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Abstract

The assessment of income inequality can be investigated looking at the solution concepts of the cooperative game theory. We propose a multi-factorial decomposition of the Atkinson index by income sources and evaluate it as a cooperative game of the social cost of inequality. This framework extends the distributive and efficient properties of the uniform rule (Sprumont [35]) in a setup with heterogeneous income sources and single-peaked preferences. We provide an axiomatic foundation of this preference-based allocation rule, called weakly uniform rule, with a further comparison with the solution concept of nucleolus. Sufficient conditions for their coincidence are therefore defined. Finally we characterize a welfare loss game expressed as the difference between the sum of inequalities generated by each source and the cost of the entire distribution. We show that income factors' contributions may increase or decrease the income inequality in the society ensuring different perspectives in terms of public policies.

Keywords: Atkinson index, cost game, uniform rule, income sources, single-peaked preferences.

JEL Classification: C71; D33, D63; D71; I32.

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1 Introduction

The use of a cooperative game approach to inequality has known an increasing interest in the recent literature on inequality measurement. Economists have recently developed some contributions to mould the general tools of cooperative game theory, in particular the Shapley value concept, into a form that could be used for distributional analysis. Their investigation was guided by a partial dissatisfaction with various aspects of the traditional decomposition methods, rethinking the instruments that may underpin the multi-factorial assessment of inequality.

Our analysis builds on the solution concepts of the *cost* cooperative games adapting an environment of unequal profiles of sources and propose a preference-based allocation rule among factors. We observe that the heterogeneity of income sources generates unequal distribution and investigate how the use of the cooperative structure may guarantee information to design inequality-reducing policy intervention.

The aim of this paper is therefore threefold. First, a welfare foundation mechanism á la Blackorby et *al.* [4] is developed and characterizes a new axiomatic decomposition of the Atkinson index [2] by income sources. We develop this measure as a characteristic function of the game in terms of social cost of inequality[1]. Second, we associate the cooperative game to the well-known problem of allocating an infinitely divisible good among agents. This model was investigated by Sprumont [35] who demonstrates the existence of a *uniform allocation* rule as the only solution concept satisfying relevant criteria with single-peaked preferences. We extend this rule focusing on bidimensional information about income sources and preferences among types. In particular we study how to allocate each income source according to a complete preference ordering and then observe how this rule performs in terms of equity and efficient properties[2].

To get immediately what we have in mind, let us suppose that society is in a original position where each type has clearly indicated his own preference among income sources[3]. The policy-maker decides on a rule that assigns in place of the income profile produced by each type, a new vector characterized by a collapse of all contributions in the single preferred entry, while setting to zero the other ones. This *weakly* allocation rule satisfies a set of purely ordinal axioms: feasibility, efficiency and anonymity previously described in the literature, see [35] and [36], plus some other properties as preference monotonicity, Lorenz dominance and equal treatment of non-preferred contributions. Our claim is that under general conditions, the extreme egalitarian solution which prescribes equal division of the aggregate worth among factor coalitions is still possible without violating the individual preferences. We then discover that the nucleolus of a cost cooperative game may coincide with this weakly uniform rule.

As a last point, we propose a (specular) welfare loss game characterized by the difference between the sum of the costs generated by each income source and the overall cost borne by the entire society. We show that the heterogeneity among income factors

[1] In the literature, the cost of inequality is interpreted as the fraction of total income which could be sacrificed with no loss of social welfare if the income sources were to be equally distributed.

[2] Dutta and Ray [12] and [13] propose a constrained egalitarian solution concept which satisfies the core-like participation constraints.

[3] Basically the choice is conducted on the basis of different employment categories, for instance, labour and capital incomes, rents, inherited wealth or more general endowments.

according to the population preferences may determine different impact in terms of public policy. Some sources (or coalitions of sources) reduces the welfare loss originated by the inequality into the society. Social preference ordering on factors are then established.

Some strands of literature are taken into account in our evaluation.

As regards the decomposition by income sources, theoretical contributions have mainly focused on Gini and Theil coefficients. Shorrocks [32] proves the possibility to derive an infinite number of decompositions without restrictions; a property which is called *natural* decomposition and is valid for the main inequality indices. According to his framework, Lerman and Yitzhaki [21] exploit this property proposing a covariance formula of the Gini coefficient *á la* Fei et al. [16][4]. This is equal to the sum of the covariances between each income source and the cumulative distribution function of total income. Taking cue from [21], we develop the Atkinson index as a unique measure satisfying some properties useful to identify the marginal contribution of each income source distributed among types in the society.

On the possibility to relate inequality to solution concepts of cooperative game, the pioneering contribution was proposed by Shorrocks [34] in 1999 (recently published in 2013) who evaluates each factor's contribution in terms of its marginal effect. This measurement captures the impact of inequality if one of the factors were not present. His solution specifically solves the inherent adding-up problem of sources repeating the exercise with the change in the order, and then averaging the results over the runs for all sources. Chantreuil and Trannoy [5] introduce an income inequality game performing axiomatic characterizations of the Shapley value. On the same field, see also Sastre and Trannoy [29], Israeli [20] and Devicienti [9] for diverse econometric applications. Charpentier and Mussard [6] introduce multiplicative games providing dual results compared to Chantreuil and Trannoy's contribution [5]. They also show the advantage to perform multiplicative game to gauge income inequality variations, quite useful for policy purposes.

Finally, for what involves the issue of fairness in the distribution, our reference points enclose models where preferences are single-peaked. The formal one is introduced by Sprumont [35]. In general the uniform rule seems to be central in solving some class of problems, for instance, changes in the amount to divide, evaluations of the preferences of types or within the population. Different extensions were proposed in the last decade. Among others, Thomson [36] demonstrate that the uniform rule is the only efficient solution satisfying no envy criteria, and Thomson [37] shows that it is the only 'selection from the no-envy and Pareto solution satisfying weak replication-invariance and one-sided welfare-domination under preference-replacement'. Otten et al. [24] require that two agents with the same preferences should receive amounts that are indifferent on the basis of these preferences. They characterize an extension of the uniform rule which satisfies monotonicity property with respect to simultaneous changes in the social endowment and preferences.[5]

The remainder of this paper is as follows: Section 2 introduces the setup of the model and the features of the multi-factor Atkinson index of inequality discussing its axiomatization and key properties. Section 3 presents the cost of inequality framework as a cooperative game, listing possible solution concepts, and particularly focusing on the ax-

[4]See Morduch and Sicular [23] and Rao [26].

[5]See also note 17 on this point.

iomatic foundation of the weakly uniform rule and its strict relation with the nucleolus. Section 4 depicts the welfare loss game and outlines the social preferences ordering helpful to implement inequality-reducing policies. Conclusions follow in Section 5.

2 A multi-factor Atkinson index of inequality

The population is finite and divided into types, $\{1, ..., N\}$. The set of income factors is finite, $\{1, ..., M\}$. Let $y_{ij} \geq 0$ denote the income that type i obtains from factor (or source) j, i.e., $\mathcal{F}_j$. The population of $N \geq 1$ types is subject to $M \geq 1$ income sources, such that $I = N \times M$ is the number of income units in the society. Each type i' s income vector is $\mathbf{y_i} = (y_{i1}, y_{i2}, \ldots, y_{iM}) \in \mathbb{R}_+^M$, while, $Y_{\mathcal{F}_j} = (y_{1j}, y_{2j}, \ldots, y_{Nj}) \in \mathbb{R}_+^N$ and $Y = \{\mathbf{y}_1, \ldots, \mathbf{y}_M\} \in \mathbb{R}_+^I$ define, respectively, the income vector of $\mathcal{F}_j$ and the set of all income vectors. For each i, there exists at least one factor $\mathcal{F}_k$, where $y_{ik} > 0$.

We propose a multi-factorial structure of the Atkinson index by taking into account the heterogeneity of sources within the society. We first introduce a normative evaluation based on social welfare function with some requirements.[6] According to Atkinson [2], a multi-factor utilitarian social welfare function is:

$$W(y_{11}, \ldots, y_{NM}) = \frac{1}{I} \sum_{j=1}^{M} \sum_{i=1}^{N} U_{ij}(y_{ij}) =$$

$$= \frac{1}{M} \left(\frac{\sum_{i=1}^{N} U_{i1}(y_{i1})}{N} + \frac{\sum_{i=N+1}^{2N} U_{i2}(y_{i2})}{N} + \cdots + \frac{\sum_{i=N(M-1)}^{I} U_{iM}(y_{iM})}{N} \right)$$

where the welfare generated by contribution$-j$ is:

$$W_j(y_{1j}, \ldots, y_{Nj}) = \frac{1}{N} \sum_{i=N(j-1)+1}^{jN} U_{ij}(y_{ij})$$

such that

$$W(y_{11}, \ldots, y_{NM}) = \sum_{j=1}^{M} \frac{W_j(y_{1j}, \ldots, y_{Nj})}{M}$$

The M different functions received an equal weight on welfare orderings. For all $\mathcal{F}_j$, $j = 1, \ldots, M$:

$$W_j(y_{1j}, \ldots, y_{Nj}) = W_j(\widehat{y}_j, \ldots, \widehat{y}_j) = U_j(\widehat{y}_j) \iff \begin{cases} \dfrac{1}{1-\epsilon} \widehat{y}_j^{1-\epsilon} = \dfrac{1}{N} \sum_{i=1}^{N} \dfrac{y_{ij}^{1-\epsilon}}{1-\epsilon} \\ \\ \log\ \widehat{y}_j = \dfrac{1}{N} \sum_{i=1}^{N} \log\ y_{ij} \end{cases} \tag{1}$$

[6] See Ebert [14].

a functional form of utility$-j$ can be defined on the basis of a unique inequality aversion parameter, $\epsilon \in (0,1)$. By (1), the multi-factor *equally distributed* equivalent (*ede*) income $\widehat{y}_j$ is similar to the standard Atkinson setting, i.e.,

$$\begin{cases} \widehat{y}_j = \left[\frac{1}{N}\sum_{i=1}^{N} y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}} & \text{if } \epsilon \in (0,\ 1) \\ \\ \widehat{y}_j = \left[\prod_{i=1}^{N} y_{ij}\right]^{\frac{1}{N}} & \text{if } \epsilon = 1 \end{cases} \tag{2}$$

for all $j = 1, \ldots, M$. Some axioms satisfy by the multi-factor decomposition by income sources:

- **Normalization**: An index of inequality is normalized if, for any egalitarian distribution $Y = (y,\ y, \ldots, y) \in \mathbb{R}_+^I$, then $\mathcal{I}_A(Y) = 0$.
- **Positive homogeneity of order 0**: An index of inequality is relative if, for any $Y \in \mathbb{R}_+^I$ and $\lambda > 0$, then $\mathcal{I}_A(Y) = \mathcal{I}_A(\lambda Y)$.
- **Symmetry**: For all $X, Y \in \mathbb{R}_+^I$, such as $Y = \Pi X \in \mathbb{R}_+^I$, where Π is a permutation matrix, then $\mathcal{I}_A(Y) = \mathcal{I}_A(X)$.
- **Population principle**: Let any $Y \in \mathbb{R}_+^I$ and $Y^{(t)}$ being obtained after concatenating Y t times. For all $t \in N^* \backslash \{1\}$, then $\mathcal{I}_A(Y) = \mathcal{I}_A(Y^{(t)})$.
- **Transfer principle** For all $X, Y \in \mathbb{R}_+^I$, such as $Y = BX \in \mathbb{R}_+^I$, where B is a bistochastic matrix, then $\mathcal{I}_A(Y) = \mathcal{I}_A(X)$.

