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Abstract

This paper seeks to extend the unidimensional notion of Lorenz dominance to the multidimensional context. It formulates a definition of a multidimensional Lorenz dominance relation (MLDR) on the set of alternative distributions of well-being in an economy by incorporating generalizations of the well-known Pigou-Dalton condition of unidimensional theory. Besides the definitional requirements, an MLDR is also desired to satisfy two other conditions which seem to be intuitively reasonable. The paper notes that the existing literature does not seem to contain an example of an MLDR with these characteristics and seeks to close this gap.

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1. Introduction

It is by now generally recognized that well-being of an individual depends not only on his or her income but also on other attributes (such as, education, health etc.). Therefore, the methods of measuring inequality in the distribution of well-being among the individuals in an economy need to be extended from the unidimensional to the multidimensional context.

As in the case of a single dimension, the method of comparing between the levels of inequality of alternative multidimensional distributions may take the approach of constructing a complete ordering over the set of distributions by proposing a scalar-valued *inequality index*.

However, since different inequality indices may lead to different complete orderings of the distributions, attention may also be given to the task of constructing orderings which may be partial but which would be more readily acceptable in some intuitive sense. In single-attribute theory the most widely used partial ordering of this type is the Lorenz partial ordering: a distribution **x** Lorenz dominates another distribution **y** if the Lorenz curve for **x** does not lie below that for **y** at any point and lies above it at, at least, one point. **x** is then interpreted to be a more desirable distribution than **y**. In the multidimensional case a main task in this approach is to extend the notion of Lorenz dominance to the multi-attribute context.

Although the economic theory of multidimensional inequality measurement as a whole is a relatively new field of research, within this field the first of the two approaches mentioned above has by now led to a sizable literature containing important contributions. For reviews see, for instance, Savaglio (2006) and Weymark (2006).

The second approach, however, seems be a relatively neglected area. In this paper we shall be concerned with multidimensional Lorenz dominance.

In this context it is convenient to describe the allocations of the different attributes to the different individuals by a matrix. We shall suppose that in a *distribution matrix* each column refers to an attribute and each row to an individual. The entries represent the allocations. If X and Y are two distribution matrices, the question under what conditions X is to be considered to Lorenz dominate Y does not seem to have an obvious and unique answer. Various suggestions have been made, many of them belonging to the mathematical literature. (See, for instance, Arnold (2005) and Koshevoy and Mosler (2007). For reviews see Savaglio (2006) and Trannoy (2006)).

This paper seeks to develop an economic approach to the problem. Towards that end it formulates a definition of a multidimensional Lorenz dominance relation (MLDR) on the set of distribution matrices. The definition uses, apart from other standard requirements, two different (and independent) generalizations of the Pigou-Dalton transfer condition of unidiensional theory viz. the Uniform Majorization (**UM**) condition due to Kolm (1977) and the Pigou-Dalton Bundle Principle (**PDBP**) introduced in Fleurbaey and Trannoy (2003).

Both **UM** and **PDBP** were origianally introduced in the literature as conditions on equity-sensitive social evaluation functions. Later they have also been stated as conditions on multidimensional inequality indices. The inequality index version of each of these conditions takes the following form: Letting I(X) denote the value of an inequality index I for any distribution matrix X, each of the conditions requires that if the distribution matrix Y is obtained from X by subjecting it to a specified type of transformation, then I(X) < I(Y). However, since the conditions require *any* inequality index to behave in the specified way, intuitively it seems reasonable to adapt these to the present context by restating them to require that X Lorenz dominates Y if Y is obtained from Y in the specified manner.

We shall also desire an MLDR to satisfy two additional conditions which seem to be intuitively reasonable. One of these is the condition of Correlation Increasing Majorization (**CIM**) introduced in the economic literature by Tsui (1999). We shall, again adapt it from the inequality index context for our purposes. The essential idea behind this condition is that greater correlation among the columns of the distribution matrix increases multidimensional inequality, however such inequality may be measured.

The other condition, to be called Conditional Equalizing Majorization (**CEM**), relates to the question whether equalizing the distribution of an attribute would lead to an unambiguous improvement in the state of distribution in the society. Given a distribution matrix X and an attribute *j*, let Y be the matrix obtained by replacing each entry in the *j*-th column of X by the arithmetic mean of the column. **CEM** requires that Y dominates X if the distribution of the *j*-th attribute in X is "sufficiently" more unequal (in a sense to be made precise) than that of each of the other attributes.

The existing literature does not seem to contain an example of an MLDR satisfying all of the conditions considered here. We seek to close this gap by suggesting such an MLDR.

Section 2 below introduces the notations, develops a definition of an MLDR and introduces the conditions of **CIM** and **CEM**. Section 3 reviews the literature to search in vain for an MLDR satisfying these conditions. Section 4 proposes a specific binary relation on the set of distribution matrices and proves that it is an MLDR possessing these characteristics. Section 5 concludes the paper.

2. Notations, Definitions and Axioms

Consider an economy with n individuals whose levels of well-being are determined by the amounts of m attributes that are allocated to them. Allocations are assumed to be non-negative. $M = \{1, 2,\ldots,m\}$ and $N = \{1, 2,\ldots,n\}$ will denote the set of attributes and the set of individuals respectively. Since we shall be concerned with inequality among the standards of living of the individuals, we assume that $n \geq 2$. However, we assume that while m is exogenously fixed, n is allowed to be any positive integer. This allows inequality comparisons to be made across populations of different sizes.

By a *distribution matrix* X we shall mean an $n\times m$ *non-negative* matrix whose (*p*-th row, *j*-th column) term, x_p^j, is the amount of attribute *j* allocated to individual *p* for all *j* in *M* and for all *p* in *N*. Thus, a distribution matrix describes a pattern of allocations of the attributes in the economy. For a distribution matrix X, $\mathbf{x}_p$ will denote its *p*-th row and $\mathbf{x}^j$ its *j*-th column.

It is assumed that in any distribution matrix the sum of each column is positive i.e. for every attribute there is a positive total amount to be distributed among the individuals.

Thus the domain of matrices under consideration is: $\mathbf{X} = \{X \, \varepsilon \, \Re^{nm}_{+} : \mu(\mathbf{x}^j) > 0, j \; 1.2,\ldots,m,$ and $n \geq 2\}$for some given and fixed positive integer *m* where, for any vector **y**, μ(**y**) denotes its arithmetic mean.

