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Abstract

This paper is concerned with the measurement of the extent of deprivation when the available information is given by a set of dichotomous variables, and the data are summarised by the count distribution, i.e. the distribution of the number of dimensions in which an individual suffers from deprivation. Next, by drawing on the expected utility framework that originates from Atkinson (1970) the individual deprivation counts are aggregated into summary measures of deprivation, which prove to admit a convenient decomposition into the mean and the dispersion of the distribution of multiple deprivations.

Keywords: Multidimensional deprivation, counting approach, partial orderings, measures of deprivation, principles of association rearrangements.

JEL Classification: D31, D63, I32.

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1. Introduction

A long tradition in social sciences has been concerned with measuring material deprivation by looking at a number of indicators of living conditions, such as the ownership of durables or the possibility to carry out certain activities like going out for a meal with friends. The typical way to summarise the information has been to count the number of dimensions in which people fail to achieve a minimum standard, hence the label of "counting approach". Counting deprivations is the simplest way to embed the association between deprivations at the individual level into an overall index of deprivation, moving away from treating the achievements in the various dimensions as independent one from the other.

As stressed by Atkinson (2003), the challenge for the counting approach is to clarify the nature of the social judgments inherent in the criteria for ranking the distributions of deprivation counts. The characterisation of these social judgments has proved elusive. One reason may be that welfare criteria have been generally understood in terms of the distributions of the original variables rather than in terms of the distribution of deprivation scores, that is the numbers of dimensions in which individuals fail to achieve the minimum standards. Such a distribution contains all the relevant information in the counting approach, which ignores by construction the levels of the achievements, or the shortfalls relative to the minimum standards, in the original variables. There is a loss of information here, but one that is offset by the capacity of the counting approach to effectively summarise available data.

Following this line of reasoning, in this paper we introduce criteria to rank distributions of deprivation counts. We set conditions on the derivatives of a social welfare function analogous to the "expected utility" type of social welfare functions explored in income inequality measurement by Kolm (1969) and Atkinson (1970), but defined in terms of the number of deprivations rather than income. Our approach parallels the discussion by Aaberge and Peluso (2011), who however characterise a class of social evaluation functions drawing on the rank-dependent measurement of income inequality introduced by Weymark (1981) and Yaari (1988). We identify first- and second-degree dominance conditions as well as a class of counting deprivation measures that encompass those proposed by Atkinson (2003), Chakravarty and D'Ambrosio (2006) and Alkire and Foster (2011).

Although we collapse the multiple dimensions of deprivation into a univariate space, one important aspect distinguishes the analysis of the number of deprivations from that of income: the identification of the poor. Is it poor only a person suffering from deprivation in all r dimensions, or is it poor everyone suffering in at least 1 dimension? These two opposite views are equally reasonable and correspond to what Atkinson (2003) calls the "intersection" and "union" views in multidimensional poverty assessment. It is also possible to take an intermediate position, where someone is classified as poor if suffering from deprivation in at least c dimensions, with $1 < c < r$ as in the "dual cut-off" approach advocated by Alkire and Foster (2011). The distinction between the intersection and union views has important implications for the shape of the social welfare function. As shown by Aaberge and Peluso (2011), it leads to define two alternative second-degree dominance criteria: the "upward" criterion when social preferences are characterised by the union view, and the "downward" criterion when they favour the intersection approach. These criteria correspond, in turn, to concave and convex social preferences, respectively. This is where the analyses of deprivations scores and incomes noticeably differ. Convex preferences are ruled out in the analysis of income distributions because they would yield a social evaluation function violating the Pigou-Dalton principle of transfers, but concave preferences are perfectly legitimate in the analysis of deprivation counts as they simply mean being relatively more concerned with the spreading of a given number of deprivations across many people than with their concentration on fewer people who are hit more, that is leaning toward the union rather than the intersection criterion.[1]

The paper is organised as follows. In Section 2 we provide the axiomatic characterisation of a family of deprivation measures. These deprivation measures generate linear orders on the set of deprivation count distributions and are shown to be decomposable into the extent and spread of deprivation counts. We also introduce second-degree upward and downward dominance criteria to capture the union and intersection views. In Section 3 we discuss the relationship between association rearrangement principles, second-degree upward and downward dominance criteria, and two families of deprivation measures. In Section 4 we deal with the measurement of poverty as distinct from deprivation, whereas in Section 5 we examine the inequality in the distribution of achievement counts. We explain how the framework can be extended to account for different weights in Section 6.

2. Ranking counting distributions

We begin with considering the simple situation where there are only two dimensions of well-being to illustrate the main issues. Assume that the achievement in dimension j, $j = 1, 2$, of individual i is Y_{ij}, and that X_{ij} is equal to 1 if individual i has an achievement below the socially minimum standard, that is the individual suffers from deprivation in dimension j, and 0 otherwise. The standard approach to multidimensional poverty measurement (e.g. Tsui, 2002; Bourguignon and Chakravarty, 2003; Alkire and Foster, 2011) would be to define some individual poverty function $\pi(Y_{i1}, Y_{i2})$ and then an appropriate aggregator function g so that the overall poverty indicator would be $P = g\left[\pi(Y_{i1}, Y_{i2})\right]$. In the case where deprivation in the two dimensions is measured by a binary indicator, the previous expression would become $P = g\left[\pi(X_{i1}, X_{i2})\right]$. In the counting approach, individual achievements are reduced to pairs of 0's and 1's, and persons are only distinguished by the number of failures. If there are only two dimensions, then there are three types of individuals characterised by the fact of being deprived in 0, 1 or 2 dimensions. This means that the overall poverty indicator can be expressed only in terms of the distribution of the number of deprivations, to some extent dispensing with the need to aggregate across the individuals.

Let $p_{jh} = \Pr\left(\left(X_1 = j\right) \cap \left(X_2 = h\right)\right)$, $p_{j+} = \Pr\left(X_1 = j\right)$ and $p_{+h} = \Pr\left(X_2 = h\right)$. Then, weighting equally the two dimensions, define the deprivation score $X = X_1 + X_2$, which can take the values $(0,1,2)$ with associated probabilities (q_0, q_1, q_2). The parameters (q_0, q_1, q_2) of the count distribution X are determined by the parameters of the simultaneous distribution of (X_1, X_2) in the following manner: $q_0 = p_{00}$, $q_1 = p_{10} + p_{01}$ and $q_2 = p_{11}$. The original simultaneous distribution and the derived count distribution are summarised in Table 2.1.

If only the marginal distributions in the left panel of the Table were known, an overall poverty indicator P could be expressed as a function g of p_{1+} and p_{+1} only, that is $P = g(p_{1+}, p_{+1})$. This is an example of a composite poverty index, which is obtained by computing first the proportions of people suffering in each dimension, and then aggregating these proportions into a composite index of deprivation. However, we could invert the order of aggregation: the synthesis of the available information would begin with aggregating across the single dimensions for each individual, and then across the individuals. If the dimensions are independent of each other, so that $p_{10} = p_{01} = 0$, the order of aggregation

[1] Convex (concave) preferences in the income space correspond to concave (convex) preferences in the space of deprivations counts, which represent "bads" (loss in wellbeing) rather than "goods" (gains in wellbeing).

does not matter and the two approaches are equivalent. If they are not, and suffering from multiple deprivations has a more than proportionate effect on people's well-being, then ignoring the impact of the association among the achievements in the various dimensions may imply missing an important aspect of hardship.[2] This implies knowing the simultaneous distribution. In such a case, we could turn to the distribution of *X* in the right panel of Table 2.1 and the overall index could account for the number of deprivations that each individual suffers from. Counting deprivations means accounting for the association between dimensions, but there are two possible ways of identifying someone as poor: either he fails in either dimension ($X = 1$), or he fails in both ($X = 2$). In the first case, we would adopt the "union criterion": the poor are those with at least one deprivation and $P = g(1 - p_{00})$. In the second case, we would favour the "intersection criterion": the poor are those with two failures and $P = g(p_{11})$.

Table 2.1. The distribution of deprivations in two dimensions and the derived distribution of deprivations scores

	X_2=0	X_2=1	
X_1=0	p_{00}	p_{01}	p_{0+}
X_1=1	p_{10}	p_{11}	p_{1+}
	p_{+0}	p_{+1}	1

	$X=X_1+X_2$
X=0	$q_0=p_{00}$
X=1	$q_1= p_{10}+p_{01}$
X=2	$q_2=p_{11}$
	1

Source: authors' elaboration.