Denoting $\mu_j = \dfrac{1}{N}\sum_{i=1}^{N} y_{ij}$, the average income of j-factor among types, the related Atkinson index is $I_A(Y_{\mathcal{F}_j}) = 1 - \dfrac{\widehat{y}_j}{\mu_j}$. More generally,

$$\mathcal{I}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_k}) = 1 - \frac{1}{k}\sum_{j=1}^{k}\frac{\widehat{y}_j}{\mu_j}$$

for $k = 2, \ldots, M$. Therefore,

Definition 1. *Given the income distribution $Y \in \mathbb{R}_+^I$ and the factors $\mathcal{F}_1, \ldots \mathcal{F}_M$, the* ***multi-factor Atkinson index of inequality*** $\mathcal{I}_A(Y_{\mathcal{F}_1,\ldots\mathcal{F}_m})$ *is:*

$$\mathcal{I}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_m}) = 1 - \frac{1}{M}\sum_{j=1}^{M}\frac{\widehat{y}_j}{\mu_j} = \begin{cases} 1 - \frac{N^{-\frac{\epsilon}{1-\epsilon}}}{M}\sum_{j=1}^{M}\left(\frac{\left[\sum_{i=1}^{N} y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}}{\sum_{i=1}^{N} y_{ij}}\right) & \text{if } \epsilon \in (0,1) \\ \\ 1 - \frac{N}{M}\sum_{j=1}^{M}\left(\frac{\left[\prod_{i=1}^{N} y_{ij}\right]^{\frac{1}{N}}}{\sum_{i=1}^{N} y_{ij}}\right) & \text{if } \epsilon = 1 \end{cases} \tag{3}$$

When $M > 1$, it can also be formulated as the arithmetic mean of the standard Atkinson [2][7]:

$$\mathcal{I}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) = \frac{\mathcal{I}_A(Y_{\mathcal{F}_1}) + \cdots + \mathcal{I}_A(Y_{\mathcal{F}_M})}{M}$$

Denote $\mathcal{P} = \{\mathcal{F}_1, \ldots, \mathcal{F}_M\}$ the set of available income sources. For each subset $S \subseteq \mathcal{P}$, $S \neq \emptyset$, then (3) is:

$$\mathcal{I}_A(Y_S) = 1 - \frac{1}{|S|} \sum_{\mathcal{F}_j \in S} \frac{\widehat{y}_j}{\mu_j} = 1 - \frac{N^{-\frac{\epsilon}{1-\epsilon}}}{|S|} \sum_{\mathcal{F}_j \in S} \left(\frac{\left[\sum_{i=1}^N y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}}{\sum_{i=1}^N y_{ij}} \right) \tag{4}$$

where $|S|$ indicates the cardinality of S. A crucial property of (4) is subadditivity,

Proposition 1. *The index $\mathcal{I}_A$ is **subadditive**, i.e., for all nonempty subsets $S, T \subset \mathcal{P}$, such that $S \cap T = \emptyset$, we have that:*

$$\mathcal{I}_A(Y_{S \cup T}) < \mathcal{I}_A(Y_S) + \mathcal{I}_A(Y_T) \tag{5}$$

Proof. We employ (4) yielding:

$$\mathcal{I}_A(Y_{S\cup T}) - \mathcal{I}_A(Y_S) - \mathcal{I}_A(Y_T) = 1 - \frac{N^{-\frac{\epsilon}{1-\epsilon}}}{|S+T|} \sum_{\mathcal{F}_j \in S\cup T} \left(\frac{\left[\sum_{i=1}^N y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}}{\sum_{i=1}^N y_{ij}} \right) - 1+$$

$$+\frac{N^{-\frac{\epsilon}{1-\epsilon}}}{|S|} \sum_{\mathcal{F}_j \in S} \left(\frac{\left[\sum_{i=1}^N y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}}{\sum_{i=1}^N y_{ij}} \right) - 1 + \frac{N^{-\frac{\epsilon}{1-\epsilon}}}{|T|} \sum_{\mathcal{F}_j \in T} \left(\frac{\left[\sum_{i=1}^N y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}}{\sum_{i=1}^N y_{ij}} \right) =$$

$$= -1 - \frac{N^{-\frac{\epsilon}{1-\epsilon}}}{|S+T|} \left[\sum_{\mathcal{F}_j \in S} \left(\frac{\left[\sum_{i=1}^N y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}}{\sum_{i=1}^N y_{ij}} \right) + \sum_{\mathcal{F}_j \in T} \left(\frac{\left[\sum_{i=1}^N y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}}{\sum_{i=1}^N y_{ij}} \right) \right] +$$

$$+\frac{N^{-\frac{\epsilon}{1-\epsilon}}}{|S|} \sum_{\mathcal{F}_j \in S} \left(\frac{\left[\sum_{i=1}^N y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}}{\sum_{i=1}^N y_{ij}} \right) + \frac{N^{-\frac{\epsilon}{1-\epsilon}}}{|T|} \sum_{\mathcal{F}_j \in T} \left(\frac{\left[\sum_{i=1}^N y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}}{\sum_{i=1}^N y_{ij}} \right) = \cdots =$$

$$= -1 + \frac{|T|}{|S+T|}\left(1 - \mathcal{I}_A(Y_S)\right) + \frac{|S|}{|S+T|}\left(1 - \mathcal{I}_A(Y_T)\right) =$$

$$= \frac{-|S+T| + |T| - |T|\mathcal{I}_A(Y_S) + |S| - |S|\mathcal{I}_A(Y_T)}{|S+T|} < 0$$

where $|S+T| = |S| + |T|$. This completes the proof of (5). □

[7]Instead when $M = 1$, this index directly collapses into the standard one [2].

Subadditivity is a desirable property for inequality measure. It implies that the inequality generated by a joint distribution of income sources is lower than the sum of inequality of those sources considered separately. The aggregation of the sources tends to shrink the level of the overall inequality. Subadditivity is in general verified by cooperative games[8]. Our idea consists of investigating the characteristics of the multi-factor Atkinson index (3) as a cost cooperative game[9] where the income sources are players of a subadditive coalitional game.

3 The cost of inequality as a cooperative game

Let us redefine $\mathcal{P} = \{\mathcal{F}_1, \ldots, \mathcal{F}_M\}$ as the set of players.[10] The characteristic function of the game $\mathcal{I}_A : 2^{\mathcal{P}} \to \mathbb{R}$ is given by (3) where $\mathcal{I}_A(Y_S)$ is computed by evaluating the *ede*-components $\widehat{y}_j$ for each coalition S.[11] Note that $(\mathcal{I}_A,\ \mathcal{P})$ is not monotone since the inequality function may increase or decrease with the numbers of factors in the coalition. It is even verifiable that in $(\mathcal{I}_A,\ \mathcal{P})$, *concavity* property does not hold as well.

By taking into account all factors $\mathcal{F}_1, \ldots \mathcal{F}_M$ in Y, (3) can be expressed as:

$$\mathcal{I}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) = 1 - \frac{1}{M}\sum_{j=1}^{M}\frac{\widehat{y}_j}{\mu_j} = \sum_{j=1}^{M}\left(\frac{1}{M} - \frac{1}{M}\frac{\widehat{y}_j}{\mu_j}\right) = \frac{1}{M}\sum_{j=1}^{M}\left(\frac{\mu_j - \widehat{y}_j}{\mu_j}\right)$$

The cost of inequality caused by source j is $C_A(Y_{\mathcal{F}_j}) = \dfrac{\mu_j - \widehat{y}_j}{\mu_j}$. For each factor j, it is generated by the difference between the average and the *ede* income required by the society to guarantee the same level of welfare among types. The multi-factor Atkinson index can then be interpreted as the average cost of inequality induced by factors $\mathcal{F}_1, \ldots, \mathcal{F}_k$ into the society, i.e.,

$$\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_k}) = \frac{1}{k}\sum_{j=1}^{k} C_A(Y_{\mathcal{F}_j}) = \mathcal{I}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_k})$$

for $k = 1, \ldots, M$. Let us investigate some properties and solution concepts of a cost allocation game $(\mathcal{C}_A,\ \mathcal{P})$[12].

[8] In the remainder of the paper, we will call them *cooperative games* instead of *TU-games* since their characteristic value is neither a utility nor a payoff function.

[9] See [27] for the foundations of cost games.

[10] In such game, coalitions of factors are supposed to be defined rigorously: $S \subseteq \mathcal{P}$ is a coalition of factors if for all $\mathcal{F}_j \in S$, there exists at least one income y_{pj} different from the arithmetic mean $\frac{\sum_{i=1}^{N} y_{ij}}{\mu_j}$. Consequently, when we evaluate inequality related to some factors, we rule out all constant factors, i.e., all elements which assign the same income to all individuals or types.

[11] The inequality is zero in case of an empty set where factors are not taken into account, i.e., $\mathcal{I}_A(Y_\emptyset) = 0$. Note that this assumption is necessary when dealing with a cooperative game for both payoff and cost functions.

[12] See [27].

3.1 Solution concepts

We first carry out a decomposition of (3) according to Banzhaf[13] and Shapley values[14]. This approach has been developed in the literature on inequality assessment. Our theoretical background relies on Shorrocks [32], [33]. The main applications are developed by Shorrocks [34] and Chantreuil and Trannoy [5][15]. This technique leads to establish the marginal contribution of each factor to the aggregate inequality level, see [6]. In particular, Shapley [31] proposes to solve the problem of outcome distribution among the players by taking into account the worth of each coalition. This value is here defined as the mathematical expectation of the inequality level induced by each factor. All orders of formation of the grand coalition are equiprobable, while, income sources enter in the coalition one by one. Each of them receives the entire saving offered to the coalition previously formed.

Definition 2. *The* ***Shapley value*** *of the game* $(\mathcal{C}_A,\ \mathcal{P})$ *is a vector*

$$\Phi(\mathcal{C}_A) = (\phi_1(\mathcal{C}_A), \dots, \phi_M(\mathcal{C}_A)) \in \mathbb{R}^M$$

such that:

$$\phi_j(\mathcal{C}_A) = \sum_{S \subseteq \mathcal{P},\ \mathcal{F}_j \in S} \frac{(M - |S|)!(|S| - 1)!}{M!} (\mathcal{C}_A(S) - \mathcal{C}_A(S \setminus \{\mathcal{F}_j\})) \tag{6}$$

for all $j = 1, \dots, M$.

An alternative solution is the Banzhaf value based on the subjective belief that each factor is equally fair to join any coalition.

Definition 3. *The* ***Banzhaf value*** *of the game* $(\mathcal{C}_A,\ \mathcal{P})$ *is a vector*

$$\beta(\mathcal{C}_A) = (\beta_1(\mathcal{C}_A), \dots, \beta_M(\mathcal{C}_A)) \in \mathbb{R}^M$$

such that:

$$\beta_j(\mathcal{C}_A) = \frac{1}{2^{M-1}} \sum_{S \subseteq \mathcal{P},\ j \in S} (\mathcal{C}_A(S) - \mathcal{C}_A(S \setminus \{j\})) \tag{7}$$

for all $j = 1, \dots, M$.

These values define the marginal contributions of factors to inequality. For example, in an elementary 3-factor case, calling the factors $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, $\phi_1(\mathcal{C}_A)$ and $\beta_1(\mathcal{C}_A)$ respectively amount to:

$$\phi_1(\mathcal{C}_A) = \frac{1}{6} \left[2(\mathcal{C}_A(Y_{\{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\}}) - \mathcal{C}_A(Y_{\mathcal{F}_2,\mathcal{F}_3})) + \mathcal{C}_A(Y_{\{\mathcal{F}_1,\mathcal{F}_2\}}) - \mathcal{C}_A(Y_{\{\mathcal{F}_2\}}) + \right.$$

$$\left. + \mathcal{C}_A(Y_{\{\mathcal{F}_1,\mathcal{F}_3\}}) - \mathcal{C}_A(Y_{\mathcal{F}_3}) + 2(\mathcal{C}_A(Y_{\mathcal{F}_1}) - \mathcal{C}_A(Y_\emptyset)) \right]$$

[13] The Banzhaf value was initially introduced in [3] as a power index for voting games and subsequently generalized to arbitrary cooperative games.

[14] The Shapley value is a world famous solution concept in Cooperative Game Theory, initially introduced in [31] and then widely employed in Election Games, Bargaining Theory and many other areas. An exhaustive overview of power indices, including axiomatization and applications, is [25].

[15] Pignataro [28] proposes an application of Shapley value in the opportunity egalitarian environment.

and

$$\beta_1(\mathcal{C}_A) = \frac{1}{4}\left[\mathcal{C}_A(Y_{\{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\}}) - \mathcal{C}_A(Y_{\mathcal{F}_2,\mathcal{F}_3}) + \mathcal{C}_A(Y_{\{\mathcal{F}_1,\mathcal{F}_2\}}) - \mathcal{C}_A(Y_{\{\mathcal{F}_2\}}) + \right.$$
$$\left. + \mathcal{C}_A(Y_{\{\mathcal{F}_1,\mathcal{F}_3\}}) - \mathcal{C}_A(Y_{\mathcal{F}_3}) + \mathcal{C}_A(Y_{\mathcal{F}_1}) - \mathcal{C}_A(Y_{\emptyset})\right]$$

It is well-known that in case of two factors, the Banzhaf value and the Shapley value perfectly coincide.

