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Abstract

In the context of the binomial decomposition of OWA functions, we investigate the parametric constraints associated with the 3-additive case in n dimensions. The resulting feasible region in two coefficients is a convex polygon with n vertices and n edges, and is strictly increasing in the dimension n. The orness of the OWA functions within the feasible region is linear in the two coefficients, and the vertices associated with maximum and minimum orness are identified.

Keywords: Generalized Gini welfare functions and inequality indices, symmetric capacities and Choquet integrals, OWA functions and orness, binomial decomposition and k-additivity.

JEL Classification: D31, D63, I31.

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1 Introduction

The generalized Gini welfare functions introduced by Weymark [53] and the associated inequality indices in Atkinson-Kolm-Sen's (AKS) framework are related by Blackorby and Donaldson's correspondence formula [5, 6], $A(\boldsymbol{x}) = \bar{x} - G(\boldsymbol{x})$, where $A(\boldsymbol{x})$ denotes a generalized Gini welfare function, $G(\boldsymbol{x})$ is the associated absolute inequality index, and $\bar{x}$ is the plain mean of the income distribution $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{D}^n$ of a population of $n \geq 2$ individuals, with $\mathbb{D} = [0, \infty)$.

The generalized Gini welfare functions [53] have the form $A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)}$ where $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$ and, as required by the principle of inequality aversion, $w_1 \geq w_2 \geq \ldots \geq w_n \geq 0$ with $\sum_{i=1}^{n} w_i = 1$. These welfare functions correspond to the S-concave class of the ordered weighted averaging (OWA) functions introduced by Yager [56], which in turn correspond [22] to the Choquet integrals associated with symmetric capacities.

The use of non-additivity and Choquet integration [16] in Social Welfare and Decision Theory dates back to the seminal work of Schmeidler [48, 49], Ben Porath and Gilboa [4], and Gilboa and Schmeidler [25, 26]. In the discrete case, Choquet integration [46, 14, 17, 27, 28, 40] corresponds to a generalization of both weighted averaging (WA) and ordered weighted averaging (OWA), which remain as special cases. For recent reviews of Choquet integration see Grabisch and Labreuche [33, 34, 35], and Grabisch, Kojadinovich, and Meyer [32].

The complex structure of Choquet capacities can be suitably described in the k-additivity framework introduced by Grabisch [29, 30], see also Calvo and De Baets [11], Cao-Van and De Baets [13], and Miranda, Grabisch, and Gil [45]. The 2-additive case, in particular, has been examined by Miranda, Grabisch, and Gil [45], and Mayag, Grabisch, and Labreuche [42, 43]. Due to its low complexity and versatility the 2-additive case is relevant in a variety of modelling contexts.

The characterization of symmetric Choquet integrals (OWA functions) has been studied by Fodor, Marichal and Roubens [22], Calvo and De Baets [11], Cao-Van and De Baets [13], and Miranda, Grabisch and Gil [45]. It is shown, see Gajdos [24], that in the k-additive case the generating function of the OWA weights is polynomial of degree k, where the weights correspond to differences between consecutive generating function values, as illustrated in (27). In the symmetric 2-additive case, in particular, the generating function is quadratic and thus the weights are equidistant, as in the classical Gini welfare function.

In this paper we review the analysis of symmetric capacities in the Möbius representation framework and we recall the binomial decomposition of OWA functions due to Calvo and De Baets [11], see also [10]. The binomial decomposition can be formulated in terms of two equivalent functional bases, the binomial Gini welfare functions and the Atkinson-Kolm-Sen (AKS) associated binomial Gini inequality indices, according to Blackorby and Donaldson's correspondence formula.

The binomial Gini welfare functions, denoted C_j with $j = 1, \ldots, n$, have null weights associated with the $j - 1$ richest individuals in the population and therefore they are progressively focused on the poorest part of the population. Correspondingly, the associated binomial Gini inequality indices, denoted G_j with $j = 1, \ldots, n$, have equal weights associated with the $j - 1$ richest individuals in the population and therefore they are progressively insensitive to income transfers within the richest part of the population.

The paper is organized as follows. In Section 2 we review the basic notions of welfare function and inequality index for populations of $n \geq 2$ individuals. In Section 3 we present the basic definitions and results on capacities and Choquet integration, with reference to the Möbius representation framework. In Section 4 we consider the context of symmetric capacities and we recall the binomial decomposition of OWA functions due to Calvo and De Baets [11], see also [10].

In Section 5 we present the main results of the paper. We examine the binomial decomposition of OWA functions focusing on the 2-additive and 3-additive cases. In particular, we investigate the parametric constraints associated with the 3-additive case in n dimensions. The resulting feasible region in two coefficients is a convex polygon with n vertices and n edges, and is strictly increasing in the dimension n. The orness of the OWA functions within the feasible region is linear in the two coefficients, and the vertices associated with maximum and minimum orness are identified. Finally, Section 6 contains some conclusive remarks.

2 Welfare functions and inequality indices

In this section we consider populations of $n \geq 2$ individuals and we briefly review the notions of welfare function and inequality index in the standard framework of averaging functions on the $\mathbb{D}^n$ domain, with $\mathbb{D} = [0, \infty)$. The income distributions in this framework are represented by points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$. In any case, most of our results hold analogously over different domains, for instance the reduced domain $[0, 1]$ or even the extended domain $\mathbb{R}$.

We begin by presenting notation and basic definitions regarding averaging functions on the domain $\mathbb{D}^n$, with $n \geq 2$ throughout the text. Comprehensive reviews of averaging functions can be found in Fodor and Roubens [23], Calvo et al. [12], Beliakov et al. [2], and Grabisch et al. [36].

Notation. Points in $\mathbb{D}^n$ are denoted $\boldsymbol{x} = (x_1, \ldots, x_n)$, with $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{0} = (0, \ldots, 0)$. Accordingly, for every $x \in \mathbb{D}$, we have $x \cdot \mathbf{1} = (x, \ldots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i = 1, \ldots, n$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in \mathbb{D}^n$, the increasing and decreasing reorderings of the coordinates of $\boldsymbol{x}$ are indicated as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \ldots, x_n\} = x_{[n]}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\} = x_{[1]}$. In general, given a permutation σ on $\{1, \ldots, n\}$, we denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Finally, the arithmetic mean is denoted $\bar{x} = (x_1 + \cdots + x_n)/n$.

Definition 1 *Let $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ be a function.*

1. *A is* monotonic *if $\boldsymbol{x} \geq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \geq A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$. Moreover, A is* strictly monotonic *if $\boldsymbol{x} > \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) > A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$.*

2. *A is* idempotent *if $A(x \cdot \mathbf{1}) = x$, for all $x \in \mathbb{D}$. On the other hand, A is* nilpotent *if $A(x \cdot \mathbf{1}) = 0$, for all $x \in \mathbb{D}$.*

3. *A is* symmetric *if $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$, for any permutation σ on $\{1, \ldots, n\}$ and all $\boldsymbol{x} \in \mathbb{D}^n$.*

4. *A is* invariant for translations *if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, A is* stable for translations *if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x}) + t$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$.*

5. *A is* invariant for dilations *if $A(t \cdot \boldsymbol{x}) = A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\mathbf{x} \in \mathbb{D}^n$. On the other hand, A is* stable for dilations *if $A(t \cdot \boldsymbol{x}) = t\, A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$.*

We introduce the majorization relation on $\mathbb{D}^n$ and we discuss the concept of income transfer following the approach in Marshall and Olkin [41], focusing on the classical results relating majorization, income transfers, and bistochastic transformations, see Marshall and Olkin [41, Ch. 4, Prop. A.1].

Definition 2 *The* majorization relation $\preceq$ *on $\mathbb{D}^n$ is defined as follows: given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we say that*

$$\boldsymbol{x} \preceq \boldsymbol{y} \quad \textit{if} \quad \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \qquad k = 1, \ldots, n \tag{1}$$

where the case $k = n$ is an equality due to $\bar{x} = \bar{y}$. As usual, we write $\boldsymbol{x} \prec \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and not $\boldsymbol{y} \preceq \boldsymbol{x}$, and we write $\boldsymbol{x} \sim \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and $\boldsymbol{y} \preceq \boldsymbol{x}$. We say that $\boldsymbol{y}$ majorizes $\boldsymbol{x}$ if $\boldsymbol{x} \prec \boldsymbol{y}$, and we say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are indifferent if $\boldsymbol{x} \sim \boldsymbol{y}$.

Another traditional reading, which reverses that of majorization, refers to the concept of Lorenz dominance: we say that $\boldsymbol{x}$ is Lorenz superior to $\boldsymbol{y}$ if $\boldsymbol{x} \prec \boldsymbol{y}$, and we say that $\boldsymbol{x}$ is Lorenz indifferent to $\boldsymbol{y}$ if $\boldsymbol{x} \sim \boldsymbol{y}$.