. We define a *weak* inequality dominance relation, *D*, on **X**. For all X and Y in **X**, if X *D* Y, this will be interpreted to mean that relative inequality in the distribution of overall well-being in the pattern of allocations described by X is not more than that in the pattern described by Y, whatever may be the specific method of measuring the degree of overall inequality. D_P and D_I will denote the asymmetric and the symmetric components of *D* respectively, i.e., for all X and Y in **X**, X D_P Y if and only if [X *D* Y and ¬(Y *D* X)] ; and X D_I Y if and only if [X *D* Y and Y *D* X].

We shall impose a number of conditions on *D*. We start with some basic conditions which are *not* related to equity considerations.

Ratio-Scale Invariance (RSI): For all $n \times m$ matrices X in **X** and for all diagonal matrices Λ with positive entries along the main diagonal, X D_I (XΛ).

Quasi-ordering (QORD): *D* is a quasi-ordering.

Anonymity (ANON): If X and Y in **X** are such that Y is obtained by a permutation of the rows of X, then X D_I Y.

Population Replication Invariance (PRI): For all X and Y in **X** such that Y is obtained by a *k*-fold replication of the population in X for some positive integer *k* i.e., for all *p* in *N*,

$$\mathbf{x}_p = \mathbf{y}_p = \mathbf{y}_{n+p} = \ldots\ldots = \mathbf{y}_{n(k-1)+p},$$

X D_I Y.

One of the first issues that arise in any multidimensional analysis is that of *commensurability* of the attributes. Commensurability requires that the attributes are measured in the same or, at least, similar (for instance, monetary) units. Since this may not be true of the original data, we make the entries in the distribution matrices independent of the scales of measurement of the different attributes. Imposing the condition of **RSI** is one way of doing this**.** It requires that if each column of a distribution matrix is multiplied by a positive constant (possibly different for the different columns), the matrix obtained is 'equivalent' to the original matrix in terms of the weak equality dominance relation *D*. The requirement also tallies with the fact that in this paper we shall be concerned with *relative* inequality.

QORD states that *D* is a reflexive and transitive relation which is not necessarily complete. **ANON** requires that the labelling of the individuals in the economy should be inconsequential. **PRI** implies that in any distribution matrix it is the proportion of the population (rather than the absolute number of individuals) getting a particular allocation of an attribute that is important.

We now turn to equity considerations. In the case where there is a single attribute ($m = 1$), the standard notion of inequality dominance is that of Lorenz dominance. For any non-negative distribution vector **x** specifying the allocations of the attribute to the n individuals, let $\mathbf{x}^{\sim}$ denote the rearrangements of **x** in non-decreasing order and let $\mu(\mathbf{x})$ denote the arithmetic mean of **x.** As per the standard Gastwirth (1971) definition of a *Lorenz curve* applied to the case of a discrete distribution, the Lorenz curve of **x** is the curve in the unit square obtained by joining

the $(n + 1)$ points $(0, 0)$ and $(k/n, (1/n) \sum_{i=1}^{k} x_i^{\sim} /\mu(\mathbf{x}))$, $k = 1, 2, \ldots, n$ by line segments, $x_i^{\sim}$ being the i-th component of $x^{\sim}$.

For the distribution vector **x**, the mapping from [0, 1] into [0, 1] described by the Lorenz curve of **x** is denoted by L_x. For all distribution vectors, **x** and **y**, **x** *Lorenz dominates* **y** if and only if $L_x(p) \geq L_y(p)$ for all p in [0, 1]. It *strictly Lorenz dominates* **y** if, in addition, $L_x(p) > L_y(p)$ for some p in [0, 1].

We shall denote the unidimensional *Lorenz dominance* relation on the set of all non-negative distribution vectors by L: for all distribution vectors **x** and **y**, **x** L **y** if and only if **x** Lorenz dominates **y**. Clearly, L is a quasi-ordering. P will denote the *strict Lorenz dominance* relation: **x** P **y** if and only if **x** strictly Lorenz dominates **y.** P coincides with the asymmetric component of L. The symmetric component of L will be denoted by I. For all distribution vectors **x** and **y**, **x** I **y** if and only if the Lorenz curve of **x** coincides with that of **y** (i.e. **x** = **y** or **x** is a permutation of **y**).

In unidimensional theory Lorenz dominance is closely related to the notion of Pigou-Dalton (PD) transfers. If the attribute in question is income, a PD transfer is an income transfer from a richer to a poorer person by an amount less than their initial income difference. The following three statements are equivalent (Hardy, Littlewood and Polya (1952) and Marshall and Olkin (1979, Ch.1)): (1) **x** Lorenz dominates **y**; (2) **x** *Pigou-Dalton majorizes* **y** i.e. **x** is obtained from **y** by a *finite* sequence of PD transfers; and (3) **x** = B**y** for some bistochastic matrix B. (A bistochasic matrix is a non-negative matrix in which each row as well as each column sums to 1.)

The literature on multidimensional inequality contains generalizations of the concept of Pigou-Dalton majorization. One of the most widely used among such generalization is the concept of Uniform Majorization UM). (See Kolm (1977).) For all $n \times m$ matrices X and Y in **X**, Y is said to uniformly majorize X if Y ≠ X and Y = BX for some bistochastic matrix B which is not a permutation matrix. Since Y = BX implies, $y^i = Bx^i$ for all i in M, y^i Pigou-Dalton majorizes x^i for each i ; and since the same matrix B is used to majorize all the columns of X, the majorization is said to be *uniform* across the attributes. A variant of this type of majorization is w-majorization formulated in Savaglio (2006, 2007, 2010) where B is required to be a row-stochastic (but not necessarily a bistochastic) matrix.

Kolm (1977) used UM to formulate an axiom regarding an equity-sensitive social evaluation function. According to this axiom, for all X and Y, if Y uniformly majorizes X, then the society considers Y to be superior to X from the distributional point of view. In the present framework we do not use a social evaluation function. However, axioms similar to the ones mentioned above can be formulated in terms of multidimensional indices of equality. Take, for instance, the concept of UM. Let f be a mapping of **X** into the real line. If f is to be an index of multidimensional equality, it is to satisfy the following axiom (called the axiom of UM): for all X and Y in **X**, if Y is a UM of X, then f (Y) > f (X). [In the literature on unidimensional distribution it is customary to define indices of *inequality*. We shall, however, couch the discussion in terms of equality indices. Dominance in terms of equality seems to be in line with the notion of Lorenz dominance.]

We wish to formulate a condition under which a distribution matrix can reasonably be said to *dominate* another. However, in analogy with the unidimensional case, the statement that Y dominates X may be interpreted to mean that, according to *any* reasonable measure of multidimensional inequality, Y would have a lower degree of inequality than X. Hence, the generalizations of the Pigou-Dalton majorization can be used to formulate suitable conditions of inequality dominance in the multidimensional context. This type of adaptation of the axiom of UM leads to the following condition on *D*.