Let us denote with $x_j \geq 0$ the share of the cost of inequality $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})$ attributed to source j. A *rule* is a function defined for $S \subseteq \mathcal{P}$ relating each problem $(\mathcal{C}_A,\ \mathcal{P})$ to a vector $\mathbf{x} = (x_1, \ldots, x_M) \in R_+^M$. The vector $\mathbf{x}$ guarantees an allocation of the cost of inequality for the grand coalition $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})$. The set of *efficient* solutions is therefore defined as the set of all vectors $\mathbf{x}$ whose coordinates sum up to the aggregate cost of inequality, i.e., $\sum_{j=1}^{m} x_j = \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_m})$. The standard *individual rationality*, i.e., *source rationality,* is verified if $x_j \leq \mathcal{C}_A(Y_{\mathcal{F}_j})$ for all $j = \{1, \ldots, M\}$. Whenever both efficiency and source rationality are ensured, an allocation is defined as an *imputation*. The set of imputations of the game $(\mathcal{C}_A,\ \mathcal{P})$ is then given by:

$$I(\mathcal{C}_A) = \left\{ \mathbf{x} \in \mathbb{R}^M \mid \sum_{\mathcal{F}_j \in \mathcal{P}} x_j = \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_m}) \text{ and } x_j \leq \mathcal{C}_A(Y_{\mathcal{F}_j}) \quad \text{for all } \mathcal{F}_j \in \mathcal{P} \right\}$$

The idea of rationality can be extended to subcoalitions (not only for each single factor). Thus a coalition of sources S where $S \neq \emptyset$ has an incentive to cooperate if S cannot improve on the allocation, i.e., coalition rationality holds such that $\sum_{j\in S} x_j \leq \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_S})$. Whenever a stronger rationality condition (*collective rationality*) holds, a further definition can be provided (see [15]):

Definition 4. *The set*

$$Core(\mathcal{C}_A) := \left\{ \mathbf{x} \in \mathbb{R}^M \mid \sum_{j \in \mathcal{P}} x_j = \mathcal{C}_A(\mathcal{P}) \text{ and } \sum_{j \in S} x_j \leq \mathcal{C}_A(S) \text{ for all } S \subseteq \mathcal{P}, S \neq \emptyset \right\}$$

is the core of the game $(\mathcal{C}_A, \mathcal{P})$, *i.e. the set of all non dominated allocations.*

This solution concept is particularly relevant since it allows for a stability of the imputation (see [25]). It implies that it is not feasible to form alternative coalitions (rather than the grand coalition) in which each income source produces lower cost. Thus the core of $\mathcal{C}_A$ refers to the notion of *Pareto optimality*: there is no imputation outside the core where each income source may reduce its effect in terms of cost of inequality without determining an increase in the cost originated by another source $\mathcal{F}_k$, $\mathcal{F}_k \neq \mathcal{F}_j$. In general, some contributions x_j might be negative since the characteristic function $\mathcal{C}_A$ is not monotone[16]. Note that in general the core of an essential constant-sum game is empty.

Schmeidler [30] introduced the concept of nucleolus due to lexicographic maximization of the minimum excess of costs over the contribution for all coalitions. In our framework, since the cost of inequality originated by different income sources is $\mathcal{C}_A(Y_S)$, we can perform

[16] Drechsel [10] and Drechsel and Kimms [11] show that if the characteristic function of the cost game is monotone, then all cost assignments in the core are non-negative.

this measure as the difference between the stand-alone cost of inequality originated by a coalition S of factors, for all $S \subseteq \mathcal{P}$, and the sum of the costs attributed to those factors in the assignment procedure. This is the so-called excess of the game:

Definition 5. *Given an allocation* $\mathbf{x} \in R^M$*, the* ***excess*** *of a coalition* S *with respect to* $\mathbf{x}$ *can be defined as:*

$$e(S, \mathbf{x}) = \mathcal{C}_A(Y_S) - \sum_{\mathcal{F}_j \in S} x_j \tag{8}$$

The excess is typically viewed as a measure of satisfaction of each coalition S (or dissatisfaction in case of payoff game). The nucleolus can be identified as the imputation that lexicographically maximizes the minimal excess among all coalitions.

Definition 6. *Given a game* $(\mathcal{C}_A, \mathcal{P})$ *and an imputation* $\mathbf{x} \in I(\mathcal{C}_A)$*, if* $\Theta(\mathbf{x})$ *is the* $(2^{|\mathcal{P}|}-2)$*-dimensional vector of all excesses* $e(S, \mathbf{x})$*, where* $S \in 2^{\mathcal{P}} \setminus \{\emptyset, \mathcal{P}\}$*, arranged in lexicographic order, the* ***nucleolus*** *of* $(\mathcal{C}_A, \mathcal{P})$ *is the unique imputation* $\mathbf{x}^*$ *that lexicographically maximizes the minimal excess in* $\Theta(\mathbf{x})$*.*

The nucleolus always exists and it is in the core of the game if the core is nonempty. We extend these characterizations on the cost of inequality game $(\mathcal{C}_A, \mathcal{P})$ to design a better indicator of each factor's marginal contribution. The aim of reducing the cost of inequality among coalitions is developed to maximize recursively the *welfare* of the worst-off treated coalitions.

3.2 The preferences-based uniform rule among factors

After introducing the solution concepts in the game, we investigate how to allocate each income factor $\mathcal{F}_j \in \mathcal{P}$ among types. We take into account that each type is endowed with a complete preference ordering, see [17] and [18]. Efficiency and equity properties of the game must be evaluated.

Suppose that each type i has a preference ordering R_i. Let us denote $u^i_j \in [0, 1]$ as the value that type i attributes to the contribution y_{ij} obtained from $\mathcal{F}_j \in \mathcal{P}$. $u^i_k R_i u^i_j$ means that u^i_k is preferred to u^i_j by type i. Type i announces her preference R_i and its ranking is known for all u^i_j. According to regularity conditions, these preferences relations are assumed to be single-peaked. For each type i, there always exists a peak value u^i_*, corresponding to a certain factor $\mathcal{F}_k \in \mathcal{P}$, such that the following 2 conditions holds according to Sprumont [35]:

1. $\forall \mathcal{F}_s, \mathcal{F}_t \in \mathcal{P} \setminus \mathcal{F}_k$, $u^i_s < u^i_t < u^i_k = u^i_*$ implies that $u^i_* R_i u^i_t R_i u^i_s$;
2. $\forall \mathcal{F}_s, \mathcal{F}_t \in \mathcal{P} \setminus \mathcal{F}_k$, $u^i_s > u^i_t > u^i_k = u^i_*$ implies that $u^i_* R_i u^i_t R_i u^i_s$;

In both cases, all the remaining contributions are arranged with the ranking induced by preference R_i. Let $R = (R_1, \ldots, R_N)$ be the system of preferences in the society. Denote $U = (u^i_j)_{i=1,\ldots,N;\ j=1,\ldots,M} \in \mathcal{M}_{N,M}(\mathbb{R}_+)$ the matrix of all evaluations u^i_j and

$V = (v^i_j) \in \mathcal{M}_{N,M}(\mathbb{R}_+)$ the matrix generated by:

$$v^i_j = \begin{cases} \sum_{l=1}^{M} u^i_l & \text{if } \ u^i_j = u^i_* \\ \\ 0 & \text{otherwise} \end{cases}$$

The matrices U and V are strictly related. The former contains the type's evaluations for each source. The latter requires the sum of the $i-$type's into the u^i_* cell, while resetting the other cells to 0 value. It therefore collects all information about preferences R_i in the single preferred contribution. Let us define $v_{j*} = \sum_{i \in N} v^i_j$, $\mathbf{v}_* = (v_{1*}, \ldots, v_{M*}) \in \mathbb{R}^M$ and $v_*(R) = \sum_{j=1}^{M} v_{j*}$[17]. When dealing with single-peaked preferences, Sprumont [35] proves that the uniform rule is the only one to satisfy the axioms of feasibility, efficiency, strategy-proofness and anonymity[18]. We give a new characterization conceived as an allocation rule of types' preferences among factor components. We denote it *weakly* uniform rule since some properties of the original rule hold, while some egalitarian axioms are also considered.

An allocation rule is efficient among agents if and only if no type gets more than her peak value, while other gets less according to matrix V. The matrix V is efficient since all types obtain a contribution higher than their initially preferred one.

Definition 7. *We say that the element $z_j \in \mathbb{R}_+$ is an* ***efficient contribution*** *if for all v_{j*} in V, we have that:*

$$\begin{cases} z_j \leq v_{j*} & \textit{if } \ \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \leq v_*(R) \ \ \textit{for all } j \in \{1, \ldots, M\} \\ \\ z_j \geq v_{j*} & \textit{if } \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \geq v_*(R) \ \ \textit{for all } j \in \{1, \ldots, M\} \end{cases}$$

We focus our attention on some specific axioms to be satisfied by this allocation rule:

- **Feasibility (F)**: An allocation $\mathbf{z} = (z_1, \ldots, z_M) \in \mathbb{R}^M$ is feasible if $\sum_{j=1}^{M} z_j = \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})$.
- **Efficiency (E):** An allocation $\mathbf{z}$ is efficient if its coordinates are efficient contributions.
- **Type anonimity (TA)**: An allocation $\mathbf{z}$ is anonymous if it does not depend on the order of types, i.e. if for all permutations $\Pi : N \longrightarrow N$ on types, the allocation $\mathbf{z}_\Pi$ obtained after applying Π is such that $\mathbf{z}_\Pi = \mathbf{z}$.
- **Preference monotonicity (PM)**: Given two factors $\mathcal{F}_j$, $\mathcal{F}_k$ such that $v_{j*} = 0$ and $v_{k*} \neq 0$, i.e. $\mathcal{F}_j$ is not preferred by any type, whereas $\mathcal{F}_k$ is preferred by one type at least, $z_j \leq z_k$.

[17] Anno and Sasaki [1] propose an alternative egalitarian rule compatible with second-best efficiency allocations.

[18] See for details, Ching [7], [8], Sprumont [35] and Thomson [36]. For alternative extension, Otten et al.[24] demonstrate that the uniform rule may coincide with the lexicographic egalitarian solution of a bargaining game.

- **Equality of treatment for non-preferred sources (ETNS)**: Given two factors $\mathcal{F}_j$, $\mathcal{F}_k$ such that $v_{j*} = v_{k*} = 0$, i.e. they are both not preferred by any type, $z_j = z_k$.
- **Lorenz dominance (LD)**: Denoting with $>_L$ the order induced by standard Lorenz domination, we establish that $\mathbf{z}$ Lorenz dominates the aggregate preferences vector $\mathbf{v}_*$ in this way: by rearranging and normalizing their coordinates in increasing order in the vectors $\widehat{\mathbf{z}} = (\widehat{z}_1, \ldots, \widehat{z}_M)$ and $\widehat{\mathbf{v}}_* = (\widehat{v}_{1*}, \ldots, \widehat{v}_{M*})$, $\widehat{\mathbf{z}} >_L \widehat{\mathbf{v}}_*$ if $\sum_{j=1}^{l} \widehat{z}_j \geq \sum_{j=1}^{l} \widehat{v}_{j*}$ for $l \in \{1, \ldots, M\}$.