In the next Sections we develop partial and complete ranking criteria and define a class of deprivation measures when we have full information about the simultaneous distribution and can hence construct the counting distribution *X*.

2.1. Partial orderings

We assume that individuals might suffer from deprivation in *r* dimensions. Let X_j equal 1 if an individual suffers from deprivation in dimension *j* and 0 otherwise. Moreover, let

$$X = \sum_{j=1}^{r} X_j$$

be a random variable with cumulative distribution function *F* and mean μ . Thus, $X = 1$ means that the individual suffers from one deprivation, $X = 2$ means that the individual suffers from two deprivations, etc. We call *X* the deprivation count. Furthermore, let $q_k = \Pr\left(X = k\right)$ which yields

$$F(k) = \sum_{j=0}^{k} q_j, \; k = 0,1,2...,r \tag{2.1}$$

and

$$\mu = \sum_{k=1}^{r} kq_k \,. \tag{2.2}$$

Although *F* is a discrete distribution function we will for notational convenience occasionally use the integration symbol when we aggregate across count distributions. For the sake of

[2] See Dutta et al. (2003) on the equivalence of results when the order of aggregation changes.

simplicity, we are assigning equal weights to all dimensions, but this assumption will be relaxed in Section 5.

As is standard in the income distribution literature, the strongest criterion is first-degree dominance defined by[3]

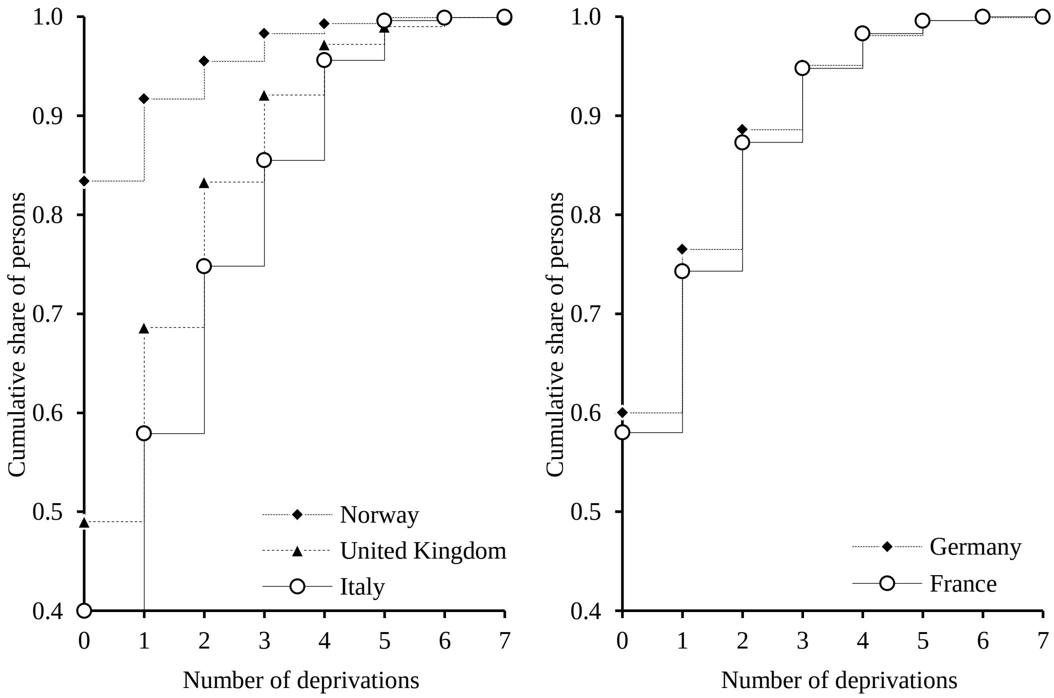
Definition 2.1. *A deprivation count distribution* F_1 *is said to first-degree dominate a deprivation count distribution* F_2 *if*

$$F_1(k) \geq F_2(k) \quad \textit{for all } k = 0,1,...,r$$

and the inequality holds strictly for some k.

If F_1 first-degree dominates F_2, then F_1 exhibits less deprivation than F_2. An example is given in Figure 2.1, where we use the material deprivation indicators in five European countries in 2012 drawn from Eurostat (2014). Figure 2.1 plots on the vertical axis the cumulative proportion of persons that suffer from deprivation in at most the number of dimensions indicated on the horizontal axis. (Figure 2.1 considers a maximum of seven deprivation items since nobody suffers from more than seven in the countries considered.)

Figure 2.1: Cumulative distributions of material deprivation scores in selected European countries in 2012



Source: authors' elaboration on data from Eurostat (2014).

The left panel shows that Norway first-degree dominates both the United Kingdom and Italy, whereas the last two countries cannot be ordered by the criterion of first-degree dominance since their distributions intersect. The United Kingdom clearly lies ahead of Italy for up to five items, but then exhibits a share of people suffering from six or seven deprivations that is more than twice the Italian level (1 vs. 0.4 per cent). The right panel of Figure 2.1 shows that also the cumulative distributions of deprivations scores for France and Germany intersect, though being much closer. The share of non-deprived is higher in Germany than in France,

[3] Lasso de la Vega (2010) and Yalonetzky (2014) also identify dominance conditions to rank deprivation count distributions.

and the same holds true when we sequentially add those with one, two and three deprivations; however, when we add people suffering from four deprivations the order reverses, and no longer changes when we consider more severe situations.[4]

This example shows that first-degree dominance might be too demanding in practice: where count distributions intersect, they can be ranked only by defining weaker dominance criteria. This implies that we have to impose stricter conditions on social preferences, taking into account that in the evaluation we might be leaning towards either the intersection or the union criteria. In the former case, we would start aggregating "from above", looking first at the proportion of those who are deprived in r dimensions, then adding the proportion of those failing in $r-1$ dimensions, and so forth; in the latter case, we would start "from below". This distinction leads naturally to the definition of two second-degree dominance criteria, as suggested by Aaberge and Peluso (2011).

Definition 2.2A. *A deprivation count distribution* F_1 *is said to second-degree downward dominate a deprivation count distribution* F_2 *if*

$$\sum_{k=s}^{r} F_1(k) \geq \sum_{k=s}^{r} F_2(k) \quad \textit{for all } s = 0,1,\ldots,r$$

and the inequality holds strictly for some s.

Definition 2.2B. *A deprivation count distribution* F_1 *is said to second-degree upward dominate a deprivation count distribution* F_2 *if*

$$\sum_{k=0}^{s} F_1(k) \geq \sum_{k=0}^{s} F_2(k) \quad \textit{for all } s = 0,1,\ldots,r$$

and the inequality holds strictly for some s.

Second-degree downward as well as upward count distribution dominance preserves first-degree dominance since first-degree dominance implies second-degree downward and upward dominance. As mentioned, the choice between the two notions of second-degree dominance is closely associated with whether we favour the union or the intersection criterion. If we were more concerned with the extent to which deprivation is diffused across the population (union criterion) than with the occurrence of multiple deprivations (intersection criterion), we should adopt second-degree upward dominance. Intuitively, we can see this in Definition 2.2B from the fact that we are making comparisons on (doubly) cumulated population proportions that start by considering the share of people who do not suffer from any deprivation, $F(0)$, and sequentially add the shares of those who suffer from 1 deprivation, then those who suffer from 2 deprivations, and so forth. In calculating the cumulative function we "go up". The opposite happens in the second case, where we aggregate "going down", thus placing more weight on the most deprived. Formally, second-degree upward dominance parallels the dominance criterion used by Atkinson (1970) for ranking income distributions. Second-degree downward dominance has no correspondent in income inequality measurement, as it would be inconsistent with the Pigou-Dalton principle of transfers. We return to this point in the empirical illustration in Section 2.5.

[4] In this example and in all subsequent empirical illustrations, we treat statistics as they were exact and we abstract from the fact that they are subject to sampling and other types of errors. Accounting for these errors would possibly lead us to conclude that neither the observed difference between France and Germany nor the upper tail intersection between France and Norway is statistically significant.

2.2. The independence axiom and complete orderings

Let ***F*** denotes the family of deprivation count distributions. The social preferences over ***F*** can be represented by the ordering $\succeq$ which is assumed to be continuous, transitive and complete, and to satisfy first-degree count distribution dominance and the following independence axiom:

Axiom (Independence). *Let* F_1 *and* F_2 *be members of* ***F***. *Then* $F_1 \succeq F_2$ *implies* $\alpha F_1 + (1-\alpha)F_3 \succeq \alpha F_2 + (1-\alpha)F_3$ *for all* $F_3 \in \boldsymbol{F}$ *and* $\alpha \in [0,1]$.