Uniform Majorization (UM):: For all X and Y in **X** such that Y is a UM of X, Y D_P X .

The recent literature on inequality has, however, pointed out a number of inadequacies of the axiom of UM. First, all attributes may not be transferable in principle. (What, for instance, do we mean by transferring educational attainments or health status?), Secondly, even when all of these are transferable, there seem to be cases in which a transfer is non-uniform across the attributes and yet there seem to be reasonable grounds for hypothesizing that it leads to an unambiguously superior state of distribution.. UM does not cover these cases. For a more detailed discussion on these two issues see, for instance, Lasso de la Vega, Urrutia and Amaia de Sarachu (2010).

In this paper in order to take these considerations into account we shall use the Pigou-Dalton Bundle Principle (PDBP) introduced by Fleurbaey and Trannoy [2003 in the context of the normative theory of inequality. (See Lasso de la Vega et. al. [2010] for an innovative use of PDBP for the purpose of deriving a multidimensional inequality index.)

Consider the case where the amounts of the attributes that are transferred are allowed to differ between attributes and are not restricted to be non-zero for all attributes. It is, however, assumed (i) that transfers from an individual q to an individual p are allowed only if q is unambiguously richer than p (i.e. q has more of every attribute than p) and (ii) that transfers preserve the relative ranks, in each dimension, of the two individuals whose allocations are altered.

Definition 2.1: For all X and Y in **X**, Y is said to be derived from X by a **Pigou-Dalton Bundle Transfer (PDBT)** if there exist *p* and *q* in *N* such that

(i) $\mathbf{x}_q > \mathbf{x}_p$;

(ii) $\mathbf{y}_q = \mathbf{x}_q - \mathbf{d}$ and $\mathbf{y}_p = \mathbf{x}_p + \mathbf{d}$ for some $\mathbf{d}$ in $\Re^m_+$ such that $\mathbf{d} \neq 0$.

(iii) $\mathbf{y}_r = \mathbf{x}_r$ for all *r* in $N - \{p, q\}$;

(iv) $\mathbf{y}_q \geq \mathbf{y}_p$.

Part (i) of Definition 2.1 states that individual *q* is unambiguously richer than individual *p* in the initial allocation matrix X. Part (ii) requires that non-negative amounts of the different attributes are transferred from individual *q* to individual *p*. The amounts or the proportions of the transfers need not be the same for all attributes. Neither is it required that some amounts of *all* attributes must be transferred i.e. it is recognized that some attributes may, by their nature, be non-transferable. It is required, however, that the transfer is non-trivial i.e. some amount of at least one attribute is transferred. Part (iii) states that all individuals other than *p* and *q* are unaffected by the transfer. Part (iv) states that after the transfer *q* remains unambiguously at least as well off as *p*.

As an illustration consider the case in which $n = 3$, $m = 2$, $X = \begin{pmatrix} 10 & 9 \\ 2 & 8 \\ 7 & 6 \end{pmatrix}$ and $Y = \begin{pmatrix} 8 & 9 \\ 4 & 8 \\ 7 & 6 \end{pmatrix}$.
In X individual 1 is unambiguously richer than individual 2. Y is obtained from X by transferring 2 units of the first attribute from individual 1 to individual 2. This is a PDBT since, as is easily checked, all parts of Definition 2.1 are satisfied.

We impose the following condition on the dominance relation *D*.

Pigou-Dalton Bundle Principle (PDBP): For all X and Y in **X** such that Y is obtained from X by a finite sequence of PDBT's, Y D_P X.

We are now ready to state the definition of a multidimensional inequality dominance relation.

Definition 2.2: A multidimensional inequality dominance relation (**MIDR**), *D*, is a binary relation on **X** satisfying **RSI**, **QORD**, **ANON**, **PRI**, **UM and PDBP**.

Since we are interested in obtaining a generalization of the unidimensional Lorenz dominance relation, *L*, it is natural to require that the dominance relation reduces to *L* if there is just one attribute.

Definition 2.3: A multidimensional Lorenz dominance relation (**MLDR**), L^M, is an MIDR on **X** such that $L^M = L$ if $m = 1$.

The antisymmetric and symmetric components of an MLDR, L^M, will be denoted by P^M and I^M respectively.

All inequality dominance relations are, by definition, concerned with equity considerations. Some basic aspects of such considerations are captured by generalizations of the Pigou-Dalton transfer principle such as **PDBP** and **UPDM.** In multi-attribute theory, however, there are still other aspects of the matter. We now state two conditions that deal with some of these aspects. The first of these focuses on the pattern of inter-relation among the attributes and its relation to multidimensional inequality.

For all X in **X** and for all p,q in N, let $x_p \wedge x_q$ denote the vector $\{\min(x_p^1, x_q^1), \min(x_p^2, x_q^2),...,\min(x_p^m, x_q^m)\}$ and $x_p \vee x_q$ the vector $\{\max(x_p^1, x_q^1), \max(x_p^2, x_q^2), ...,\max(x_p^m, x_q^m)\}$.

Definition 2.4: For all X and Y in **X** such that X is not equal to Y or a row permutation of Y, X is said to be obtained from Y by a Correlation Increasing Transfer (CIT) if there exist p and q in N such that
(i) $x_p = y_p \wedge y_q$;

(ii) $x_q = y_p \vee y_q$; and

(iii) $x_r = y_r$ for all r in $N - \{p, q\}$.

We shall desire L^M to satisfy the following condition:

Correlation Increasing Majorisation (CIM): For all X and Y in **X** such that Y is obtained from X by a finite sequence of CIT's, X P^M Y.

The basic idea behind **CIM** is that greater correlation among the different columns of the distribution matrix implies greater inequality, irrespective of how inequality is measured. It was introduced in the economic literature by Tsui (1999) in the context of inequality measurement. In the statistical literature it was proposed by Boland and Proschan (1988). The concept of CIT on which it is based was studied in Atkinson and Bourguignon (1982) and in Epstein and Tanny (1980).

The acceptability of a condition depends on its intuitive plausibility. **CIM** seems to have a strong intuitive appeal. Consider, for instance, the following example. Let $n = 2 = m$.
Let Y = $\begin{pmatrix} 9 & 7 \\ 6 & 3 \end{pmatrix}$ and X = $\begin{pmatrix} 9 & 3 \\ 6 & 7 \end{pmatrix}$. Y is obtained by a switch of the entries in the second column of X. It is easily checked that this is a CIT. If it is now asked whether we should consider X to Lorenz dominate Y (i.e. whether X should be judged to display a lower degree of equality as per any measure of inequality), there seems to be intuitive grounds for an affirmative answer. In X individual 1 has a higher allocation of attribute 1 than individual 2. But this is at least partially compensated for by the fact that w.r.t. attribute 2 it is individual 2 who has a lower allocation. In Y, however, the effect of the lower allocation of attribute 1 to allocation 2 is compounded by the fact that individual 2 faces the same predicament w.r.t. attribute 2 i.e. there is a compounding of inequalities across the attributes.