Before proceeding, we define the *multi-factor version* of Sprumont's uniform rule:

Definition 8. *Given a cost game* $(\mathcal{C}_A,\ \mathcal{P},\ (R_i)_{i\in N})$*, and the types' preferences in the matrices* U *and* V*, we denote with* $\Psi = (\Psi_1, \ldots, \Psi_M)$ *the* ***weakly uniform rule****, where:*

$$\Psi_j\left(\mathcal{C}_A,\ R\right) = \begin{cases} \min\{v_{j*}, \lambda(R)\} \ \textit{if} \ \ \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \leq v_*(R) \\ \\ \max\{v_{j*}, \mu(R)\} \ \textit{if} \ \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \geq v_*(R) \end{cases}$$

and where $\lambda(R)$ *and* $\mu(R)$ *are respectively the solutions to the equations*

$$\sum_{j=1}^{m} \min\{v_{j*}, \lambda(R)\} = \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \quad \textit{and} \quad \sum_{j=1}^{m} \max\{v_{j*}, \mu(R)\} = \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})$$

This rule simply suggests that whenever there is too little to share about cost $\mathcal{C}_A$ among types, i.e., $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}m}) \leq v_*(R)$, it assigns a positive amount $\lambda(R)$ to factors with preferred contributions (peaks) above this amount, while leaving other factors to get their initial values. Viceversa, it assigns $\mu(R)$ when there is too much to share. This appears to be a way to take efficiently into account the value assigned to each factor by types' preferences.

Theorem 1. *The weakly uniform rule* Ψ *satisfies* ***F****,* ***E****,* ***TA****,* ***PM****,* ***ETNS*** *and* ***LD****.*

Proof. The axiom **F** is satisfied by Ψ by Definition 8. **E** is satisfied as well because in the former case $\Psi_j = \min\{v_{j*}, \lambda(R)\}$ implies $\Psi_j \leq v_{j*}$, whereas in the latter case $\Psi_j = \max\{v_{j*}, \lambda(R)\}$ implies $\Psi_j \geq v_{j*}$. As far as **TA** is concerned, we have to prove the invariance of Ψ with respect to any permutation of preference between types. Each permutation corresponds to a swap between the rows of matrix U, and since any matrix $\widetilde{U}$ obtained from a permutation over N leads to a matrix $\widetilde{V}$ containing the sums of all entries of U, also $\widetilde{V}$ can be obtained from V by swapping its rows. Hence, the vector $\widetilde{v}_*(R)$ coincides with $v_*(R)$ because it contains the sum of all entries of U as well. Consequently, the solutions to the two problems coincide.

The validity of axiom **PM** requires some more investigation: if $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \leq v_*(R)$, it trivially holds, because if factor $\mathcal{F}_j$ is not preferred by any type, $v_{j*} = 0$ entails $\min\{0,\ \lambda(R)\} = 0$, and $\Psi_j = 0$ cannot exceed any other coordinate. On the other hand, if $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \geq v_*(R)$, then the coordinate Ψ_j corresponding to each non-preferred source is $\mu(R)$, which cannot exceed $\max\{v_{k*}, \mu(R)\}$ which is Ψ_k for each preferred factor. The axiom **ETNS** holds as well, because when $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \leq v_*(R)$ all the non-preferred sources correspond to coordinates of Ψ which are equal to 0, whereas in the opposite case they are all equal to $\mu(R)$, hence the weakly uniform rule always assigns the same share to them.

What remains to prove is the Lorenz dominance (axiom **LD**). If we rearrange the coordinates of Ψ and of $\mathbf{v}_*$ in increasing order, achieving the vectors $\widehat{\Psi} = (\widehat{\Psi}_1, \ldots, \widehat{\Psi}_M)$ and $\widehat{\mathbf{v}}_* = (\widehat{v}_{1*}, \ldots, \widehat{v}_{M*})$, we can consider the cases of non-preferred sources separately:

- if $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \leq v_*(R)$, for all the non-preferred factors, the contributions are 0 in both vectors, so they do not affect the respective sums. If we denote the preferred ones with $\mathcal{F}_p, \ldots, \mathcal{F}_M$, the related $\widehat{\Psi}_j$ may be either $\lambda(R)$ or $\widehat{v}_{j*}$. If $p > r \geq 0$ is the number of sources whose contributions are equal to $\lambda(R)$ we will have:

$$\frac{1}{\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})}\left(r\lambda(R) + \sum_{j=r+1}^{l} \widehat{v}_{j*}\right) \geq \frac{1}{v_*(R)}\sum_{j=p}^{l}\widehat{v}_{j*} \iff$$

$$\iff \sum_{j=p}^{M}\widehat{v}_{j*}\left(r\lambda(R) + \sum_{j=r+1}^{l} \widehat{v}_{j*}\right) \geq \left(r\lambda(R) + \sum_{j=r+1}^{M} \widehat{v}_{j*}\right)\sum_{j=p}^{l}\widehat{v}_{j*} \iff$$

$$\iff r\lambda(R)\sum_{j=l+1}^{M} \widehat{v}_{j*} \geq 0$$

  which holds for all $l = p, \ldots, M-1$.

- if $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \geq v_*(R)$, for each non-preferred factor

$$\frac{\widehat{\Psi}_j}{\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})} = \frac{\mu(R)}{\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})} \geq 0$$

  hence the Lorenz dominance criterion is verified; if we indicate the preferred ones with $\mathcal{F}_p, \ldots, \mathcal{F}_M$, we have:

$$\frac{1}{\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})}\left((p-1)\mu(R) + \sum_{j=p}^{l} \widehat{v}_{j*}\right) \geq \frac{1}{v_*(R)}\sum_{j=p}^{l}\widehat{v}_{j*} \iff$$

$$\iff \sum_{j=p}^{M}\widehat{v}_{j*}\left((p-1)\mu(R) + \sum_{j=p}^{l} \widehat{v}_{j*}\right) \geq \left((p-1)\mu(R) + \sum_{j=p}^{M} \widehat{v}_{j*}\right)\sum_{j=p}^{l}\widehat{v}_{j*} \iff$$

$$\iff (p-1)\mu(R)\sum_{j=l+1}^{M} \widehat{v}_{j*} \geq 0$$

  which holds for all $l = p, \ldots, M-1$.

So, **LD** is verified too. □

The *weakly* uniform rule attributes an equal share to *not preferred* factors. This share is lower than the one imposed to the preferred factors. The rule guarantees an allocation which is less unequal than the vector collecting aggregated preferences. The application of this procedure to vector $\mathbf{v}$, as a result of the aggregation process of preferences, involves some relevant properties collected below.

Proposition 2. *If $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) > v_*(R)$, and if*

1. *there exist $s \geq 1$ factors $\mathcal{F}_1, \ldots, \mathcal{F}_s$ such that $v_{1*}, \ldots, v_{s*} = 0$;*
2. $\dfrac{\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) - v_*(R)}{s} < \min v_{j*}$ *for all $v_{j*} \neq 0$;*

then $\Psi_j = v_{j*}$ for all $\mathcal{F}_j \notin \{\mathcal{F}_1, \ldots, \mathcal{F}_s\}$, and $\Psi_j = \dfrac{\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) - v_*(R)}{s}$ for all $j = 1, \ldots, s$.

Proof. If $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) > v_*(R) = \sum_{\mathcal{F}_j \notin \{\mathcal{F}_1,\ldots,\mathcal{F}_s\}} v_{j*}$, we are supposed to determine $\Psi_j = \max\{v_{j*}, \mu(R)\}$ subject to $\sum_{j=1}^{M} \max\{v_{j*}, \mu(R)\} = \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})$. Because $\mu(R)$ is nonnegative, the equation boils down to:

$$s\mu(R) + \sum_{\mathcal{F}_j \notin \{\mathcal{F}_1,\ldots,\mathcal{F}_s\}} \max\{v_{j*}, \mu(R)\} = \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})$$

Suppose that the assertion is false, hence there exists $\mathcal{F}_k \notin \{\mathcal{F}_1, \ldots, \mathcal{F}_s\}$ such that $\Psi_k = \mu(R)$, which means that $\mu(R) > v_{k*}$. The equation becomes

$$(s+1)\mu(R) + \sum_{\mathcal{F}_j \notin \{\mathcal{F}_1,\ldots,\mathcal{F}_s\} \setminus \mathcal{F}_k} \max\{v_{j*}, \mu(R)\} = \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})$$

implying

$$(s+1)\mu(R) + v_*(R) - v_{k*} = \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \iff \frac{\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) - v_*(R)}{s} = \frac{(s+1)\mu(R) - v_{k*}}{s}$$

By the second assumption, $\dfrac{(s+1)\mu(R) - v_{k*}}{s} < v_{k*}$, but on the other hand $\mu(R) > v_{k*}$ entails $\dfrac{(s+1)\mu(R) - v_{k*}}{s} > \dfrac{(s+1)v_{k*} - v_{k*}}{s} = v_{k*}$, which is a contradiction. □

Proposition 3. *If $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_m}) < v_*(R)$, and if there exist $s \geq 1$ factors $\mathcal{F}_1, \ldots, \mathcal{F}_s$ such that $v_{1*}, \ldots, v_{s*} = 0$, then $\Psi_j = 0$ for all $j = 1, \ldots, s$.*

Proof. It is straightforward to see that for all factors $\mathcal{F}_j$ such that $v_{j*} = 0$, the nonnegativity of $\lambda(R)$ implies $\min\{0,\ \lambda(R)\} = 0$, hence $\Psi_j = 0$. □

Proposition 2 ensures that when the overall cost exceeds the aggregate values of factors, not preferred sources still obtain a positive share although smaller than the one achieved by the other factors. Proposition 3 claims that when the aggregate values attached to factors exceed the cost, not-preferred sources do not receive any share. What follows is an example helpful to clarify the procedure:

Example 1. *Imagine a society composed by 3 different types and 4 different factors A, B, C, D. The respective evaluations are collected in the following* 3×4 *matrix:*

$$U = \begin{pmatrix} u_A^1 & u_B^1 & u_C^1 & u_D^1 \\ u_A^2 & u_B^2 & u_C^2 & u_D^2 \\ u_A^3 & u_B^3 & u_C^3 & u_D^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0.1 & 0.15 \\ 0.1 & 0 & 0.2 & 0 \\ 0.1 & 0 & 0.2 & 0 \end{pmatrix}$$

Consider a preference system such that in compliance with R_1*, type 1 has a peak value* $u_*^1 = u_D^1 = 0.15$*, and by* R_2 *and* R_3 *the remaining peak values are* $u_*^2 = u_C^2 = 0.2$ *and* $u_*^3 = u_C^3 = 0.2$*. In this case, the matrix* V *reads as:*

$$V = \begin{pmatrix} 0 & 0 & 0 & 0.25 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0 \end{pmatrix}$$

and we have that

$$v_{A*} = 0, \qquad v_{B*} = 0, \qquad v_{C*} = 0.6, \qquad v_{D*} = 0.25$$

$$\mathbf{v}_* = (0,\ 0,\ 0.6,\ 0.25), \qquad v_*(R) = 0 + 0 + 0.6 + 0.25 = 0.85$$

If the overall cost of inequality is $\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_4}) = 0.9$*, we can explicitly compute the value of* Ψ*. Since* $0.85 < \mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_4}) = 0.9$*, we determine* $\mu(R)$ *such that*

$$\max\{0,\ \mu(R)\} + \max\{0,\ \mu(R)\} + \max\{0.6,\ \mu(R)\} + \max\{0.25,\ \mu(R)\} = 0.9$$

The unique value for which such equation is verified is $\mu(R) = 0.025$*, entailing the following weakly uniform rule:*

$$\Psi = (0.025,\ 0.025,\ 0.6,\ 0.25)$$

We can even calculate the value of Ψ *in the minimization case. If we suppose to have the same matrices* U *and* V *and the overall cost of inequality is* $\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_4}) = 0.8$*, in this case we must find* $\lambda(R)$ *such that:*

$$\min\{0,\ \lambda(R)\} + \min\{0,\ \lambda(R)\} + \min\{0.6,\ \lambda(R)\} + \min\{0.25,\ \lambda(R)\} = 0.8$$

The unique solution is $\lambda(R) = 0.55$*, yielding the following weakly uniform rule:*

$$\Psi = (0,\ 0,\ 0.55,\ 0.25)$$

The next results intend to investigate the occurrence of egalitarian solutions to the problem in the two cases.

