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Abstract

The theoretical literature on inequality of opportunity formulates basic properties that measures of inequality of opportunity should have. Standard methods for the measurement of inequality of opportunity require the construction of counterfactual outcome distributions through statistical methods. We show that, when standard parametric procedures are used to construct the counterfactuals, the specification used determines whether the resulting measures of inequality of opportunity satisfy the basic properties.

Keywords: Counterfactuals, inequality measurement, opportunities.

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1 Introduction

Theories of equality of opportunity (Dworkin (1981a), Dworkin (1981b), Arneson (1989), Cohen (1989), Roemer (1993), Roemer (1998)) put individual responsibility in the forefront in the assessment of the distribution of outcomes. Individuals' outcomes such as their income level, education attainment or health status, are determined by two kinds of factors. On the one hand, there are circumstances, factors that are beyond individuals' responsibility. On the other hand, there are efforts, factors for which individuals are responsible. Inequalities that are due to circumstances are deemed ethically unacceptable while those arising from efforts are not considered offensive. Hence, the outcome inequalities associated with these two factors should be treated differently. Economists developed social criteria and ways to measure inequality of opportunity based on the dichotomy between circumstances and efforts. Recent overviews of this literature are Ferreira and Peragine (2015), Roemer and Trannoy (2015) and Ramos and Van de gaer (2015).

The number of studies measuring inequality of opportunity has been increasing rapidly over the last few years (see again the overviews mentioned above). Still, many theorists feel uneasy with the tools used, as they rely on statistical procedures which obscure how the resulting measure of inequality of opportunity responds to changes in the data. It is the purpose of the present paper to investigate this issue.

More in particular, we follow closely Fleurbaey and Peragine (2013), and formulate two desirable properties that a measure of inequality of opportunity should have. The idea that inequalities that are due to unequal circumstances are offensive is reflected in the compensation principle: reducing inequalities between individuals that have the same efforts (such that the resulting inequalities are due to circumstances) should decrease inequality of opportunity. The idea that inequalities that are due to differences in efforts are not offensive is reflected in the utilitarian reward principle: reducing inequalities between individuals that have the same circumstances (such that the resulting inequalities are due to efforts) should not affect inequality of opportunity.

In empirical work, the measurement of inequality of opportunity usually relies on constructed counterfactuals. In the direct measurement approach, one measures the inequality in a counterfactual distribution that contains only inequalities that are due to circumstances. In the indirect approach, one measures the difference between actual inequality and the inequality in a counterfactual that contains only inequalities that are due to efforts.

The counterfactuals can either be constructed non-parametrically or parametrically. The non-parametric methodology has been developed by Checchi and Peragine (2010) and was applied to university access by Brunori et al. (2012). It constructs counterfactuals as group averages. In the direct approach each individual is assigned the average outcome obtained by those having the same circumstances as he, in the indirect approach he is assigned the average outcome obtained by those having the same efforts as he. The parametric methodology specifies and estimates a functional form between the outcome under study, on the one hand, and circumstances, effort and a random term, on the other hand. These estimates are then used to construct the counterfactuals. For instance, following Fleurbaey and Schokkaert (2009), one can estimate an equation between the outcome, circumstances, and a random term, set the random term equal to zero, and use the resulting "predicted" outcome (a function of individuals' circumstances only), as a counterfactual in the direct approach. Alternatively, as suggested by Bourguignon et al. (2007), one can set circumstances equal to a reference value (which is the same for all individuals), and plug in the estimated residual to obtain a counterfactual that only reflects differences in outcome that are due to differences in efforts. This counterfactual can be used in the indirect approach.

The functional form used in parametric empirical analyses usually depends on the nature of the dependent variable and on the specification that one is used to in the literature. When interested in inequality of opportunity for income, a loglinear specification is almost universally used (see, e.g. Bourguignon et al. (2007), Ferreira and Gignoux (2011), Hassine (2012), Marrero and Rodríguez (2012), Singh (2012), Niehues and Peichl (2014). When the outcome of interest is binary or categorical, nonlinear probability models are commonly used. Foguel and Veloso (2014) and Trannoy et al. (2010), for instance, use a logit model to study inequality of access to education, and self-assessed health status, respectively, while Rosa Dias (2009) employs an ordered probit to examine self-assessed health status. For inequality in PISA scores, Ferreira and Gignoux (2014) use a simple linear specification.

We show that the non-parametric methods introduced by Checchi and Peragine (2010) satisfy one of the two basic properties. For parametric methods, we argue that, when using a parametric approach to construct the counterfactuals, the statistical procedure and functional form chosen are crucial to determine the properties of the resulting measure of inequality of opportunity. We show that they determine whether the measure of inequality of opportunity satisfies the compensation or the utilitarian reward principle and use our results to reflect upon the state of the art.

The structure of the paper is as follows. Section 2 introduces the notation and the way counterfactuals are constructed in the empirical literature. Section 3 formulates the theoretical basic principles, compensation and utilitarian reward. Section 4 presents the results when the counterfactual is constructed on the basis of a linear least squares estimate (Section 4.1), a loglinear least squares estimate (Section 4.2) and a non-parametric estimate (Section 4.3). Section 4.4 develops a framework with binary outcomes. The conclusion is contained in Section 5.

2 Measuring inequality of opportunity using counterfactuals

Let $N = \{1, \ldots, n\}$, be the set of individuals. In theoretical work, N is the population; in empirical work N is typically a sample of individuals drawn from a population. The purpose of our work is to see how empirical approaches perform when used in the theorists' set-up. Therefore, it is best to think about N as the population. Up to section 4.4 we assume that the outcome we observe for individual i is such that Pigou-Dalton transfers of this outcome can be meaningfully defined. We call this outcome individual i's income, $y_i \in \mathbb{R}_{++}$. Following the literature on equality of opportunity, we want to compensate her for some of these characteristics, while we want to hold her accountable for other characteristics. The former are brought together in the vector of circumstances $c_i \in \mathbb{R}^{d^C}$, the latter in the vector of efforts $e_i \in \mathbb{R}^{d^R}$. Define

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, C = \begin{bmatrix} c_1' \\ \vdots \\ c_n' \end{bmatrix}, E = \begin{bmatrix} e_1' \\ \vdots \\ e_n' \end{bmatrix}.$$

Throughout we will assume that C and E are given and that the researcher is fully informed about the values of Y, C and E. This is not an innocuous assumption. The consequences of unobserved circumstances and efforts is discussed elsewhere, see, e.g., Roemer and Trannoy (2015) or Ramos and Van de gaer (2015), and is not the focus of the present paper, where we analyze a set-up that is as close as possible to that used in theoretical work. A dataset d is a triplet (Y, C, E). The set of all possible data sets is

$$D = \left\{ (Y, C, E) : Y \in \mathbb{R}^n, C \in \mathbb{R}^{n \times d^C}, E \in \mathbb{R}^{n \times d^E} \right\}.$$

A measure of inequality of opportunity is a function $M : d \in D \mapsto \mathbb{R}^n$ such that $M(d) > M(d')$ means that inequality of opportunity in d is higher than in d'.

Let the function $I(\cdot) : Z \in \mathbb{R}^n_{++} \to \mathbb{R}_+$ be a measure of inequality, i.e., a function satisfying two properties. First, it satisfies the Pigou-Dalton transfer principle, meaning that, for all $Y, \widetilde{Y} \in \mathbb{R}^n_{++}$ that are such that there exists $\delta \in \mathbb{R}_{++}$ and $i, j \in N : \widetilde{y}_i = y_i - \delta \geq \widetilde{y}_j = y_j + \delta$, and for all $k \notin \{i, j\} : \widetilde{y}_k = y_k$, $I(\widetilde{Y}) < I(Y)$. Second, it satisfies anonymity, meaning that, for all $Y, \widetilde{Y} \in \mathbb{R}^n_{++}$ that are such that the vector $\widetilde{Y}$ can be obtained from the vector Y by permuting its incomes, $I(\widetilde{Y}) = I(Y)$.

The two most popular ways to measure inequality of opportunity that rely on counterfactual income distributions are the direct and the indirect approach.

Direct measures determine the amount of inequality of opportunity as the inequality in a counterfactual income distribution Y^c in which all inequalities due to differences in effort have been eliminated, such that the remaining inequality is solely due to differences in circumstances:

$$M^D(d) = I\left(Y^c(d)\right). \tag{1}$$

Indirect measures determine the amount of inequality of opportunity by comparing the inequality in the actual distribution of income, Y, to the inequality in a counterfactual income distribution where there is no inequality of opportunity Y^{EO}. This results in the measure

$$M^I(d) = I\left(Y\right) - I\left(Y^{EO}(d)\right). \tag{2}$$

To compute either of these measures a counterfactual distribution of income has to be constructed using the information in the dataset d. This can be done either with a parametric or a non-parametric approach. Three types of specifications have been used to construct counterfactual distributions parametrically. Individual i 's income, y_i, is assumed to depend on her circumstances c_i and her efforts, e_i, such that

$$y_i = g(c_i, e_i) \quad \text{where} \quad g : \mathbb{R}^{d^C} \times \mathbb{R}^{d^R} \to \mathbb{R}_{++}.$$

As the functional form g is unknown, the parametric approach imposes a functional form to estimate the equation, yielding the function

$$\widehat{g}(c_i, e_i, u_i) \quad \text{where} \quad \widehat{g} : \mathbb{R}^{d^C} \times \mathbb{R}^{d^R} \times \mathbb{R} \to \mathbb{R}_{++}. \tag{3}$$

The effect of specification errors go into the estimated random term, $\widehat{u}_i$, which is defined implicitly by the equation $y_i = \widehat{g}(c_i, e_i, \widehat{u}_i)$. Its estimate is determined by the chosen functional form $\widehat{g}$ and the dataset d. Some counterfactuals treat it as a circumstance, others as an effort (see below). Other counterfactuals are based on estimates of incomes as a function of, alternatively circumstances and random variation, or efforts and random variation:

$$\widehat{g}^C\left(c_i, u_i^C\right) \quad \text{where} \quad \widehat{g}^C : \mathbb{R}^{d^C} \times \mathbb{R} \to \mathbb{R}_{++}, \tag{4}$$

$$\widehat{g}^E\left(e_i, u_i^E\right) \quad \text{where} \quad \widehat{g}^E : \mathbb{R}^{d^R} \times \mathbb{R} \to \mathbb{R}_{++}. \tag{5}$$

In the first (second) equation, the effect of omitted efforts (circumstances) are taken over by circumstances (efforts) to the extent that these two are correlated. The rest of their effect as well as specification errors go into the estimated random variation, $\widehat{u}_i^C$ ($\widehat{u}_i^E$), which is defined implicitly by the equation $y_i = \widehat{g}^C\left(c_i, \widehat{u}_i^C\right)$ ($y_i = \widehat{g}^E\left(e_i, \widehat{u}_i^E\right)$).