The independence axiom requires that the ordering of distributions is invariant with respect to certain changes in the distributions being compared. If F_1 is weakly preferred to F_2, then the independence axiom states that any mixture on F_1 is weakly preferred to the corresponding mixture on F_2. The intuition is that identical mixing interventions on the count distributions do not affect their ranking. Thus, the independence axiom requires the ordering relation to be invariant with respect to aggregation of sub-populations across deprivation counts. If for a specific population the counting distribution F_1 is weakly preferred to the counting distribution F_2, then mixing this population with another population would not change the ranking of the count distributions.

We can now prove the following theorem:

Theorem 2.1. *A preference relation* $\succeq$ *on* ***F*** *satisfies continuity, transitivity, completeness, first-degree count distribution dominance and independence if and only if there exists a continuous and non-decreasing real function* γ *defined on the unit interval, such that for all* $F_1, F_2 \in \boldsymbol{F}$

$$F_1 \succeq F_2 \quad \Leftrightarrow \quad \sum_{k=1}^{r} \gamma(k) q_{1k} \leq \sum_{k=1}^{r} \gamma(k) q_{2k},$$

where q_{ik}, *with* $i = 1,2$ *and* $k = 1,2,...,r$, *is the proportion of people suffering from k deprivations in distribution i. Moreover,* γ *is unique up to a positive affine transformation.*

The proof of Theorem 2.1 is analogues to the proof of the expected utility theory for choice under risk. We refer to Fishburn (1982) for a detailed proof.

2.3. A summary measure of deprivation

As demonstrated by Theorem 2.1, the independence axiom provides a justification for the following family of deprivation measures,[5]

(2.3) $$d_\gamma(F) = \sum_{k=1}^{r} \gamma(k) q_k$$

[5] Chakravarty and D'Ambrosio (2006) provide an alternative axiomatic justification of (2.3) with a convex γ for measuring social exclusion. They also prove that second-degree downward dominance implies a convex γ and is preserved under a "favourable composite change", which is an intervention principle that is closely related to the Pigou-Dalton principle of transfers. This principle differs from the association rearrangement principles motivated by the measurement of multidimensional poverty and discussed in Section 3.

where $\gamma(k)$, with $\gamma(0)=0$ and $\gamma(r)=r$, is a non-decreasing function that represents the social preferences. Since the family of measures $d_\gamma(F)$ represents an ordering relation defined on the family of deprivation count distributions, it can be considered as a social evaluation function which select the count distribution that minimises $d_\gamma(F)$. Thus, $d_\gamma(F)$ provides a normatively justified measure of the welfare loss due to the extent of deprivation exhibited by the distribution *F* and attains its maximum value $\gamma(r)$ when all individuals suffer from deprivation in all dimensions. The family of deprivation measures defined by (2.3) is analogue to the family of social welfare functions (and associated measures of inequality) introduced by Kolm (1969) and Atkinson (1970).

It follows by straightforward calculation that $d_\gamma(F)$ admits the following decomposition:

(2.4) $$d_\gamma(F)=\sum_{k=1}^{r}\gamma(k)q_k=\begin{cases}\gamma(\mu)+\delta_\gamma(F) & \textit{when } \gamma \textit{ is convex}\\ \gamma(\mu)-\delta_\gamma(F) & \textit{when } \gamma \textit{ is concave,}\end{cases}$$

where

(2.5) $$\delta_\gamma(F)=\begin{cases}\sum_{k=0}^{r}\left(\gamma(k)-\gamma(\mu)\right)q_k & \textit{when } \gamma \textit{ is convex}\\ \sum_{k=0}^{r}\left(\gamma(\mu)-\gamma(k)\right)q_k & \textit{when } \gamma \textit{ is concave,}\end{cases}$$

This decomposition offers an identification of the separate contributions to the overall deprivation from the mean and the spread of the deprivations, where the latter is captured by a measure $\delta_\gamma(F)$ of left- or right-spread (left- and right-tail heaviness) when γ is concave or convex.[6] By inserting for $\gamma(k)=k^2$ (convex) and $\gamma(k)=2r-k^2$ (concave) in (2.4) and (2.5) we find that $\delta_\gamma(F)$ coincides with the variance, and the left- and right-tail measures of spread are symmetric.[7] When $\gamma(k)=k$ for all *k*, then $d_\gamma(F)=\mu$. As can be observed from (2.4) and (2.5), $0\le d_\gamma(F)\le\gamma(\mu)$ when γ is concave and $\gamma(\mu)\le d_\gamma(F)\le\gamma(r)$ when γ is convex. If it is desirable to derive $[0,1]$-normalised measures of deprivation, these inequalities provide the required information. By choosing $\gamma(k)=1$ for $k=1,2,...,r$, we get $d_\gamma(F)=1-q_0$, which means that the union criterion can be considered as a limiting case of the d_γ-family of deprivation measures for concave γ. By contrast, when $\gamma(k)=0$ *for* $k=0,1,...,r-1$ and $\gamma(r)=1$ then $d_\gamma(F)=q_r$, which means that the intersection criterion can be considered as a limiting case of d_γ-family of deprivation measures for convex γ.

The deprivation measures defined by (2.3) are decomposable by population subgroups, in the sense that overall deprivation can be expressed as a weighted average of the deprivation for each subgroup,

[6] Fernández-Ponce et al. (1998) and Shaked and Shanthikumar (1998) provide a discussion on how to compare the right-spread variability of distribution functions.
[7] The variance plays a similar role for this measure as does the Gini measure of dispersion for the dual measure proposed by Aaberge and Peluso (2011).

(2.6) $$d_\gamma(F) = \sum_{j=1}^{s} a_j \left(\sum_{k=1}^{r} \gamma(k) q_{kj} \right),$$

where a_j is the proportion of people belonging to subgroup *j*, $j = 1, 2, ..., s$ and q_{kj} is the proportion of people in subpopulation *j* that suffer from *k* deprivations.

The measure d_γ generalises the counting measure proposed by Atkinson (2003, p. 62) for a bivariate distribution ($r = 2$). Atkinson's measure A_θ can be written as

(2.7) $$A_\theta = 2^{-\theta}\left[p_{1+} + p_{+1} + 2(2^{\theta-1} - 1) p_{11} \right] = 2^{-\theta}\left(p_{1+} + p_{+1} \right) + \left(1 - 2^{1-\theta} \right) p_{11} = 2^{-\theta} q_1 + q_2,$$

by making use of the notation of Table 2.1 and after dividing through the original formula by 2^θ. We can obtain (2.7) from (2.3) by inserting $\gamma(k) = (k/r)^\theta$ and $r = 2$. The parameter θ varies from 0 to infinity and is introduced by Atkinson to capture alternative views on the importance of multiple deprivations. (Strictly speaking, both extreme values are inconsistent with the assumed continuity of the function γ, and should be seen as limiting cases.) When $\theta \to 0$, the index counts all people with at least one deprivation, regardless of their number for each individual: $A_0 = p_{1+} + p_{+1} - p_{11} = q_1 + q_2$. When $\theta = 1$, people with two deprivations are counted twice and A_1 gives the simple mean of the headcount rates in the two dimensions, providing the same result as with a composite index. As θ goes to infinity, the index tends to coincide with the proportion of people deprived on both dimensions: $A_\infty \to p_{11}$. As the original Atkinson's counting deprivation index, its generalisation to more than two dimensions obtained by inserting $\gamma(k) = (k/r)^\theta$ in (2.3) embodies, as limiting cases, both the union criterion (A_0) and the intersection criterion (A_∞). This index characterises a family of deprivation measures that may be seen as the analogue of the poverty measures proposed by Foster et al. (1984), referred to as the *FGT* measures.

The decomposition of the overall measures of deprivation in terms of the mean and the dispersion of the distributions of (transformed) deprivation counts is analogue to the mean-inequality decomposition of the social welfare functions derived from the expected utility theory. However, differently from the income inequality analysis, the structure of the decomposition of the deprivation measures is linearly additive rather than multiplicative and depends on whether social preferences are associated with the union or the intersection criterion. In the former case the deprivation measures fall and social welfare rises when the dispersion of deprivation across the population goes up, meaning that more people are affected by few or no deprivations. Even though they allow for the decomposition in terms of mean and dispersion of deprivation, the summary measures $d_\gamma(F)$ are silent about the role played by each dimension. Thus, the information provided by these summary measures should be complemented with estimates of the proportions of people who suffer from deprivation in each of the dimensions. This information reveals whether deprivation is concentrated on few or many dimensions.