Given an income distribution $\boldsymbol{x} \in \mathbb{D}^n$, with mean income $\bar{x}$, it holds that $\bar{x} \cdot \mathbf{1} \preceq \boldsymbol{x}$ since $k\, \bar{x} \geq \sum_{i=1}^{k} x_{(i)}$ for $k = 1, \ldots, n$. The majorization is strict, $\bar{x} \cdot \mathbf{1} \prec \boldsymbol{x}$, when $\boldsymbol{x}$ is not a uniform income distribution. In such case, $\bar{x} \cdot \mathbf{1}$ is Lorenz superior to $\boldsymbol{x}$. Moreover, for any income distribution $\boldsymbol{x} \in \mathbb{D}^n$ with mean income $\bar{x}$ it holds that $\boldsymbol{x} \preceq (0, \ldots, 0, n\bar{x})$, which is strict when $\boldsymbol{x} \neq \mathbf{0}$.

The majorization relation is a partial preorder, a necessary condition for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ to be comparable is that $\bar{x} = \bar{y}$, and $\boldsymbol{x} \sim \boldsymbol{y}$ if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ differ by a permutation. In general, $\boldsymbol{x} \preceq \boldsymbol{y}$ if and only if there exists a bistochastic matrix $\mathbf{C}$ (non-negative square matrix of order n where each row and column sums to one) such that $\boldsymbol{x} = \mathbf{C}\boldsymbol{y}$. Moreover, $\boldsymbol{x} \prec \boldsymbol{y}$ if the bistochastic matrix $\mathbf{C}$ is not a permutation matrix.

A particular case of bistochastic transformation is the so-called transfer, also called T-transformation.

Definition 3 *Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we say that $\boldsymbol{x}$ is derived from $\boldsymbol{y}$ by means of a* transfer *if, for some pair $i, j = 1, \ldots, n$ with $y_i \leq y_j$, we have*

$$x_i = (1 - \varepsilon)\, y_i + \varepsilon y_j \qquad x_j = \varepsilon y_i + (1 - \varepsilon)\, y_j \qquad \varepsilon \in [0, 1] \tag{2}$$

and $x_k = y_k$ for $k \neq i, j$. These formulas express an income transfer, from a richer to a poorer individual, of an income amount $\varepsilon(y_j - y_i)$. The transfer obtains $\boldsymbol{x} = \boldsymbol{y}$ if $\varepsilon = 0$, and exchanges the relative positions of donor and recipient in the income distribution if $\varepsilon = 1$, in which case $\boldsymbol{x} \sim \boldsymbol{y}$. In the intermediate cases $\varepsilon \in (0, 1)$ the transfer produces an income distribution $\boldsymbol{x}$ which is Lorenz superior to the original $\boldsymbol{y}$, that is $\boldsymbol{x} \prec \boldsymbol{y}$.

In general, for the majorization relation $\preceq$ and income distributions $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, it holds that $\boldsymbol{x} \preceq \boldsymbol{y}$ if and only if $\boldsymbol{x}$ can be derived from $\boldsymbol{y}$ by means of a finite sequence of transfers. Moreover, $\boldsymbol{x} \prec \boldsymbol{y}$ if any of the transfers is not a permutation.

Definition 4 *Let $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ be a function. In relation with the majorization relation $\preceq$, the notions of* Schur-convexity *(S-convexity) and* Schur-concavity *(S-concavity) of the function A are defined as follows:*

1. *A is* S-convex *if $\boldsymbol{x} \preceq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \leq A(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$*

2. *A is* S-concave *if $\boldsymbol{x} \preceq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \geq A(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$.*

Moreover, the S-convexity *(resp.* S-concavity*) of a function A is said to be* strict *if $\boldsymbol{x} \prec \boldsymbol{y}$ implies $A(\boldsymbol{x}) < A(\boldsymbol{y})$ (resp. $A(\boldsymbol{x}) > A(\boldsymbol{y})$). Notice that S-convexity (S-concavity) implies symmetry, since $\boldsymbol{x} \sim \boldsymbol{x}_\sigma \;\Rightarrow\; A(\boldsymbol{x}) = A(\boldsymbol{x}_\sigma)$.*

Definition 5 *A function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is an n-*ary averaging function *if it is monotonic and idempotent. An averaging function is said to be* strict *if it is strictly monotonic. Note that monotonicity and idempotency implies that* $\min(\boldsymbol{x}) \leq A(\boldsymbol{x}) \leq \max(\boldsymbol{x})$*, for all $\boldsymbol{x} \in \mathbb{D}^n$.*

For simplicity, the n-arity is omitted whenever it is clear from the context. Particular cases of averaging functions are weighted averaging (WA) functions, ordered weighted averaging (OWA) functions, and Choquet integrals, which contain the former as special cases.

Definition 6 *Given a weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$, with $\sum_{i=1}^n w_i = 1$, the* Weighted Averaging (WA) function *associated with $\boldsymbol{w}$ is the averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as*

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_i . \tag{3}$$

Definition 7 *Given a weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$, with $\sum_{i=1}^n w_i = 1$, the* Ordered Weighted Averaging (OWA) function *associated with $\boldsymbol{w}$ is the averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as*

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)} . \tag{4}$$

The traditional form of OWA functions as introduced by Yager [56] is as follows, $A(\boldsymbol{x}) = \sum_{i=1}^n \tilde{w}_i \, x_{[i]}$ where $\tilde{w}_i = w_{n-i+1}$. In [57, 58] the theory and applications of OWA functions are discussed in detail.

Definition 8 *Let A the OWA function associated with the weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$. The* orness *of A is defined as*

$$\text{Orness}\,(A) = \frac{1}{n-1} \sum_{i=1}^{n} (i-1)\, w_i \, . \tag{5}$$

The orness of A coincides with the value $A(\boldsymbol{x}_0)$, where $x_i^0 = (i-1)/(n-1)$,

$$\text{Orness}\,(A) = \frac{1-1}{n-1}\, w_1 + \frac{2-1}{n-1}\, w_2 + \ldots + \frac{(n-1)-1}{n-1}\, w_{n-1} + \frac{n-1}{n-1}\, w_n \, . \tag{6}$$

The following are two classical results particulary relevant in our framework. The first result regards a form of dominance relation between weighting structures and OWA functions, see for instance Bortot and Marques Pereira [10].

Proposition 1 *Consider two OWA functions* $A, B : \mathbb{D}^n \longrightarrow \mathbb{D}$ *associated with weighting vectors* $\boldsymbol{u} = (u_1, \ldots, u_n) \in [0,1]^n$ *and* $\boldsymbol{v} = (v_1, \ldots, v_n) \in [0,1]^n$*, respectively. It holds that* $A(\boldsymbol{x}) \leq B(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \mathbb{D}^n$ *if and only if*

$$\sum_{i=1}^{k} u_i \geq \sum_{i=1}^{k} v_i \qquad \text{for} \quad k = 1, \ldots, n \tag{7}$$

where the case $k = n$ *is an equality due to weight normalization.*

The next result regards the relation between the weighting structure and the S-convexity or S-concavity of the OWA function, see for instance Bortot and Marques Pereira [10].

Proposition 2 *Consider an OWA function* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *associated with a weighting vector* $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$*. The OWA function* A *is S-convex if and only if the weights are non decreasing,* $w_1 \leq \ldots \leq w_n$*, and* A *is strictly S-convex if and only if the weights are increasing,* $w_1 < \ldots < w_n$*. Analogously, the OWA function* A *is S-concave if and only if the weights are non increasing,* $w_1 \geq \ldots \geq w_n$*, and* A *is strictly S-concave if and only if the weights are decreasing,* $w_1 > \ldots > w_n$*.*

We will now review the basic concepts and definitions regarding welfare functions and inequality indices. Certain properties which are generally considered to be inherent to the concepts of welfare and inequality are now accepted as basic axioms for welfare and inequality measures, see for instance Kolm [38, 39]. The crucial axiom in this field is the *Pigou-Dalton transfer principle*, which states that welfare (inequality) measures should be non-decreasing (non-increasing) under transfers. This axiom translates directly into the properties of S-concavity and S-convexity in the context of symmetric functions on $\mathbb{D}^n$. In fact, a function is S-concave (S-convex) if and only if it is symmetric and non-decreasing (non-increasing) under transfers, see for instance Marshall and Olkin [41].

Definition 9 *An averaging function* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *is a* welfare function *if it is continuous, idempotent, and S-concave. The welfare function is said to be* strict *if it is a strict averaging function which is strictly S-concave.*

Due to monotonicity and idempotency, a welfare function is non decreasing over $\mathbb{D}^n$ but increasing along the diagonal $\boldsymbol{x} = x \cdot \mathbf{1} \in \mathbb{D}^n$, with $x \in \mathbb{D}$. Moreover, notice that S-concavity implies symmetry. Due to S-concavity, a welfare function ranks any Lorenz superior income distribution with the same mean as $\boldsymbol{x}$ as no worse than $\boldsymbol{x}$, whereas a strict welfare function ranks it as better.

Given a welfare function A, the *uniform equivalent income* $\tilde{x}$ associated with an income distribution $\boldsymbol{x}$ is defined as the income level which, if equally distributed among the population, would generate the same welfare value, $A(\tilde{x} \cdot \mathbf{1}) = A(\boldsymbol{x})$. The uniform equivalent concept has been originally proposed by Chisini [15] in the general context of averaging functions, see for instance Bennet et al. [3]. In the welfare context the uniform equivalent income has been considered by Atkinson [1], Kolm [37], and Sen [50] and further elaborated by Blackorby and Donaldson [5, 6, 7] and Blackorby, Donaldson, and Auersperg [8].