For the purpose of stating our second additional condition on L^M it will be convenient to use the notion of scaled distribution matrices. For any distribution matrix X, X* denote the matrix obtained by dividing each entry in X by the arithmetic mean of the column containing it. (Similarly, for a distribution vector **x**, **x*** will its scaled version.) By **RSI,** X L^M Y if and only if X* L^M Y*.

Suppose that a distribution matrix X has a column x^j which is strictly Lorenz dominated by all columns. Thus, x^j is unambiguously the most unequally distributed column in X. We ask whether replacing x^j by the column vector $\mu(\mathbf{x}^j)\mathbf{1}_n$ (i.e. making the distribution of the relevant attribute perfectly equal) would lead to an unambiguous improvement in the state of distribution in the economy. Thus, the question is whether the matrix obtained from X by the proposed operation dominates X. If X is *comonotonic*, an affirmative answer would seem to be reasonable. [An $n \times m$ distribution matrix X is comonotonic if either $x_1^j \geq x_2^j \geq \ldots \geq x_n^j$ for all j in M or the same is true with all the weak inequalities reversed.] Indeed, in this special case, equalization of *any* column of X (rather than that of only the most unequally distributed column) which is not equally distributed to start with can reasonably be considered to be an unambiguous improvement.

In the general case, however, the answer is less obvious. Consider first the case where n = 2 = m. Let X and Y be such that X* = $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ and Y* = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Since Y* represents the situation where every attribute is equally distributed and since this is not true for X*, it seems reasonable to require that Y strictly dominates X. Consider now the non-comonotonic matrix Z for which Z* = $\begin{pmatrix} 1-d & 2 \\ 1+d & 0 \end{pmatrix}$ where d is a real number in (0, 1]. The matrix W* = $\begin{pmatrix} 1-d & 1 \\ 1+d & 1 \end{pmatrix}$ is obtained by equalizing the most unequal column (to wit, the second) in Z*. If d is "small", W* would be "close" to the matrix of perfect equality Y* while Z* would be "close" to X*. For such a value of d, therefore, it may, again, seem reasonable to require that W* dominates Z* i.e. that W dominates Z.. However, if we now go on increasing d, there would come a stage where the requirement would lose intuitive appeal. For instance, in the extreme case where d = 1, Z* and W* become $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ respectively. The requirement that W* dominates Z* can now be questioned. Indeed, if the society considers the two attributes to be equally important i.e. if it attaches equal weights (or "prices") to them, then it is Z* which would seem to represent a situation of perfect over-all equality, balancing the inequalities in the two attributes perfectly by setting them against each other. In any case the requirement that W* dominates Z* is now unreasonable.

In general, therefore, the question seems to be not only whether the column of the matrix that is to be equalized is the most unequally distributed among all the columns but also whether it is "sufficiently" unequal i.e. whether the distribution in each of the other columns is "sufficiently" more equal. One way of formalizing the question in the context of the above example is ask whether d is such that the (scaled) distribution vector $\begin{pmatrix} 1-d \\ 1+d \end{pmatrix}$(= **c**, say) is "closer" to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$(= **a**, say) than to $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$(= **b**, say). Whether or not this is true cannot be checked on the basis of the notion of (undimensional) Lorenz dominance. However, one simple procedure (which can be stated in terms of Lorenz dominance) would be to examine whether **c** Lorenz dominates [(**a** + **b**) / 2]. If this is the case, the stipulation that W* dominates Z* would seem to be acceptable.

More generally, let $\mathbf{X}^1$ denote the set of all distribution matrices (vectors) where $m = 1$. Let the binary relation P^* on $\mathbf{X}^1$ be defined as follows: for all n-vectors **x** and **y** in $\mathbf{X}^1$, **x** P^* **y** if and only if **x*** P [(**y*** + $\mathbf{1}_n$) / 2] where P is the asymmetric component of L.

It may be noted that, for any **x** and **y**, [**x** P^* **y**] implies **y** $\neq \mathbf{1}_n$ and that, for any such **y,** [(**y*** + $\mathbf{1}_n$) / 2] P **y*.** Since P is transitive, it follows that, for all **x** and **y**, [**x** P^* **y**] implies [**x** P **y**]. The converse, however, is not true. Thus, $P^* \subset P$.

For any $n \times m$ matrix X in **X** and for any j in M, let $X^{-j,\mu}$ denote the matrix obtained from X by replacing $\mathbf{x}^j$ by $\mu(\mathbf{x}^j)\mathbf{1}_n$.

The following condition on L^M is proposed.

Conditional Equalizing Majorizaation (CEM): For any $n \times m$ matrix X in **X** if j in M is such that x^i P^* x^j for all i in M such that $i \neq j$, then $X^{-j,\mu}$ P^M X.

CEM states that if a distribution matrix X contains a column x^j which is "sufficiently more unequally distributed " than each of the other columns (in the sense that each of the other columns strictly Lorenz dominates the arithmetic mean of x^{j*} and the equal distribution $\mathbf{1}_n$), then the matrix obtained from X by equalizing x^j strictly dominates X.

In course of our work below we shall see that there exist MLDR's which satisfy **CIM** but violate **CEM**. There also are MLDR's that violate **CIM** but satisfy **CEM**. Thus, **CIM** and **CEM** are independent conditions.

In this paper we look for binary relations on **X** which are MLDR's as per Definition 2.3 and which also satisfy **CIM** and **CEM**.

The reader will notice that while **UM** and **PDBP** have been stated as parts of the definition of an MLDR, **CIM** and **CEM** have only been given the status of additionally desired properties. [This is in deference to the fact that in the inequality index literature there are indices which have come to be accepted as multidimensional inequality indices but which do not necessarily satisfy (the inequality index version of) **CIM**.] The results of this paper can easily be suitably restated if these additional conditions are included in the definitional requirements of an MLDR.

3. "Candidate" MLDR's

The existing literature contains a number of specific suggestions regarding the construction of MLDR's. In this Section we review some of these suggestions and assess their acceptability in terms of the conditions stated in Section 2.

Examples of Suggested MLDR's:

(1) Directional Lorenz Majorization (L^1): L^1 is the binary relation on **X** such that, for all X and Y in **X**, X L^1 Y if and only if X is a *directional Lorenz majorization* of Y i.e. (Xw) L (Yw) for all w in $\Re^m$.