The following parametric counterfactuals have been proposed for the direct approach:

$$y_i^{c1} = \widehat{g}^C\left(c_i, 0\right), \tag{6}$$

$$y_i^{c2} = \widehat{g}^E\left(\overline{e}, \widehat{u}_i^E\right), \tag{7}$$

$$y_i^{c3} = \widehat{g}\left(c_i, \overline{e}, 0\right), \tag{8}$$

$$y_i^{c4} = \widehat{g}\left(c_i, \overline{e}, \widehat{u}_i\right). \tag{9}$$

In (7) - (9), $\overline{e}$ is a vector of reference values for efforts. Use of counterfactual (6) implies that one measures the inequality that is due to circumstances, including the indirect correlation between circumstances and efforts. Counterfactual (7) measures all inequalities that are due to circumstances and random terms u_i that are not correlated with effort. In (6) and (8), differences in u_i are treated as efforts (i.e. inequalities due to differences in u_i are legitimate), while in (7) and (9) they are treated as circumstances. [1] Counterfactual, y^{c1} has become the most popular; it was used, e.g., by Rosa Dias (2009), Ferreira and Gignoux (2011), Ferreira and Gignoux (2014), Brunori et al. (2012), Marrero and Rodríguez (2012), Foguel and Veloso (2014) and Niehues and Peichl (2014). Fleurbaey and Schokkaert (2009) suggest to use y^{c3} and y^{c4}. Pistolesi (2009) used y^{c4}. We are unaware of any application of y^{c2}.

Non-parametric procedures rely on averaging. Let $N_{i\cdot} = \{k \in N \text{ such that } c_k = c_i\}$. The non-parametric counterfactual for the direct approach becomes

$$y_i^{c5} = \frac{1}{|N_{i\cdot}|} \sum_{k \in N_{i\cdot}} y_k, \tag{10}$$

which is the average income of all those having the same circumstances as individual i. It was proposed in Van de gaer (1993) and developed in Checchi and

[1] As in actual applications it is unclear whether the u_i should be treated as a circumstance or effort, Fleurbaey and Schokkaert (2009) suggest to compute inequality of opportunity in both cases. This holds, of course also when using an indirect measure of inequality of opportunity.

Peragine (2010).

For the indirect measurement approach the following parametric counterfactuals have been proposed:

$$y_i^{EO1} = \widehat{g}^E(e_i, 0), \tag{11}$$

$$y_i^{EO2} = \widehat{g}^C\left(\overline{c}, \widehat{u}_i^C\right), \tag{12}$$

$$y_i^{EO3} = \widehat{g}(\overline{c}, e_i, 0), \tag{13}$$

$$y_i^{EO4} = \widehat{g}(\overline{c}, e_i, \widehat{u}_i). \tag{14}$$

In (12) - (14), $\overline{c}$ is a vector of reference values for circumstances. The inequality in counterfactual (11) measures the inequality that is due to the direct effect of efforts as well as the part that is due to the correlation between efforts and circumstances. The inequality in counterfactual (12) reflects all inequalities that are due to efforts and random terms u_i that are not correlated with circumstances. In (11) and (13), differences in u_i are treated as circumstances, while in (12) and (14) they are treated as efforts. Counterfactual, y^{EO1} was used in Trannoy et al. (2010), y^{EO2} was used in Bourguignon et al. (2007), Singh (2012) and Hassine (2012). Fleurbaey and Schokkaert (2009) suggested y^{EO3} and y^{EO4}. Brunori et al. (2012) applied y^{EO3} and y^{EO4}. The latter was also applied by Bourguignon et al. (2007) and Pistolesi (2009).

Also in the indirect approach a non-parametric counterfactual can be constructed. Let $N_{\cdot i} = \{k \in N \text{ such that } e_k = e_i\}$. The non-parametric counterfactual for the indirect approach becomes

$$y_i^{EO5} = \frac{1}{|N_{\cdot i}|} \sum_{k \in N_{\cdot i}} y_k, \tag{15}$$

which is the average income of all those having the same effort as individual i. Checchi and Peragine (2010) proposed this counterfactual.

3 A measurement perspective

Both the theorists and empirists use the information in the dataset d to construct a measure of inequality of opportunity. The empirists use the dataset d in order to obtain a good estimate of relationship (3), (4) or (5), construct the counterfactuals and compute the direct or indirect measure of inequality of opportunity, respectively (1) or (2).

The theorists take a measurement theory perspective. It requires that the measure of inequality of opportunity responds to changes *in the dataset* in a way compatible to the intuitions prescribed by equality of opportunity principles. Let the set of all datasets compatible with Y, C and E be

$$\Delta = \left\{ (X, C, E) : X \in \mathbb{R}^n_{++} \text{ and} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \right\}.$$

The domain $\Delta \subset D$. It keeps not only the set of individuals fixed, but also their characteristics C and E, as well as the total income. It allows us to state how the measure of inequality of opportunity should respond to income redistributions between individuals. Several axioms embodying desirable properties involving certain types of Pigou-Dalton transfers have been proposed in the literature (see, e.g., Fleurbaey and Peragine (2013)).

The first axiom, Compensation, requires that, when 2 individuals i and j have the same efforts, and i has a higher level of income than j, then, transferring an amount of money δ from i to j without resulting in i having a lower income than j, decreases inequality of opportunity, as their income difference, which is entirely due to circumstances, diminishes. [2]

COM (Compensation): For all $d = (Y, C, E)$ and $\widetilde{d} = (\widetilde{Y}, C, E) \in \Delta$ that are such

[2] Fleurbaey and Peragine (2013) labelled this axiom "Ex-Post Compensation".

that there exist $\delta \in \mathbb{R}_{++}$ and $i, j \in N : e_i = e_j$ and $\widetilde{y}_i = y_i - \delta \geq \widetilde{y}_j = y_j + \delta$, and for all $k \notin \{i, j\} : \widetilde{y}_k = y_k$, we have $M(\widetilde{d}) < M(d)$.

The idea that people are responsible for their efforts can be expressed by a second axiom, Utilitarian Reward, which also makes a statement about the effect of a particular kind of Pigou-Dalton transfer on inequality of opportunity. It requires that, when individuals i and j have the same circumstances and i has a higher income than j, then, as the difference in incomes is due to efforts, transferring an amount of money δ from i to j without resulting in i having a lower income than j, should not affect inequality of opportunity. [3]

UR (Utilitarian Reward): For all $d = (Y, C, E)$ and $\widetilde{d} = (\widetilde{Y}, C, E) \in \Delta$ that are such that there exists $\delta \in \mathbb{R}_{++}$ and $i, j \in N : c_i = c_j$ and $\widetilde{y}_i = y_i - \delta \geq \widetilde{y}_j = y_j + \delta$, and for all $k \notin \{i, j\} : \widetilde{y}_k = y_k$, we have $M(\widetilde{d}) = M(d)$.

We know from the literature on fair compensation that it is very difficult to reconcile compensation and reward principles (see, e.g., Bossert (1995), Fleurbaey (1995), Fleurbaey (2008) or Fleurbaey and Maniquet (2011)). The same is true here: COM and UR are incompatible. To see this, consider four individuals: $N = \{1, 2, 3, 4\}$, $c_1 = c_2$, $c_3 = c_4$, $e_1 = e_3$, $e_2 = e_4$, incomes y_1, y_2, y_3 and y_4 and the set of all datasets Δ^0 compatible with Y, C and E specified above. Take a dataset $d^0 \in \Delta^0$ such that $y_1 < y_4 < y_3 < y_2$. In what follows we consider Pigou-Dalton transfers of a fixed size $\delta \leq (y_3 - y_4)/2$ such that the transfer always satisfies the conditions in the axioms we are using. First, do a Pigou-Dalton transfer from individual 2 to 1, resulting in dataset d'. By UR, inequality of opportunity has not changed. Next, do a Pigou-Dalton transfer from individual 3 to 4, resulting in dataset d''. By UR, inequality of opportunity has to be the same in datasets d^0, d' and d''. Now, start again from dataset d^0, and do a Pigou-Dalton transfer from

[3] This principle has also been called "Utilitarianism for Equal Circumstances" by Fleurbaey (2008). He provides a critical discussion of this reward principle and compares it with alternative principles.

individual 3 to 1, resulting in dataset d'''. Next, do an additional Pigou-Dalton transfer from individual 2 to 4, and we are back in dataset d''. By COM, inequality of opportunity in dataset d^0 is larger than in d''', where it is larger than in economy d'', which contradicts our previous finding.

Hence we know that there does not exist any measure of inequality of opportunity that satisfies both COM and UR. The purpose of this paper is to see whether, or under which conditions, the procedures that are most frequently used in the empirical literature to measure inequality of opportunity based on counterfactuals satisfy one of these two basic properties.

It turns out that the properties of the direct and indirect measures of inequality of opportunity depend on the functional form and the statistical procedure chosen to estimate the equation used to compute the counterfactual. In Sections 4.1 and 4.2 we assume that the predicted values are generated by an ordinary least squares regression and in section 4.3 we investigate the non-parametric alternatives. In Section 4.4 we reformulate the framework to deal with binary responses and discuss nonlinear probability models.

4 Results

4.1 Linear least squares

Define average income $\mu_Y = \frac{1}{n}\sum_{j=1}^{n} y_j$, the average value of circumstance k, $\mu_{Ck} = \frac{1}{n}\sum_{j=1}^{n} c_{jk}$ and μ_C the d^C dimensional vector containing μ_{Ck} as k-th element for all $k \in \left\{1, \dots, d^C\right\}$. The average value for effort q, $\mu_{Eq} = \frac{1}{n}\sum_{j=1}^{n} e_{jq}$, and μ_E is the d^E dimensional vector containing the μ_{Eq} as q-th element for all $q \in \left\{1, \dots, d^E\right\}$. Let ι be the $n-$ dimensional vector of ones, $X^C = C - \iota(\mu_C)'$,

$X^E = E - \iota(\mu_E)'$, $X^{CE} = \begin{bmatrix} X^C & X^E \end{bmatrix}$, $Y^D = Y - \iota\mu_Y$,

$$U^A = \begin{bmatrix} u_1^A \\ \vdots \\ u_n^A \end{bmatrix} \text{ and } V^A = \begin{bmatrix} v_1^A \\ \vdots \\ v_n^A \end{bmatrix}, \text{ where } A \in \{C, E, CE\}.$$

The equations that have to be estimated to construct the counterfactuals are, in deviational form,

$$Y^D = X^A\beta^A + U^A. \tag{16}$$

The corresponding least squares estimator of the coefficient vectors β^A are

$$b^A = \left((X^A)'X^A\right)^{-1}(X^A)'Y^D \text{ for } A \in \{C, E, CE\}. \tag{17}$$

Performing a Pigou-Dalton transfer from observation i to j will change the least squares estimates of the coefficient vector (see Lemma A2 in the Appendix A1), and thus the constructed counterfactuals (see Lemma A3 in Appendix A1) and, thereby, the values of the direct and indirect measures of inequality of opportunity. We want to verify whether the way these measures respond to Pigou-Dalton transfers is in accordance with the prescriptions in the Axioms COM and UR. We find that some measures obey one of the Axioms, others do not and still others do so under some assumptions. In particular the following assumptions turn out to be helpful.