Due to the idempotency of A, we obtain $\tilde{x} = A(\boldsymbol{x})$. Since $\bar{x} \cdot \mathbf{1} \preceq \boldsymbol{x}$ for any income distribution $\boldsymbol{x} \in \mathbb{D}^n$, S-concavity implies $A(\bar{x} \cdot \mathbf{1}) \geq A(\boldsymbol{x})$ and therefore

$A(\boldsymbol{x}) \leq \bar{x}$ due to the idempotency of the welfare function. In other words, the mean income $\bar{x}$ and the uniform equivalent income $\tilde{x}$ are related by $0 \leq \tilde{x} \leq \bar{x}$.

We now define the notion of absolute inequality index, introduced by Kolm [38, 39] and developed by Blackorby and Donaldson [6], Blackorby, Donaldson, and Auersperg [8], and Weymark [53]. Following Kolm, inequality measures are described as absolute when they are invariant for additive transformations (translation invariance).

Definition 10 *A function $G : \mathbb{D}^n \longrightarrow \mathbb{D}$ is an* absolute inequality index *if it is continuous, nilpotent, S-convex, and invariant for translations. The absolute inequality index is said to be* strict *if it is strictly S-convex.*

In relation with the properties of the majorization relation discussed earlier, it holds that: over all income distributions $\boldsymbol{x} \in \mathbb{D}^n$ with the same mean income $\bar{x}$, a welfare function has minimum value $A(0, \dots, 0, n\bar{x})$, and an absolute inequality index has maximum value $G(0, \dots, 0, n\bar{x})$.

In the AKS framework introduced by Atkinson [1], Kolm [37], and Sen [50], a welfare function which is stable for translations induces an associated absolute inequality index by means of the correspondence formula $A(\boldsymbol{x}) = \bar{x} - G(\boldsymbol{x})$, see Blackorby and Donaldson [6]. The welfare function and the associated inequality index are said to be *ethical*, see also Sen [51], Blackorby, Donaldson, and Auersperg [8], Weymark [53], Blackorby and Donaldson [9], and Ebert [20].

Definition 11 *Given a welfare function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ which is stable for translations, the associated* Atkinson-Kolm-Sen (AKS) absolute inequality index *$G : \mathbb{D}^n \longrightarrow \mathbb{D}$ is defined as*

$$G(\boldsymbol{x}) = \bar{x} - A(\boldsymbol{x}) \tag{8}$$

The fact that A is stable for translations ensures the translational invariance of G. The absolute inequality index can be written as $G(\boldsymbol{x}) = \bar{x} - \tilde{x}$ and represents the per capita income that could be saved if society distributed incomes equally without any loss of welfare.

In the AKS framework, a welfare function A which is stable for both translations and dilations is associated with both absolute and relative inequality indices G and G_R, respectively, with $G(\boldsymbol{x}) = \bar{x}\, G_R(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{D}^n$. In what follows we will omit the term "absolute" when referring to G.

A class of welfare functions which plays a central role in this paper is that of the generalized Gini welfare functions introduced by Weymark [53], see also Mehran [44], Donaldson and Weymark [18, 19], Yaari [54, 55], Ebert [21], Quiggin [47], Ben-Porath and Gilboa [4].

Definition 12 *Given a weighting vector $\boldsymbol{w} = (w_1, \dots, w_n) \in [0, 1]^n$, with $w_1 \geq \dots \geq w_n \geq 0$ and $\sum_{i=1}^n w_i = 1$, the* generalized Gini welfare function *associated with $\boldsymbol{w}$ is the function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as*

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{(i)} \tag{9}$$

and, in the AKS framework, the associated generalized Gini inequality index *is defined as*

$$G(\boldsymbol{x}) = \bar{x} - A(\boldsymbol{x}) = -\sum_{i=1}^{n} (w_i - \frac{1}{n})\, x_{(i)}\,. \tag{10}$$

The generalized Gini welfare functions, which are strict if and only if $w_1 > \ldots > w_n > 0$, are clearly stable for both translations and dilations. For this reason they have a natural role within the AKS framework and Blackorby and Donaldson's correspondence formula.

An important particular case of the AKS generalized Gini framework is the classical Gini welfare function $A^c_G(\boldsymbol{x})$ and the associated classical Gini inequality index $G^c(\boldsymbol{x}) = \bar{x} - A^c_G(\boldsymbol{x})$,

$$A^c_G(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{2(n-i)+1}{n^2} x_{(i)} \tag{11}$$

where the coefficients of $A^c(\boldsymbol{x})$ have unit sum, $\sum_{i=1}^{n}(2(n-i)+1) = n^2$, and

$$G^c(\boldsymbol{x}) = -\sum_{i=1}^{n} \frac{n-2i+1}{n^2} x_{(i)} \tag{12}$$

where the coefficients of $G^c(\boldsymbol{x})$ have zero sum, $\sum_{i=1}^{n}(n-2i+1) = 0$. The classical Gini inequality index G^c is traditionally defined as

$$G^c(\boldsymbol{x}) = \frac{1}{2n^2} \sum_{i,j=1}^{n} |x_i - x_j| \,. \tag{13}$$

but in our framework it is convenient to express it as in (12), see [10].

3 Capacities and Choquet integrals

In this section we present a brief review of the basic facts on Choquet integration, focusing on the Möbius representation framework. For recent reviews of Choquet integration see [33, 32, 34, 35] for the general case, and [45, 42, 43] for the 2-additive case in particular.

Consider a finite set of interacting individuals $N = \{1, 2, \ldots, n\}$. Any subsets S, $T \subseteq N$ with cardinalities $0 \le s,\ t \le n$ are usually called coalitions. The concepts of capacity and Choquet integral in the definitions below are due to [16, 52, 17, 27, 28].

Definition 13 *A* capacity *on the set N is a set function $\mu : 2^N \longrightarrow [0,1]$ satisfying*

(i) $\mu(\emptyset) = 0$, $\mu(N) = 1$ (boundary conditions)

(ii) $S \subseteq T \subseteq N \quad \Rightarrow \quad \mu(S) \le \mu(T)$ (monotonicity conditions).

Definition 14 *Let μ be a capacity on N. The* Choquet integral $\mathcal{C}_\mu : \mathbb{D}^n \longrightarrow \mathbb{D}$ *with respect to μ is defined as*

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \left[\mu(A_{(i)}) - \mu(A_{(i+1)})\right] x_{(i)} \qquad \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{D}^n \tag{14}$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$. Moreover, $A_{(i)} = \{(i), \ldots, (n)\}$ and $A_{(n+1)} = \emptyset$.

Definition 15 *Let μ be a capacity on the set N. The* Möbius transform $m_\mu : 2^N \longrightarrow \mathbb{R}$ *associated with the capacity μ is defined as*

$$m_\mu(T) = \sum_{S \subseteq T} (-1)^{t-s} \mu(S) \qquad T \subseteq N \tag{15}$$

where s and t denote the cardinality of the coalitions S and T, respectively.

Conversely, given the Möbius transform m_μ, the associated capacity μ is obtained as

$$\mu(T) = \sum_{S \subseteq T} m_\mu(S) \qquad T \subseteq N\,. \tag{16}$$

In the Möbius representation, the boundary conditions take the form

$$m_\mu(\emptyset) = 0 \qquad \sum_{T \subseteq N} m_\mu(T) = 1 \tag{17}$$

and the monotonicity conditions can be expressed as follows: for each $i = 1, \ldots, n$ and each coalition $T \subseteq N \setminus \{i\}$, the monotonicity condition is written as

$$\sum_{S \subseteq T} m_\mu(S \cup \{i\}) \geq 0 \qquad T \subseteq N \setminus \{i\} \qquad i = 1, \ldots, n\,. \tag{18}$$

This form of the monotonicity conditions derives from the original monotonicity conditions in Definition 13, expressed as $\mu(T \cup \{i\}) - \mu(T) \geq 0$ for each $i \in N$ and $T \subseteq N \setminus \{i\}$.

Defining a capacity μ on a set N of n elements requires $2^n - 2$ real coefficients, corresponding to the capacity values $\mu(T)$ for $T \subseteq N$. In order to control exponential complexity, Grabisch [29] introduced the concept of k-additive capacities.

Definition 16 *A capacity μ on the set N is said to be* k-additive *if its Möbius transform satisfies $m_\mu(T) = 0$ for all $T \subseteq N$ with $t > k$, and there exists at least one coalition $T \subseteq N$ with $t = k$ such that $m_\mu(T) \neq 0$.*

In the k-additive case, with $k = 1, \ldots, n$, the capacity μ is expressed as follows in terms of the Möbius transform m_μ,

$$\mu(T) = \sum_{S \subseteq T,\, s \leq k} m_\mu(S) \qquad T \subseteq N \tag{19}$$

and the boundary and monotonicity conditions (17) and (18) take the form

$$m_\mu(\emptyset) = 0 \qquad \sum_{T \subseteq N,\, t \leq k} m_\mu(T) = 1 \tag{20}$$

$$\sum_{S \subseteq T,\, s \leq k-1} m_\mu(S \cup \{i\}) \geq 0 \qquad T \subseteq N \setminus \{i\} \qquad i = 1, \ldots, n\,. \tag{21}$$

Finally, we examine the particular case of symmetric capacities and Choquet integrals, which play a crucial role in this paper.