2.4. The relationship between partial and complete orderings

An interesting question is to define the restrictions on γ that guarantee that d_γ ranks F_1 to be preferable to F_2 or vice versa. The answer is given by Theorems 2.2A and 2.2B, whose proofs require the following Lemma 1.

Lemma 1. *Let H be the family of bounded, continuous and non-negative functions on* $\left[0,1\right]$ *which are positive on* $\left\langle 0,1\right\rangle$ *and let g be an arbitrary bounded and continuous function on* $\left[0,1\right]$. *Then* $\int g(t)h(t)dt > 0$ *for all* $h \in H$ *implies* $g(t) \geq 0$ *for all* $t \in \left[0,1\right]$ *and the inequality holds strictly for at least one* $t \in \left\langle 0,1\right\rangle$.

Let ω_1 be a subset of the d_γ-family, defined as follows:

$$\omega_1 = \left\{\gamma : \gamma(k) > 0, \gamma''(k) > 0 \text{ for all } k \in \left[0, r\right], \text{ and } \gamma'(0) = 0\right\}.$$

Note that $\gamma'(0) = 0$ can be considered as a normalisation condition. The following result provides a characterisation of the second-degree downward distribution dominance.

Theorem 2.2A. *Let F_1 and F_2 be members of* ***F***. *Then the following statements are equivalent:*

(i) *F_1 second-degree downward dominates F_2;*

(ii) $d_\gamma(F_1) < d_\gamma(F_2)$ *for all* $\gamma \in \omega_1$.

Proof. Using integration by parts, we get:

$$d_\gamma(F_2) - d_\gamma(F_1) = \int_0^r \gamma(k)d(F_2(k) - F_1(k)) = \int_0^r \gamma'(k)(F_1(k) - F_2(k))\,\mathrm{d}k$$

$$= -\gamma'(0)\int_0^r (F_1(k) - F_2(k))dk + \int_0^r \gamma''(s)\int_s^r (F_1(k) - F_2(k))dkds = \int_0^r \gamma''(s)\int_s^r (F_1(k) - F_2(k))dkds.$$

Thus, if (i) holds then $d_\gamma(F_1) < d_\gamma(F_2)$ for all $\gamma \in \omega_1$. To prove the converse statement we restrict the preference functions to $\gamma \in \omega_1$. Hence,

$$d_\gamma(F_2) - d_\gamma(F_1) = \int_0^r \gamma''(s)\int_s^r (F_1(k) - F_2(k))dkds,$$

and the result is obtained by applying Lemma 1.

Theorem 2.2A shows that restricting the preference function γ to be increasing and convex ensures the equivalence between second-degree downward dominance and the ranking derived by using the d_γ-measures. If, by contrast, we take γ to be increasing and concave, then Theorem 2.2B provides the analogue result of Theorem 2.2A for upward dominance. Let ω_2 be a subset of the d_γ-family defined by

$$\omega_2 = \left\{\gamma : \gamma'(k) > 0, \gamma''(k) < 0 \text{ for } k \in \left\langle 0, r\right\rangle, \text{ and } \gamma'(r) = 0\right\}.$$

Theorem 2.2B. *Let F_1 and F_2 be members of* ***F***. *Then the following statements are equivalent:*

(i) *F_1 second-degree upward dominates F_2;*

(ii) $d_\gamma(F_1) < d_\gamma(F_2)$ *for all* $\gamma \in \omega_2$.

Proof. Using integration by parts, we get:

$$d_\gamma(F_2) - d_\gamma(F_1) = \int_0^r \gamma(k)d(F_2(k) - F_1(k)) = \int_0^r \gamma'(k)(F_1(k) - F_2(k))\,\mathrm{d}\,k$$

$$= -\gamma'(r)\int_0^r (F_1(k) - F_2(k))dt - \int_0^r \gamma''(s)\int_0^s (F_1(k) - F_2(k))dkds = -\int_0^r \gamma''(s)\int_0^s (F_1(k) - F_2(k))dkds.$$

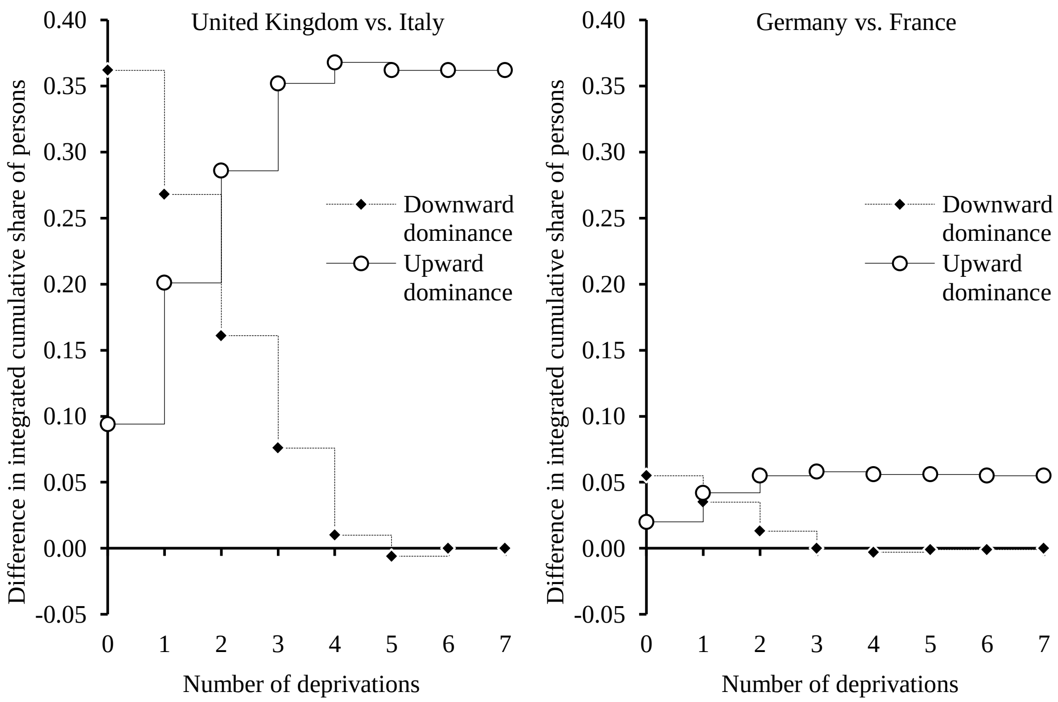
By using arguments like those in the proof of Theorem 2.2A the results of Theorem 2.2B are obtained.

Even though members of ω_1 and ω_2 are strictly convex and strictly concave, respectively, for brevity's sake we shall refer to them as convex and concave functions.

2.5. An empirical illustration

An empirical application can help illustrating the two notions of second-degree dominance as well as the summary measures of deprivations. As seen, a prime concern with the extent to which deprivation is diffused across the population (union criterion) should lead us to adopt second-degree upward dominance, whereas a concern with the occurrence of multiple deprivations (intersection criterion) should lead to favour second-degree downward dominance. Is this sufficient to rank count distributions in empirical applications? Not always. This can be seen by reconsidering the previous comparisons of Italy and the United Kingdom, and of France and Germany, where neither country in each comparison was found to first-degree dominate the other.

Figure 2.2: Second-degree dominance for material deprivation scores in selected European countries in 2012



Source: authors' elaboration on data from Eurostat (2014).

In Figure 2.2 we plot the difference between the integrated cumulative distributions considered by Definitions 2.2A and 2.2B for each pair of countries. If we integrate going up as in Definition 2.2B, the United Kingdom and Germany second-degree (upward) dominate Italy and France, respectively: the lower proportions of people who do not suffer from any deprivation give the first two countries an advantage that is not offset by their worst results for the incidence of people deprived in many dimensions. On the other hand, if we integrate going down as in Definition 2.2A, the difference between the integrated cumulative distributions changes from positive to negative and no country second-degree (downward) dominates the other in either comparison. The distribution of deprivation scores enables social evaluators favouring the union perspective to rank the United Kingdom and Germany ahead of Italy and France, but do not allow social evaluators supporting the intersection perspective to draw unambiguous conclusions. In such a case, higher degree criteria are needed, although they could still provide a partial ordering.