C1M: Circumstances are one-dimensional and the transfer goes from someone with better circumstances to someone with worse circumstances: $d^C = 1$ and $c_{i1} > c_{j1}$.

E$\boldsymbol{\mu}$: Reference efforts are equal to average efforts: $\bar{e} = \mu_E$.

C$\boldsymbol{\mu}$: Reference circumstances are equal to average circumstances: $\bar{c} = \mu_C$.

COR0: Circumstances and efforts are not correlated.

One-dimensionality of circumstances is a very strong assumption to make. However, observe that in the literature on intergenerational mobility, one often draws lessons about inequality of opportunity (see O'Neill et al. (2000) and, for a recent example Chetty et al. (2014)), which requires a one-dimensional view of circumstances. In addition, C1M requires monotonicity: the transfer goes from an individual with better circumstances and higher income to someone with worse circumstances and lower income. The determination of the value for reference efforts or circumstances is a tedious issue in the literature. [4] In much of the empirical literature, reference values are simply set to their sample means, without much justification, see, e.g., Bourguignon et al. (2007), Pistolesi (2009), Ferreira and Gignoux (2011) or Singh (2012). There has been some debate about the implications of the correlation between efforts and circumstances. Roemer (1993) and Roemer (1998) argued that normatively relevant effort has to be measured such that it is, by construction, not correlated with circumstances since it is very hard to hold people responsible for efforts if they are correlated with circumstances, which are, by definition, not under individual control. Others, e.g., Rawls (1971), Dworkin (1981a), Dworkin (1981b), Van Parijs (1995) and Fleurbaey (2008) have argued that people should be responsible for their tastes, even if these are correlated with their circumstances.

The following Proposition lists the properties of the measures of inequality of opportunity.

PROPOSITION 1: Using the linear specification (16) and least squares as the estimation method, the following Table gives sufficient conditions for the measures of inequality of opportunity to satisfy COM or UR.

[4] There are not many theoretical results about the consequences of the choice of reference values –for an exception see Luttens and Van de gaer (2007). To solve the arbitrariness of the choice of reference value Ramos and Van de gaer (2015) propose an averaging procedure. García-Gómez et al. (2013) propose to minimize the extent to which the theoretical principles are violated.

Table 1: Conditions for the measures to satisfy COM and UR

Measure	COM	UR
Panel A: Direct measures		
(a) $y_l^{c1} = \widehat{g}^C(c_i, 0)$	C1M	+
(b) $y_l^{c2} = \widehat{g}^E\left(\overline{e}, \widehat{u}_i^E\right)$	+	-
(c) $y_l^{c3} = \widehat{g}(c_i, \overline{e}, 0)$	Eμ and C1M COR0 and C1M	Eμ and COR0
(d) $y_l^{c4} = \widehat{g}(c_i, \overline{e}, \widehat{u}_i)$	COR0	-
Panel B: Indirect measures		
(e) $y_l^{EO1} = \widehat{g}^E(e_i, 0)$	+	-
(f) $y_l^{EO2} = \widehat{g}^C\left(\overline{c}, \widehat{u}_i^C\right)$	Cμ and C1M	+
(g) $y_l^{EO3} = \widehat{g}(\overline{c}, e_i, 0)$	Cμ and COR0	-
(h) $y_l^{EO4} = \widehat{g}(\overline{c}, e_i, \widehat{u}_i)$	Cμ and C1M	COR0

Notes: A "+" means that the measure always satisfies the Axiom in the column; a "-" means that the measure does not satisfy the Axiom under any of the Assumptions considered.

Table 1 makes clear that, if one wants a measure to satisfy COM, one can use measure (b) or (e); if one wants a measure to satisfy UR, one can use (a) or (f). Absence of correlation between efforts and circumstances helps to establish properties of the inequality measures in case a counterfactual is used that relies on b^{CE}. The use of average values as reference values by itself does not help to guarantee that an inequality of opportunity measure has desirable properties, but in conjunction with other assumptions the properties of the measure can sometimes be established. One-dimensionality of circumstances and monotonicity helps often to guarantee that measures satisfy COM. We verified that one-dimensionality of efforts does not help measures to satisfy UR. On the contrary, it ensures that inequality in the counterfactual is affected (for direct measures) and is affected differently from inequality in the actual distribution of income (for indirect measures). If one only has information on circumstances, such that only counterfactuals based on b^C can be constructed, measures that satisfy UR are available, but no measure that satisfies COM is available. The opposite occurs in case one only has information on efforts. Finally, there are only three measures of inequality of opportunity in Table 1 that are not dominated by any other, in the sense that there does not exist another measure that requires weaker assumptions to satisfy the axioms. These are measures (a), (b) and (e).

4.2 Loglinear least squares

The equations that have to be estimated to construct the counterfactuals are now

$$log(Y) - \iota\mu_{log(Y)} = X^A\alpha^A + V^A, \tag{18}$$

where $\mu_{log(Y)} = \frac{1}{n}\sum_{i=1}^{n} log(y_i)$ and the corresponding least squares estimator of α^A is

$$a^A = \left((X^A)'X^A\right)^{-1}(X^A)'\left[log(Y) - \iota\mu_{log(Y)}\right] \text{ for } A \in \{C, E, CE\}. \tag{19}$$

These estimates can be used to determine the counterfactuals (6)-(14) under the assumption of loglinearity and the use of least squares (see Lemma A4 in Appendix A2 for a proof). Again, Pigou-Dalton transfers influence the estimated coefficients (Lemma A5 in Appendix A2) and counterfactuals (Lemma A6 in Appendix A2). We can verify whether the way the inequality of opportunity measures respond to Pigou-Dalton transfers is in accordance with the prescriptions in the Axioms COM and UR. The result is stated in Proposition 2.

PROPOSITION 2: Using the loglinear specification (18) and least squares as the estimation method, none of the measures of inequality of opportunity satisfies COM nor UR.

We have seen that, when measuring inequality of opportunity for income, a loglinear specification is standard (see the references in the Introduction). Proposition 2 suggests that, with a loglinear specification, none of the resulting measures satisfies any of the basic principles that a measure of inequality of opportunity should have. This is bad news for what has become standard practice. However, if one were to claim that the relevant outcome is not income, but the log of income (for instance because it might be assumed that individual utilities can be measured that way) and a measure of inequality in the distribution of the log of (counterfactual) incomes is used in (1) or (2), then the transfers in the axioms should apply to transfers in log income, which means that we are in the framework of Section 4.2, and the results from Proposition 1 apply. [5]

More precisely, consider the factor proportional transfer principle, in which, with $\delta > 0$ the income of the rich individual is divided by a factor $1 + \delta$, while the income of the poor individual is multiplied by $1 + \delta$, without resulting in a post transfer income for the rich individual that is lower than the post transfer income of the poor individual. [6] Replacing the standard Pigou-Dalton transfer in the COM and UR axioms by such factor proportional transfers, it becomes clear that use of the loglinear specification (18) and a measure of inequality in the distribution of the log of (counterfactual) incomes in (1) or (2) means that the results from Table 1 apply to these factor proportional transfer versions of the COM and UR axioms. Observe that it is crucial that the inequality measure is defined on the distribution of the log of incomes. This is, as far as we are aware of, never done in the literature. The mean log deviation is commonly used, but it is not suited, as it is the average value of the log of mean income divided by income. The standard deviation of the log of incomes, or any standard inequality measure defined on the distribution of the log of incomes are alternatives that are more coherent with the basic intuitions of inequality of opportunity measurement.

[5] Actually, the empirical literature finds estimates close to one of the constant elasticity of marginal utility (see e.g. Layard et al. (2008), Gandelman and Hernández-Murillo (2013)), which could give empirical support to a loglinear specification as the best approximation to modeling individual utility as a function of income.

[6] This principle is different from the proportional transfer principles considered in Fleurbaey and Michel (2001). We discuss this in Appendix C.

When measuring inequality of opportunity for income, a loglinear specification is almost universally used, and usually one uses the direct measure with counterfactual (6). The loglinear specification is motivated by its empirical fit; the use of a specification involving only circumstances is motivated by the absence of data on efforts. The results in this paper provide an additional and important motivation for this approach: if, in the axioms, the transfers are defined as factor proportional transfers, then none of the other approaches in the literature satisfies the COM and UR principle under weaker assumptions. Moreover, it dominates the only other approach that only needs information on incomes and circumstances (the indirect measure with counterfactual (12)), which requires in addition to one-dimensionality of circumstances, that reference circumstances are put equal to their empirical averages in order to satisfy COM.

That the standard procedure is not dominated by any other, and that it always satisfies UR (with factor proportional transfers) are strong arguments in favor of the standard practice. Without information on efforts, it is the best one can do, but if circumstances are not one-dimensional it does not satisfy COM. Hence, it gives priority to the UR principle above the COM principle. Our results show that there exist measures that satisfy factor propotional transfer versions of the COM axiom, such as the direct measure with counterfactual (7) or the indirect measure with counterfactual (11), but they require information on efforts.

4.3 Non-parametric approaches

The properties of the non-parametric procedures to construct counterfactuals are easy to derive. It is clear that the counterfactual defined in (10) is unchanged if a Pigou-Dalton transfer occurs between two individuals having the same circumstances. Hence it immediately follows that when this counterfactual is used in the direct approach (1), the resulting measure of inequality of opportunity satisfies UR always and hence cannot satisfy COM. The counterfactual defined in (15) is unchanged if a Pigou-Dalton transfer occurs between two individuals having the same efforts. Hence, as the inequality in the actual income distribution has decreased, it immediately follows that when this counterfactual is used in the indirect approach (2), the resulting measure of inequality of opportunity satisfies COM always and hence cannot satisfy UR. We summarize the results in the following Proposition.

PROPOSITION 3: The non-parametric counterfactual (10) in the direct approach always satisfies UR and never satisfies COM. The non-parametric counterfactual (15) in the indirect approach always satisfies COM and never satisfies UR.