Definition 17 *A capacity μ is said to be* symmetric *if it depends only on the cardinality of the coalition considered, in which case we use the simplified notation*

$$\mu(T) = \mu(t) \qquad \textit{where} \qquad t = |T|\,. \tag{22}$$

Accordingly, for the Möbius transform m_μ associated with a symmetric capacity μ we use the notation

$$m_\mu(T) = m_\mu(t) \qquad \text{where} \qquad t = |T|\,. \tag{23}$$

In the symmetric case, the expression (16) for the capacity μ in terms of the Möbius transform m_μ reduces to

$$\mu(t) = \sum_{s=1}^{t} \binom{t}{s} m_\mu(s) \qquad\qquad t = 1, \ldots, n \tag{24}$$

and the boundary and monotonicity conditions (17) and (18) take the form

$$m_\mu(0) = 0 \qquad\qquad \sum_{s=1}^{n} \binom{n}{s} m_\mu(s) = 1 \tag{25}$$

$$\sum_{s=1}^{t} \binom{t-1}{s-1} m_\mu(s) \geq 0 \qquad\qquad t = 1, \ldots, n\,. \tag{26}$$

The monotonicity conditions correspond to $\mu(t) - \mu(t-1) \geq 0$ for $t = 1, \ldots, n$.

The Choquet integral (14) with respect to a symmetric capacity μ reduces to an Ordered Weighted Averaging (OWA) function [22, 56],

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} [\mu(n-i+1) - \mu(n-i)]x_{(i)} = \sum_{i=1}^{n} w_i\, x_{(i)} = A(\boldsymbol{x}) \tag{27}$$

where the weights $w_i = \mu(n-i+1) - \mu(n-i)$ satisfy $w_i \geq 0$ for $i = 1, \ldots, n$ due to the monotonicity of the capacity μ, and $\sum_{i=1}^{n} w_i = 1$ due to the boundary conditions $\mu(0) = 0$ and $\mu(n) = 1$. Comprehensive reviews of OWA functions can be found in [57] and [58].

The weighting structure of the OWA function (27) is of the general form $w_i = f(\frac{n-i+1}{n}) - f(\frac{n-i}{n})$ where f is a continuous and increasing function on the unit interval, with $f(0) = 0$ and $f(1) = 1$. Gajdos [24] shows that the OWA function A is associated with a k-additive capacity μ, with $k = 1, \ldots, n$, if and only if f is polynomial of order k. In fact, in (24), the k-additive case is obtained simply by taking $m_\mu(k+1) = \ldots = m_\mu(n) = 0$, and the binomial coefficient of the Möbius value $m_\mu(k)$ corresponds to $t(t-1)\ldots(t-k+1)/k!$, which is polynomial of order k in the coalition cardinality t.

4 The binomial decomposition

We now consider OWA functions $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ and we recall the binomial decomposition of OWA functions due to Calvo and De Baets [11], with the addition of a uniqueness result, see also [10].

We begin by introducing the convenient notation

$$\alpha_j = \binom{n}{j} m_\mu(j) \qquad j = 1, \ldots, n\,. \tag{28}$$

In this notation the upper boundary condition (25) reduces to

$$\sum_{j=1}^{n} \alpha_j = 1 \tag{29}$$

and the monotonicity conditions (26) take the form

$$\sum_{j=1}^{i} \frac{\binom{i-1}{j-1}}{\binom{n}{j}}\, \alpha_j \geq 0 \qquad\qquad i = 1, \ldots, n\,. \tag{30}$$

Definition 18 *The* binomial OWA functions $C_j : \mathbb{D}^n \longrightarrow \mathbb{D}$*, with* $j = 1, \ldots, n$*, are defined as*

$$C_j(\boldsymbol{x}) = \sum_{i=1}^{n} w_{ji}\, x_{(i)} = \sum_{i=1}^{n} \frac{\binom{n-i}{j-1}}{\binom{n}{j}}\, x_{(i)} \qquad j = 1, \ldots, n \tag{31}$$

where the binomial weights w_{ji}*,* $i, j = 1, \ldots, n$ *are null when* $i + j > n + 1$ *according to the usual convention that* $\binom{p}{q} = 0$ *when* $p < q$*, with* $p, q = 0, 1, \ldots$

Except for $C_1(\boldsymbol{x}) = \bar{x}$, the binomial OWA functions C_j, $j = 2, \ldots, n$ have an increasing number of null weights, in correspondence with $x_{(n-j+2)}, \ldots, x_{(n)}$. The weight normalization of the binomial OWA functions, $\sum_{i=1}^{n} w_{ji} = 1$ for $j = 1, \ldots, n$, is due to the column-sum property of binomial coefficients,

$$\sum_{i=1}^{n} \binom{n-i}{j-1} = \sum_{i=0}^{n-1} \binom{i}{j-1} = \binom{n}{j} \qquad j = 1, \ldots, n\,. \tag{32}$$

Proposition 3 *Any OWA function* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *can be written uniquely as*

$$A(\boldsymbol{x}) = \alpha_1 C_1(\boldsymbol{x}) + \alpha_2 C_2(\boldsymbol{x}) + \ldots + \alpha_n C_n(\boldsymbol{x}) \tag{33}$$

where the coefficients α_j*,* $j = 1, \ldots, n$ *are subject to conditions (29) and (30). In the binomial decomposition the* k*-additive case, with* $k = 1, \ldots, n$*, is obtained simply by taking* $\alpha_{k+1} = \ldots = \alpha_n = 0$*.*

Example 1 Consider the case $n = 3$. Using the boundary condition $\alpha_1 + \alpha_2 + \alpha_3 = 1$ as in (29), we can write the monotonicity conditions (30) only in terms of α_2, α_3 as follows,

$$\begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ \alpha_3 \leq 1 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \tag{34}$$

and the corresponding feasible region is illustrated in Fig. 1.
The origin in Fig. 1 is associated with $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$, which corresponds in the binomial decomposition (33) to $A(\boldsymbol{x}) = C_1(\boldsymbol{x}) = \bar{x}$.

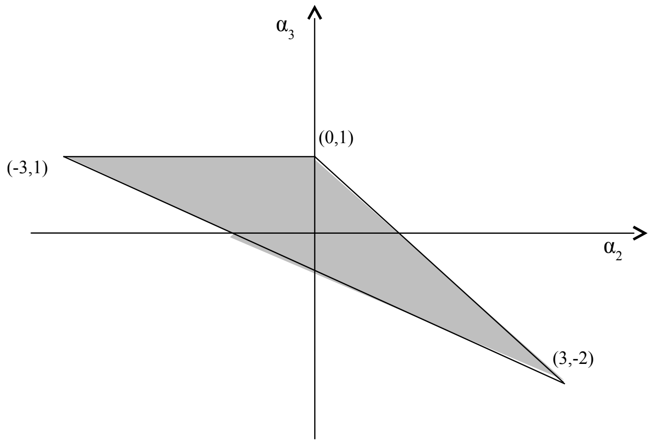


Figure 1: Feasible region associated with conditions (34).

Given that the coefficients α_j, $j = 1, \dots, n$ are constrained by the boundary and monotonicity conditions (29) and (30), the binomial decomposition (33) does not express a free (vector space) linear combination of the binomial OWA functions C_j, $j = 1, \dots, n$, or even a simple convex combination of the binomial OWA functions, as the boundary condition $\alpha_1 + \ldots + \alpha_n = 1$ might suggest. In fact, the monotonicity conditions allow for negative α values, as illustrated by the feasible region in Fig. 1.

The following interesting result concerning the cumulative properties of binomial weights is due to Calvo and De Baets [11], see also Bortot and Marques Pereira [10].

Proposition 4 *The binomial weights $w_{ji} \in [0, 1]$, with $i, j = 1, \dots, n$, have the following cumulative property,*

$$\sum_{k=1}^{i} w_{j-1,k} \leq \sum_{k=1}^{i} w_{jk} \qquad i = 1, \dots, n \quad j = 2, \dots, n\,. \tag{35}$$

Given that binomial weights have the cumulative property (35), Proposition 1 implies that the binomial OWA functions C_j, $j = 1, \dots, n$ satisfy the relations $\bar{x} = C_1(\boldsymbol{x}) \geq C_2(\boldsymbol{x}) \geq \ldots \geq C_n(\boldsymbol{x}) \geq 0$, for any $\boldsymbol{x} \in \mathbb{D}^n$.