For the same countries considered so far, Table 2.1 shows the estimates for several summary measures of deprivation. The generalised Atkinson-type class of indices d_θ^{GA} is defined as:

(2.8) $$d_\theta^{GA} = r^{-\theta}\sum_{k=1}^{r} k^\theta q_k .$$

For $\theta = 1$, the previous expression gives the mean headcount ratio, which equals the ratio μ / r. For $\theta = 2$, it coincides with the convex version of the variance-type measure of deprivation $d_2^{V,convex}$ multiplied by r^{-2}, while the concave version ($\gamma(k) = 2rk - k^2$) is

(2.9) $$d_2^{V,concave} = 2r\mu - \sum_{k=1}^{r} k^2 q_k = 2r\mu - r^2 d_2^{GA} = 2r\mu - d_2^{V,convex} .$$

Table 2.1. Indices of material deprivations in selected European countries in 2012

Index		Germany	France	Italy	United Kingdom	Norway	Germany vs. France	United Kingdom vs. Italy
Linear indices								
Mean deprivations		0.822	0.877	1.471	1.109	0.320	-6.3	-24.6
Mean headcount ratio		0.091	0.097	0.163	0.123	0.036	-6.3	-24.6
Concave indices								
d_θ^{GA}	$\theta \to 0$	0.400	0.420	0.604	0.510	0.166	-4.8	-15.6
	$\theta = 0.1$	0.340	0.358	0.523	0.436	0.140	-5.0	-16.6
	$\theta = 0.5$	0.184	0.195	0.303	0.241	0.074	-5.7	-20.4
	$\theta = 0.9$	0.104	0.111	0.184	0.140	0.041	-6.2	-23.8
$d_2^{V,concave}$		12.550	13.399	21.883	16.747	4.914	-6.3	-23.5
Convex indices								
d_θ^{GA}	$\theta = 1.1$	0.080	0.086	0.146	0.109	0.031	-6.3	-25.3
	$\theta = 2$	0.028	0.029	0.057	0.040	0.010	-5.9	-30.0
	$\theta = 3$	0.011	0.011	0.023	0.016	0.004	-3.6	-31.6
	$\theta = 4$	0.005	0.005	0.010	0.007	0.002	0.4	-30.1
	$\theta = 8$	0.001	0.001	0.001	0.001	0.000	20.6	-13.5
	$\theta = 9$	0.0003	0.0002	0.0005	0.0005	0.0001	42.8	2.3
	$\theta = 20$	7.6×10^{-06}	1.3×10^{-06}	7.8×10^{-06}	9.4×10^{-06}	6.6×10^{-06}	479.9	20.9
$d_2^{V,convex} = r^2 d_2^{GA}$		2.246	2.387	4.595	3.215	0.846	-5.9	-30.0

Source: authors' elaboration on data from Eurostat (2014).

Norway shows the lowest mean number of deprivations followed by Germany and France, rather close each other, the United Kingdom, and finally Italy. The mean headcount ratio ranges between 3.6 per cent in Norway and 16.3 per cent in Italy. With a concave index, we always find that deprivation is lower in Germany than in France and in the United Kingdom than in Italy, which is not surprising in the light of the results on second-degree upward dominance reported above. On the other hand, the lack of second-degree downward dominance in these same comparisons is noticeable in the fact that the rankings reverse as the functions become more convex. For instance, the generalised Atkinson-type deprivation index turns out to be lower in France than in Germany for values of θ higher than 4. The French overall deprivation is below the German level whenever we favour the intersection criterion and weight somebody suffering from 2h deprivations at least 16 ($=2^4$) times somebody suffering from h deprivations (as the index d_4^{GA} assigns each person with h deprivations a weight equal to h^4). Since the United Kingdom fares much better than Italy except than in the occurrence of very severe deprivation (6 or more items), the ranking between the two countries changes only for high values of θ, which correspond to an extreme aversion to the worst conditions of deprivations. Finally, note that the generalised Atkinson-type deprivation index approaches the proportion of people experiencing at least one deprivation (union criterion) as θ tends to 0 and the proportion of people suffering from the maximum number of deprivations (intersection criterion) as θ goes to infinity; as nobody lacks all nine items, in the latter case the index converges to zero in all countries.

3. Association rearrangements

In many respects, the discussion so far has proceeded as in the case of a single variable, whereas the key feature of the multivariate case is the pattern of association across dimensions. It is then natural to ask how social welfare responds to a change in the distribution of deprivations across the population, though the total number of deprivations remains the same. The most common approach for evaluating multidimensional measures of poverty and inequality is to consider how social welfare varies after a “marginal-free change” in the association between two variables, which is a change that does not affect the marginal distributions.[8] In the real world, however, the condition of marginal-free changes may be too restrictive, as policies may reduce deprivation in one dimension at the cost of increasing deprivation in another. We hence adopt a more general approach and we require that only the mean number of deprivations but not the marginal distributions be kept fixed. (The latter implies the former, but not vice versa.) It follows that we need a measure of association that is invariant with regard to changes in the marginal distributions, unlike the correlation coefficient. This is the case of the cross-product κ introduced by Yule (1900). In the 2x2 distribution of Table 2.1, Yule’s measure is defined by

$$\kappa = \frac{p_{00}\,p_{11}}{p_{01}\,p_{10}}, \tag{3.11}$$

[8] For instance, Bourguignon and Chakravarty (2003, 2009) and Atkinson (2003) use the principle of marginal-free correlation increasing shifts as a basis for making a normative judgement of poverty measures derived from continuous variables (attributes) rather than from deprivation scores. They distinguish whether the poverty measure increases or decreases because of a correlation increasing shift, and consider the associated attributes to be substitutes (one attribute can compensate for the lack of the other) in the former case and to be complements in the latter.

which is invariant to the transformation $p_{ij} \to a_i b_j p_{ij}$, that is does not change if the marginal distributions (p_{0+}, p_{1+}) and (p_{+0}, p_{+1}) change. This association measure, together with the marginal distributions, provides complete information on the distribution. Note that $\kappa \in [0, \infty\rangle$, $\kappa = 1$ if X_1 and X_2 are independent, $\kappa = 0$ if there is perfect negative association ($p_{00} = 0$ and/or $p_{11} = 0$), and $\kappa \to \infty$ if there is perfect positive association ($p_{01} = 0$ and/or $p_{10} = 0$).

Following Aaberge and Peluso (2011), we relax the marginal-free condition by introducing an association increasing/decreasing rearrangement principle that relies on the condition of fixed overall mean number of deprivations rather than on the condition of fixed proportions of people suffering from each deprivation. Marginal-free arrangements are special cases of this alternative rearrangement principle.[9]

Definition 3.1. *Consider a 2x2 table with parameters* $(p_{00}, p_{01}, p_{10}, p_{11})$ *where* $\sum_i \sum_j p_{ij} = 1$ *. The change* $(p_{00} + \varepsilon, p_{01}, p_{10} - 2\varepsilon, p_{11} + \varepsilon)$ *is said to provide a mean preserving positive association increasing (decreasing) rearrangement if* $\varepsilon > 0$ *(* $\varepsilon < 0$ *) and* $\kappa > 1$*, and a mean preserving negative association increasing (decreasing) rearrangement if* $\varepsilon < 0$ *(* $\varepsilon > 0$ *) and* $\kappa < 1$.

It follows from Definition 3.1 that a mean preserving rearrangement reduces the number of people deprived according to indicator X_1 at the cost of increasing the number of people deprived according to indicator X_2 when $\varepsilon > 0$ and vice versa when $\varepsilon < 0$.