Proposition 4. *If* $\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_M}) > v_*(R)$*, and for all* $j = 1, \dots, M$*, there exists a type* i *such that* $u_j^i = u_*^i \neq 0$*, i.e.,* $v_{j*} \neq 0$ *for each factor* $\mathcal{F}_j$*, then*

$$\frac{\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_M})}{M} \geq \max v_{j*} \implies \Psi_1 = \Psi_2 = \cdots = \Psi_M = \frac{\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_M})}{M}$$

Proof. Since $\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_M}) > v_*(R)$, there must be at least one $k \in \{1,\dots,M\}$ such that $v_{k*} < \mu(R)$ so that the equation becomes $\mu(R) + \sum_{j\neq k} \max\{v_{j*}, \mu(R)\} = \mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_M})$. If $\mu(R) = v_{j*}$ for all $j \neq k$, the proof is complete. If there exists $l \in \{1,\dots,M\}$, $l \neq k$, such that $\mu(R) \neq v_{l*}$, then $\max\{v_{l*}, \mu(R)\}$ must be equal to $\mu(R)$. By iterating the process, we find that each v_{j*} must be smaller than $\mu(R)$. Since each coordinate Ψ_j is equal to $\dfrac{\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_M})}{M}$ in the egalitarian solution, the sufficient condition to achieve it is $\dfrac{\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_M})}{M} \geq \max v_{j*}$. □

Proposition 5. *If $\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_M}) < v_*(R)$, and if for all $j = 1,\dots,M$ there exists a type i such that $u^i_j = u^i_* \neq 0$, i.e. $v_{j*} \neq 0$ for each factor $\mathcal{F}_j$, then*

$$\frac{\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_M})}{M} \leq \min v_{j*} \implies \Psi_1 = \Psi_2 = \cdots = \Psi_M = \frac{\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_M})}{M}$$

Proof. It suffices to repeat the proof of Proposition 4 with simple modifications. □

The example below illustrates a setup where the weakly uniform rule is egalitarian.

Example 2. *Suppose now 4 different types and 4 different factors A, B, C, D in the society. The respective evaluations are collected in the following 4×4 matrix:*

$$U = \begin{pmatrix} u^1_A & u^1_B & u^1_C & u^1_D \\ u^2_A & u^2_B & u^2_C & u^2_D \\ u^3_A & u^3_B & u^3_C & u^3_D \\ u^4_A & u^4_B & u^4_C & u^4_D \end{pmatrix} = \begin{pmatrix} 0 & 0.03 & 0.04 & 0.03 \\ 0.1 & 0 & 0 & 0.06 \\ 0.05 & 0.02 & 0.03 & 0.01 \\ 0.07 & 0.01 & 0.06 & 0.04 \end{pmatrix}$$

Consider a preference system such that in compliance with their respective preferences, R_1, the peak values are $u^1_* = 0.04$, $u^2_* = 0.1$, $u^3_* = 0.01$ and $u^4_* = 0.01$, leading to the following matrix:

$$V = \begin{pmatrix} 0 & 0 & 0.1 & 0 \\ 0.16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.11 \\ 0 & 0.18 & 0 & 0 \end{pmatrix}$$

and to

$$\mathbf{v}_* = (0.16,\ 0.18,\ 0.1,\ 0.11), \qquad v_*(R) = 0.16 + 0.18 + 0.1 + 0.11 = 0.55$$

If the cost of inequality is $\mathcal{C}_A(Y_{\mathcal{F}_1,\dots,\mathcal{F}_4}) = 0.8$, Proposition 6 is verified and we determine $\mu(R)$ such that

$$\max\{0.16,\ \mu(R)\} + \max\{0.18,\ \mu(R)\} + \max\{0.1,\ \mu(R)\} + \max\{0.11,\ \mu(R)\} = 0.8$$

i.e. $\mu(R) = 0.2$, entailing the following weakly uniform rule:

$$\Psi = (0.2,\ 0.2,\ 0.2,\ 0.2)$$

In the minimization case if $\mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_4}) = 0.39$*, then Proposition 7 holds. In this case the equation:*

$$\min\{0.16,\ \lambda(R)\} + \min\{0.18,\ \lambda(R)\} + \min\{0.1,\ \lambda(R)\} + \min\{0.11,\ \lambda(R)\} = 0.39$$

is solved by $\lambda(R) = 0.097$*, yielding the following weakly uniform rule:*

$$\Psi = (0.097,\ 0.097,\ 0.097,\ 0.097)$$

3.3 The nucleolus vs the weakly uniform rule

Now we want to shed some light on the connections between the nucleolus and the weakly uniform rule. The aim is to link the costs induced by the coalitions and the types' preferences on factors. The subadditivity of the inequality cost $\mathcal{C}_A(\cdot)$ is the sufficient property to satisfy the solution concept. In general the nucleolus is computed by solving a standard sequence of linear programs (see [19] and [22]):

$$\begin{aligned} & & \max \quad & \alpha_i \\ \text{s.t.} \quad e(S,\mathbf{x}) & \geq & & \alpha_i \quad \text{for all } S \in 2^{\mathcal{P}} \\ \sum_{j=1}^{M} x_j & = & & \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M}) \end{aligned} \tag{9}$$

where $\alpha_i \in \mathbb{R}$ are the minimizers of each optimization program to be maximized. To avoid the treatment of all cases in the implementation, we fix our attention on some clear-cut applications. Initially, we consider the 2−factor case, where we observe that the nucleolus coincides with the intersection point between the two excess lines, as shown by the example below.

Example 3. *Consider the cost game* $\mathcal{C}_A : 2^{\mathcal{P}} \longrightarrow \mathbb{R}$ *on the factor set* $\mathcal{P} = \{\mathcal{F}_1,\ \mathcal{F}_2\}$ *such that:*

$$\mathcal{C}_A(Y_{\mathcal{F}_1,\mathcal{F}_2}) = 0.8, \qquad \mathcal{C}_A(Y_{\mathcal{F}_1}) = 0.7, \qquad \mathcal{C}_A(Y_{\mathcal{F}_2}) = 0.6$$

Let us define $\mathbf{x} = (x_1,\ 0.8 - x_1)$ *as the general allocation vector. The conditions verifying individual rationality are:*

$$\begin{cases} x_1 \leq 0.7 \\ 0.8 - x_1 \leq 0.6 \end{cases} \qquad \Longrightarrow \qquad 0.2 \leq x_1 \leq 0.7 \tag{10}$$

and the two relevant excesses:

$$e(\mathcal{F}_1,\ \mathbf{x}) = 0.7 - x_1, \qquad e(\mathcal{F}_2,\ \mathbf{x}) = -0.2 + x_1$$

The coordinate x_1 *can be computed as the intersection of the two lines.* $x_1 = 0.45$ *verifies (10) and corresponds to the imputation which maximizes the minimal excess, i.e.* $\mathbf{x} = (0.45,\ 0.35)$*, as can be observed in the figure.*

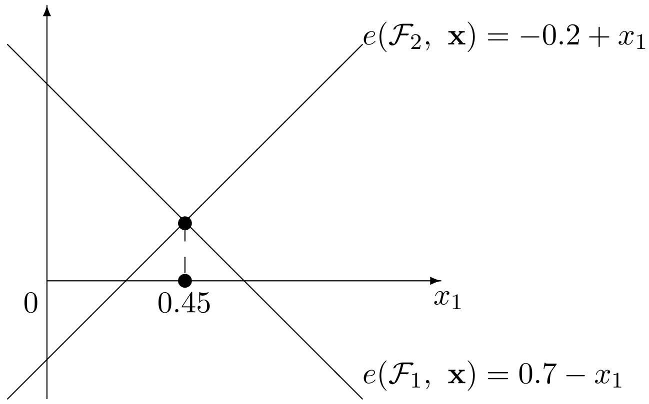

Figure 1: *The nucleolus of* $(\mathcal{C}_A, \mathcal{P})$ *is given by* $\mathbf{x} = (0.45,\ 0.35)$.

We can derive an alternative formula for the nucleolus:

$$\mathbf{x} = \left(\frac{\mathcal{C}_A(Y_{\mathcal{F}_1,\mathcal{F}_2}) + \mathcal{C}_A(Y_{\mathcal{F}_1}) - \mathcal{C}_A(Y_{\mathcal{F}_2})}{2},\ \frac{\mathcal{C}_A(Y_{\mathcal{F}_1,\mathcal{F}_2}) + \mathcal{C}_A(Y_{\mathcal{F}_2}) - \mathcal{C}_A(Y_{\mathcal{F}_1})}{2} \right) \tag{11}$$

We characterize the relation between it and the weakly uniform rule as follows[19]:

Proposition 6. *Given a 2-factor cost of inequality game, where* $\mathbf{v}_* = (v_{1*}, v_{2*}) \in \mathbb{R}^2$ *is the vector of the aggregated preferences and* $v_*(R) = \sum_{j=1}^{2} v_{j*}$*, the nucleolus* $\mathbf{x}$ *and the weakly uniform rule* Ψ *coincide if and only if one of the following conditions holds:*

1. $v_{1*} = \dfrac{\mathcal{C}_A(Y_{\mathcal{F}_1,\mathcal{F}_2}) + \mathcal{C}_A(Y_{\mathcal{F}_1}) - \mathcal{C}_A(Y_{\mathcal{F}_2})}{2}$;
2. $v_{2*} = \dfrac{\mathcal{C}_A(Y_{\mathcal{F}_1,\mathcal{F}_2}) + \mathcal{C}_A(Y_{\mathcal{F}_2}) - \mathcal{C}_A(Y_{\mathcal{F}_1})}{2}$.

Proof. It immediately follows from the procedure described in Subsection 3.2 and from formula (11). □

We show that the nucleolus and the weakly uniform rule do coincide as follows in the case of 3-factor cost of inequality..

Example 4. *Consider the subadditive cost game* $\mathcal{C}_A : 2^{\mathcal{P}} \longrightarrow \mathbb{R}$ *on* $\mathcal{P} = \{\mathcal{F}_1,\ \mathcal{F}_2,\ \mathcal{F}_3\}$ *such that:*

$$\mathcal{C}_A(Y_{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3}) = 0.75, \quad \mathcal{C}_A(Y_{\mathcal{F}_1,\mathcal{F}_2}) = 0.6, \quad \mathcal{C}_A(Y_{\mathcal{F}_1,\mathcal{F}_3}) = 0.5, \quad \mathcal{C}_A(Y_{\mathcal{F}_2,\mathcal{F}_3}) = 0.7$$

$$\mathcal{C}_A(Y_{\mathcal{F}_1}) = 0.2, \qquad \mathcal{C}_A(Y_{\mathcal{F}_2}) = 0.45, \qquad \mathcal{C}_A(Y_{\mathcal{F}_3}) = 0.4$$

[19]Note that from (11), the nucleolus is egalitarian if and only if the cost game is symmetric, i.e., $\mathcal{C}_A(Y_{\mathcal{F}_1}) = \mathcal{C}_A(Y_{\mathcal{F}_2})$.

In a society influenced by factors $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, consider 3 different types with the following contribution and preferences:

$$U = \begin{pmatrix} u^1_{\mathcal{F}_1} & u^1_{\mathcal{F}_2} & u^1_{\mathcal{F}_3} \\ u^2_{\mathcal{F}_1} & u^2_{\mathcal{F}_2} & u^2_{\mathcal{F}_3} \\ u^3_{\mathcal{F}_1} & u^3_{\mathcal{F}_2} & u^3_{\mathcal{F}_3} \end{pmatrix} = \begin{pmatrix} 0.07 & 0.045 & 0.01 \\ 0.2 & 0.05 & 0.1 \\ 0.15 & 0.04 & 0.1 \end{pmatrix}$$

Suppose that types' preference system is: type 1 has the peak value $u^1_* = u^1_{\mathcal{F}_1} = 0.07$, type 2 has $u^2_* = u^2_{\mathcal{F}_2} = 0.05$ and type 3 has $u^3_* = u^3_{\mathcal{F}_2} = 0.04$. The matrix V is:

$$V = \begin{pmatrix} 0.125 & 0 & 0 \\ 0 & 0.35 & 0 \\ 0 & 0.29 & 0 \end{pmatrix}$$

hence $v_*(R) = 0.765 > \mathcal{C}_A(Y_{\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3})$. The related weakly uniform rule, computed as in the previous Subsection, amounts to $\Psi = (0.125,\ 0.35,\ 0.275)$.