Proposition 5 *The orness of the binomial OWA functions C_j, with $j = 1, \dots, n$, is given by*

$$\text{Orness}\,(C_j) = \frac{n-j}{(n-1)(j+1)} \qquad j = 1, \dots, n\,. \tag{36}$$

Proof: From the definition of C_j (31) and the general definition of orness (5), we have

$$\text{Orness}\,(C_j) = C_j(\boldsymbol{x}_0) = \sum_{i=1}^{n} \frac{\binom{n-i}{j-1}}{\binom{n}{j}} \, \frac{i-1}{n-1} \qquad j = 1, \dots, n\,. \tag{37}$$

Using the formula

$$\sum_{i=1}^{n} \binom{n-i}{j-1}(i-1) = \binom{n}{j+1} \qquad j = 1, \dots, n \tag{38}$$

and substituting in (37), we obtain

$$\text{Orness}\,(C_j) = \frac{1}{n-1}\frac{\binom{n}{j+1}}{\binom{n}{j}} = \frac{n-j}{(n-1)(j+1)} \qquad j = 1,\ldots,n\,. \tag{39}$$

Notice that the orness of the binomial OWA function is strictly decreasing with respect to $j = 1, \ldots, n$, from Orness $(C_1) = 1/2$ to Orness $(C_n) = 0$. □

Proposition 6 *In relation with the binomial decomposition, the orness of an OWA function* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *is given by*

$$\text{Orness}\,(A) = \sum_{j=1}^{n} \frac{(n-j)}{(n-1)(j+1)}\,\alpha_j \tag{40}$$

Proof: Considering the binomial decomposition as in Proposition 3,

$$\text{Orness}\,(A) = A(\boldsymbol{x}_0) = \sum_{j=1}^{n} \alpha_j\, C_j(\boldsymbol{x}_0) = \sum_{j=1}^{n} \frac{(n-j)}{(n-1)(j+1)}\,\alpha_j \tag{41}$$

where we have used that $C_j(\boldsymbol{x}_0) = \text{Orness}\,(C_j)$ as in Proposition 5. □

Summarizing, the binomial decomposition (33) holds for any OWA function A in terms of the binomial OWA functions C_j, $j = 1, \ldots, n$ and the corresponding coefficients α_j, $j = 1, \ldots, n$ subject to conditions (29) and (30).

Consider the binomial OWA functions C_j with $j = 1, \ldots, n$. The binomial weights w_{ji}, $i, j = 1, \ldots, n$ as in (31) have regularity properties which have interesting implications at the level of the functions C_j, $j = 1, \ldots, n$, see [10].

Proposition 7 *The binomial weights* $w_{ji} \in [0, 1]$*, with* $i, j = 1, \ldots, n$*, have the following properties,*

- *i.* *for* $j = 1$ $\quad 1/n = w_{11} = w_{12} = \ldots = w_{1,n-1} = w_{1n}$
- *ii.* *for* $j = 2$ $\quad 2/n = w_{21} > w_{22} > \ldots > w_{2,n-1} > w_{2n} = 0$
- *iii.* *for* $j = 3, \ldots, n$ $\quad j/n = w_{j1} > w_{j2} > \ldots > w_{j,n-j+2} = \ldots = w_{jn} = 0$

The functions C_j, $j = 1, \ldots, n$ are continuous, idempotent, and stable for translations, where the latter two properties follow immediately from $\sum_{i=1}^{n} w_{ji} = 1$ for $j = 1, \ldots, n$. Moreover, given that binomial weights are non increasing, $w_{j1} \geq w_{j2} \geq \ldots \geq w_{jn}$ for $j = 1, \ldots, n$, Proposition 2 implies that the functions C_j, $j = 1, \ldots, n$ are S-concave, with strict S-concavity applying only to C_2.

In relation with these properties, we conclude that the functions C_j, $j = 1, \ldots, n$, which we hereafter call *binomial Gini welfare functions*, are generalized Gini welfare functions on the income domain $\boldsymbol{x} \in \mathbb{D}^n$.

Definition 19 *Consider the binomial Gini welfare functions* $C_j : \mathbb{D}^n \longrightarrow \mathbb{D}$*, with* $C_j(\boldsymbol{x}) = \sum_{i=1}^{n} w_{ji} x_{(i)}$ *for* $j = 1, \ldots, n$*. The* binomial Gini inequality indices $G_j : \mathbb{D}^n \longrightarrow \mathbb{D}$*, with* $j = 1, \ldots, n$*, are defined as*

$$G_j(\boldsymbol{x}) = \bar{x} - C_j(\boldsymbol{x}) \qquad j = 1, \ldots, n \tag{42}$$

which means that

$$G_j(\boldsymbol{x}) = -\sum_{i=1}^{n} v_{ji} x_{(i)} = -\sum_{i=1}^{n} \left[w_{ji} - \frac{1}{n} \right] x_{(i)} \qquad j = 1, \ldots, n \tag{43}$$

where the coefficients v_{ji}, $i, j = 1, \ldots, n$ are equal to $-1/n$ when $i + j > n + 1$, since in such case the binomial weights w_{ji} are null. The weight normalization of the binomial Gini welfare functions, $\sum_{i=1}^{n} w_{ji} = 1$ for $j = 1, \ldots, n$, implies that $\sum_{i=1}^{n} v_{ji} = 0$ for $j = 1, \ldots, n$.

The binomial Gini inequality indices G_j, $j = 1, \ldots, n$ are continuous, nilpotent, and invariant for translations, where the latter two properties follow immediately from $\sum_{i=1}^{n} v_{ji} = 0$ for $j = 1, \ldots, n$. Moreover, the G_j are S-convex: given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we have that $\boldsymbol{x} \preceq \boldsymbol{y} \Rightarrow C_j(\boldsymbol{x}) \geq C_j(\boldsymbol{y}) \Rightarrow G_j(\boldsymbol{x}) \leq G_j(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$, due to the S-concavity of the C_j, $j = 1, \ldots, n$.

In fact, the binomial Gini inequality indices G_j, $j = 1, \ldots, n$ in (42) correspond to the Atkinson-Kolm-Sen (AKS) absolute inequality indices associated with the binomial welfare functions C_j, $j = 1, \ldots, n$, in the spirit of Blackorby and Donaldson's correspondence formula. Together, as we discuss below, the binomial Gini welfare functions C_j and the binomial Gini inequality indices G_j, $j = 1, \ldots, n$ can be regarded as two equivalent functional bases for the class of generalized Gini welfare functions and inequality indices.

In analogy with the binomial weights w_{ji}, $i, j = 1, \ldots, n$, their inequality counterparts v_{ji}, $i, j = 1, \ldots, n$ have interesting regularity properties, which follow directly from Proposition 7.

Proposition 8 *The coefficients $v_{ji} \in [-1/n, (n-1)/n]$, with $i, j = 1, \ldots, n$, have the following properties,*

i. *for* $j = 1$ $\quad 0 = v_{11} = v_{12} = \ldots = v_{1,n-1} = v_{1n}$

ii. *for* $j = 2$ $\quad 1/n = v_{21} > v_{22} > \ldots > v_{2,n-1} > v_{2n} = -1/n$

iii. *for* $j = 3, \ldots, n$ $\quad \frac{j-1}{n} = v_{j1} > v_{j2} > \ldots > v_{j,n-j+2} = \ldots = v_{jn} = -1/n$

Notice that $C_1(\boldsymbol{x}) = \bar{x}$ and $G_1(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, $C_2(\boldsymbol{x})$ has $n-1$ positive linearly decreasing weights and one null last weight, and the associated $G_2(\boldsymbol{x})$ has linearly increasing coefficients and is in fact proportional to the classical Gini index, $G_2(\boldsymbol{x}) = \frac{n}{n-1} G^c(\boldsymbol{x})$. The remaining $C_j(\boldsymbol{x})$, $j = 3, \ldots, n$, have $n - j + 1$ positive non-linear decreasing weights and $j - 1$ null last weights, and the associated $G_j(\boldsymbol{x})$, $j = 3, \ldots, n$ have $n - j + 2$ non-linear increasing weights and $j - 1$ equal last weights.

Therefore, the only strict binomial welfare function is $C_1(\boldsymbol{x}) = \bar{x}$ and the only strict binomial inequality index is $G_2(\boldsymbol{x}) = \frac{n}{n-1} G^c(\boldsymbol{x})$. In the remaining $G_j(\boldsymbol{x})$, $j = 3, \ldots, n$ the last $j - 1$ coefficients coincide and thus they are non strict absolute inequality indices, in the sense that they are insensitive to income transfers involving only the $j - 1$ richest individuals of the population.