(2) Lorenz Majorization by Non-negative Weights (L^2): L^2 is such that, for all X and Y in **X**, X L^2 Y if and only if X is a *majorization* of *Y by non-negative weights* i.e. (Xw) L (Yw) for all w in the set of non-negative *m*- dimensional real vectors $\Re_+{}^m$.

(3) Lorenz Majorization by Equal Weights (L^3): L^3 is such that, for all X and Y in **X**, X L^3 Y if and only if X is a *majorization* of Y *by equal weights* i.e. (Xw) L (Yw) where w is the *m*-vector in which each entry is 1/*m*.

(4) Columnwise Lorenz Majorization (L^4): For all X and Y in **X**, X L^4 Y if and only if X is a *columnwise majorization* of Y i.e. x^i L y^i for all i in M i.e. (Xw) L (Yw) for all *m*-vectors w such that $w_i = 1$ for some i in M and $w_j = 0$ for all j in M such that $j \neq i$.

For L^1 and L^2 see, for instance, Bhandari (1988), Kolm (1977), List (1999) and Koshevoy and Mosler (2006). L^1, L^2 and L^4 are also mentioned by Arnold (2005) among ‘candidate definitions’ of multi-attribute Lorenz dominance. L^3 has been included in the list in view of the fact the use of equal weights seems to be widespread in empirical research on multidimensional inequality because of its computational simplicity.

Clearly, $L^1 \subset L^2 \subset L^3$. Also, $L^2 \subset L^4$. But L^3 and L^4 are not nested.

(5) Majorization of Lorenz Zonoids (L^Z): While the MLDR’s illustrated in Examples 1 through 4 are suggested multi-attribute analogues of Lorenz dominance in the single-attribute case, they do not suggest a multi-attribute Lorenz curve. It would seem that a more satisfactory approach would be to proceed in more direct analogy with the single-attribute case i.e. to first suggest an extension of the concept of Lorenz curve to the case of multiple attributes and then to define Lorenz dominance for this case in terms of dominance relations between the generalized curves for different distribution matrices. Because of the mathematical difficulties inherent in this approach, progress along these lines has been slow. Arnold (1983) and Taguchi (1972) were among the early attempts in this direction. In recent statistical literature a more satisfactory definition of such a multi-attribute analogue has emerged. See Koshevoy (1995) for the case of empirical distributions and Koshevoy and Mosler (2007) and Mosler (2002) for extension to the case of random variables and other developments.

(More recently, Sarabia and Jorda (2013) has used the definition proposed in Arnold (1983) to obtain closed expressions for bivariate Lorenz curves. However, their formulation involves specific assumptions regarding the underlying bivariate distributions.)

The Koshevoy-Mosler approach is based on the notion of a *Lorenz zonoid*. First define the *lift zonoid* of an $n \times m$ matrix X, Z(X) (say), as the Minkowski sum of the n line segments $[0_{m+1}, ((1/n), (x_p/n))]$, $p = 1, 2, \ldots, n$, in $\Re^{m+1}$. It is a convex set. The *Lorenz zonoid* of a distribution matrix X, Z*(X) (say), is then defined as the lift zonoid of X*, the scaled version of X. Thus, for all X, Z*(X) = Z(X*). For details see the references cited above.

Koshevoy and Mosler (2007) introduced the following *strict dominance* relation: for all $n \times m$ matrices in X and Y, X strictly Lorenz dominates Y if and only if $Z^*(X) \subset Z^*(Y)$.

As shown by the authors, for all $n\times m$ matrices X and Y, $Z^*(X) \subset Z^*(Y)$ if and only if $(X^*w)\ P\ (Y^*w)$ for all w in $\Re^m$. We shall denote this strict dominance relation by P^Z. It is easily seen that P^Z does not coincide with P^1 but is more restrictive i.e. $P^Z \subset P^1$.

Hence, we can obtain a "candidate" MLDR by constructing a quasi-ordering whose asymmetric component would coincide with P^Z. We shall consider the relation L^Z defined as follows. $L^Z = P^Z \cup I^Z$ where I^Z is the relation on **X** such that, for all X and Y in **X**, X I^Z Y if and only if $Z^*(X) = Z^*(Y)$.

(6) Majorization of Extended Lorenz Zonoids (L^{eZ}): Koshevoy and Mosler (2007) also introduced the concept of the *extended Lorenz zonoid.* Define the *extended lift zonoid* of a distribution matrix X, eZ (X), as the lift zonoid augmented by all points that are below a point in the lift zonoid Z(X) w.r.t. the first coordinate and above the point w.r.t. the other m coordinates:
$eZ(X) = \{(v_0, v_1,.....,v_m) : v_0 \leq z_0, v_j \geq z_j, j = 1, 2, ..., m, \text{ for some } (z_0, z_1, ..., z_m)\ \varepsilon\ Z(X)\}$.

The *extended Lorenz Zonoid* of a distribution matrix X, eZ*X), is the extended lift zonoid of the scaled version of X i.e. $eZ^*(X) = eZ\ (X^*)$.

In similarity with the case of Lorenz zonoids a strict dominance relation P^{eZ} (say) can be defined in terms of strict set set inclusion of extended Lorenz zonoids: for all admissible X and Y, X P^{eZ} Y if and only if $eZ^*(X) \subset eZ^*(Y)$. However, it has been shown that, for all X and Y, $eZ^*(X) \subset eZ^*(Y)$ if and only if $(X^*w)\ P\ (Y^*w)$ for all w in $\Re_+^m$. Thus, P^{eZ} is not the asymmetric component P^2 of L^2 but is, rather, a subset of it.

We shall consider the acceptability, as MLDR, of a relation L^{eZ} on **X** whose asymmetric component would coincide with P^{eZ} . We define L^{eZ} to be $P^{eZ} \cup I^{eZ}$ where I^{eZ} is the relation on **X** for which, for all X and Y in **X**, X I^{eZ} Y if and only if $eZ^*(X) = eZ^*(Y)$.

(7) Majorization by data-driven weights (L^w): The idea of using data-driven weights for the purpose of majorization of distribution matrices has also been pursued in the literature. The following criterion was suggested in Banerjee (2014).

For any pair (X, Y) of $n \times m$ matrices in **X** first define the pair (X_0, Y_0) as follows: If X and Y are such that
(i) $\mu(\mathbf{x}^j) = \mu(\mathbf{y}^j)$ for all j in M; and
(ii) for some non-empty subset N' of N, $\mathbf{x}_p = \mathbf{y}_p$ for all p in N',
then X_0 and Y_0 are the $(n - n') \times m$ matrices (where n' is the cardinality of N') obtained from X and Y respectively by deleting the common rows. In all other cases $(X_0, Y_0) = (X, Y)$.