Aaberge and Peluso (2011) show how to extend Definition 3.1 to *r* dimensions. As the standard subscript notation becomes cumbersome for more than two dimensions, they simplify the notation to p_{ijm}, where *i* and *j* represent two arbitrary chosen deprivation dimensions and *m* represents the remaining $r-2$ dimensions. The Yule's measure κ_{ijm} is defined by

(3.2) $$\kappa_{ijm} = \frac{p_{iim} p_{jjm}}{p_{ijm} p_{jim}},$$

where *m* is a $(r-2)$-dimensional vector of any combination of zeroes and ones. In this case, the association is defined by $r(r-1)/2$ cross-products. Aaberge and Peluso (2011) introduce the following generalisation of Definition 3.1:

Definition 3.2A. *Consider a 2x2x…x2 table formed by s dichotomous variables with parameters* $(p_{iim}, p_{ijm}, p_{jim}, p_{jjm})$ *where* $\sum_i \sum_j \sum_m p_{ijm} = 1$ *and* $\kappa_{ijm} > 1$*. The following change* $(p_{iim} + \varepsilon, p_{ijm}, p_{jim} - 2\varepsilon, p_{jjm} + \varepsilon)$ *is said to provide a mean preserving positive association increasing (decreasing) rearrangement if* $\varepsilon > 0$ *(* $\varepsilon < 0$ *).*

Definition 3.2B. *Consider a 2x2x…x2 table formed by s dichotomous variables with parameters* $(p_{iim}, p_{ijm}, p_{jim}, p_{jjm})$ *where* $\sum_i \sum_j \sum_m p_{ijm} = 1$ *and* $\kappa_{ijm} < 1$*. The following*

[9] Note that the multinomial distribution defined by the parameters p_{00}, p_{10}, p_{01} and $p_{11} (= 1 - p_{00} - p_{10} - p_{01})$ can alternatively be described by the marginal distributions $(p_{0+}, p_{1+} = 1 - p_{0+})$ and $(p_{+0}, p_{+1} = 1 - p_{+0})$, and the cross-product *κ*.

change $(p_{iim}+\varepsilon, p_{ijm}, p_{jim}-2\varepsilon, p_{jjm}+\varepsilon)$ *is said to provide a mean preserving negative association increasing (decreasing) rearrangement if* $\varepsilon<0$ *(* $\varepsilon>0$ *).*

Theorems 3.1A below demonstrates that social preferences favouring second-degree downward dominance imply that overall deprivation rises after a mean preserving positive association increasing rearrangement as well as a mean preserving negative association decreasing rearrangement. By contrast, Theorem 3.1B proves that preferences favouring second-degree upward dominance consider such rearrangement as a reduction in the overall deprivation. Moreover, it follows directly from the decomposition (2.4) that the principles of mean preserving association increasing/decreasing rearrangement are equivalent to the mean preserving spread/contraction defined by

Definition 3.3. *Let* F_1 *and* F_2 *be members of the family* ***F*** *of count distributions based on r deprivations and assume that they have equal means. Then* F_2 *is said to differ from* F_1 *by a mean preserving spread (contraction) if* $\delta_\gamma(F_2)>\delta_\gamma(F_1)$ *for all convex* γ *(* $\delta_\gamma(F_2)<\delta_\gamma(F_1)$ *for all concave* γ *).*

Definition 3.3 is equivalent to a sequence of the mean preserving spread introduced by Rothschild and Stiglitz (1970).

Recalling that all members of the set ω_1 are increasing convex functions, and all members of ω_2 are increasing concave functions, it is then possible to prove Theorems 3.1A and 3.1B.

Theorem 3.1A. *Let* F_1 *and* F_2 *be members of the family* ***F*** *of count distributions based on r deprivations and assume that they have equal means. Then the following statements are equivalent:*

(i) $d_\gamma(F_1)<d_\gamma(F_2)$ *for all* $\gamma\in\omega_1$*;*

(ii) F_2 *can be obtained from* F_1 *by a sequence of mean preserving positive association increasing rearrangements when* $\kappa>1$ *for both* F_1 *and* F_2*, a sequence of mean preserving negative association decreasing rearrangements when* $\kappa<1$ *for both* F_1 *and* F_2*, and a combination of mean preserving positive association increasing and negative association decreasing rearrangements when* $\kappa>1$ *for either* F_1 *or* F_2*;*

(iii) F_2 *can be obtained from* F_1 *by a mean preserving spread.*

Proof. The equivalence between (i) and (iii) follows from the equivalence between (i) of Theorem 2.2A and (ii) of Theorem 3.1A, which was proved by Aaberge and Peluso (2011). The equivalence between (i) and (iii) follows directly from the second terms of equations (2.4).

Theorem 3.1B. *Let* F_1 *and* F_2 *be members of the family* ***F*** *of count distributions based on r deprivations and assume that they have equal means. Then the following statements are equivalent:*

(i) $d_\gamma(F_1)<d_\gamma(F_2)$ *for all* $\gamma\in\omega_2$*;*

(ii) F_2 *can be obtained from* F_1 *by a sequence of mean preserving positive association decreasing rearrangements when* $\kappa>1$ *for both* F_1 *and* F_2*, a sequence of mean*

preserving negative association increasing rearrangements when $\kappa < 1$ *for both* F_1 *and* F_2*, and a combination of mean preserving positive association decreasing and negative association increasing rearrangements when* $\kappa > 1$ *for either* F_1 *or* F_2*;*

(iii) F_2 *can be obtained from* F_1 *by a mean preserving contraction.*

Proof. The proof of Theorem 3.1B is analogous to the proof of Theorem 3.1A.

Following the distinction made by Bourguignon and Chakravarty (2003, 2009) and Atkinson (2003), the results of Theorem 3.1A (3.1B) justify the use of d_γ for convex γ (concave γ) when the attributes associated with the deprivation indicators can be considered as substitutes (complements). Theorems 3.1A and 3.1B show that d_γ satisfies the mean preserving association rearrangement principles, where a distinction is made between whether a rearrangement comes from a distribution characterised by positive or negative association. Consider the specific subfamily of two-dimensional deprivation measures discussed by Atkinson (2003) and defined by (2.8), and assume that there is positive association between the two deprivations ($\kappa > 1$). The d_γ-function associated with the family A_θ is concave for $\theta < 1$ and convex for $\theta > 1$, and approaches the union condition when $\theta \to 0$ and the intersection condition when $\theta \to \infty$. Theorem 3.1B states that a sequence of mean preserving positive association decreasing rearrangements raises the overall deprivation A_θ if $\theta < 1$. Is it reasonable to suppose that the overall deprivation rises as we observe a reduction in the positive association between deprivations in the two attributes? After all, the share of people suffering from deprivation for both attributes falls, while the total number of deprivations does not vary. The answer is positive if we regard the two attributes as complements, which means that we rule out any trade-off between them, and we dislike the fact that more people are deprived more than the fact that fewer people are hit more.

4. Poverty

4.1. Threshold-specific measures

So far, we have been concerned with the distribution of deprivation counts, irrespective of how many people are regarded as poor when deprivation and poverty are considered as distinct concepts. In terms of the classical distinction made by Sen (1976), we have focused only on the "aggregation" of the characteristics of deprivation into an overall measure of deprivation, ignoring the first step concerning the "identification" of the poor. The emphasised contrast between the union and the intersection criteria suggests, however, that there is some leeway in defining who is poor. For instance, Bourguignon and Chakravarty (2003) and Tsui (2002) adopt the more extensive union criterion and define people to be (multidimensional) poor if they suffer from at least one deprivation. In this case deprivation and poverty come to coincide. On the other hand, the European Union regards as severally materially deprived all persons who cannot afford at least four out of nine amenities, moving midway between the union and the (strict) intersection views. Alkire and Foster (2011) formalise what they label the "dual cut-off" identification system, where the dimension-specific thresholds are integrated with a further threshold that identifies the minimum number of deprivations to be classified as poor. If a person is poor when he or she is deprived in at least *c*, $1 \le c \le r$, dimensions, the headcount ratio is uniquely determined by the count distribution *F* and is defined by

(4.1) $$H(c) = 1 - F(c-1) = \sum_{k=c}^{r} q_k \,.$$

In the case of the European indicator of severe material deprivation, *c* equals 4. As the choice of a specific cut-off *c* is arbitrary, it is useful to check the sensitivity of the ranking of distributions to *c* by treating $H(c)$ as a function of *c*, henceforth labelled “headcount curve”. As evident from (4.1), the condition of first-degree dominance of headcount curves is equivalent to first-degree dominance of the associated count distributions. If $c > 1$, first-degree dominance for headcount curves is a less demanding condition than that for the overall count distribution, as it ignores what happens to those that suffer from deprivation in fewer than *c* dimensions. Moreover, the second-degree dominance results of Theorems 3.1A and 3.1B are also valid for the headcount curve, which means that $H(c)$ satisfies the principle of association increasing/decreasing rearrangements when this principle is restricted to be applied among the poor.

To complement the information provided by the headcount ratio, we may employ the measures defined by (2.3) as overall measures of poverty for the conditional count distribution $F(k;c)$ defined by

(4.2) $$F(k;c) = \Pr(X \le k \,|\, X \ge c) = \frac{F(k) - F(c-1)}{1 - F(c-1)} = \frac{\sum_{j=c}^{k} q_j}{\sum_{j=c}^{r} q_j}, \quad k = c, c+1, ..., r \,,$$

with mean given by

(4.3) $$\mu(c) = \frac{\sum_{j=c}^{r} j q_j}{\sum_{j=c}^{r} q_j} \,.$$

Expressions (2.4) and (2.5) show that the overall measures of poverty for $F(k;c)$ admits a decomposition into the mean (or a function of the mean) and a measure of dispersion. An analogue to the *FGT* family of poverty measures is obtained by inserting $\gamma(k) = k^{\theta}$ in expression (2.3).