4.4 Binary outcomes

In this section we analyze binary response models. Observations are now brought together in the n-dimensional vector Y^b which contains, as $i-$ the element, the binary outcome $y_i^b \in \{0,1\}$ of individual i. A dataset d^b is a triplet $\big(Y^b, C, E\big)$. The set of all possible data sets is

$$D^b = \left\{(Y^b, C, E) : Y^b \in \{0,1\}^n, C \in \mathbb{R}^{n \times d^C}, E \in \mathbb{R}^{n \times d^E}\right\}.$$

A measure of inequality of opportunity is a function $M^b : d^b \in D^b \to \mathbb{R}_+$ such that $M^b(d^b) > M^b(d'^b)$ means that inequality of opportunity is higher in d^b than in d'^b. Again we will formulate desirable properties, taking the set of individuals, their characteristics and the total (binary) outcome as given. That is, we work with the domain

$$\Delta^b = \left\{(X^b, C, E) : X^b \in \{0,1\}^n \text{ and } \sum_{i=1}^n x_i^b = \sum_{i=1}^n y_i^b\right\},$$

where $\Delta^b \subset D^b$. Due to the binary nature of the outcomes, the concept of a Pigou-Dalton transfer no longer makes sense such that the COM and UR axioms have to be redefined. We propose definitions that rely on switching binary outcomes of two individuals with the same circumstances. Binary compensation requires that, if, as a result, after the switch all individuals that have the same efforts as the individuals whose outcomes were switched obtain the same binary outcome, then the switch decreased inequality of opportunity.

BCOM (Binary Compensation): For all $d^b = (Y^b, C, E)$ and $\widetilde{d^b} = (\widetilde{Y}^b, C, E) \in \Delta^b$ that are such that there exist $i, j \in N$ with $c_i = c_j$ and $y_i^b \neq y_j^b$, and $\widetilde{y}_i^b = y_j^b$ and $\widetilde{y}_j^b = y_i^b$, while for all $k \notin \{i, j\} : \widetilde{y}_k^b = y_k^b$, for all $l : e_l = e_i$, $\widetilde{y}_l^b = \widetilde{y}_i^b$, and for all $m : e_m = e_j$, $\widetilde{y}_m^b = \widetilde{y}_j^b$, we have $M^b(\widetilde{d^b}) < M^b(d^b)$.

Binary utilitarian reward requires that a switch of binary outcomes between individuals having the same circumstances has no effect on inequality of opportunity.

BUR (Binary Utilitarian Reward): For all $d^b = (Y^b, C, E)$ and $\widetilde{d^b} = (\widetilde{Y}^b, C, E) \in \Delta^b$ that are such that there exist $i, j \in N$ with $c_i = c_j$ and $\widetilde{y}_i^b = y_j^b$ and $\widetilde{y}_j^b = y_i^b$, while for all $k \notin \{i, j\} : \widetilde{y}_k^b = y_k^b$, we have $M^b(\widetilde{d^b}) = M^b(d^b)$.

From the two definitions, it is obvious that both principles are incompatible: the switch described in BCOM is a valid switch in BUR, but contrary to what BUR prescribes, it decreases inequality of opportunity instead of not affecting it.

To measure inequality of opportunity with binary outcomes, one usually relies on parametric estimates. The parameter estimates of a nonlinear probability model are then used to construct counterfactual distributions of probabilities. Direct measures of inequality of opportunity are based on counterfactuals similar to (6)-(9), indirect measures of inequality of opportunity are based on counterfactuals similar to (11)-(14). There exist many different specifications of nonlinear probability models, such as the probit and the logit model. Due to the highly nonlinear nature of these models, expressions that compute the consequences of switches in binary outcomes between individuals with the same circumstances on the counterfactuals are intractable. By focussing on a simple case we can nevertheless prove the following Proposition (see Appendix A.4).

PROPOSITION 4: Using any non-linear probability model, irrespective of the inequality measure chosen, (a) a direct measure based on counterfactual (6) or an indirect measure based on counterfactuals (12), (13) or (14) cannot satisfy BCOM, (b) a direct measure based on the counterfactuals (7), (8) or (9) or an indirect measure based on counterfactal (11) cannot satisfy BUR, (c) a direct measure based on counterfactual (6) always satisfies BUR.

Once the choice between a direct and an indirect measure has been made, and the counterfactual has been specified, the Proposition tells us which of the two basic axioms BCOM or BUR the resulting measure of inequality of opportunity cannot satisfy. Moreover, the direct measure based on counterfactual (6) always satisfies BUR, as the switch in outcomes between two individual with the same circumstances does not change the dataset used in the estimation as only information on binary outcomes and circumstances are used.

It is of course also possible to use for binary outcomes the non-parametric counterfactuals (10) and (15) in the direct and indirect measurement approach, respectively. The results for these approaches are given in the following proposition.

PROPOSITION 5: The non-parametric counterfactual (10) in the direct approach always satisfies BUR and never satisfies BCOM. The non-parametric counterfactual (15) in the indirect approach always satisfies BCOM and never satisfies BUR.

5 Conclusion

The theoretical literature on inequality of opportunity formulates some basic properties that measures of inequality of opportunity should have. The principle of compensation pleads for a reduction of inequality between individuals that have the same efforts. Utilitarian reward says that a transfer between individuals having the same circumstances should not affect inequality of opportunity. These principles can be easily formulated into requirements on measures of inequality of opportunity that can be computed on the basis of datasets.

The empirical literature tries to quantify the amount of inequality of opportunity. It has evolved in ways disconnected from these principles. This is especially so in the dominant part of the literature, which measures inequality of opportunity by means of counterfactuals, estimated through statistical procedures that are common to model the outcome of interest, such as income, education or health. As a result, the theoretical properties of the measures are obscured, and many theorists feel uneasy with the empirical work.

Bridging the gap between the theoretical and the empirical literature was one of the main goals of this paper. We have shown that some counterfactuals based on estimates from a linear least square specification yield measures that satisfy some of the desirable properties. This set of counterfactuals can be extended if additional assumptions, which have been previously used or discussed in the literature, are imposed. Contrary to this, no single measure of inequality of opportunity satisfies the desirable properties when counterfactuals are based on estimates from a loglinear specification. This is worrying, as this specification is used a lot when one is interested in inequality of opportunity for income. However, provided one assumes logarithmic utility, and transfers are defined in terms of utility, and a measure of inequality of the distribution of log of incomes is used, the previous results are restored. This is, provided a right inequality measure is used, an additional motivation for what has become the standard approach in the measurement of inequality of opportunity for income, which measures the inequality in a counterfactual distribution, constructed on the basis of a loglinear least squares regression of incomes on circumstances only. In terms of incomes, this approach always satisfies the factor proportional transfer version of the utilitarian reward principle, and provided circumstances are one-dimensional and monotonic, it also satisfies the factor proportional transfer version of compensation. It is not dominated by any of the other approaches we considered. When a binary response model is used to construct the counterfactuals, we have shown that the choice of measure (direct / indirect) and of the counterfactual imply which of the two properties one is giving up for sure. Finally, irrespective of whether outcomes are real numbers or binary, when non-parametric averaging procedures are used to construct counterfactuals in the direct approach, the resulting measure of inequality of opportunity always satisfies utilitarian reward. Doing the same for the counterfactual in the indirect approach results in a measure of inequality of opportunity that satisfies compensation.

If one believes that the theoretical properties of measures of inequality of opportunity are important, the choice of the functional form and the statistical procedure used to estimate the counterfactual, and the choice of the inequality measure should not be based exclusively on its convenience, goodness of fit or what one is used to in a particular context. One should also be aware that the properties of the resulting measure of inequality of opportunity are affected by these choices.

Our paper is a first analysis of the issues involved, and it has several shortcomings. First, the only reward principle we considered was utilitarian reward. Liberal reward is the most prominent reward principle in the axiomatic literature on fair allocations (see, e.g., Bossert (1995) and Fleurbaey (1995)), and fair social orderings (see, e.g. Fleurbaey and Maniquet (2005), Fleurbaey and Maniquet (2008) and Fleurbaey and Maniquet (2011)). It states that government taxes and transfers should respect differences in incomes that are due to differences in responsibility. Hence, to incorporate the idea of liberal reward, one also needs information on net transfers. It is well known that also natural reward is incompatible with compensation (see Bossert (1995) and Fleurbaey (1995)). The axiomatic literature proceeded to formulate weakened versions of compensation and liberal reward and formulated redistribution mechanisms that satisfy these weakened versions (see, e.g. Bossert and Fleurbaey (1996)). Devooght (2008) and Almas et al. (2011) propose to use the income resulting from these redistribution mechanism to define a norm income distribution and to measure inequality of opportunity by aggregating the deviations of individual's actual incomes from their norm incomes. The computation of the norm incomes relies on counterfactuals that are estimated using similar methods as the ones described here. Hence similar issues to the ones analyzed here also arise for that approach. Second, we only considered the most commonly used econometric models found in the literature. Researchers are using more and more advanced techniques to construct counterfactuals. Which, if any, of the basic properties the resulting measures of inequality of opportunity have is an important topic for future work.

A Proofs

A.1 Notation

Define, for all individuals $l \in N$ the following vectors:

$$
\begin{aligned}
x_l^C &= \left[c_{l,1} - \mu_{C1} \dots c_{l,d^C} - \mu_{Cd^C}\right]', \\
x_l^E &= \left[e_{l,1} - \mu_{E1} \dots e_{l,d^E} - \mu_{Ed^E}\right]', \\
x_l^{C\overline{C}} &= \left[c_{l,1} - \overline{c}_1 \dots c_{l,d^C} - \overline{c}_{d^C}\right]', \\
x_l^{E\overline{E}} &= \left[e_{l,1} - \overline{e}_1 \dots e_{l,d^E} - \overline{e}_{d^E}\right]'.
\end{aligned}
$$

Hence, x_l^C ($x_l^{C\overline{C}}$) is the d^C dimensional vector of the deviation of circumstances of individual l from their mean (reference) values, and x_l^E ($x_l^{E\overline{E}}$) is the d^E dimensional vector of the deviation of his efforts from their mean (reference) values. Next, define the $d^C + d^E$ dimensional vectors

$$
\begin{aligned}
x_l^{CE} &= \left[c_{l,1} - \mu_{C1} \dots c_{l,d^C} - \mu_{Cd^C} \quad e_{l,1} - \mu_{E1} \dots e_{l,d^E} - \mu_{Ed^E}\right]', \\
x_l^{C\overline{E}} &= \left[c_{l,1} - \mu_{C1} \dots c_{l,d^C} - \mu_{Cd^C} \quad \overline{e}_1 - \mu_{E1} \dots \overline{e}_{d^E} - \mu_{Ed^E}\right]', \\
x_l^{\overline{C}E} &= \left[\overline{c}_1 - \mu_{C1} \dots \overline{c}_{d^C} - \mu_{Cd^C} \quad e_{l,1} - \mu_{E1} \dots e_{l,d^E} - \mu_{Ed^E}\right]', \\
x_l^{0E} &= \left[0 \dots 0 \quad e_{l,1} - \overline{e}_1 \dots e_{l,d^E} - \overline{e}_{d^E}\right]', \\
x_l^{C0} &= \left[c_{l,1} - \overline{c}_1 \dots c_{l,d^C} - \overline{c}_{d^C} \quad 0 \dots 0\right]',
\end{aligned}
$$

and the n dimensional vectors

$$
\begin{aligned}
E_i &= [0 \dots 0 \quad 1 \quad 0 \dots 0]', \\
\widetilde{E}_i &= [0 \dots 0 \quad 1/y_i \quad 0 \dots 0]', \\
\iota &= [1 \dots 1]',
\end{aligned}
$$

such that E_i has zeros everywhere, except for its $i-$ element which equals 1, $\widetilde{E}_i$ has zeros everywhere, except for its $i-$ element which equals $1/y_i$ and all elements in ι are equal to 1.