We determine the nucleolus of the game. The candidate nucleolus can be written as $\mathbf{x} = (x_1,\ x_2,\ 0.75 - x_1 - x_2)$, and we are supposed to maximize the minimum of the 6 involved excesses or, in other words, to maximize α_1 subject to the following system of inequalities:

$$\begin{cases} 0.6 - x_1 - x_2 \geq \alpha_1 \\ -0.25 + x_2 \geq \alpha_1 \\ -0.05 + x_1 \geq \alpha_1 \\ 0.2 - x_1 \geq \alpha_1 \\ 0.45 - x_2 \geq \alpha_1 \\ -0.35 + x_1 + x_2 \geq \alpha_1 \end{cases}$$

and to the types' rationality constraints:

$$\begin{cases} x_1 \leq 0.2 \\ x_2 \leq 0.45 \\ x_1 + x_2 \geq 0.35 \end{cases}$$

$\alpha_1 = 0.075$ is the value such that $x_1^* = 0.125$. The following step consists in solving the next constrained optimization problem:

$$\begin{cases} 0.475 - x_2 \geq \alpha_2 \\ -0.25 + x_2 \geq \alpha_2 \\ 0.45 - x_2 \geq \alpha_2 \\ -0.225 + x_2 \geq \alpha_2 \end{cases}$$

subject to $0.225 \leq x_2 \leq 0.45$.

The solution is $\alpha_2 = 0.1$ corresponding to $x_2^* = 0.35$, which belongs to the interval satisfying the individual rationality condition. Hence the nucleolus of the game turns out to be $\mathbf{x} = (0.125,\ 0.35,\ 0.275)$.

The above Example provides some further information about the possible coincidence between the weakly uniform rule and the nucleolus. Basically, when $v_*(R) > \mathcal{C}_A(Y_{\mathcal{F}_1,\ldots,\mathcal{F}_M})$, if the sequence of linear programs returns $M-1$ coordinates of the vector $\mathbf{v}_*$, the remaining one must be equal to the corresponding coordinate of the nucleolus.

4 Welfare loss game

Since $(\mathcal{C}_A,\ \mathcal{P})$ is a cost cooperative game, we convert it into a cost *savings* game. We therefore take into account a cooperative structure $(\mathcal{L}_A,\ \mathcal{P})$, involving the multi-factor Atkinson inequality index $I_A(.)$ of coalition S for $\forall S \subseteq \mathcal{P}$:

$$I_A(Y_S) = 1 - \frac{\widehat{y}_S}{\mu_S} = \begin{cases} 1 - \dfrac{|S|^{-\frac{\epsilon}{1-\epsilon}}\, N^{-\frac{\epsilon}{1-\epsilon}} \left[\sum_{j=1}^{|S|}\sum_{i=1}^{N} y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}}{\sum_{j=1}^{|S|}\sum_{i=1}^{N} y_{ij}} & \text{if} \quad \epsilon \in (0,\ 1) \\ 1 - \dfrac{|S|\, N \left[\prod_{j=1}^{|S|}\prod_{i=1}^{N} y_{ij}\right]^{\frac{1}{T}}}{\sum_{j=1}^{|S|}\sum_{i=1}^{N} y_{ij}} & \text{if} \quad \epsilon = 1 \end{cases} \tag{12}$$

hence, the *ede*-income is:

$$\begin{cases} \widehat{y}_S = \left[\dfrac{1}{|S|\, N} \sum_{j=1}^{|S|}\sum_{i=1}^{N} y_{ij}^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}} & \text{if} \quad \epsilon \in (0,\ 1) \\ \widehat{y}_S = \left[\prod_{j=1}^{|S|}\prod_{i=1}^{N} y_{ij}\right]^{\frac{1}{S}} & \text{if} \quad \epsilon = 1 \end{cases}$$

We consider $\widehat{y}_S$ as the *ede* income associated to coalition S. $\mathcal{K}_A(Y_S) := \dfrac{\mu_S - \widehat{y}_S}{\mu_S}$ is the cost of inequality associated to S, for $\forall S \subseteq \mathcal{P}$.

Definition 9. *We call* ***welfare loss function*** *the following characteristic value:*

$$\mathcal{L}_A(S) = \sum_{j \in S} \mathcal{C}_A(Y_{\mathcal{F}_j}) - \mathcal{K}_A(Y_S)$$

for all $S \subseteq \mathcal{P}$.

where $\mathcal{L}_A(\emptyset) = 0$. Modeling welfare is crucial since it allows us to measure the heterogeneity of factor components in the income distribution. It is defined as the difference between the costs of inequality originated by sources $\mathcal{F}_j$ and the aggregate cost $\mathcal{K}_A$ for each $S \subseteq \mathcal{P}$. We investigate some basic properties of $\mathcal{L}_A(S)$. First, we should remark that $\mathcal{L}_A(\{j\}) = 0$, i.e., it trivially vanishes at each $1-$factor coalition. We can characterize its positivity property as:

Proposition 7. *If* $\dfrac{\widehat{y}_S}{\mu_S} \geq \min\left\{\dfrac{\widehat{y}_1}{\mu_1}, \ldots, \dfrac{\widehat{y}_{|S|}}{\mu_{|S|}}\right\}$, *then* $\mathcal{L}_A(S)$ *is positive.*

Proof.

$$\mathcal{L}_A(S) = \sum_{j=1}^{|S|}\left(1-\frac{\widehat{y}_j}{\mu_j}\right) - 1 + \frac{\widehat{y}_S}{\mu_S} = |S| - 1 - \frac{\widehat{y}_1}{\mu_1} - \cdots - \frac{\widehat{y}_{|S|}}{\mu_{|S|}} + \frac{\widehat{y}_S}{\mu_S}$$

where it suffices that $\frac{\widehat{y}_S}{\mu_S}$ be larger than one of the remaining fractions to ensure positivity. □

We assess the marginal contribution of factor $\mathcal{F}_j$ to the welfare loss function. First we evaluate the trivial case where $S=\emptyset$. For each $\mathcal{F}_j \in \mathcal{P}$, it follows that:

$$\mathcal{L}_A(\emptyset \cup \mathcal{F}_j) - \mathcal{L}_A(\emptyset) = C_A(Y_{\mathcal{F}_j}) - K_A(Y_{\mathcal{F}_j}) = 0$$

For each coalition S and factor $\mathcal{F}_j \notin S$, the *ede* and the arithmetic mean are characterized by:

$$\begin{cases} \widehat{y}_{S\cup\mathcal{F}_j} = \left(\dfrac{|S|\widehat{y}_S^{1-\epsilon}+\widehat{y}_j^{1-\epsilon}}{|S|+1}\right)^{\frac{1}{1-\epsilon}} \\ \\ \mu_{S\cup\mathcal{F}_j} = \dfrac{|S|\mu_S+\mu_j}{|S|+1} \end{cases} \tag{13}$$

After some manipulations, this leads to:

$$\begin{cases} |S|\widehat{y}_{S\cup\mathcal{F}_j}^{1-\epsilon} + \widehat{y}_{S\cup\mathcal{F}_j}^{1-\epsilon} = |S|\widehat{y}_S^{1-\epsilon} + \widehat{y}_j^{1-\epsilon} \\ \\ |S|\mu_{S\cup\mathcal{F}_j} + \mu_{S\cup\mathcal{F}_j} = |S|\mu_S + \mu_j \end{cases} \iff \begin{cases} |S|\left(\widehat{y}_{S\cup\mathcal{F}_j}^{1-\epsilon} - \widehat{y}_S^{1-\epsilon}\right) = \widehat{y}_j^{1-\epsilon} - \widehat{y}_{S\cup\mathcal{F}_j}^{1-\epsilon} \\ \\ |S|(\mu_{S\cup\mathcal{F}_j} - \mu_S) = \mu_j - \mu_{S\cup\mathcal{F}_j} \end{cases} \tag{14}$$

It implies that only 4 possible cases may occur:

1. $\begin{cases} \widehat{y}_j > \widehat{y}_{S\cup\mathcal{F}_j} > \widehat{y}_S \\ \mu_j < \mu_{S\cup\mathcal{F}_j} < \mu_S \end{cases}$

2. $\begin{cases} \widehat{y}_j < \widehat{y}_{S\cup\mathcal{F}_j} < \widehat{y}_S \\ \mu_j > \mu_{S\cup\mathcal{F}_j} > \mu_S \end{cases}$

3. $\begin{cases} \widehat{y}_j > \widehat{y}_{S\cup\mathcal{F}_j} > \widehat{y}_S \\ \mu_j > \mu_{S\cup\mathcal{F}_j} > \mu_S \end{cases}$

4. $\begin{cases} \widehat{y}_j < \widehat{y}_{S\cup\mathcal{F}_j} < \widehat{y}_S \\ \mu_j < \mu_{S\cup\mathcal{F}_j} < \mu_S \end{cases}$

The marginal contribution of each source to the welfare loss is strictly positive under some conditions.

Proposition 8. *Given a factor $\mathcal{F}_j \in \mathcal{P}$ and a coalition $S \subseteq \mathcal{P} \setminus \mathcal{F}_j$, if any of the following conditions holds:*

1. $\begin{cases} \widehat{y}_j > \widehat{y}_{S\cup\mathcal{F}_j} > \widehat{y}_S \\ \\ \mu_j < \mu_{S\cup\mathcal{F}_j} < \mu_S \end{cases}$

2. $\begin{cases} \widehat{y}_j < \widehat{y}_{S\cup\mathcal{F}_j} < \widehat{y}_S \\ \\ \mu_j > \mu_{S\cup\mathcal{F}_j} > \mu_S \end{cases}$

then $\mathcal{L}_A(S \cup \mathcal{F}_j) > \mathcal{L}_A(S)$.

Proof. The marginal contribution of $\mathcal{F}_j$ to any coalition $S \subseteq \mathcal{P} \setminus \mathcal{F}_j$ is expressed as follows:

$$\mathcal{L}_A(S \cup \mathcal{F}_j) - \mathcal{L}_A(S) = C_A(Y_{\mathcal{F}_j}) - K_A(Y_{S\cup\mathcal{F}_j}) + K_A(Y_S) =$$

$$= 1 - \frac{\widehat{y}_j}{\mu_j} - 1 + \frac{\widehat{y}_{S\cup\mathcal{F}_j}}{\mu_{S\cup\mathcal{F}_j}} + 1 - \frac{\widehat{y}_S}{\mu_S} = 1 - \frac{\widehat{y}_j}{\mu_j} + \frac{\widehat{y}_{S\cup\mathcal{F}_j}}{\mu_{S\cup\mathcal{F}_j}} - \frac{\widehat{y}_S}{\mu_S} \tag{15}$$

We are going to carry out a separate analysis for each of them.

1. Since $\frac{\widehat{y}_{S\cup\mathcal{F}_j}}{\mu_{S\cup\mathcal{F}_j}} > \frac{\widehat{y}_S}{\mu_S}$, this entails:

$$\mathcal{L}_A(S \cup \mathcal{F}_j) - \mathcal{L}_A(S) > 1 - \frac{\widehat{y}_j}{\mu_j} \geq 0$$

2. Since $\frac{\widehat{y}_{S\cup\mathcal{F}_j}}{\mu_{S\cup\mathcal{F}_j}} > \frac{\widehat{y}_j}{\mu_j}$, this entails:

$$\mathcal{L}_A(S \cup \mathcal{F}_j) - \mathcal{L}_A(S) > 1 - \frac{\widehat{y}_S}{\mu_S} \geq 0$$

□

In the remaining two cases, the sign of the contributions is not uniquely determined. The next Example describes a situation where also negative contributions are generated.