Example 2 In dimensions $n = 2, 3, 4, 5, 6$ the weights $w_{ij} \in [0, 1]$, $i, j = 1, \ldots, n$ of the binomial Gini welfare functions C_j, $j = 1, \ldots, n$ and the coefficients $-v_{ij} \in [-(n-1)/n, 1/n]$, $i, j = 1, \ldots, n$ of the binomial Gini inequality indices G_j, $j = 1, \ldots, n$ are as follows,

$n = 2$

$$C_1: \quad (\tfrac{1}{2}, \tfrac{1}{2}) \qquad G_1: \quad (0, 0)$$
$$C_2: \quad (1, 0) \qquad G_2: \quad (-\tfrac{1}{2}, \tfrac{1}{2})$$

$n = 3$

$$C_1: \quad (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}) \qquad G_1: \quad (0, 0, 0)$$
$$C_2: \quad (\tfrac{2}{3}, \tfrac{1}{3}, 0) \qquad G_2: \quad (-\tfrac{1}{3}, 0, \tfrac{1}{3})$$
$$C_3: \quad (1, 0, 0) \qquad G_3: \quad (-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3})$$

$n = 4$

$$C_1: \quad (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) \qquad G_1: \quad (0, 0, 0, 0)$$
$$C_2: \quad (\tfrac{3}{6}, \tfrac{2}{6}, \tfrac{1}{6}, 0) \qquad G_2: \quad (-\tfrac{3}{12}, -\tfrac{1}{12}, \tfrac{1}{12}, \tfrac{3}{12})$$
$$C_3: \quad (\tfrac{3}{4}, \tfrac{1}{4}, 0, 0) \qquad G_3: \quad (-\tfrac{2}{4}, 0, \tfrac{1}{4}, \tfrac{1}{4})$$
$$C_4: \quad (1, 0, 0, 0) \qquad G_4: \quad (-\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4})$$

$n = 5$

$$C_1: \quad (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}) \qquad G_1: \quad (0, 0, 0, 0, 0)$$
$$C_2: \quad (\tfrac{4}{10}, \tfrac{3}{10}, \tfrac{2}{10}, \tfrac{1}{10}, 0) \qquad G_2: \quad (-\tfrac{2}{10}, -\tfrac{1}{10}, 0, \tfrac{1}{10}, \tfrac{2}{10})$$
$$C_3: \quad (\tfrac{6}{10}, \tfrac{3}{10}, \tfrac{1}{10}, 0, 0) \qquad G_3: \quad (-\tfrac{4}{10}, -\tfrac{1}{10}, \tfrac{1}{10}, \tfrac{2}{10}, \tfrac{2}{10})$$
$$C_4: \quad (\tfrac{4}{5}, \tfrac{1}{5}, 0, 0, 0) \qquad G_4: \quad (-\tfrac{3}{5}, 0, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5})$$
$$C_5: \quad (1, 0, 0, 0, 0) \qquad G_5: \quad (-\tfrac{4}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5})$$

$n = 6$

$$C_1: \quad (\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}) \qquad G_1: \quad (0, 0, 0, 0, 0, 0)$$
$$C_2: \quad (\tfrac{5}{15}, \tfrac{4}{15}, \tfrac{3}{15}, \tfrac{2}{15}, \tfrac{1}{15}, 0) \qquad G_2: \quad (-\tfrac{5}{30}, -\tfrac{3}{30}, -\tfrac{1}{30}, \tfrac{1}{30}, \tfrac{3}{30}, \tfrac{5}{30})$$
$$C_3: \quad (\tfrac{10}{20}, \tfrac{6}{20}, \tfrac{3}{20}, \tfrac{1}{20}, 0, 0) \qquad G_3: \quad (-\tfrac{20}{60}, -\tfrac{8}{60}, \tfrac{1}{60}, \tfrac{7}{60}, \tfrac{10}{60}, \tfrac{10}{60})$$
$$C_4: \quad (\tfrac{10}{15}, \tfrac{4}{15}, \tfrac{1}{15}, 0, 0, 0) \qquad G_4: \quad (-\tfrac{15}{30}, -\tfrac{3}{30}, \tfrac{3}{30}, \tfrac{5}{30}, \tfrac{5}{30}, \tfrac{5}{30})$$
$$C_5: \quad (\tfrac{5}{6}, \tfrac{1}{6}, 0, 0, 0, 0) \qquad G_5: \quad (-\tfrac{4}{6}, 0, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6})$$
$$C_6: \quad (1, 0, 0, 0, 0, 0) \qquad G_6: \quad (-\tfrac{5}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6})$$

The binomial Gini welfare functions C_j, $j = 1, \ldots, n$ have null weights associated with the $j-1$ richest individuals in the population and therefore, as j increases from 1 to n, they behave in analogy with poverty measures which progressively focus on the poorest part of the population. Correspondingly, the binomial Gini inequality indices G_j, $j = 1, \ldots, n$ have equal coefficients associated with the $j-1$ richest individuals in the population and therefore, as j increases from 1 to n, they are progressively insensitive to income transfers within the richest part of the population.

5 The binomial decomposition: 2-additive and 3-additive cases

In this section we use the boundary condition (29) to write the binomial decomposition in Proposition 3 only in terms of $\alpha_2, \ldots, \alpha_n$, plus the corresponding binomial Gini welfare functions $C_j(\boldsymbol{x})$ and the associated binomial Gini inequality indices $G_j(\boldsymbol{x}) = \bar{x} - C_j(\boldsymbol{x})$, with $j = 2, \ldots, n$.

Proposition 9 *Any OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ can be written uniquely as*

$$\begin{aligned} A(\boldsymbol{x}) &= (1 - \alpha_2 - \ldots - \alpha_n)\,\bar{x} + \alpha_2 C_2(\boldsymbol{x}) + \ldots + \alpha_n C_n(\boldsymbol{x}) \\ &= \bar{x} - \alpha_2 G_2(\boldsymbol{x}) - \ldots - \alpha_n G_n(\boldsymbol{x}) \end{aligned} \tag{44}$$

where the coefficients α_j, $j = 2, \ldots, n$ are subject to the boundary and monotonicity (BM) conditions

$$\sum_{j=2}^{n} \Big[1 - n\frac{\binom{i-1}{j-1}}{\binom{n}{j}}\Big]\alpha_j \leq 1 \qquad\qquad i = 1, \ldots, n\,. \tag{45}$$

Notice that $C_1(\boldsymbol{x}) = \bar{x}$ and $G_1(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathbb{D}^n$. Therefore the absence of G_1 in (44) is in any case immaterial.

Proof: The expression of the binomial decomposition (44) is obtained directly from (33) in Proposition 3 by substituting for $\alpha_1 = 1 - \alpha_2 - \alpha_3 - \ldots - \alpha_n$, as in the boundary condition (29).

Consider now the monotonicity conditions (30). Substituting for $\alpha_1 = 1 - \alpha_2 - \alpha_3 - \ldots - \alpha_n$, we obtain

$$\begin{aligned} &\frac{1}{n} + \left[\frac{\binom{i-1}{1}}{\binom{n}{2}} - \frac{1}{n}\right]\alpha_2 + \left[\frac{\binom{i-1}{2}}{\binom{n}{3}} - \frac{1}{n}\right]\alpha_3 + \ldots + \left[\frac{\binom{i-1}{i-1}}{\binom{n}{i}} - \frac{1}{n}\right]\alpha_i \\ &-\frac{1}{n}(\alpha_{i+1} + \ldots + \alpha_n) \geq 0 \qquad\qquad i = 1, \ldots, n \end{aligned} \tag{46}$$

which correspond to the following n combined boundary and monotonicity (BM) conditions in terms of the $n-1$ coefficients α_j, $j = 2, \ldots, n$,

$$\sum_{j=2}^{n} \Big[1 - n\frac{\binom{i-1}{j-1}}{\binom{n}{j}}\Big]\alpha_j \leq 1 \qquad\qquad i = 1, \ldots, n\,. \tag{47}$$

The first and the last of these BM conditions are always of the form $\alpha_2 + \alpha_3 + \ldots + \alpha_n \leq 1$ and $\alpha_2 + 2\,\alpha_3 + \ldots + (n-1)\,\alpha_n \geq -1$, respectively. $\square$

In the binomial welfare and inequality decomposition (44) the level of k-additivity of the generalized Gini welfare function A is controlled by the coefficients α_2, ..., α_n subject to the conditions (45). As k-additivity increases, the binomial decomposition of A includes an increasing number of binomial Gini welfare functions and inequality indices which are progressively insensitive to income transfers within the richest part of the population. Moreover, the binomial Gini inequality indices are increasingly stronger, $0 = G_1(\boldsymbol{x}) \leq G_2(\boldsymbol{x}) \leq \ldots \leq G_n(\boldsymbol{x}) \leq 1$ for any $\boldsymbol{x} \in \mathbb{D}^n$, in correspondence with the analogous but inverse ordering of binomial Gini welfare functions obtained after Proposition 4.

5.1 The 2-additive case

We now examine the binomial decomposition of OWA functions (44) in the 2-additive case, focusing on the particular form of the BM conditions (45).

In the 2-additive case, with $n \geq 2$, the BM conditions (45) take the form

$$\left[1 - \frac{n(i-1)}{\binom{n}{2}}\right]\alpha_2 \leq 1 \qquad i = 1, \ldots, n \tag{48}$$

which reduce to

$$-1 \leq \alpha_2 \leq 1 \tag{49}$$

corresponding to the first and last of the n conditions (48), the others been dominated by these two. Notice that in the 2-additive case the BM conditions are independent of n.

As an immediate consequence of Proposition 3 and Proposition 9, we have the following result.