A.2 Proof of Proposition 1

Lemma A1. Under the assumption of linearity and using the least squares estimator, the counterfactuals (6)-(14) become

$$y_l^{c1} = \mu_Y + (x_l^C)'b^C, \tag{A.1}$$

$$y_l^{c2} = \mu_Y + y_l^D - (x_l^{E\overline{E}})'b^E, \tag{A.2}$$

$$y_l^{c3} = \mu_Y + (x_l^{C\overline{E}})'b^{CE}, \tag{A.3}$$

$$y_l^{c4} = \mu_Y + y_l^D - (x_l^{0E})'b^{CE}, \tag{A.4}$$

$$y_l^{EO1} = \mu_Y + (x_l^E)'b^E, \tag{A.5}$$

$$y_l^{EO2} = \mu_Y + y_l^D - (x_l^{C\overline{C}})'b^C, \tag{A.6}$$

$$y_l^{EO3} = \mu_Y + (x_l^{\overline{C}E})'b^{CE}, \tag{A.7}$$

$$y_l^{EO4} = \mu_Y + y_l^D - (x_l^{C0})'b^{CE}. \tag{A.8}$$

Proof of Lemma A1. Equations (A.1),(A.3), (A.5) and (A.7) are straightforward. The others are only slightly more complicated. We prove (A.2); the proof of the others is analogous. From (7), with the linear specification, and, for all individuals $l \in N$, $x_l^{\overline{E}\mu_E} = [\overline{e}_1 - \mu_{E1} \ldots \overline{e}_{d^E} - \mu_{Ed^E}]'$, we have

$$y_l^{c2} = \mu_Y + (x^{\overline{E}\mu_E})'b^E + \widehat{u}_l^R$$

$$
\begin{aligned}
&= \mu_Y + (x^{\overline{E}\mu_E})' b^E + y_l^D - (x_l^E)' b^E \\
&= \mu_Y + y_l^D - \left((x_l^E)' - (x^{\overline{E}\mu_E})' \right) b^E \\
&= \mu_Y + y_l^D - (x_l^{E\overline{E}})' b^E,
\end{aligned}
$$

which is expression (A.2).□

After having performed a Pigou Dalton transfer δ from observation i to j, we obtain a new estimate of the coefficients in the regression equation, denoted by $\widetilde{b}^A$. The following Lemma relates the new estimated coefficient vector to b^A.

Lemma A2. The coefficient estimate after a Pigou-Dalton transfer δ from observation i to j results in a new estimate

$$\widetilde{b}^A = b^A + \delta \left((X^A)' X^A \right)^{-1} (X^A)' (E_j - E_i),$$

Proof of Lemma A2. Define $\widetilde{Y} = Y + \delta(E_j - E_i)$, which is the vector Y after a Pigou-Dalton transfer of an amount δ from observation j to i. After the transfer we estimate the equation in deviational form

$$\widetilde{Y}^D = \widetilde{Y} - \iota \mu_Y = X^A \widetilde{\beta}^A + \widetilde{U}^A. \tag{A.9}$$

For the least squares estimate $\widetilde{b}^A = \left((X^A)' X^A \right)^{-1} (X^A)' \widetilde{Y}^D$ we obtain

$$\left((X^A)' X^A \right)^{-1} (X^A)' Y^D + \delta \left((X^A)' X^A \right)^{-1} (X^A)' (E_j - E_i),$$

from which the expression in the Lemma follows immediately. □

Using this Lemma, defining $\Sigma_A = \frac{1}{n} (X^A)' X^A$, the covariance matrix of the variables in X^A, and adding a tilde to denote the counterfactuals after the transfer, it is easy to obtain the following expressions for the effect of the Pigou-Dalton transfers on the counterfactuals.

Lemma A3. The change in the estimated counterfactuals of a Pigou-Dalton transfer δ from observation i to j is

$$\widetilde{y}_l^{c1} - y_l^{c1} = \frac{\delta}{n}(x_l^C)'\Sigma_C^{-1}(x_j^C - x_i^C), \tag{A.10}$$

$$\widetilde{y}_l^{c2} - y_l^{c2} = \widetilde{y}_l^D - y_l^D - \frac{\delta}{n}(x_l^{E\overline{E}})'\Sigma_E^{-1}(x_j^E - x_i^E), \tag{A.11}$$

$$\widetilde{y}_l^{c3} - y_l^{c3} = \frac{\delta}{n}(x_l^{C\overline{E}})'\Sigma_{CE}^{-1}(x_j^{CE} - x_i^{CE}), \tag{A.12}$$

$$\widetilde{y}_l^{c4} - y_l^{c4} = \widetilde{y}_l^D - y_l^D - \frac{\delta}{n}(x_l^{0E})'\Sigma_{CE}^{-1}(x_j^{CE} - x_i^{CE}), \tag{A.13}$$

$$\widetilde{y}_l^{EO1} - y_l^{EO1} = \frac{\delta}{n}(x_l^E)'\Sigma_E^{-1}(x_j^E - x_i^E), \tag{A.14}$$

$$\widetilde{y}_l^{EO2} - y_l^{EO2} = \widetilde{y}_l^D - y_l^D - \frac{\delta}{n}(x_l^{C\overline{C}})'\Sigma_C^{-1}(x_j^C - x_i^C), \tag{A.15}$$

$$\widetilde{y}_l^{EO3} - y_l^{EO3} = \frac{\delta}{n}(x_l^{\overline{C}E})'\Sigma_{CE}^{-1}(x_j^{CE} - x_i^{CE}), \tag{A.16}$$

$$\widetilde{y}_l^{EO4} - y_l^{EO4} = \widetilde{y}_l^D - y_l^D - \frac{\delta}{n}(x_l^{C0})'\Sigma_{CE}^{-1}(x_j^{CE} - x_i^{CE}). \tag{A.17}$$

Proof of Proposition 1. Due to the similarity of the proofs, we first prove parts (a) and (e), followed by parts (b) and (f), (c) and (g), and, finally, parts (d) and (h) of the Proposition.

Consider part (a) of the Proposition and Equation (A.10).
Observe that, as $\sum_{l=1}^n x_l^C = 0$, $\sum_{l=1}^n \widetilde{y}_l^{c1} = \sum_{l=1}^n y_l^{c1}$, such that the mean of the counterfactual has not changed, and there is no need to normalize the counterfactual distribution to study the effects of a Pigou-Dalton transfer.

Take $x_j^C = x_i^C$. From (A.10), $\widetilde{y}_l^{c1} = y_l^{c1}$, such that the counterfactual has not changed. Hence, UR is satisfied. Take $x_j^E = x_i^E$. Under C1M, $d^C = 1$ and let $c_{i1} > c_{j1}$. From Lemma B1 (see Appendix B), we immediately have that the counterfactual for those observations for which $c_{l1} > (<)\mu_{C1}$ decrease (increase). Hence, $I(Y^{c1})$ decreases and the measure satisfies COM. When $d^C > 1$, and maintaining all other assumptions, it is not possible to generalize the statement in the Proposition about COM; see Lemma B2 (in Appendix B).

Consider part (e) of the Proposition, and Equation (A.14).
Observe that, as $\sum_{l=1}^{n} x_l^E = 0$, $\sum_{l=1}^{n} \widetilde{y}_l^{EO1} = \sum_{l=1}^{n} y_l^{EO1}$, such that the mean of the counterfactual has not changed, and there is no need to normalize the counterfactual distribution to study the effects of a Pigou-Dalton transfer.

Take $x_j^E = x_i^E$. From (A.14), $\widetilde{y}_l^{EO1} = y_l^{EO1}$, such that the counterfactual has not changed. However, the Pigou-Dalton transfer decreases the inequality in the income vector Y, such that $I(Y)$ decreases, and thus $I(Y) - I\left(Y^{EO1}\right)$ decreases. Hence COM is satisfied. Take $x_j^C = x_i^C$. In case $d^E = 1$ and $e_{i1} > e_{j1}$, from Lemma B1, we have that the counterfactual for those observations for which $e_{l1} > (<)\mu_{E1}$ decrease (increase). Hence the transfer decreases inequality in Y^{EO1}. However, it also decreases the inequality in Y in a different manner, and thus the effect on $I(Y) - I\left(Y^{EO1}\right)$ is ambiguous, such that UR is not even satisfied with one-dimensional effort.

Consider part (b) of the Proposition and Equation (A.11). We have

$$\frac{\sum_{l=1}^{n} \widetilde{y}_l^{c2}}{n} = \frac{\sum_{l=1}^{n} y_l^{c2}}{n} - \frac{\delta}{n}(\mu_E - \overline{e})'\Sigma_E^{-1}(x_j^E - x_i^E). \tag{A.18}$$

Take $x_j^E = x_i^E$. From (A.18), the mean of the counterfactual has not changed. We then see from (A.11) that $\widetilde{y}_l^{c2} - y_l^{c2} = \widetilde{y}_l^D - y_l^D$, such that the inequality in the counterfactual declines and the measure satisfies COM. Take $x_j^C = x_i^C$. If $\overline{e} = \mu_E$, from (A.18), the mean of the counterfactual has not changed, and no normalization of the counterfactual is necessary to analyze the consequences of the Pigou-Dalton transfer. With this reference value, with one dimensional efforts, and assuming that $e_{i1} > e_{j1}$, from Lemma B1, the change in the counterfactual is larger (smaller) than the change in the actual income distribution for those with $e_{l1} > (<)\mu_{E1}$, and inequality of opportunity changes. Hence the measure does not even satisfy UR under Eμ with one dimensional efforts.