Example 5. *Let us consider a setup with 2 heterogeneous types and 4 different factors, A, B, C, D. The set of all feasible coalitions is:*

$$2^{\mathcal{P}} = \{\{A,B,C,D\},\{A,B,C\},\{A,B,D\},\{A,C,D\},\{B,C,D\}$$

$$\{A,B\},\{A,C\},\{A,D\},\{B,C\},\{B,D\},\{C,D\},\{A\},\{B\},\{C\},\{D\},\emptyset\}$$

If the aversion parameter is $\epsilon = \frac{1}{2}$*, the ede incomes* $\widehat{y}_S$ *are:*

$$\widehat{y}_S = \left[\frac{1}{2|S|} \sum_{j=1}^{|S|} \sum_{i=1}^{2} y_{ij}^{\frac{1}{2}} \right]^2 \tag{16}$$

whereas μ_S *are the arithmetic means. The table below collects the income profiles:*

	Factor A	***Factor B***	***Factor C***	***Factor D***
Type 1	***400***	***900***	***3600***	***100***
Type 2	***1600***	***2500***	***100***	***400***

First, we simply compute all coalitions' arithmetic means:

$$\mu_{\{A,B,C,D\}} = \frac{400 + 1600 + 900 + 2500 + 3600 + 100 + 100 + 400}{8} = 1200$$

$$\mu_{\{A,B,C\}} = 1516.\overline{6}, \qquad \mu_{\{A,B,D\}} = 983.\overline{3}$$

$$\mu_{\{A,C,D\}} = 1033.\overline{3}, \qquad \mu_{\{B,C,D\}} = 1266.\overline{6}$$

$$\mu_{\{A,B\}} = 1350, \qquad \mu_{\{A,C\}} = 1425, \qquad \mu_{\{A,D\}} = 625$$

$$\mu_{\{B,C\}} = 1775, \qquad \mu_{\{B,D\}} = 975, \qquad \mu_{\{C,D\}} = 1050$$

$$\mu_A = 1000, \qquad \mu_B = 1700, \qquad \mu_C = 1850, \qquad \mu_D = 250$$

and ede incomes:

$$\widehat{y}_{\{A,B,C,D\}} = \left[\frac{1}{8} (20 + 40 + 30 + 50 + 60 + 10 + 10 + 20) \right]^2 = 900$$

$$\widehat{y}_{\{A,B,C\}} = 1225, \qquad \widehat{y}_{\{A,B,D\}} = 802.\overline{7}$$

$$\widehat{y}_{\{A,C,D\}} = 711.\overline{1}, \qquad \widehat{y}_{\{B,C,D\}} = 900$$

$$\widehat{y}_{\{A,B\}} = 1225, \qquad \widehat{y}_{\{A,C\}} = 1056.25, \qquad \widehat{y}_{\{A,D\}} = 506.25$$

$$\widehat{y}_{\{B,C\}} = 1406.25, \qquad \widehat{y}_{\{B,D\}} = 756.25, \qquad \widehat{y}_{\{C,D\}} = 625$$

$$\widehat{y}_A = 900, \qquad \widehat{y}_B = 1600, \qquad \widehat{y}_C = 1225, \qquad \widehat{y}_D = 225$$

The values of the welfare loss are:

$$\mathcal{L}_A(S) = |S| - \sum_{j=1}^{|S|} \frac{\widehat{y}_{\{j\}}}{\mu_j} - 1 + \frac{\widehat{y}_S}{\mu_S}$$

such that

$$\mathcal{L}_A(\{A,B,C,D\}) = 4 - \frac{900}{1000} - \frac{1600}{1700} - \frac{1225}{1850} - \frac{225}{250} - 1 + \frac{900}{1200} = 0.3468$$

$$\mathcal{L}_A(\{A,B,C\}) = 0.3045, \qquad \mathcal{L}_A(\{A,B,D\}) = 0.0752$$

$$\mathcal{L}_A(\{A,C,D\}) = 0.226, \qquad \mathcal{L}_A(\{B,C,D\}) = 0.2073$$
$$\mathcal{L}_A(\{A,B\}) = 0.0663, \qquad \mathcal{L}_A(\{A,C\}) = 0.1791, \qquad \mathcal{L}_A(\{A,D\}) = 0.01$$
$$\mathcal{L}_A(\{B,C\}) = 0.189, \qquad \mathcal{L}_A(\{B,D\}) = -0.065, \qquad \mathcal{L}_A(\{C,D\}) = 0.0331$$
$$\mathcal{L}_A(\{A\}) = 0, \qquad \mathcal{L}_A(\{B\}) = 0, \qquad \mathcal{L}_A(\{C\}) = 0, \qquad \mathcal{L}_A(\{D\}) = 0$$

Some negative contributions emerge:

$$\mathcal{L}_A(\{B,D\}) - \mathcal{L}_A(\{B\}) = -0.065 < 0$$
$$\mathcal{L}_A(\{B,D\}) - \mathcal{L}_A(\{D\}) = -0.065 < 0$$

meaning that negativity occurs when $S = \{B\}$ *and* $j = \{D\}$*, and the verified conditions are:*

$$\begin{cases} \widehat{y}_D < \widehat{y}_{\{B,D\}} < \widehat{y}_B \\ \\ \mu_D < \mu_{\{B,D\}} < \mu_B \end{cases}$$

i.e. case 4 *in the previous list. If we swapped* S *with* j*, we would recover case 3 with* $|S| = 1$.

4.1 An inequality-reducing policy: a preference ordering among factors

The possible occurrence of a negative contribution of some sources is a relevant feature. We can observe the impact of each factor to the welfare of the society. In accordance with the preference ordering on $\mathcal{P}$, the policy maker is able to choose a policy aimed at reducing the level of inequality.

Definition 10. *If there exists a coalition* $S \in 2^{\mathcal{P}}$*,* $|S| \geq 2$*, such that* $\mathcal{L}_A(T) \leq 0$ *for all* $T \subseteq S$*, we denote with* $\mathcal{P}^-$ *the* $\mathcal{L}_A^-$***-subset****, i.e. the subset of* $\mathcal{P}$ *with factors belonging to* S*. If no such coalition* S *exists, then* $\mathcal{P}^- = \emptyset$*. We denote with* $\mathcal{P}^+ = \mathcal{P} \setminus \mathcal{P}^-$ *the* $\mathcal{L}_A^+$***-subset****.*

According to Definition 10, in the above Example, the subsets are respectively $\mathcal{P}^- = \{B,D\}$ and $\mathcal{P}^+ = \{A,C\}$. If $\mathcal{L}_A$ maps each coalition of $2^{\mathcal{P}}$ to a positive value, then the $\mathcal{L}_A^+$-subset is $\mathcal{P}$ and the $\mathcal{L}_A^-$-subset is empty.

We determine an *impact* ranking of sources in the $\mathcal{L}_A^-$-subset. We evaluate the coalitions of factors of $\mathcal{L}_A^+$-subset and the ones of $\mathcal{L}_A^-$-subset, i.e., the values of $\mathcal{L}_A(S \cup \mathcal{F}_j)$, where $S \subseteq \mathcal{P}^+$ and $\mathcal{F}_j \in \mathcal{P}^-$. The $\mathcal{L}_A^-$-subset is either empty or contains at least two factors. We claim a preference ordering on $\mathcal{P}$ in accordance with their contribution to welfare loss.

Definition 11. $\succ_{\mathcal{L}_A}$ *is a social preference ordering on* $\mathcal{P}$ *such that:*

1. $\mathcal{F}_g \succ_{\mathcal{L}_A} \mathcal{F}_j$ *if* $\mathcal{F}_g \in \mathcal{P}^-$ *and* $\mathcal{F}_j \in \mathcal{P}^+$*;*
2. $\mathcal{F}_g \succ_{\mathcal{L}_A} \mathcal{F}_j$ *if* $\mathcal{F}_g, \mathcal{F}_j \in \mathcal{P}^-$ *and if, for all* $S \subseteq \mathcal{P}^+$*,* $\mathcal{L}_A(S \cup \mathcal{F}_g) < \mathcal{L}_A(S \cup \mathcal{F}_j)$*;*
3. $\mathcal{F}_g \succ_{\mathcal{L}_A} \mathcal{F}_j$ *if* $\mathcal{F}_g, \mathcal{F}_j \in \mathcal{P}^+$ *and there exists a coalition* S*,* $|S| > 1$*, such that* $\mathcal{F}_g \in S$ *and* $\mathcal{F}_j \notin S$*, with* $\mathcal{L}_A(S) < \mathcal{L}_A(T)$ *for all* $T \ni \mathcal{F}_j$*,* $|T| > 1$*.*

The following properties can be simply derived:

- If $\mathcal{L}_A(\cdot)$ is a positive game, $\mathcal{P}^- = \emptyset$ and $\mathcal{P}^+ = \mathcal{P}$. If $\mathcal{P}^-$ is not empty, then it necessarily contains at least 2 factors.
- If S is the maximal coalition in $\mathcal{P}^-$, $\mathcal{L}_A(S \cup \mathcal{F}_j) - \mathcal{L}_A(S)$ is strictly positive, i.e., the contribution of each $\mathcal{F}_j \in \mathcal{P}^+$ on the $\mathcal{L}_A^-$-subset triggers a welfare loss.
- The preference ordering $\succ_{\mathcal{L}_A}$is partial since two factors $\mathcal{F}_i, \mathcal{F}_j$ could be into $\mathcal{P}^+$ and included in the coalition S such that $\mathcal{L}_A(S) < \mathcal{L}_A(T)$ for all $T \ni \mathcal{F}_j$. In this case, they cannot be compared on the basis of $\succ_{\mathcal{L}_A}$.
- If $\mathcal{F}_g, \mathcal{F}_j \in \mathcal{P}^-$ and for each $S \subseteq \mathcal{P}^+$, $\mathcal{L}_A(S \cup \mathcal{F}_g) = \mathcal{L}_A(S \cup \mathcal{F}_j)$, no preference exists between $\mathcal{F}_g$ and $\mathcal{F}_j$. This happens when the $\mathcal{L}_A^+$-subset is empty, i.e. when $S \cup \mathcal{F}_g = \mathcal{F}_g$ and the same happens for $\mathcal{F}_j$, so that welfare loss functions vanish evaluated at these points.

On the basis of the preference ordering to $\mathcal{P}^-$, a connection between the $\mathcal{L}_A^-$-subset and the characteristics of its factors can be established. The following Proposition allows us to refine our interpretation of $\succ_{\mathcal{L}_A}$.

Proposition 9. *If $\mathcal{F}_g, \mathcal{F}_j \in \mathcal{P}^-$, then $\mathcal{F}_g \succ_{\mathcal{L}_A} \mathcal{F}_j$ if, for all $S \subseteq \mathcal{P}^+$, we have:*

$$\mathcal{C}_A(Y_{\mathcal{F}_j}) - \mathcal{C}_A(Y_{\mathcal{F}_g}) > \mathcal{K}_A(Y_{S\cup\mathcal{F}_j}) - \mathcal{K}_A(Y_{S\cup\mathcal{F}_g}) \tag{17}$$

Proof. If $\mathcal{F}_g, \mathcal{F}_j \in \mathcal{P}^-$, then $\mathcal{F}_g \succ_{\mathcal{L}_A} \mathcal{F}_j$ if the following difference is positive for all S belonging to the $\mathcal{L}_A^+$-subset:

$$\mathcal{L}_A(S \cup \mathcal{F}_j) - \mathcal{L}_A(S \cup \mathcal{F}_g) = \cdots = 1 - \frac{\widehat{y}_j}{\mu_j} - 1 + \frac{\widehat{y}_{S\cup\mathcal{F}_j}}{\mu_{S\cup\mathcal{F}_j}} - 1 + \frac{\widehat{y}_g}{\mu_g} + 1 - \frac{\widehat{y}_{S\cup\mathcal{F}_g}}{\mu_{S\cup\mathcal{F}_g}} =$$

$$= \mathcal{C}_A(Y_{\mathcal{F}_j}) - \mathcal{K}_A(Y_{S\cup\mathcal{F}_j}) - \mathcal{C}_A(Y_{\mathcal{F}_g}) + \mathcal{K}_A(Y_{S\cup\mathcal{F}_g}) > 0$$

which yields the condition (17), after rearranging terms. □

Such inequality enlightens the nature of this preference ordering. (17) holds when the reduction in the cost of inequality (related to the entire inequality) generated by $\mathcal{F}_j$ is higher than the reduction produced by $\mathcal{F}_g$. A sufficient condition can be provided such that the ordering $\succ_{\mathcal{L}_A}$is connected to the Shapley and Banzhaf values of $\mathcal{L}_A(\cdot)$.