Proposition 10 *Any 2-additive OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ can be written uniquely as*

$$A(\boldsymbol{x}) = (1 - \alpha_2)\,\bar{x} + \alpha_2 C_2(\boldsymbol{x}) = \bar{x} - \alpha_2 G_2(\boldsymbol{x}) \tag{50}$$

where $C_2(\boldsymbol{x})$ is the binomial Gini welfare function

$$C_2(\boldsymbol{x}) = \sum_{i=1}^{n} w_{2i}\, x_{(i)} = \sum_{i=1}^{n} \frac{2(n-i)}{n(n-1)}\, x_{(i)}\,, \tag{51}$$

$G_2(\boldsymbol{x})$ is the binomial Gini inequality index

$$G_2(\boldsymbol{x}) = -\sum_{i=1}^{n} v_{2i}\, x_{(i)} = -\sum_{i=1}^{n} \frac{n-2i+1}{n(n-1)}\, x_{(i)}\,, \tag{52}$$

and the coefficient α_2 is subject to the conditions (49), $-1 \leq \alpha_2 \leq 1$.

Given that G_2 is proportional to the classical absolute Gini inequality index

$$G_2(\boldsymbol{x}) = \frac{n}{n-1}\, G^c(\boldsymbol{x}) \tag{53}$$

any 2-additive OWA function can be written as

$$A(\boldsymbol{x}) = \bar{x} - \frac{n}{n-1}\, \alpha_2 G^c(\boldsymbol{x}) \tag{54}$$

where α_2 is a free parameter subject to the conditions (49).

The strict case $\alpha_2 > 0$ in (54) corresponds to the well-known Ben Porath and Gilboa's formula [4] for Weymark's generalized Gini welfare functions, with linearly decreasing (inequality averse) weight distributions, see also [31].

In particular, with $\alpha_2 = (n-1)/n$ in (54), we obtain the classical Gini welfare function

$$A(\boldsymbol{x}) = A_G^c(\boldsymbol{x}) \qquad \alpha_2 = \frac{n-1}{n}\,. \tag{55}$$

Other interesting parametric choices for α_2 could be $\alpha_2 = (n-l)/n$ with $l = 0, 1, \ldots, n$. In the case $l = 0$ all the Choquet capacity structure lies in the non-additive Möbius values $m_\mu(2)$, the case $l = 1$ corresponds to the classical absolute Gini inequality index, and the remaining cases correspond to increasingly weak structure being associated with the values $m_\mu(2)$, towards the additive case $l = n$. In other words, the parametric choices associated with $l = 0, 1, \ldots, n$ correspond to an interpolation between $A(\boldsymbol{x}) = \bar{x} = C_1(\boldsymbol{x})$ (with $l = n$) and $A(\boldsymbol{x}) = C_2(\boldsymbol{x})$ (with $l = 0$) through the intermediate (with $l = 1$) case $A(\boldsymbol{x}) = A^c(\boldsymbol{x})$, the classical Gini welfare function.

Proposition 11 *Considering the binomial decomposition (50), the orness of the 2-additive OWA function associated with coefficient α_2 is given by*

$$\text{Orness}\,(A) = \frac{1}{2} - \frac{1}{6}\,\frac{n+1}{n-1}\,\alpha_2 \tag{56}$$

where the coefficient α_2 is subject to the conditions (49), $-1 \leq \alpha_2 \leq 1$.

5.2 The 3-additive case

We now examine the binomial decomposition of OWA functions (44) in the 3-additive case, focusing on the particular form of the BM conditions (45).

In the 3-additive case, with $n \geq 3$, the BM conditions (45) take the form

$$\left[1 - n\frac{\binom{i-1}{1}}{\binom{n}{2}}\right]\alpha_2 + \left[1 - n\frac{\binom{i-1}{2}}{\binom{n}{3}}\right]\alpha_3 \leq 1 \qquad i = 1, \ldots, n\,. \tag{57}$$

In contrast with the 2-additive case, notice that in the 3-additive case the BM conditions depend on n.

As an immediate consequence of Proposition 3 and Proposition 9, we have the following result.

Proposition 12 *Any 3-additive OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ can be written uniquely as*

$$A(\boldsymbol{x}) = (1 - \alpha_2 - \alpha_3)\,\bar{x} + \alpha_2 C_2(\boldsymbol{x}) + \alpha_3 C_3(\boldsymbol{x}) = \bar{x} - \alpha_2 G_2(\boldsymbol{x}) - \alpha_3 G_3(\boldsymbol{x}) \tag{58}$$

where $C_2(\boldsymbol{x})$ and $G_2(\boldsymbol{x})$ are as in (51) and (52), $C_3(\boldsymbol{x})$ is the binomial Gini welfare function

$$C_3(\boldsymbol{x}) = \sum_{i=1}^{n} w_{3i}\, x_{(i)} = \sum_{i=1}^{n} \frac{3(n-i)(n-i-1)}{n(n-1)(n-2)}\, x_{(i)}\,, \tag{59}$$

$G_3(\boldsymbol{x})$ is the binomial Gini inequality index

$$G_3(\boldsymbol{x}) = -\sum_{i=1}^{n} v_{3i}\, x_{(i)} = -\sum_{i=1}^{n} \frac{2n^2 - 2 + 3i - 6in + 3i^2}{n(n-1)(n-2)}\, x_{(i)}\,, \tag{60}$$

and the coefficients α_2 and α_3 are subject to the BM conditions (57).

Notice that in G_3 the last 2 coefficients coincide ($v_{3,n-1} = v_{3n} = -1/n$) and thus G_3 is a non strict absolute inequality index, in the sense that it is insensitive to income transfers involving the 2 richest individuals in the population.

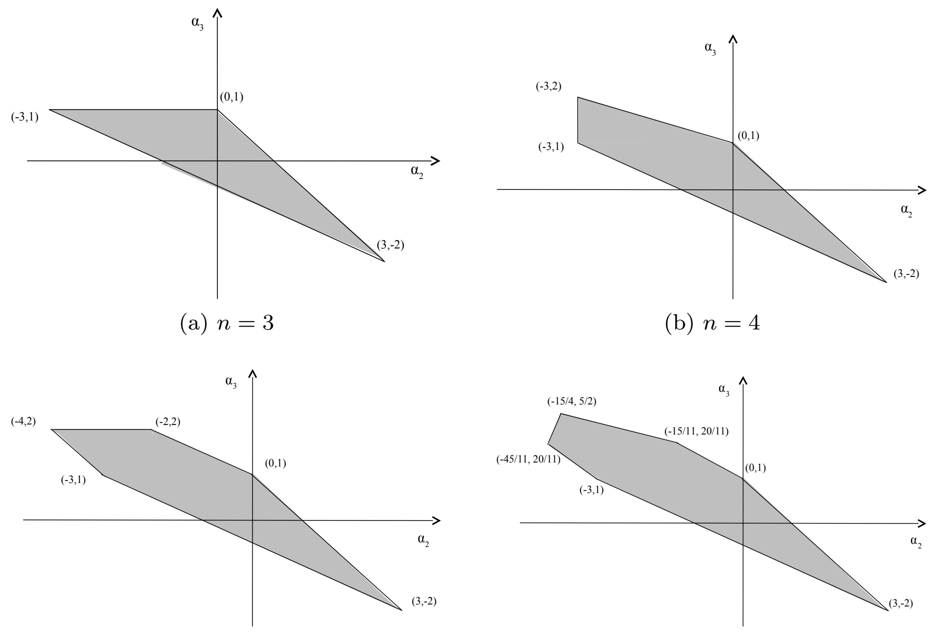
We now illustrate the 3-additive case for populations of size $n = 3, 4, 5, 6$. The feasible regions in Fig. 2 refer to the binomial decomposition of 3-additive OWA functions in Proposition 12.

Example 3 Consider the 3-additive case for $n = 3, 4, 5, 6$. We have the following BM conditions (57) in terms of the two coefficients α_2 and α_3,

$$n = 3 \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ \alpha_3 \leq 1 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \qquad n = 4 \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ \alpha_2 + 3\alpha_3 \leq 3 \\ \alpha_2 \geq -3 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \tag{61}$$

$$n = 5 \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ \alpha_2 + 2\alpha_3 \leq 2 \\ \alpha_3 \leq 2 \\ \alpha_2 + \alpha_3 \geq -2 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \qquad n = 6 \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ 3\alpha_2 + 5\alpha_3 \leq 5 \\ 2\alpha_2 + 7\alpha_3 \leq 10 \\ 2\alpha_2 - \alpha_3 \geq -10 \\ 3\alpha_2 + 4\alpha_3 \geq -5 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \tag{62}$$

and the corresponding feasible regions are illustrated in Fig. 2.



(a) $n = 3$ (b) $n = 4$ (c) $n = 5$ (d) $n = 6$

Figure 2: Feasible regions associated with conditions (61) and (62).

In relation to the binomial decomposition of OWA functions in the 3-additive case as illustrated in Fig. 2, we observe that the increasing dimension $n = 3, 4, 5, 6$ has the effect of extending the feasible region associated with the BM constraints. This effect emerges clearly when comparing the feasible regions in Fig. 2.