Consider part (f) of the Proposition and Equation (A.15). We have

$$\frac{\sum_{l=1}^{n} \widetilde{y}_l^{EO2}}{n} = \frac{\sum_{l=1}^{n} y_l^{EO2}}{n} - \frac{\delta}{n}(\mu_C - \overline{c})'\Sigma_C^{-1}(x_j^C - x_i^C). \tag{A.19}$$

Take $x_j^C = x_i^C$. From (A.19), the mean of the counterfactual has not changed. We then see, from (A.15) that $\widetilde{y}_l^{EO2} - y_l^{EO2} = \widetilde{y}_l^D - y_l^D$, such that the inequality in the counterfactual declines by the same amount as the inequality in the actual income distribution and the measure satisfies UR. Take $x_j^E = x_i^E$. If $\overline{c} = \mu_C$, from (A.19), the mean of the counterfactual has not changed, and no normalization of the counterfactual is necessary to analyze the consequences of the Pigou-Dalton transfer. With this reference value, with one dimensional circumstances and assuming that $c_{i1} > c_{j1}$ (i.e. under C1M), from Lemma B1, and Equation (A.15), we have that the change in the counterfactual is larger (smaller) than the change in the actual distribution for those with $c_{l1} > (<)\mu_{C1}$, such the indirect measure of inequality of opportunity decreases and the measure satisfies COM. As shown in Lemma B2, it is not possible to generalize this result to a situation with more than one circumstance. If $\overline{c} \neq \mu_C$, it follows from (A.19) that the mean of the counterfactual has changed. Under C1M, $c_{i1} > c_{j1}$, such that, if $\overline{c}_1 < (>)\mu_{C1}$, the mean increases (decreases), and this can counter the effect on inequality of opportunity that arises from the fact that, assuming that $c_{i1} > c_{j1}$, from Lemma B1 and Equation (A.15), we have that the change in the counterfactual is larger (smaller) than the change in the actual distribution for those with $c_{l1} > (<)\overline{c}_1$. A similar issue occurs in the following cases if the mean of the counterfactual changes; for that reason, we focus on cases where the mean remains constant.

The counterfactuals in (c), (d),(g) and (h) of the Proposition rely on estimates of b^{CE}, such that Σ_{CE}, the estimated covariance matrix of circumstances and efforts plays a role. Define the following matrix

$$\Sigma_{CE}^{-1} = \begin{bmatrix} A_{CC} & A_{CE} \\ A_{EC} & A_{EE} \end{bmatrix}.$$

In case circumstances and efforts are not correlated, their covariance is zero, and Σ_{CE} is block diagonal. The inverse of a block-diagonal matrix is also block diagonal, such that, if efforts and circumstances are not correlated, $A_{CE} = (A_{EC})'$ contains only zeros.

Consider part (c) of the Proposition and Equation (A.12). Observe,

$$\frac{\sum_{l=1}^{n} \widetilde{y}_l^{c3}}{n} = \frac{\sum_{l=1}^{n} y_l^{c3}}{n} + \frac{\delta}{n}(\overline{e} - \mu_E)' \left[A_{EC}(x_j^C - x_i^C) + A_{EE}(x_j^E - x_i^E) \right]. \quad \text{(A.20)}$$

Take $\overline{e} = \mu_E$. The mean of the counterfactual has not changed. First, with $x_j^C = x_i^C$, from (A.12),

$$\widetilde{y}_l^{c3} = y_l^{c3} + \frac{\delta}{n}(x_l^C)' A_{CE} \left[x_j^E - x_i^E \right].$$

If, in addition, $A_{CE} = 0$, we get $\widetilde{y}_l^{c3} = y_l^{c3}$: the transfer has no effect on the counterfactual Y^{c3}. Hence, in this case, the measure satisfies UR. Second, with $x_j^E = x_i^E$, from (A.12),

$$\widetilde{y}_l^{c3} = y_l^{c3} + \frac{\delta}{n}(x_l^C)' A_{CC} \left[x_j^C - x_i^C \right], \quad \text{(A.21)}$$

If in addition $d^C = 1$ and $c_{i1} > c_{j1}$, by Lemma B1, the counterfactual for those observations for which $c_{l1} > (<)\mu_{C1}$ decrease (increase), such that $I(Y^{c3})$ decreases and the measure satisfies COM. When $d^C > 1$, and maintaining all other assumptions, it does not satisfy COM -see also Lemma B2.

Take $\bar{e} \neq \mu_E$, $A_{EC} = 0$ and $x_j^E = x_i^E$. From (A.20), the mean of the counterfactual has not changed, and, from (A.12), we obtain again (A.21), and the same conclusion follows: under C1M the measure satisfies COM.

Consider part (g) of the Proposition and Equation (A.16). Observe,

$$\frac{\sum_{l=1}^{n} \tilde{y}_l^{EO3}}{n} = \frac{\sum_{l=1}^{n} y_l^{EO3}}{n} + \frac{\delta}{n}(\bar{c} - \mu_C)' \left[A_{CC}(x_j^C - x_i^C) + A_{CE}(x_j^E - x_i^E) \right]. \quad (A.22)$$

Take $\bar{c} = \mu_C$. The mean of the counterfactual has not changed. First, with $x_j^E = x_i^E$, from (A.16),

$$\tilde{y}_l^{EO3} = y_l^{EO3} + \frac{\delta}{n}(x_l^E)' A_{EC} \left[x_j^C - x_i^C \right].$$

If, in addition, $A_{EC} = 0$, we get $\tilde{y}_l^{EO3} = y_l^{EO3}$: the transfer has no effect on the counterfactual. However, $I(Y)$ falls, hence $I(Y) - I(Y^{EO4})$ decreases, and the measure satisfies COM. Second, with $x_j^C = x_i^C$, from (A.16),

$$\tilde{y}_l^{EO3} = y_l^{EO3} + \frac{\delta}{n}(x_l^E)' A_{EE} \left[x_j^E - x_i^E \right]. \quad (A.23)$$

If, in addition $d^E = 1$ and $e_{i1} > e_{j1}$, from Lemma B1, we have that the predicted values for those observations for which $e_{l1} > (<)\mu_{E1}$ decrease (increase). Hence the transfer decreases inequality in Y^{EO3}. However, since the inequality in Y decreases in a different way, the effect on $I(Y) - I(Y^{EO3})$ cannot be determined and the measure does not even satisfy UR in the one-dimensional case.

Take $\bar{c} \neq \mu_C$, $A_{CE} = 0$ and $x_j^C = x_i^C$. From (A.22), the mean of the counterfactual has not changed, and, from (A.12), we obtain again (A.23), and the same conclusion follows: the measure does not satisfy UR even in the one-dimensional case.

Consider part (d) of the Proposition and Equation (A.13). Observe,

$$\frac{\sum_{l=1}^{n}\widetilde{y}_l^{c4}}{n} = \frac{\sum_{l=1}^{n}y_l^{c4}}{n} - \frac{\delta}{n}(\mu_E - \overline{e})'\left[A_{EC}(x_j^C - x_i^C) + A_{EE}(x_j^E - x_i^E)\right]. \quad \text{(A.24)}$$

Take $\overline{e} = \mu_E$. The mean of the counterfactual has not changed. First, with $x_j^C = x_i^C$, from (A.13),

$$\widetilde{y}_l^{c4} - y_l^{c4} = \widetilde{y}_l^D - y_l^D - \frac{\delta}{n}(x_l^E)'A_{EE}(x_j^E - x_i^E).$$

With this reference value, with one dimensional efforts, and assuming that $e_{i1} > e_{j1}$, from Lemma B1, the change in the counterfactual is larger (smaller) than the change in the actual income distribution for those with $e_{l1} > (<)\mu_{E1}$, and inequality of opportunity changes. Hence the measure does not satisfy UR. Second, with $x_j^E = x_i^E$, from (A.13),

$$\widetilde{y}_l^{c4} - y_l^{c4} = \widetilde{y}_l^D - y_l^D - \frac{\delta}{n}(x_l^E)'A_{EC}(x_j^C - x_i^C). \quad \text{(A.25)}$$

If, in addition, $A_{EC} = 0$, $\widetilde{y}_l^{c4} - y_l^{c4} = \widetilde{y}_l^D - y_l^D$, such that inequality in the counterfactual decreased and the measure satisfies COM.

Take $\overline{e} \neq \mu_E$, $A_{EC} = 0$ and $x_j^E = x_i^E$. From (A.24), the mean of the counterfactual has not changed, and, from (A.13), we obtain $\widetilde{y}_l^{c4} - y_l^{c4} = \widetilde{y}_l^D - y_l^D$, meaning that inequality decreased. Hence the measure satisfies COM.

Consider part (h) of the Proposition and Equation (A.17). Observe

$$\frac{\sum_{l=1}^{n}\widetilde{y}_l^{EO4}}{n} = \frac{\sum_{l=1}^{n}y_l^{EO4}}{n} - \frac{\delta}{n}(\mu_C - \overline{c})'\left[A_{CC}(x_j^C - x_i^C) + A_{CE}(x_j^E - x_i^E)\right]. \quad \text{(A.26)}$$

Take $\overline{c} = \mu_C$. The mean of the counterfactual has not changed. First, with

$x_j^E = x_i^E$, from (A.17),

$$\widetilde{y}_l^{EO4} - y_l^{EO4} = \widetilde{y}_l^D - y_l^D - \frac{\delta}{n}(x_l^C)' A_{CC}(x_j^C - x_i^C).$$

If in addition $d^C = 1$ and $c_{i1} > c_{j1}$, by Lemma B1, the change in the counterfactual for those observations for which $c_{l1} > (<)\mu_{C1}$ is larger (smaller) than the change in the actual income distribution. Hence inequality in the counterfactual increases more than the inequality in the actual income distribution and the measure satisfies COM. When $d^C > 1$, and maintaining all other assumptions, it does not satisfy COM -see also Lemma B2. Second, with $x_j^C = x_i^C$, from (A.17),

$$\widetilde{y}_l^{EO4} - y_l^{EO4} = \widetilde{y}_l^D - y_l^D - \frac{\delta}{n}(x_l^C)' A_{CE}(x_j^E - x_i^E).$$

If, in addition, $A_{CE} = 0$, from (A.17), $\widetilde{y}_l^{EO4} - y_l^{EO4} = \widetilde{y}_l^D - y_l^D$, and the measure of inequality of opportunity has not changed. Hence the measure satisfies UR.