As an alternative, Alkire and Foster (2011) propose to combine the headcount ratio $H(c)$ and the conditional mean $\mu(c)$ and introduce the adjusted headcount ratio defined by

(4.4) $$M_1(c) = \frac{H(c)\mu(c)}{r} = \frac{1}{r}\sum_{k=c}^{r} k q_k \,,$$

which is the ratio of the total number of deprivations experienced by the poor to the maximum number of deprivations that could be experienced by the entire population. For $c = 1$, the index $M_1(c)$ coincides with the Atkinson-type primal measure of deprivation d_1^{GA}. Alkire and Foster (2011, p. 482) underline that both the identification of the poor and the adjusted headcount ratio are invariant to monotonic transformations applied to the deprivation variables and the respective thresholds. Moreover, the index $M_1(c)$ increases if a poor person becomes deprived in an additional dimension (dimensional monotonicity), is decomposable by population subgroups, and can be broken down by indicator as it is the (weighted) average of the deprivations headcount ratios for each dimension computed considering only the poor at the numerator (so-called "censored headcount ratios"). On the other hand, this index is indifferent to changes in the way deprivations are distributed across the poor.

A general family of adjusted poverty measures that take into account not only the average deprivation experienced by the poor, $\mu(c)$, but also the distribution of deprivations across the poor can be derived from the d_γ-measure defined by (2.3)

(4.5) $$M_\gamma(c) = \frac{H(c)d_\gamma(c)}{r},$$

where the d_γ -index for $F(k;c)$ is given by:

(4.6) $$d_\gamma(c) = \frac{\sum_{k=c}^{r}\gamma(k)q_k}{\sum_{k=c}^{r}q_k}.$$

Inserting (4.6) into (4.5) yields:

(4.7) $$M_\gamma(c) = \frac{1}{r}\sum_{k=c}^{r}\gamma(k)q_k .$$

Such a measure may weight differently poor persons according to the number of deprivations from which they suffer. Setting $\gamma(k) = k^\theta$ into (4.7) yields the general family of adjusted *FGT* measures for count data

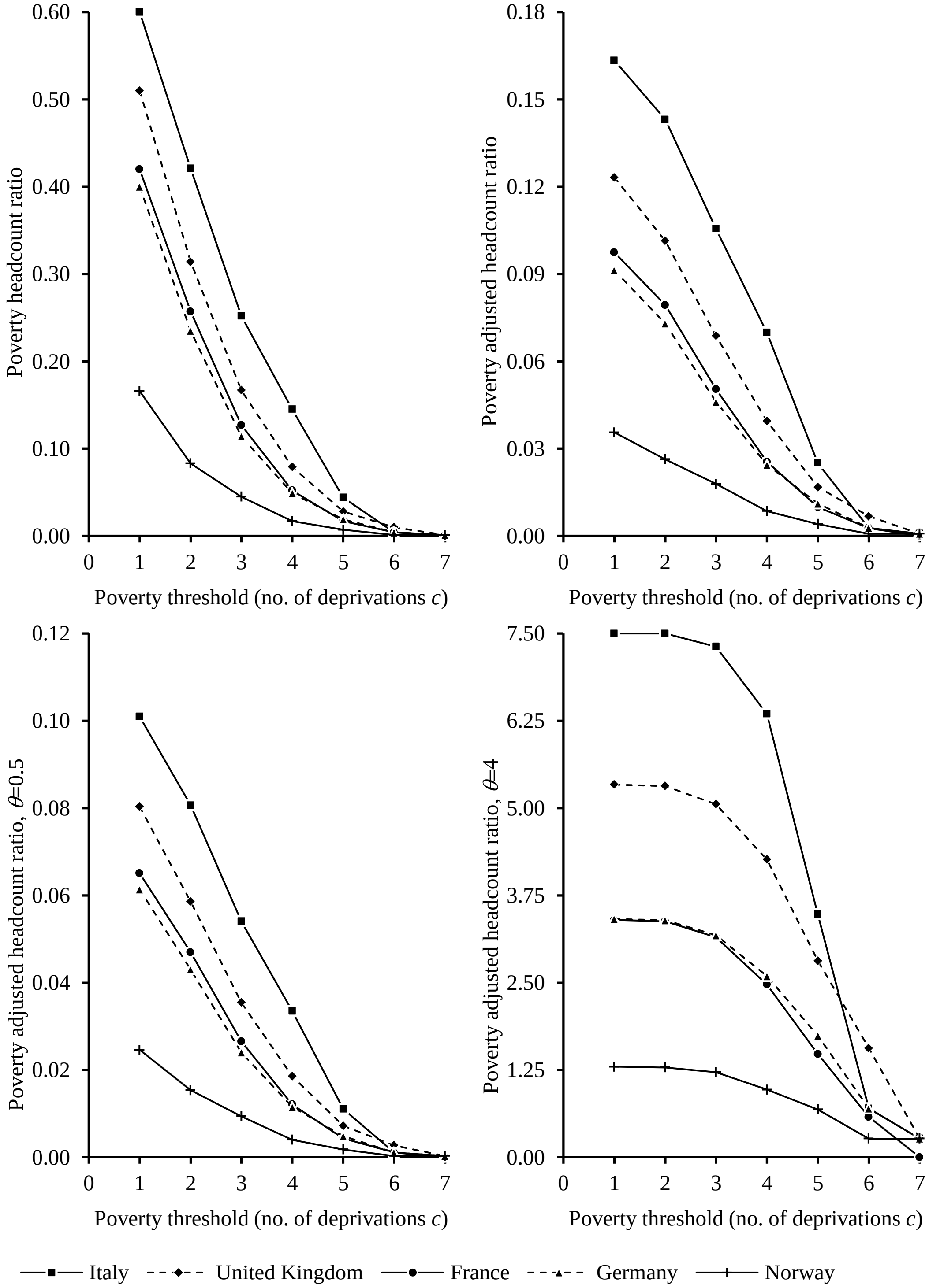
(4.8) $$M_\theta(c) = \frac{1}{r}\sum_{k=c}^{r}k^\theta q_k, \quad \theta > 0,$$

which encompasses (4.4) for $\theta = 1$. When $\theta \to 0$, the adjusted *FGT* measure reaches its minimum value $H(c)/r$, which ignores altogether any cumulative effect of multiple deprivations. As θ rises, greater weight is placed on those who suffer from deprivation in several dimensions.

The adjusted headcount ratio $M_1(c)$ proposed by Alkire and Foster (2011) provides the theoretical basis for the Multidimensional Poverty Index (*MPI*) developed by Alkire and Santos (2010). The *MPI* has replaced the *HPI* in the reports of the United Nations Development Programme since 2010 in order to capture "... how many people experience overlapping deprivations and how many deprivations they face on average" (UNDP, 2010, p. 95). The *MPI* considers ten dichotomous indicators for three dimensions: health, education and living standards. Dimensions, and indicators within each dimension, are equally weighted, and the cut-off *c* for the number of (weighted) deprivations is set at three out of a maximum of ten.

Figure 4.1 compares how poverty headcount ratios change as we vary the poverty cut-off using the deprivation indicators in the five European countries considered earlier.

Figure 4.1. Poverty headcount and adjusted headcount ratios for different poverty cut-offs in selected European countries in 2012



Source: authors' elaboration on data from Eurostat (2014).

The proportion of poor people, shown in the top-left panel, fell by three fourths in Italy and around nine tenths in the other countries as the poverty cut-off is raised from one deprivation (union criterion) to four deprivations (the European criterion). Censoring at four deprivations implies excluding from measured poverty a substantial fraction of population suffering from one, two or three deprivations: 15 per cent in Norway and 46 per cent in Italy, accounting for 76 and 57 per cent of all deprivations, respectively. However, the ranking of countries does not change. It changes by setting the cut-off at five deprivations, when

Germany and France reverse their order, and again at six deprivations, when the United Kingdom becomes the country with the highest share of poor people. In the top-right panel, the ranking is the same for the adjusted headcount ratio $M_1(c)$, except for a better position granted to France by its lower average intensity of deprivation ($\mu(c)/r$) when the cut-off is set at six deprivations. The bottom panels show results for the adjusted *FGT* measure $M_\theta(c)$: lowering the weights of multiple deprivations ($\theta = 0.5$; left panel) does not modify the sorting produced by the adjusted headcount ratio, whereas significantly raising them ($\theta = 4$; right panel) steadily switches the positions of Germany and France, as seen in Section 2.5. This comparison reveals that varying the poverty cut-off has a considerable impact on measured poverty, whereas adjusting the headcount ratio for the deprivations experienced by the poor seems to have minor effects, unless their distribution is taken into account.