Now, for any matrix X, let X^ denote its *comonotonization* i.e. the comonotonic matrix obtained by rearranging, if necessary, the entries in each column of X. Again, for any X, let X* be its scaled version. Let C(X) denote the covariance matrix of X

The suggested criterion L^w is defined to be such that, for all X and Y in **X**, X L^w Y if and only if [(X_0*) w(X_0^*)] L [(Y_0*)w(Y_0^*)] where, for all X in **X**, w(X_0^*) is the first eigen vector (i.e. the eigen vector associated with the maximal eigen value) of C[(X_0^*) ^]. □

None of the seven binary relations on **X** mentioned above, however, is an MLDR satisfying **CIM** and **CEM**. To show this we first establish that, among these relations, L^2 and L^4 are the only ones that satisfy Definition 2.3 of an MLDR.

All of the seven relations satisfy **QORD**, **ANON** and **PRI**. Moreover, all of them coincide with the unidimensional Lorenz dominance relation when m = 1. However, L^3 violates **RSI**. For instance, consider the case where $n = 2 = m$, X = $\begin{pmatrix} 4 & 0 \\ 2 & 4 \end{pmatrix}$ and Λ = $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. **RSI** requires that X I^3 (XΛ) = $\begin{pmatrix} 8 & 0 \\ 4 & 4 \end{pmatrix}$i.e. it requires that, for w = $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, Xw either equals or is a row permutation of (XΛ)w. However, Xw = $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and (XΛ)w = $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$, violating the requirement. The same argument shows that **RSI** is also violated by L^1 and L^2.

It may be noted, however, that slightly restated versions of these relations would avoid this problem. Redefine L^3 in terms of the *scaled* versions of the matrices: for all X and Y in **X**, X L^3 Y if and only if (X*w) L (Y*w) where w is the m-vector in which each entry is $1/m$. In what follows all references to L^3 will assume that it has been so redefined. L^1 and L^2 will also be assumed to have been similarly restated.

L^4 is easily seen to satisfy **RSI**.

It is seen, however, that even the restated version of L^1 (which allows negative entries in w) violates **UM** and **PDBP**.

L^2 and L^4, however, satisfy these conditions.

L^3 is also easily seen to satisfy **PDBP**. However, it violates **UM.** Consider, for instance, X = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and Y = $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and note that X = BY where B is the bistochastic matrix $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. However, if $w_1 = ½ = w_2$, Xw = $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = Yw.

Consider now the relations L^Z and L^{eZ}. Since these are stated in terms of scaled matrices, **RSI** is satisfied. However, they fail to satisfy **UM.** The same examples of X, Y and w as in the preceding paragraph suffice to establish this.

Finally, while L^w satisfies **RSI** by construction and is known to satisfy **PDBP**, it violates **UM**. To show this, consider, again, the matrices X and Y as specified for the case of L^3 above. It can be checked that, in this example, (X, Y) = (X_0, Y_0) and that the first eigen vectors of C(X_0^) and C(Y_0^) are the same, to wit, $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ = w_0 (say). Again, therefore, Xw_0

$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ = Yw_0 so that **UM** is violated. This completes the demonstration of the fact that, among the binary relations on **X** reviewed above, L^2 and L^4 are the only ones that are MLDR's as per the definition developed in the previous Section.

It is easily seen that L^4 satisfies **CEM** also. However, it violates **CIM**. The example of L^4, therefore, shows that **CEM** does not imply **CIM**. On the other hand, it can be checked that L^2 satisfies **CIM.** However, it violates **CEM**. Consider, for instance, the case where $n = 2 = m$, X = $\begin{pmatrix} 4/3 & 1 \\ 2/3 & 1 \end{pmatrix}$ and Y = $\begin{pmatrix} 4/3 & 0 \\ 2/3 & 2 \end{pmatrix}$. Since $\begin{pmatrix} 4/3 \\ 2/3 \end{pmatrix}$ P [(1/2)$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ + (1/2)$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$], **CEM** requires that (X P^2 Y) where P^2 is the asymmetric component of L^2 i.e.it requires that [(Xw L Yw) for all w ≥ 0] and ¬[(Yw L Xw) for all w ≥ 0]. However, if w = (2/3, 1/3), the requirement is seen to be violated. Thus, **CIM** does not imply **CEM**.

4. An MLDR satisfying CIM and CEM

Since the existing literature does not seem to contain an example of an MLDR that satisfies **CIM** and **CEM**, in this Section we attempt to suggest one that does.

The suggested relation is defined indirectly in terms of unidimensional relative *equality* indices.

Definition 4.1: A *Unidimensional Relative Equality Index* (UREI), E, is a mapping from $\mathbf{X}^1$ into $\Re_+$ satisfying
(i) Anonymity[i.e., for all **x** and **y** in $\mathbf{X}^1$ such that **y** is a permutation of **x**, E(**x**) = E(**y**)];
(ii) Mean Independence [i.e., for all **x** in $\mathbf{X}^1$, E(k**x**) = E(**x**) for all positive scalars k];
(iii) Population Invariance [i.e., for all **x** and **y** in $\mathbf{X}^1$ such that **y** is a k-fold replication of **x** for any positive scalar k, E(**x**) = E(**y**)];
(iv) the Pigou-Dalton (PD) condition [i.e., for all **x** and **y** in $\mathbf{X}^1$ such that **x** is a Pigou-Dalton majorization of **y**, E(**x**) > E(**y**)] .

Let **E** be the set of all UREI's.

In view of mean independence E(**x**) = E(x*) for E in **E** and for all **x** in $\mathbf{X}^{1.}$

There is an equivalence result regarding unidimensional Lorenz dominance and dominance in terms of all UREI's.

Lemma 4.1: For any **x** and **y** in $\mathbf{X}^1$ the following statements are equivalent:
(1) E(**x**) ≥ E(**y**) for all I in **I**.
(2) **x** L **y.**

The result in Lemma 4.1 is due to Foster (1985). (Also see Chakravarty (1990, pp. 35 – 36). We have, however, stated here in terms of equality (rather than inequality) indices. (For closely elated results see Eichhorn and Gehrig (1982), Fields and Fei (1978) and Kurabayashi and Yatsuka (1977).)

In the multidimensional case for any E in **E** and for any given vector w of attribute weights we first construct a multidimensional analogue of E. For convenience we shall, however, assume that, for all *n*-vectors **x** in $\mathbf{X}^1$ such that $\mathbf{x} = \mu(\mathbf{x})\mathbf{1}_n$, E(**x**) = 1 (i.e. E takes the value 1 for perfectly equal distributions) and that w is in the set W = {w: w ε $\Re^m_+$ and $\sum_{j=1}^{m} w_j = 1$} (i.e. the attribute weights are non-negative and they sum to 1).