Proposition 10. *If $\mathcal{F}_g, \mathcal{F}_j \in \mathcal{P}^-$ and for all coalitions $S \subseteq \mathcal{P}$, $\mathcal{L}_A(S \cup \mathcal{F}_g) < \mathcal{L}_A(S \cup \mathcal{F}_j)$ holds, then*

$$\begin{cases} \phi_g(\mathcal{L}_A) < \phi_j(\mathcal{L}_A) \\ \beta_g(\mathcal{L}_A) < \beta_j(\mathcal{L}_A) \end{cases} \quad \textit{and} \quad \mathcal{F}_g \succ_{\mathcal{L}_A} \mathcal{F}_j \tag{18}$$

Proof. If for all coalitions $S \subseteq \mathcal{P}$, $\mathcal{L}_A(S \cup \mathcal{F}_g) < \mathcal{L}_A(S \cup \mathcal{F}_j)$ holds, then

$$\mathcal{L}_A(S \cup \mathcal{F}_g) - \mathcal{L}_A(S) + \mathcal{L}_A(S) - \mathcal{L}_A(S \cup \mathcal{F}_j) < 0 \iff$$

$$\iff \mathcal{L}_A(S \cup \mathcal{F}_g) - \mathcal{L}_A(S) < \mathcal{L}_A(S \cup \mathcal{F}_j) - \mathcal{L}_A(S)$$

consequently computing the Shapley and the Banzhaf values yields:

$$\begin{cases} \phi_g(\mathcal{L}_A) < \phi_j(\mathcal{L}_A) \\ \beta_g(\mathcal{L}_A) < \beta_j(\mathcal{L}_A) \end{cases}$$

And since the assumption holds for all $S \in \mathcal{P}^+$, then by definition of $\succ_{\mathcal{L}_A}$ we have that $\mathcal{F}_g \succ_{\mathcal{L}_A} \mathcal{F}_j$. □

In compliance with Definition 11, in the above Example, $D \succ_{\mathcal{L}_A} B$ since $\mathcal{L}_A(\{A, C, D\}) < \mathcal{L}_A(\{A, B, C\})$, $\mathcal{L}_A(\{A, D\}) < \mathcal{L}_A(\{A, B\})$ and $\mathcal{L}_A(\{C, D\}) < \mathcal{L}_A(\{B, C\})$. Moreover, $\mathcal{L}_A(\{A, D\})$ is the minimum positive value attained by $\mathcal{L}_A$, $A \succ_{\mathcal{L}_A} C$. Consequently, the preference ordering induced by $\mathcal{L}_A$ on the income factors is:

$$D \succ_{\mathcal{L}_A} B \succ_{\mathcal{L}_A} A \succ_{\mathcal{L}_A} C$$

Note that it is neither coherent with the order on arithmetic means, where $\mu_C > \mu_B > \mu_A > \mu_D$, nor coherent with the order on *ede* incomes, where $\widehat{y}_B > \widehat{y}_C > \widehat{y}_A > \widehat{y}_D$.

Instead, the Shapley value of $\mathcal{L}_A$ yields:

$$\Phi(\mathcal{L}_A) = (\phi_A, \phi_B, \phi_C, \phi_D) = (0.09341,\ 0.07618,\ 0.16187,\ 0.01492)$$

which points out that by arranging the Shapley value coordinates in the increasing order, the result is coherent with $\succ_{\mathcal{L}_A}$and is valid for each source, not only the ones contained in the $\mathcal{L}_A^-$-subset as shown by Proposition 10.

The same effect takes place when computing the Banzhaf value of $\mathcal{L}_A$, i.e.:

$$\beta(\mathcal{L}_A) = (\beta_A, \beta_B, \beta_C, \beta_D) = (0.03477,\ 0.02876,\ 0.05756,\ 0.0039)$$

4.2 Type and source preferences

Finally we propose a further criterion on inequality-reducing policy on the basis of the type preferences suggested in Subsection 3.2. By Definition 11, $\succ_{\mathcal{L}_A}$ is not a complete ordering. We therefore propose an ordering $\succ^V_{\mathcal{L}_A}$ by taking into account the matrices U and V which collects the preferences of types. Suppose that types' single-peaked preferences are revealed by U and V. Intuitively, a policy maker which intends to be compliant with such preferences should choose the suitable factors according to the revealed information, i.e., the ones attaining higher values in V. On the other hand, when each pair of factors are comparable, the ordering induced by $\succ^V_{\mathcal{L}_A}$ must coincide with the order induced by $\succ_{\mathcal{L}_A}$.

Definition 12. $\succ^V_{\mathcal{L}_A}$ *is a preference ordering on $\mathcal{P}$ such that, for any factors $\mathcal{F}_g$,$\mathcal{F}_j$:*

1. *if $\mathcal{F}_g \succ_{\mathcal{L}_A} \mathcal{F}_j$, then $\mathcal{F}_g \succ^V_{\mathcal{L}_A} \mathcal{F}_j$;*
2. *if neither $\mathcal{F}_g \succ_{\mathcal{L}_A} \mathcal{F}_j$ nor $\mathcal{F}_j \succ_{\mathcal{L}_A} \mathcal{F}_g$ hold, then $\mathcal{F}_g \succ^V_{\mathcal{L}_A} \mathcal{F}_j$ if and only if $v_{g*} > v_{j*}$.*

As suggested by Definition 12, this order is not always complete whether neither $\mathcal{F}_g \succ_{\mathcal{L}_A} \mathcal{F}_j$ nor $\mathcal{F}_j \succ_{\mathcal{L}_A} \mathcal{F}_g$ hold and $v_{g*} = v_{j*}$, factors $\mathcal{F}_g$ and $\mathcal{F}_j$ are equivalent (or indifferent) with respect to order $\succ^V_{\mathcal{L}_A}$.

5 Concluding remarks

The novel results in this paper point out that a new approach can be adopted and generates a more complete understanding of the distribution of income sources. This investigation makes a well-known preference ranking of types compatible with the egalitarian redistribution of resources. Our aim is not only to take into account the marginal contributions in the typical Banzhaf or Shapley fashions, but we determine an allocation rule exploiting the information that society guarantees about income sources and types' preferences-based contributions.

In particular our idea is to propose a rule able to safeguard some egalitarian properties, to reduce the cost of inequality among sources and to satisfy efficiency in the income distributions without violating the types' preferences. A solution concept which we called *weakly allocation rule* is therefore developed and axiomatized by relying on the well-known properties of the *traditional* uniform rule (feasibility, efficiency and anonymity) plus new axioms on preferences monotonicity, Lorenz dominance and equal treatment for not-preferred contributions. Sufficient conditions are established under which this uniform rule may coincide with the preference-based nucleolus of the game.

This analysis has some policy consequences. It allows for an evaluation of each income source to discover its impact on income inequality and welfare of the society. We arrange a preference-based scheme on the welfare loss game in order to rank income sources. In other words, a refinement of the policy maker's preference according to types' choices would increase equality and shrink meanwhile their possible disapproval about potential policy reallocations in the society. We provide some hints about this social ordering since potential extensions are left for future research.

References

[1] Anno, H., Sasaki H. *Second-best efficiency of allocation rules: strategy-proofness and single-peaked preferences with multiple commodities*, Economic Theory (2013), 54: 693-716

[2] Atkinson, A.B. *On the measurement of inequality*, Journal of Economic Theory (1970), 2: 244-263.

[3] Banzhaf, J.F. *Weighted Voting Doesn't Work: A Mathematical Analysis*, Rutgers Law Review (1965), 19: 317-343.

[4] Blackorby, C., Donaldson, D. and Auersperg, M. *A new procedure for the measurement of inequality within and among population subgroups*, (1981), Canadian Journal of Economics, 14: 665-685.

[5] Chantreuil, F., Trannoy A. *Inequality decomposition values: the trade-off between marginality and efficiency*, Journal of Economic Inequality (2013), 11 (1): 83-98.

[6] Charpentier, A., Mussard, S. *Income inequality games*, Journal of Economic Inequality (2011), 9: 529-554.

[7] Ching, S. *A simple characterization of the uniform rule*, Economics Letters (1992), 40: 57-60.

[8] Ching, S. *An alternative characterization of the uniform rule*, Social Choice and Welfare (1994), 11: 131-136.

[9] Devicienti F. *Shapley value decompositions of changes in wage distributions: a note*, Journal of Economic Inequality (2010), 8: 35-45.

[10] Drechsel, J. *Cooperative Lot Sizing Games in Supply Chains*, Lecture Notes in Economics and Mathematical Systems, 644, Springer Berlin Heidelberg (2010).

[11] Drechsel, J., Kimms, A., *The subcoalition-perfect core of cooperative games*, Annals of Operations Research (2010), 181: 591-601.

[12] Dutta, B., Ray D. *A Concept of Egalitarianism Under Participation Contraints*, Econometrica (1989), 57: 615-635.

[13] Dutta, B., Ray D. *Constrained egalitarian allocations*, Games and economic behaviour (1994), 3: 403-422.

[14] Ebert U. *Measurement of inequality: an attempt at unification and generalization*, Social Choice and Welfare (1988), 5: 147-169.

[15] Faigle, U., Kern, W., Kuipers, J.: *On the computation of the nucleolus of a cooperative game*, Int. J. of Game Theory (2001), 30: 79-98.

[16] Fei J., Ranis G. and Kuo W.Y. *Growth and the Family distribution of Income by factor components*. Quarterly Journal of Economics (1980), 92 (1): 451-473

[17] Fleurbaey M. *Assessing risky social situations*, Journal of Political Economy (2010), 118: 649-680.

[18] Fleurbaey M., Zuber S. *Inequality aversion and separability in risk evaluation*, Economic Theory (2013), 54: 675-692.

[19] Kopelowitz, A. *Computation of the kernels of simple games and the nucleolus of n-person games*, (No. RM-31), Hebrew Univ. Jerusalem (Israel) Dept. of Mathematics, 1967.

[20] Israeli O. *A Shapley-based decomposition of the R-Square of a linear regression*, Journal of Economic Inequality (2007), 5 (2): 199-212.

[21] Lerman, R. and Yitzhaki, S. *Income inequality by income sources: a new approach and application to the United States*, Review of Economics and Statistics (1985), LXVII , 1: 151-156.

[22] Maschler, M., Peleg, B., Shapley, L. S. *Geometric properties of the kernel, nucleolus, and related solution concepts*, Mathematics of Operations Research (1979), 4(4): 303-338.

[23] Morduch, J., Sicular, T. *Rethinking inequality decomposition with evidence from rural China*, Economic Journal (2002), 112 (476): 93-106.

[24] Otten, G.J., Peters H., Volij O. *Two Characterizations of the Uniform Rule for Division Problems with Single-Peaked Preferences*, Economic Theory (1996), 7: 291-306.

[25] Owen, G. *Game Theory*, III edition, New York: Academic Press (1995).

[26] Rao, V. M. *Two decompositions of concentration ratio*, Journal of the Royal Statistical Society, Series A (1969), 132: 418-425.

[27] Peyton-Young, H. *Cost Allocation: Methods, Principles, Applications*, New York:. Elsevier Sci (1985).

[28] Pignataro, G. *Measuring equality of opportunity by Shapley value*, Economics Bulletin (2010), 30(1): 786-798.

[29] Sastre, M., Trannoy A. *Shapley inequality decomposition by factor components: some methodological issues*, Journal of Economics (2002), 9: 51-90.

[30] Schmeidler, D.: *The nucleolus of a characteristic function game*, SIAM Journal of Applied Mathematics (1969), 17(6): 1163-1170.

[31] Shapley, L.S.: *A Value for n-person Games*, in Contributions to the Theory of Games, volume II (H.W. Kuhn and A.W. Tucker eds.), Annals of Mathematical Studies, Princeton University Press (1953), 28: 307-317.

[32] Shorrocks, A.F. *Inequality decomposition by factor components*, Econometrica (1984), 50: 193-211.

[33] Shorrocks, A.F. *Inequality decomposition by population subgroups*, Econometrica (1984), 52: 1369-1385.

[34] Shorrocks A.F. *Decomposition procedures for distributional analysis: a unified framework based on the Shapley value,* Journal of Economic Inequality (2013), 11 (1): 99-126.

[35] Sprumont, Y. *The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Allocation Rule*, Econometrica (1991), 59 (2): 509-519.

[36] Thomson, W. *Consistent solutions to the problem of fair division when preferences are single-peaked*, Journal of Economic Theory (1994), 63: 219-245.

[37] Thomson, W. *The replacement principle in private good economies with single-peaked preferences*, Journal of Economic Theory (1997), 76: 145-168.