4.2. Threshold-free measures

Setting the poverty threshold at a minimum number of deprivations *c* is admittedly arbitrary. This raises a number of problems. Firstly, empirical results depend on the value of *c*, as just seen. Secondly, poverty measures do not obey the association rearrangement and the mean preserving spread principles. Thirdly, the information about the deprivations of people who suffer from less than *c* deprivations is ignored. Can we identify a measure of poverty which is threshold-free? In order to address these issues, we draw on the approach proposed by Aaberge and Atkinson (2013) for measuring financial poverty and define the weighted average Δ_ξ of poverty headcounts $H(c)$

(4.9)
$$\Delta_\xi(H) = \sum_{c=1}^{r} \xi(c) H(c),$$

where the weighting function $\xi(c)$ increases with the number of deprivations *c*. As *c* varies from 1 to *r*, the expression $\Delta_\xi(H)$ accounts for all possible poverty thresholds and hence identifies a class of threshold-free measures of poverty.

The poverty measures $\Delta_\xi(H)$ are closely connected with the overall measures of deprivation $d_\gamma(F)$ defined earlier. Replacing the expression (4.1) for *H*(*c*) in (4.9) yields:

(4.10)
$$\Delta_\xi(H) = \sum_{c=1}^{r} \xi(c) \sum_{j=c}^{r} q_j = \sum_{c=1}^{r} \left(\sum_{j=1}^{c} \xi(j) \right) q_c \,.$$

Comparing (4.10) with (2.3), we find that $\Delta_\xi(H) = d_\gamma(F)$ if and only if $\gamma(k) = \sum_{j=1}^{k} \xi(j)$, which yields $\xi(k) = \gamma(k) - \gamma(k-1)$ and $\sum_{k=1}^{r} \xi(k) = \gamma(r)$. Thus, the family of overall measures of deprivation $d_\gamma(F)$ can also be interpreted as a family of threshold-free measures of poverty. For example, the following family of weighting functions $\xi(k) = (k/r)^\theta - ((k-1)/r)^\theta$ for $\Delta_\xi(H)$ corresponds to the weighting functions $\gamma(k) = (k/r)^\theta$ for the *FGT* subfamily of $d_\gamma(F)$. Thus, the empirical results presented in Section 2.5 can be seen as estimates of threshold-free poverty measures.

5. Inequality

The analytical framework discussed in the previous Sections can be easily adapted to measure the inequality in the distribution of deprivation scores, or more intuitively achievement scores. Let ρ_j be the proportion of people whose achievements are above the attribute-specific thresholds in *j* dimensions and $G(k)=\sum_{j=0}^{k}\rho_j$ be the cumulative proportion of people who have an achievement score not higher than *k*. Similarly to the discussion for the distribution of deprivation counts in Section 2.2, we can define the following social evaluation function:

$$w_\eta(G)=\sum_{k=0}^{r}\eta(k)\rho_k\,. \tag{5.1}$$

where η is a non-negative and non-decreasing concave function capturing the preferences of a social evaluator who supports the independence axiom for orderings defined on the set of *G*-distributions and $0\le w_\eta(G)\le\eta(\nu)$, where ν is the mean achievement score. The proportional shortfall of $w_\eta(G)$ relative to its maximum value gives the measure $J_\eta(G)$ of the inequality in the achievements in the *r* dimensions:

$$J_\eta(G)=1-\frac{w_\eta(G)}{\eta(\nu)}=1-\frac{\sum_{k=0}^{r}\eta(k)p_k}{\eta(\nu)}\,, \tag{5.2}$$

Note that $G(k)=1-F(r-k-1)$, where *F* is the count distribution of deprivations, and $\nu+\mu=r$, where μ is the mean number of deprivations discussed in Section 2. The sum of the mean number of achievements and the mean number of deprivations is necessarily equal to the number of attributes. By taking $\eta(k)=\gamma(r)-\gamma(r-k)$ and $\rho_k=q_{r-k}$, we get

$$w_\eta(G)=\sum_{k=0}^{r}\eta(k)q_{r-k}=\sum_{k=0}^{r}\eta(r-k)q_k=\sum_{k=0}^{r}\big(\gamma(r)-\gamma(k)\big)q_k=\gamma(r)-d_\gamma(F) \tag{5.3}$$

and $\eta(\nu)=\gamma(r)-\gamma(\mu),$ which yield the following alternative expression for $J_\eta(G)$:

$$J_\eta(G)=\frac{d_\gamma(F)-\gamma(\mu)}{\gamma(r)-\gamma(\mu)}=\frac{\delta_\gamma(F)}{\gamma(r)-\gamma(\mu)}\,, \tag{5.4}$$

where $\delta_\gamma(F)$ is defined by (2.5) and γ is a non-decreasing convex function such that $\gamma(\mu)\le d_\gamma(F)\le\gamma(r)$. Thus, inequality in the count distribution of achievements can be expressed in terms of the social evaluation of deprivation. Note that the notion of inequality is closely associated with the intersection view of deprivation, whereas it is in conflict with the union view.

Similarly to the equally distributed equivalent income introduced by Atkinson (1970), it is possible to define the equally distributed equivalent achievement count defined by

(5.5) $$k^* = \eta^{-1}(w_\eta(G)),$$

where $0 \le k^* \le \nu$. Using (5.3) and (2.4), (5.5) becomes

(5.6) $$k^* = \eta^{-1}\left(\gamma(r) - d_\gamma(F)\right) = \eta^{-1}\left(\gamma(r) - \gamma(\mu) - \delta_\gamma(F)\right)$$

which yields the following alternative family of inequality measures

(5.7) $$\Psi_\eta(G) = 1 - \frac{k^*}{\nu} = 1 - \frac{\eta^{-1}\left(\gamma(r) - d_\gamma(F)\right)}{\nu}.$$

Expression (5.6) demonstrates that k^* increases with decreasing mean number of deprivations and with decreasing spread of the deprivations. Hence, a social evaluator would consider an increase in the spread of deprivations as a welfare loss.

6. Accounting for different weights

Until now, we have not considered the cases of unequal weighting of the dimensions. However, the results provided by Theorems 2.1, 2.2A, 2.2B, 3.1A and 3.1B remain valid for the distribution of weighted deprivation counts. To account for different weights, we can apply the procedure suggested by Alkire and Foster (2011) to replace the deprivation count for each person by the sum of the associated weights.

Replacing the outcome 1 for dimensions 1 and 2 by the weights w_1 and w_2, respectively, in the two-dimensional case, the distribution of deprivations in two dimensions shown in Table 2.1 generates the following Table 6.1.

Table 6.1. The distribution of weighted deprivations in two dimensions

		X_1^*		
		0	w_2	
X_2^*	0	p_{00}	p_{01}	p_{0+}
	w_1	p_{10}	p_{11}	p_{1+}
		p_{+0}	p_{+1}	1

Source: authors' elaboration.

By assuming that $w_1 \le w_2$, the variable X^* defined by $X^* = X_1^* + X_2^* = w_1 X_1 + w_2 X_2$ can be considered as a weighted counting variable. The distribution F^* of X^* is given by:

(6.1) $$F^*(z) = \begin{cases} p_{00} & if\ z = 0 \\ p_{00} + p_{10} & if\ z = w_1 \\ p_{00} + p_{10} + p_{01} & if\ z = w_2 \\ 1 & if\ z = w_1 + w_2 \end{cases}$$

Using integration by parts we get the following expression for d_γ:

(6.2) $$d_\gamma(F) = \int \gamma(z)dF(z) = \int \gamma'(z)\big((1-F(z)\big)dz\,,$$

where γ' is the derivative of γ. Inserting F^* for F in (6.2) yields

(6.3) $$\begin{gathered} d_\gamma(F^*) = \gamma(w_1)\big(1-p_{00}\big) + \big(\gamma(w_2)-\gamma(w_1)\big)\big(1-p_{00}-p_{10}\big) + \\ \big(\gamma(w_1+w_2)-\gamma(w_2)\big)\big(1-p_{00}-p_{10}-p_{01}\big). \end{gathered}$$

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