Take $\bar{c} \neq \mu_C$, $A_{CE} = 0$ and $x_j^C = x_i^C$. From (A.26), the mean of the counterfactual has not changed, and, from (A.13), we obtain again $\widetilde{y}_l^{EO4} - y_l^{EO4} = \widetilde{y}_l^D - y_l^D$. Hence the measure satisfies UR.□

A.3 Proof of Proposition 2

Lemma A4. Under the assumption of loglinearity and using the least squares estimator, the counterfactuals (6)-(14) are defined by

$$log(y_l^{c1}) = \mu_{log(Y)} + (x_l^C)'\alpha^C, \tag{A.27}$$

$$log(y_l^{c2}) = log(y_l) - (x_l^{E\overline{E}})'\alpha^E, \tag{A.28}$$

$$log(y_l^{c3}) = \mu_{log(Y)} + (x_l^{C\overline{E}})'\alpha^{CE}, \tag{A.29}$$

$$log(y_l^{c4}) = log(y_l) - (x_l^{0E})'\alpha^{CE}, \tag{A.30}$$

$$log(y_l^{EO1}) = \mu_{log(Y)} + (x_l^E)'\alpha^E, \tag{A.31}$$

$$log(y_l^{EO2}) = log(y_l) - (x_l^{C\overline{C}})'\alpha^C, \tag{A.32}$$

$$log(y_l^{EO3}) = \mu_{log(Y)} + (x_l^{\overline{C}E})'\alpha^{CE}, \tag{A.33}$$

$$log(y_l^{EO4}) = log(y_l) - (x_l^{C0})'\alpha^{CE}. \tag{A.34}$$

Consider the effect of a Pigou-Dalton transfer from observation i to j. The following Lemma relates the new estimated coefficient vector $\widetilde{a}^A$ to a^A.

Lemma A5. The coefficient estimate after a Pigou-Dalton transfer δ from observation i to j results in a new estimate

$$\widetilde{a}^A = a^A + \delta \left((X^A)'X^A\right)^{-1} (X^A)' \left(\widetilde{E}_j - \widetilde{E}_i\right).$$

Proof of Lemma A5. Taking a first order approximation of the incomes after the Pigou-Dalton transfer around the incomes before the transfer, we can define

$$log(\widetilde{Y}) = log(Y) + \delta(\widetilde{E}_j - \widetilde{E}_i),$$

which is the vector log of incomes after a Pigou-Dalton transfer of an amount δ from observation j to i. After the transfer we estimate the equation in deviational form

$$log(\widetilde{Y}) - \iota\mu_{log(\widetilde{Y})} = X^A\widetilde{\alpha}^A + \widetilde{V}^A. \tag{A.35}$$

Hence, the least squares estimate $\widetilde{\alpha}^A = \left((X^A)'X^A\right)^{-1} (X^A)' \left[log(\widetilde{Y}) - \iota\mu_{log(\widetilde{Y})}\right]$. Observe that, using the same first order approximation as before,

$$\mu_{log(\widetilde{Y})} = \mu_{log(Y)} + \frac{\delta}{n}\left[\frac{1}{y_j} - \frac{1}{y_i}\right],$$

such that we have

$$\begin{aligned}\widetilde{a}^A &= \left((X^A)'X^A\right)^{-1}(X^A)'\left[log(Y)+\delta(\widetilde{E}_j-\widetilde{E}_i)-\iota\left[\mu_{log(Y)}+\frac{\delta}{n}(\frac{1}{y_j}-\frac{1}{y_i})\right]\right]\\ &= a^A+\delta\left((X^A)'X^A\right)^{-1}(X^A)'\left[\widetilde{E}_j-\widetilde{E}_i+\frac{1}{n}\iota(\frac{1}{y_j}-\frac{1}{y_i})\right].\end{aligned}$$

Now, observe that $(X^A)'\iota = 0$, a vector of zeros, such that the last term drops out and the expression in the Lemma follows immediately. □

Using Lemma A.5, it is straightforward to prove Lemma A.6.

Lemma A6. The change in the estimated counterfactuals of a Pigou-Dalton transfer δ from observation i to j is

$$log(\widetilde{y}_l^{c1})-log(y_l^{c1})=\frac{\delta}{n}(\frac{1}{y_j}-\frac{1}{y_i})+\frac{\delta}{n}(x_l^C)'\Sigma_C^{-1}(\frac{x_j^C}{y_j}-\frac{x_i^C}{y_i}), \quad (A.36)$$

$$log(\widetilde{y}_l^{c2})-log(y_l^{c2})=log(\widetilde{y}_l)-log(y_l)-\frac{\delta}{n}(x_l^{E\overline{E}})'\Sigma_E^{-1}(\frac{x_j^E}{y_j}-\frac{x_i^E}{y_i}), \quad (A.37)$$

$$log(\widetilde{y}_l^{c3})-log(y_l^{c3})=\frac{\delta}{n}(\frac{1}{y_j}-\frac{1}{y_i})+\frac{\delta}{n}(x_l^{C\overline{E}})'\Sigma_{CE}^{-1}(\frac{x_j^{CE}}{y_j}-\frac{x_i^{CE}}{y_i}), \quad (A.38)$$

$$log(\widetilde{y}_l^{c4})-log(y_l^{c4})=log(\widetilde{y}_l)-log(y_l)-\frac{\delta}{n}(x_l^{0E})'\Sigma_{CE}^{-1}(\frac{x_j^{CE}}{y_j}-\frac{x_i^{CE}}{y_i}), \quad (A.39)$$

$$log(\widetilde{y}_l^{EO1})-log(y_l^{EO1})=\frac{\delta}{n}(\frac{1}{y_j}-\frac{1}{y_i})+\frac{\delta}{n}(x_l^E)'\Sigma_E^{-1}(\frac{x_j^E}{y_j}-\frac{x_i^E}{y_i}), \quad (A.40)$$

$$log(\widetilde{y}_l^{EO2})-log(y_l^{EO2})=log(\widetilde{y}_l)-log(y_l)-\frac{\delta}{n}(x_l^{C\overline{C}})'\Sigma_C^{-1}(\frac{x_j^C}{y_j}-\frac{x_i^C}{y_i}), \quad (A.41)$$

$$log(\widetilde{y}_l^{EO3})-log(y_l^{EO3})=\frac{\delta}{n}(\frac{1}{y_j}-\frac{1}{y_i})+\frac{\delta}{n}(x_l^{\overline{C}E})'\Sigma_{CE}^{-1}(\frac{x_j^{CE}}{y_j}-\frac{x_i^{CE}}{y_i}), \quad (A.42)$$

$$log(\widetilde{y}_l^{EO4})-log(y_l^{EO4})=log(\widetilde{y}_l)-log(y_l)-\frac{\delta}{n}(x_l^{C0})'\Sigma_{CE}^{-1}(\frac{x_j^{CE}}{y_j}-\frac{x_i^{CE}}{y_i}). \quad (A.43)$$

Proof of Proposition 2. The left hand side in the equations of Lemma A6 give the percentage change in the estimated counterfactual.

First, observe that, in (A.36), (A.38), (A.40) and (A.42), the first term is the same for all observations, and so has no effect on the inequality of the counterfactual, provided a relative measure of inequality is used. The problem to sign the effect of the transfer on the inequality measure is that the second term will be different for different observations, and will never disappear (not even when the transfer is between individuals having the same circumstances and/ or efforts - except, when, in addition they have the same income level, but in that case there is no Pigou-Dalton transfer). Hence, these measures satisfy neither COM nor UR.

Second, observe that in (A.37), (A.39) (A.41) and (A.43), the first term, $log(\widetilde{y}_l) - log(y_l)$, is zero for all observations, except for i and j. Again, however, the problem is that the other term never vanishes, making it impossible to assess the effect of the transfer on the inequality in the counterfactual income distributions. Hence we obtain Proposition 2. $\square$

A.4 Proof of Proposition 4

The proof is based on a simple special case. Consider $N = \{1, 2, 3, 4\}$, with $c_1 = c_2 = \underline{c}$, $c_3 = c_4 = c^*$, $e_1 = e_3 = \underline{e}$ and $e_2 = e_4 = e^*$. Consider the initial and alternative distribution of binary outcomes, given in Panel (a) of Table 2. All the probabilities necessary for the construction of the counterfactual can be readily obtained and are the same irrespective of which nonlinear probability model one specifies. They are listed in Panel (b), and the counterfactual probability distributions for counterfactuals (6)-(14) are given in Table 5. The derivation of these counterfactual probability distributions is straightforward for the counterfactuals (6), (8), (11) and (13) as they do not depend on the estimated random variation. Moreover, in the example, the estimated random variation is zero for counterfactuals that rely on the empirical specification that includes both circumstances and efforts, such that also the counterfactuals (9) and (14) become obvious. Counterfacuals (7) and (12) are constructed in Tables 3 and 4, respectively.

Table 2: Binary Compensation and Utilitarian Reward

Panel (a) Distribution of binary outcomes				
	Initial situation		Alternative	
	$\underline{e}$	e^*	$\underline{e}$	e^*
$\underline{c}$	1	0	1	0
c^*	0	1	1	0

Panel (b) Corresponding probabilities		
	Initial situation	Alternative
$P(y=1\|\underline{c},\underline{e})$	1	1
$P(y=1\|\underline{c},e^*)$	0	0
$P(y=1\|c^*,\underline{e})$	0	1
$P(y=1\|c^*,e^*)$	1	0
$P(y=1\|\underline{c})$	1/2	1/2
$P(y=1\|c^*)$	1/2	1/2
$P(y=1\|\underline{e})$	1/2	1
$P(y=1\|e^*)$	1/2	0

Table 3: Counterfactual (7)

i	y_i^b	$\widehat{g}^E(e_i,0)$	$\widehat{u}_i^E$	$\widehat{g}^E(\underline{e},\widehat{u}_i^E)$	$\widehat{g}^E(e^*,\widehat{u}_i^E)$
Panel (a) Initial situation					
1	1	1/2	1/2	1	1
2	0	1/2	-1/2	0	0
3	0	1/2	-1/2	0	0
4	1	1/2	1/2	1	1
Panel (b) Alternative situation					
1	1	1	0	1	0
2	0	0	0	1	0
3	1	1	0	1	0
4	0	0	0	1	0

Table 4: Counterfactual (12)

i	y_i^b	$\widehat{g}^C(c_i, 0)$	$\widehat{u}_i^C$	$\widehat{g}^C(\underline{c}, \widehat{u}_i^C)$	$\widehat{g}^C(c^*, \widehat{u}_i^C)$
Panel (a) Initial situation					
1	1	1/2	1/2	1	1
2	0	1/2	-1/2	0	0
3	0	1/2	-1/2	0	0
4	1	1/2	1/2	1	1
Panel (b) Alternative situation					
1	1	1/2	1/2	1	1
2	0	1/2	-1/2	0	0
3	1	1/2	1/2	1	1
4	0	1/2	-1/2	0	0

Define the binary relations "$\succ$", "$\sim$" and "$\prec$" to mean "is less unequal than", "is as unequal as" and "is more unequal than" based on an inequality measure (which by definition, see Section 3 satisfies the Pigou-Dalton principle of transfers and anonymity), respectively. The third column in the Table 5 orders the counterfactual probability distributions using these binary relations.