In this case, however, the contribution of an attribute (*i*, say) toward the over-all degree of equality is not simply equal to $E(x^i)$ which is its "direct"(or "own") contribution. The indirect effects of the attribute through its interactions with the other attributes are to be taken into account. Let $E^{ij}(X)$ denote the contribution that attribute *i* makes toward over-all equality in conjunction with attribute *j*. One aspect of such interactions between attributes is sought to be captured by considerations such as those underlying the condition of **CIM**. (Essentially, this aspect relates to the point that the magnitude of the interaction effect should be sensitive to rank correlations between the distributions of the attributes.) However, the issue is more general. For instance, even when $\mathbf{x}^i$ and $\mathbf{x}^j$ are comonotonic, the magnitude of this effect should be allowed to change when $\mathbf{x}^i$ changes to, say, $\mathbf{y}^i$ but $\mathbf{y}^i$ and $\mathbf{x}^j$ are, again, comonotonic.

For simplicity, however, we shall consider the special case where, for all X in **X** and for all *i* and *j* in *M*, $E^{ij}(X) = E[(\mathbf{x}^i + \mathbf{x}^j) / 2]$. This assumption incorporates some intuitively plausible features of such interdependence. For instance, the effect on equality of the "interaction" of an attribute with itself coincides with its "own" contribution i.e. $E^{ii}(X) = E(\mathbf{x}^i)$ for all *i* in *M*. Moreover, the interaction effects are symmetric: $E^{ij}(X) = E^{ji}(X)$ for all *i* and *j* in *M*. (On the other hand, it restricts $E^{ij}(X)$ to be independent of $\mathbf{x}^k$ for all *k* in *M* such that $i \neq k \neq j$.)

For all non-negative vectors of attribute weights w, for all E in **E**, for all X in **X** and for all *i* in *M*, $E^M_w(X)$ will denote the degree of multidimensional equality in the economy while $E^{Mi}_w(X)$ will denote the total (i.e. direct and indirect) contribution of attribute *i* toward multidimensional equality. The proposed procedure is based on the following two assumptions. For all admissible w

(1) $E^{Mi}_w(X) = \sum_{j=1}^{m} w_j E[(\mathbf{x}^i + \mathbf{x}^j) / 2]$ for all *i* in *M* ; and

(2) $E^{M}_w(X) = A \sum_{i=1}^{m} E^{Mi}_w(X)$ where A is a positive constant such that $E^M_w(X) = 1$ if $\mathbf{x}^j$ is equally distributed for all *j* in *M*..

(1) states that, for any given attribute, its total contribution to multidimensional equality is the weighted average of the contributions generated through its interaction with all the attributes (including itself). The contribution generated through attribute *j* is weighted by w_j for all *j* in *M*.

(2) states that the degree of multidimensional equality in the economy is proportional to the sum of the contributions of the attributes. The restriction on A stated in (2) obviously implies that A = (1/*m*).

Consider now the binary relation, L^M on **X** defined as follows:

Definition 4.2: For all X and Y in **X**, X L^M Y if and only if $E^M_w(X^*) \geq E^M_w(Y^*)$ for all E in **E** and for all w in W where, for all such w and E and for all X in **X**, $E^M_w(X)$ is as defined in (1) and (2) above.

P^M and I^M will denote the asymmetric and symmetric components of L^M respectively.

Proposition 4.1: L^M is an MLDR satisfying **CIM** and **CEM**.

Proof:
I. L^M is an MLDR:

We first show that L^M is an MLDR as per Definition 2.3. **RSI** and **QORD** are easily checked. To check **ANON** let X and Y in **X** be such that Y is a row permutation of X. Then $\mu(\mathbf{x}^j) = \mu(\mathbf{y}^j)$ for all *j* in *M* ; and, for all *i* and *j* in *M*, $[(\mathbf{x}^{*i} + \mathbf{x}^{*j})/2]$ is a permutation of $[(\mathbf{y}^{*i} + \mathbf{y}^{*j})/2]$. Since any E in **E** satisfies Anonymity, it follows that, for all admissible w and E**,** $E^M_w(X^*) = E^M_w(Y^*)$. Hence, X I^M Y. To see that L^M satisfies **PRI**, let X and Y in **X** be such that Y is obtained by a *k* -fold population replication of X for a positive integer *k*. For all *i* and *j* in *M*, $[(\mathbf{y}^{*i} + \mathbf{y}^{*j})/2]$ is now a *k*-fold replication of $[(\mathbf{x}^{*i} + \mathbf{x}^{*j})/2]$. The result now follows from the fact that any E in **E** satisfies Population Invariance.

To prove that L^M satisfies **PDBP** i.e. to show that if X and Y in **X** are such that Y is obtained from X by a *finite* sequence of PDBT's, then Y L^P X, first suppose that Y is obtained from X by a *single* PDBT. Recall that, according to Definition 2.1 of PDBT, this implies that there exist *q* and *p* in *N* such that $\mathbf{x}_q > \mathbf{x}_p$ and that positive amounts of one or more attributes are transferred from individual *q* to individual *p* subject to the restriction that $\mathbf{y}_q \geq \mathbf{y}_p$.

It can be seen that, under these conditions, for any pair of attributes *i* and *j*, $[(\mathbf{y}^{*i} + \mathbf{y}^{*j})/2]$ equals $[(\mathbf{x}^{*i} + \mathbf{x}^{*j})/2]$ if the PDBT under consideration does not involve transfers of the attributes *i* and *j*. If it involves a transfer of at least one of these two attributes, the former vector is a Pigou-Dalton majorization of the latter. Thus, for all E in **E**, $E[(\mathbf{y}^{*i} + \mathbf{y}^{*j})/2] \geq E[(\mathbf{x}^{*i} + \mathbf{x}^{*j})/2]$ with strict inequality holding whenever positive amounts of either (or both) of the two attributes are transferred.

It follows that, for all admissible w and E, $E^M_w(Y^*) \geq E^M_w(X^*)$. Hence, Y L^M X. Moreover, since, by hypothesis, at least one of the attributes is transferred, the same argument also shows that it is not the case that $E^M_w(X^*) \geq E^M_w(Y^*)$ for all w and E. Thus, $\neg$[X L^M Y]. Therefore, Y L^P X.

If, now, Y is obtained from X by a finite sequence of PDBT's rather than a single such transfer, the same conclusion is reached by a repeated application of this argument.