Proposition 13 *Considering the binomial decomposition (58), the orness of the* 3*-additive OWA function* A *associated with coefficients* α_2 *and* α_3 *is given by*

$$\text{Orness}\,(A) = \frac{1}{2} - \frac{1}{6}\,\frac{n+1}{n-1}\,\alpha_2 - \frac{1}{4}\,\frac{n+1}{n-1}\,\alpha_3 \tag{63}$$

where the coefficients α_2 and α_3 are subject to the BM conditions (57).

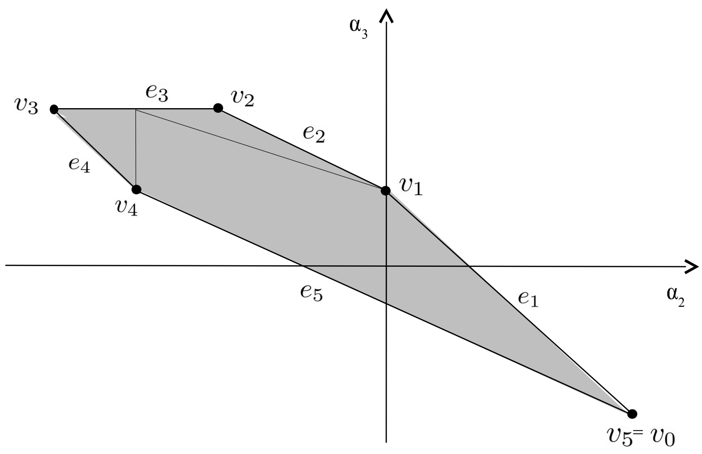


Figure 3: Vertices and edges of the feasible region in the case $n = 5$.

Proposition 14 *Consider the feasible region associated with the 3-additive BM conditions (57) in dimension $n \geq 3$. The feasible region is convex and contains n vertices and n edges as illustrated in Fig. 3 for the particular case $n = 5$. The coordinates of vertex $v_0 = v_n$ are $(3, -2)$, and the coordinates of vertex v_i, with $i = 1, \dots, n-1$, are given by*

$$\alpha_2^{(i)} = -\frac{3(i-1)(n-1)}{n^2-1-3i(n-i)} \qquad \alpha_3^{(i)} = \frac{(n-1)(n-2)}{n^2-1-3i(n-i)}\,. \tag{64}$$

Proof: The feasible region is obtained as the intersection of n linear inequality constraints and thus it is convex. The coordinates of vertex v_i, with $i = 1, \dots, n-1$, are easily obtained by jointly solving the equations associated with the BM conditions i and $i+1$ in (57). □

Proposition 15 *Consider the feasible region associated with the 3-additive BM conditions (57) in dimension $n \geq 3$. The feasible region is strictly increasing with n, and the following holds:*

1. *The vertex v_i in dimension n, with $i = 2, \dots, n-2$, lies on the edge e_{i+1} in dimension $n+1$, with $n \geq 4$.*
2. *The vertex v_i in dimension n, with $i = 2, \dots, n-2$, is external to edge e_i in dimension $n-1$, with $n \geq 4$.*

Proof: The fact that the feasible region is strictly increasing in n is a direct consequence of the two statements, particularly the latter. We now prove each one separately.

1. For instance, vertex v_2 in dimension $n = 4$ lies on the edge e_3 in dimension $n+1 = 5$, as illustrated in Fig. 3 . Consider the coordinates $\alpha_2^{(i)}, \alpha_3^{(i)}$ of

vertex v_i in dimension n as in (64), with $i = 2, \dots, n-2$. The fact that it lies on the edge e_{i+1} in dimension $n+1$ can be written as

$$\left[1 - (n+1)\frac{\binom{i}{1}}{\binom{n+1}{2}}\right]\alpha_2^{(i)} + \left[1 - (n+1)\frac{\binom{i}{2}}{\binom{n+1}{3}}\right]\alpha_3^{(i)} = 1 \tag{65}$$

where we refer to BM condition $i+1$ in dimension $n+1$, see (57). This equation can be verified straightforwardly.

2. For instance, vertex v_2 in dimension $n = 5$ is external to edge e_2 in dimension $n - 1 = 4$, as illustrated in Fig. 3. Consider the coordinates $\alpha_2^{(i)}, \alpha_3^{(i)}$ of vertex v_i in dimension n as in (64), with $i = 2, \dots, n-2$. The fact that it is external to edge e_i in dimension $n-1$ can be written as

$$\left[1 - (n-1)\frac{\binom{i-1}{1}}{\binom{n-1}{2}}\right]\alpha_2^{(i)} + \left[1 - (n-1)\frac{\binom{i-1}{2}}{\binom{n-1}{3}}\right]\alpha_3^{(i)} > 1 \tag{66}$$

where we refer to BM condition i in dimension $n-1$, see (57). This inequality reduces to

$$\frac{6(i-1)(n-i-1)}{(n-2)(n-3)(3i^2 - 3in + n^2 - 1)} > 0 \tag{67}$$

which holds since the lowest value of both $i-1$ and $n-i-1$ is 1 for $i = 2, \dots, n-2$, and the lowest value of $(3i^2 - 3in + n^2 - 1)$ is $(n^2-4)/4$ corresponding to $i = n/2$. Notice that here $n \geq 4$. □

Proposition 16 *Consider the feasible region associated with the* 3*-additive BM conditions (57) in dimension* $n \geq 3$*. We recall that the orness of the* 3*-additive OWA function* A *is linear in the coefficients* α_2 *and* α_3*,*

$$\text{Orness}\,(A) = \frac{1}{2} - \frac{1}{2}\,\frac{n+1}{n-1}\Big(\frac{1}{3}\,\alpha_2 + \frac{1}{2}\,\alpha_3\Big)\,. \tag{68}$$

It follows that the minimum and maximum orness values correspond to vertices of the feasible region. The critical vertex associated with minimum orness value is $m = v_i$ *with* $i = \text{floor}\,(h^-(n))$ *or* $i = \text{ceiling}\,(h^-(n))$*, and the critical vertex associated with maximum orness value is* $M = v_j$ *with* $j = \text{floor}\,(h^+(n))$ *or* $j = \text{ceiling}\,(h^+(n))$*, where*

$$h^{\pm}(n) = \frac{3n \pm \sqrt{3(n^2-4)}}{6} \qquad n \geq 4 \tag{69}$$

and $h^-(3) = 1$ *and* $h^+(3) = 2$*. In this way* $1 \leq h^{\pm}(n) \leq n-1$ *and therefore the critical vertices associated with minimum and maximum orness are among* $v_1, \dots, v_{n-1}$*, with* $n \geq 3$*.*

Proof: According to Proposition 13 the level lines of the orness function are

$$\text{Orness}\,(A) = \frac{1}{2} - \frac{1}{2}\,\frac{n+1}{n-1}\Big(\frac{1}{3}\,\alpha_2 + \frac{1}{2}\,\alpha_3\Big) = c \qquad c \in [0,1] \tag{70}$$

$$2\alpha_2 + 3\alpha_3 = 6(1-2c)\,\frac{n-1}{n+1} \qquad c \in [0,1] \tag{71}$$

with slope $-2/3$ independent of n. The orness of the 3-additive OWA function A associated with the coordinates $\alpha_2^{(i)}$, $\alpha_3^{(i)}$ of the vertex v_i, with $i = 1, \ldots, n-1$, is given by

$$\text{Orness}\,(A) = \frac{-2 + 6i^2 + i(2-4n) - n + n^2}{4(-1 + 3i^2 - 3in + n^2)} = \text{orness}\,(i)\,. \tag{72}$$

Considering i to be a continuous variable, the critical points of orness (i) correspond to $i = h^{\pm}(n)$, where $i = h^-(n)$ corresponds to the minimum orness value and $i = h^+(n)$ corresponds to the maximum orness value. The actual vertex associated with minimum orness value is thus $m = v_i$ with $i = \text{floor}\,(h^-(n))$ or $i = \text{ceiling}\,(h^-(n))$, and the actual vertex associated with maximum orness value is thus $M = v_j$ with $j = \text{floor}\,(h^+(n))$ or $j = \text{ceiling}\,(h^+(n))$.

The vertex $v_0 = v_n$ is in any case excluded because its orness $1/2$ is always intermediate between orness $(i = 1) = 1/2 - (n+1)/4(n-1)$ associated with v_1 and orness $(i = n-1) = 1/2 + (n+1)/4(n-1)$ associated with v_{n-1}. □

6 Conclusions

In the context of the binomial decomposition of OWA functions, in terms of the binomial Gini welfare functions $C_j(\boldsymbol{x})$ and the associated binomial Gini inequality indices $G_j(\boldsymbol{x}) = \bar{x} - C_j(\boldsymbol{x})$, with $j = 2, \ldots, n$, we have investigated the parametric constraints associated with the 3-additive case in n dimensions. The resulting feasible region in the two coefficients α_2 and α_3 is a convex polygon with n vertices and n edges, and is strictly increasing in the dimension n.

The orness of the OWA functions within the feasible region is linear in the coefficients α_2 and α_3, and the vertices associated with maximum and minimum orness have been identified.

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