Table 5: Ordering counterfactual distributions

	Initial situation		Alternative
(6)	(1/2,1/2,1/2,1/2)	$\sim$	(1/2,1/2,1/2,1/2)
(7, $\underline{e}$)	(1,0,0,1)	$\prec$	(1,1,1,1)
(7, e^*)	(1,0,0,1)	$\prec$	(0,0,0,0)
(8,9, $\underline{e}$)	(1,1,0,0)	$\prec$	(1,1,1,1)
(8,9, e^*)	(0,0,1,1)	$\prec$	(0,0,0,0)
(11)	(1/2,1/2,1/2,1/2)	$\succ$	(1,0,1,0)
(12, $\underline{c}$)	(1,0,0,1)	$\sim$	(1,0,1,0)
(12, c^*)	(1,0,0,1)	$\sim$	(1,0,1,0)
(13,14, $\underline{c}$)	(1,0,1,0)	$\sim$	(1,0,1,0)
(13,14, c^*)	(0,1,0,1)	$\sim$	(1,0,1,0)

Both direct and indirect measures can only satisfy BUR if the initial situation and the alternative have the same level of inequality. Direct measures can only satisfy BCOM if the initial situation is more unequal than the alternative. Indirect measures can only satisfy BCOM if the initial situation is less unequal than the alternative.□

A.5 Proof of Proposition 5

From the definition of (10), it is clear that switching binary outcomes among individuals having the same circumstances does not affect the counterfactual, such that the indirect measue of inequality does not change and BUR is satisfied.

From the definition of (15), it follows that the switch of binary outcomes between individual i and j in BCOM changes the counterfactual for all those that are either in $N_{\cdot i}$ or $N_{\cdot j}$. Without loss of generality, suppose that after the switch all those in $N_{\cdot i}$ have 0, and all those in $N_{\cdot j}$ have 1. Then, for all $l \in N_{\cdot i}$, we have that $\widetilde{y}_l^{E05} = 0$ and $y_l^{E05} = \frac{1}{|N_{\cdot i}|}$, and for all $m \in N_{\cdot j}$, we have that $\widetilde{y}_m^{E05} = 1$ and $y_m^{E05} = 1 - \frac{1}{|N_{\cdot j}|}$. Observe that Y^{E05} can always be obtained from $\widetilde{Y}^{E05}$ after a finite sequence of transfers from those with better outcomes (those in $N_{\cdot j}$) to those with worse outcomes (those in $N_{\cdot i}$), such that $I(\widetilde{Y}^{EO5}) > I(Y^{EO5})$ and, as $\widetilde{Y}$ is a permutation of Y, such that $I(\widetilde{Y}) = I(Y)$, we have that $I(Y) - I(\widetilde{Y}^{EO5}) < I(Y) - I(Y^{EO5})$. Hence the BCOM switch decreases inequality of opportunity. □

B Interpretation of Σ_A^{-1}

Let Σ_A, with typical element s_{ij}, be the empirical covariance matrix between K variables. The inverse of the covariance matrix, Σ_A^{-1}, known as the concentration or precision matrix, can be interpreted as follows (see Kwan (2014)).

Consider the following regressions

$$
\begin{aligned}
X_1^A &= X_2^A \eta_{12} + X_3^A \eta_{13} + \ldots + X_K^A \eta_{1K} + \epsilon_1 \\
X_2^A &= X_1^A \eta_{21} + X_3^A \eta_{23} + \ldots + X_K^A \eta_{2K} + \epsilon_2 \\
&\vdots \\
X_K^A &= X_1^A \eta_{K1} + X_2^A \eta_{K2} + \ldots + X_{K-1}^A \eta_{K(K-1)} + \epsilon_K,
\end{aligned}
$$

where the ϵ_k are random noise. The OLS estimation of these equations yields the estimated values for the coefficients, $\widehat{\eta}_{kl}$; $R_1^2, \ldots, R_K^2$ are the coefficients of determination of the equations. It can then be shown that $Z = \Sigma_A^{-1}$ equals

$$
\begin{bmatrix}
1/\left[s_{11}(1-R_1^2)\right] & -\widehat{\eta}_{12}/\left[s_{11}(1-R_1^2)\right] & \ldots & -\widehat{\eta}_{1K}/\left[s_{11}(1-R_1^2)\right] \\
-\widehat{\eta}_{21}/\left[s_{22}(1-R_2^2)\right] & 1/\left[s_{22}(1-R_2^2)\right] & \ldots & -\widehat{\eta}_{2K}/\left[s_{22}(1-R_2^2)\right] \\
\vdots & \vdots & \ddots & \vdots \\
-\widehat{\eta}_{K1}/\left[s_{KK}(1-R_K^2)\right] & -\widehat{\eta}_{K2}/\left[s_{KK}(1-R_K^2)\right] & \ldots & 1/\left[s_{KK}(1-R_K^2)\right]
\end{bmatrix},
$$

or, alternatively, the elements in Z can be found as

$$
\begin{aligned}
z_{ii} &= 1/\left[s_{ii}(1-R_i^2)\right] \\
z_{ij} &= -\sqrt{z_{ii}}\sqrt{z_{jj}}\rho_{ij,[V-\{X_i,X_j\}]} \text{ for } j \neq i,
\end{aligned}
$$

where $\rho_{ij,[V-\{X_i,X_j\}]}$ is the partial correlation coefficient between X_i and X_j, given all K conditioning variables (except X_i and X_j).

In the formulas of the propositions terms like the following play a crucial role:

$$(x_l^A)'\Sigma_A^{-1}(x_j^A - x_i^A) \text{ with } A \in \{C, E\}\,.$$

One can wonder whether it is possible to determine the sign of this expression in general.

Lemma B1. In case $K = 1$, if $x_{i1}^A \geq x_{j1}^A$, then
(a) for all l that are such that $x_{l1}^A \geq 0$, we have that $(x_l^A)'\Sigma_A^{-1}(x_j^A - x_i^A) \leq 0$;
(b) for all l that are such that $x_{l1}^A \leq 0$, we have that $(x_l^A)'\Sigma_A^{-1}(x_j^A - x_i^A) \geq 0$.

Proof of Lemma B1. Follows immediately from the fact that, with $K = 1$,

$$(x_l^A)'\Sigma_A^{-1}(x_j^A - x_i^A) = \frac{1}{(\sigma_{X1})^2}(x_{l1}^A)(x_{j1}^A - x_{i1}^A),$$

where $(\sigma_{X1})^2$ is the variance of X_1. □

Lemma B2. In case $K \geq 1$, and (i) all elements of Σ_A^{-1} are non-negative, (ii) all elements in the vector x_i^A are at least as large as the corresponding element in the vector x_j^A, then,
(a) for all l that are such that all elements in x_l^A are positive, we have that $(x_l^A)'\Sigma_A^{-1}(x_j^A - x_i^A) \leq 0$;
(b) for all l that are such that all elements in x_l^A are negative, we have that $(x_l^A)'\Sigma_A^{-1}(x_j^A - x_i^A) \geq 0$.

Proof of Lemma B2. Remember that $Z = \Sigma_A^{-1}$, such that

$$(x_l^A)'\Sigma_A^{-1}(x_j^A - x_i^A) = \Sigma_{p=1}^K \Sigma_{t=1}^K x_{lp}^A z_{pt}(x_{jt}^A - x_{it}^A).$$

Under the condition (i) all $z_{pt} \geq 0$, and under condition (ii), $x_{jt}^A - x_{it}^A \leq 0$, such that under the antecedent of (a) the expression becomes non-positive; under the antecedent of (b) the expression becomes non-negative. □

It is worth noting that the condition (i) of Lemma B2 requires that the partial correlations between the K variables have to be non-positive. This is a very strong assumption to make. The most plausible assumption is probably that for $A = C$ ($A = E$) circumstances (efforts) are positively correlated, which would make the off-diagonal elements in Z negative, and the sign of the expressions indeterminate.

C Alternative proportional transfer principles

Let y_i be individual i's income before the transfer, y_j individual j's income before the transfer, $\widetilde{y}_i$ individual i's income after the transfer, and $\widetilde{y}_j$ individual j's income after the transfer. Throughout we require

$$y_i > \widetilde{y}_i \geq \widetilde{y}_j > y_j,$$

such that the transfer goes from individual i to individual j, and also after the transfer i has at least as much income as j. Different proportional transfer principles impose different conditions on the transfers.

The **Factor Proportional Transfer Principle** (this paper) requires that, with $A > 1$,

$$\widetilde{y}_i = \frac{y_i}{A} \text{ and } \widetilde{y}_j = y_j \cdot A. \tag{C.1}$$

The **Proportional Transfer Principle** (Fleurbaey and Michel (2001, p.4)) requires that, with $\delta > 0$,

$$\widetilde{y}_i = y_i(1-\delta) \text{ and } \widetilde{y}_j = y_j(1+\delta). \tag{C.2}$$

The **Proportional Ex-Post Transfer Principle** (Fleurbaey and Michel (2001, p.4)) requires that, with $\delta > 0$,

$$\widetilde{y}_i = \frac{y_i}{1+\delta} \text{ and } \widetilde{y}_j = \frac{y_j}{1-\delta}. \tag{C.3}$$

Assuming that the transfer described in the principle is desirable (because it decreases inequality), the following Proposition formulates the logical relationship between the three transfer principles.

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Proposition C. The Proportional Transfer Principle is stronger than the Factor Proportional Transfer Principle, which is stronger than the Proportional Ex-Post Transfer Principle.

Proof of Proposition C.

(a) Comparison of (C.2) and (C.1). Consider the case where the transfer implies the same transfer in favor of the poor individual, i.e. $A = (1 + \delta)$. In that case, the income after transfer for the rich person under (C.1) is higher than the income of the rich person under (C.2), as

$$1 - \delta^2 = (1 - \delta)(1 + \delta) < 1 \Longleftrightarrow \frac{y_i}{A} = \frac{y_i}{1 + \delta} > y_i(1 - \delta).$$

Hence, all transfers that are acceptable under (C.2) are also acceptable under (C.1), but the reverse does not hold true.

(b) Comparison of (C.1) and (C.3). Consider the case where the transfer implies the same transfer in favor of the poor individual, i.e. $A\prime = 1/(1 - \delta)$. In that case, the income after transfer for the rich person under (C.3) is higher than the income of the rich person under (C.1), as

$$1 - \delta^2 = (1 - \delta)(1 + \delta) < 1 \Longleftrightarrow \frac{y_i}{1 + \delta} > (1 - \delta)y_i = \frac{y_i}{A\prime}.$$

Hence, all transfers that are acceptable under (C.1) are also acceptable under (C.3), but the reverse does not hold true. $\square$

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