To show that L^M satisfies **UM**, let X and Y in **X** be such that X = BY where B is a bistochastic (but not a permutation) matrix. Then X* = BY*. For all *j* in *M*, therefore, $\mathbf{x}^{*j} = B\mathbf{y}^{*j}$. Hence, for all *i* and *j* in *M*, $[(\mathbf{x}^{*i} + \mathbf{x}^{*j})/2] = B[(\mathbf{y}^{*i} + \mathbf{y}^{*j})/2]$ i.e. $[\mathbf{x}^{*i} + \mathbf{x}^{*j})/2]$ Pigou-

.Dalton majorizes $[(\mathbf{y}^{*i} + \mathbf{y}^{*j})/2]$. For any E in **E**, therefore, $E[(\mathbf{x}^{*i} + \mathbf{x}^{*j})/2] > E[(\mathbf{y}^{*i} + \mathbf{y}^{*j})/2]$ for all i and j in M. It follows that X L^P Y.

This completes the proof of the fact that L^M is an MIDR as per Definition 2.2. To show that it is an MLDR, it only remains to check that if m =1, $L^M = L$. This, however, is a consequence of Lemma 4.1

II. L^M satisfies **CIM**.

Let X and Y in **X** be such that Y is obtained from a finite sequence of CIT's but is not a row permutation of X. To show that X L^P Y, it suffices to prove this for the case where Y is obtained from X by a single CIT. In this case Y* is obtained from X* by a CIT. Definition 2.4 of a CIT is seen to imply that, for all i and j in M, one of the following two statements is true:

(i) $[(\mathbf{x}^{*i} + \mathbf{x}^{*j})/2]$ is a permutation of $[(\mathbf{y}^{*i} + \mathbf{y}^{*j})/2]$.

(ii) $[(x^{*i} + \mathbf{x}^{*j})/2]$ is a Pigou-Dalton majorization of $\{(\mathbf{y}^{*i} + \mathbf{y}^{*j})/2]$.

Moreover, since by hypothesis Y is not a permutation of X, it follows that (ii) is true for some i and some j in M. Hence, for any E in **E**, $E[(\mathbf{x}^{*i} + \mathbf{x}^{*j})/2] \geq E[(\mathbf{y}^{*i} + \mathbf{y}^{*j})/2]$ for all i and j in M, with strict inequality holding for at least one pair (i, j). It is then easily checked that X L^M Y but $\neg$[Y L^M X]. Hence, X L^P Y.

III. L^M satisfies **CEM**.

Let X in **X** and j in M be such that $\mathbf{x}^i\ P^*\ \mathbf{x}^j$ for all i in M such that $i \neq j$. To establish that Y L^P X where Y = $X^{-j,\ \mu}$, we need to prove that, for all admissible w and E, $E^M_w(Y^*) \geq E^M_w(X^*)$ and that, for some admissible w and E, the inequality is strict. We show that, in fact, $E^M_w(Y^*) > E^M_w(X^*)$ for such w and E. However, we indicate the proof of this assertion for the special case in which $n = 2 = m$ since the argument generalizes in a straightforward manner.

For this purpose let X in **X** be such that $X^* = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and let $\mathbf{x}^1\ P^*\ \mathbf{x}^2$ i.e. $\begin{pmatrix} a \\ b \end{pmatrix}$ P $\begin{pmatrix} (c+1)/2 \\ (d+1)/2 \end{pmatrix}$. Let Y be the matrix obtained from X by equalizing $\mathbf{x}^2$. Hence, $Y^* = \begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix}$. For any admissible w and E, it can be checked that, since the components of w sum to 1, $E^M_w(Y^*) > E^M_w(X^*)$ if $w_1E(\mathbf{y}^1) + w_2E(\mathbf{y}^2) + E\ [(\mathbf{y}^1 + \mathbf{y}^2)/2] > w_1E(\mathbf{x}^1) + w_2E(\mathbf{x}^2) + E[(\mathbf{x}^1 + \mathbf{x}^2)/2]$.

Now, $[\mathbf{x}^1\ P^*\ \mathbf{x}^2]$ precludes the possibility that $c = 1 = d$. Hence, if $a = 1 = b$, the desired inequality follows trivially.

Assume, therefore, w.l.o.g., that $a > b$.

Since E($\mathbf{y}^1$) = E($\mathbf{x}^1$) and $\mathbf{y}^{*2} = \begin{pmatrix}1\\1\end{pmatrix}$ (so that $\mathbf{y}^2\ P\ \mathbf{x}^2$ and, therefore, E($\mathbf{y}^2$) > E($\mathbf{x}^2$)), to prove the desired inequality it suffices to show that E[($\mathbf{y}^1$ + $\mathbf{y}^2$)/2] > E[($\mathbf{x}^1$ + $\mathbf{x}^2$)/2] for all E in **E** i.e. that [($\mathbf{y}^1$ + $\mathbf{y}^2$)/2] *P* [($\mathbf{x}^1$ + $\mathbf{x}^2$)/2]. Consider first the case where $c > d$. Since $c + d = 2$, $c > 1 > d$. We have: [($\mathbf{x}^1$ + $\mathbf{x}^2$)/2] = $\begin{pmatrix}(a+c)/2\\(b+d)/2\end{pmatrix}$ and, in the case under consideration, [(a + c)/2] > [(b + d)/2]. On the other hand, [($\mathbf{y}^1$ + $\mathbf{y}^2$)/2] = $\begin{pmatrix}(a+1)/2\\(b+1)/2\end{pmatrix}$ and [(a + 1)/2] > [(b + 1)/2]. The desired result is obtained by noting that [(b + 1)/2] > [(b + d)/2]. In the case where $c < d$, first note that the hypothesis that $\mathbf{x}^1\ P^*\ \mathbf{x}^2$ implies $\mathbf{x}^1\ P\ \mathbf{x}^2$ so that (a – b) < (d – c) and, therefore, [(b + d)/2] > [(a + c)/2]. Hence, the desired result will follow if [(b + 1)/2] > [(a + c)/2] i.e. if (a – b) < (1 – c). Since $d = 2 - c$, this inequality is seen to be implied by the hypothesis which requires a < [(d + 1)/2] and b > [(c + 1)/2]. □

5. Conclusion

In this paper we have sought to assess the different multidimensional Lorenz dominance relations that have been suggested in the literature. For this purpose we have formulated a definition of such a relation by using a number of conditions which seem to reflect the basic requirements of an inequality dominance relation. Two additional conditions which seem to be intuitively reasonable requirements have also been stated. It is seen, however, that none of the relations that have so far been suggested satisfies all of these conditions. The question, therefore, arises as to whether there exists a relation possessing all of the desired characteristics. We have sought to give an affirmative answer to the question by proposing a new dominance relation.

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