcal{I}(X) < \mathcal{I}(Y)$ for all $\mathcal{I}$ satisfying SPREAD.

Likewise if $Y \prec^{bp} X$ then one can obtain X from Y through CUL. If the CIPD takes place between individuals i and j as in the definition of CIPD above, then if $i, j \in N_U$ and i is wealthier than j:

$$BGG^X(\frac{k}{H}) = \frac{\mu_U(\frac{H-k}{H}) - \mu_L(\frac{H-k}{H})}{4\mu(\frac{H-k}{H})} \quad \forall k \in [0, i-1] \cup [j, H] \tag{17}$$

$$= \frac{\mu_U(\frac{H-k}{H}) + \frac{\delta}{H-k} - \mu_L(\frac{H-k}{H})}{2[\mu_U(\frac{H-k}{H}) + \frac{\delta}{H-k} + \mu_L(\frac{H-k}{H})]} \quad \forall k \in [i, j-1] \tag{18}$$

More specifically:

$$BGG^X(\frac{k}{H}) = BGG^Y(\frac{k}{H}) \quad \forall k \in [0, i-1] \cup [j, H] \tag{19}$$

$$= BGG^Y(\frac{k}{H})\frac{4\mu(\frac{H-k}{H})}{2[\mu_U(\frac{H-k}{H}) + \frac{\delta}{H-k} + \mu_L(\frac{H-k}{H})]} + \frac{\frac{\delta}{H-k}}{2[\mu_U(\frac{H-k}{H}) + \frac{\delta}{H-k} + \mu_L(\frac{H-k}{H})]} \quad \forall k \in [i, j-1] \tag{20}$$

Now, with simple derivation it is also easy to show that $\delta > 0$ increases the value of 18 (or 20). If $i, j \in N_L$ and person i is closer to the median:

$$BGG^X(\frac{k}{H}) = \frac{\mu_U(\frac{H-k}{H}) - \mu_L(\frac{H-k}{H})}{4\mu(\frac{H-k}{H})} \quad \forall k \in [0, j-1] \cup [i, H] \tag{21}$$

$$= \frac{\mu_U(\frac{H-k}{H}) + \frac{\delta}{H-k} - \mu_L(\frac{H-k}{H})}{2[\mu_U(\frac{H-k}{H}) - \frac{\delta}{H-k} + \mu_L(\frac{H-k}{H})]} \quad \forall k \in [j, i-1] \tag{22}$$

More specifically:

$$BGG^X(\frac{k}{H}) = BGG^Y(\frac{k}{H}) \quad \forall k \in [0, j-1] \cup [i, H] \tag{23}$$

$$= BGG^Y(\frac{k}{H})\frac{4\mu(\frac{H-k}{H})}{2[\mu_U(\frac{H-k}{H}) + \frac{\delta}{H-k} + \mu_L(\frac{H-k}{H})]} + \frac{\frac{\delta}{H-k}}{2[\mu_U(\frac{H-k}{H}) - \frac{\delta}{H-k} + \mu_L(\frac{H-k}{H})]} \quad \forall k \in [j, i-1] \tag{24}$$

Again, clearly $\delta > 0$ increases the value of 21 (or 24). Therefore, since $Y \prec^{bp} X$ and BGG^X was obtained from BGG^Y using CIPD, it has to be the case that $\mathcal{I}(X) > \mathcal{I}(Y)$ for all $\mathcal{I}$ satisfying CUL. ■

Finally, following Chakravarty (2009), an analogue to theorem 1 can be derived whereby $X \prec^{bp} Y$ if and only if $\mathcal{I}(X) < \mathcal{I}(Y)$ for all relative bipolarization indices satisfying SYM, SPREAD, CUL *and POP*. The key for the proof is to realize that the BGG curves are invariant to population replications.

4 Conclusion

This paper introduced a relative bipolarization quasi-ordering based on between-group Gini (BGG) curves. The curves provide an intuitive illustration of different bipolarization situations within the confines of a 0.5x1 rectangle in quadrant I of the Cartesian coordinate system. While the intersection of the BGG curve with the vertical axis provides a measure of the maximum relative spread between the means of the top and bottom half of the distribution (i.e. when everybody is accounted for), the negative slope of the curve depends on the degree of clustering within both halves. The higher the clustering the less steep the slope becomes.

As shown above, indices satisfying the key axioms of relative bipolarization rank distributions consistently if and only if the BGG curves do not cross. Therefore the BGG curve quasi-ordering is actually identical to the quasi-ordering of relative bipolarization curves characterized by Chakravarty (2009).

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