Working Paper Series

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ECINEQ WP 2016 - 424

ECINEQ 2016 - 424
December 2016

www.ecineq.org

A head-count measure of rank mobility and its directional decomposition*

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Abstract

We propose and characterize a new measure of rank mobility. The index is given by the head count of those whose positions change in the move from one period to the next divided by population size. The interpretation of this head-count ratio is straightforward and intuitive. In addition, we illustrate how the measure can be decomposed into an index of upwards mobility and an index of downwards mobility. The axioms used in our characterization results are appealing and easy to justify.

Keywords: Rank mobility, directional mobility, decomposability.

JEL Classification: D63.

*Financial support from the Fonds de Recherche sur la Société et la Culture of Québec, the Social Sciences and Humanities Research Council of Canada, the Netherlands Organisation for Scientific Research (NWO) under the grants: i) Open Competitie (OC: 400-09-354) and ii) Innovational Research Incentives Scheme (VENI 2013: 451-13-017), and the Fonds National de la Recherche Luxembourg is gratefully acknowledged.

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1 Introduction

The measurement of mobility is an issue that is, by now, well-established in the area of economic and social index numbers. The fundamental issue to be addressed is the design of measures that reflect the extent to which members of a society (or, at a more aggregate level, population subgroups or countries) move across social or economic boundaries from one period to the next. A crucial aspect that distinguishes mobility from most other criteria that are used to assess the performance of a society (such as income inequality or poverty) is that mobility is difficult—if not impossible—to define without any reference to intertemporal considerations. Of course, intertemporal approaches to the measurement of inequality, poverty and other social phenomena have been explored but they can also be defined without any difficulties in a single-period setting; in contrast, there is no mobility without movement. As a consequence, the arguments of a mobility measure are pairs of indicators of economic or social status—one indicator for each of the time periods under consideration.

According to Fields (2008), six aspects of mobility can be identified in the literature. These are time independence, positional movement, share movement, non-directional income movement, directional income movement, and equalizer of longer-term incomes. See, for instance, Maasoumi (1998), Fields and Ok (1999) and Jäntti and Jenkins (2014) for comprehensive surveys.

While the majority of earlier contributions deal with mobility in the context of income distributions, there has been an increasing interest in the notion of rank mobility—that is, the positional movement of individuals, households or countries in economic or social hierarchies. Rank-based measures are widely applied in empirical research (see, for example, Dickens, 1999) but, as far as we are aware, only few contributions such as D'Agostino and Dardanoni (2009), Cowell and Flachaire (2011) and Bossert, Can and D'Ambrosio (2016) investigate them from a theoretical perspective. The notion of rank mobility certainly plays a role in Cowell and Flachaire (2011) but its axiomatic analysis is not the focus of that article. Cowell and Flachaire (2011) propose a flexible approach that is based on a general measure of distance between individual statuses. Absolute status levels may or may not be directly observable—much of their notion is based on the status of individuals relative to the position of others. Focusing on measures of rank mobility is particularly relevant in the framework of indicators of progress of countries beyond GDP. Notable examples of the latter are the Human Development Index of the United Nations and the more recent Better Life Index of the OECD. What attracts the attention of policy makers is not the value of these composite indicators but the positions of the countries and the changes in the rankings over time.

D'Agostino and Dardanoni (2009) and Bossert, Can and D'Ambrosio (2016) propose rank-mobility measures that are based on two dominant measures of non-parametric rank correlation, namely, Spearman's (1904) ρ index and Kendall's (1938) τ index. D'Agostino and Dardanoni (2009) characterize rank-mobility preorders that are linked to Spearman's ρ index; Bossert, Can and D'Ambrosio (2016), on the other hand, focus on mobility measures that have their foundation in Kendall's τ index. The latter is at the core of the Kemeny distance, which is one of the most prominent distance measures for orderings; see Kemeny

(1959) and Kemeny and Snell (1962). The Kemeny distance is characterized in Kemeny and Snell (1962). As pointed out by Can and Storcken (2013), this axiomatization involves a redundant axiom, an observation that allows Can and Storcken (2013) to improve Kemeny and Snell's (1962) result in a very substantial manner.

In this paper, we depart from the approach based on measures of rank correlation followed by D'Agostino and Dardanoni (2009) and Bossert, Can and D'Ambrosio (2016). We develop an axiomatic framework from first principles rather than confine ourselves to tools that have their origins in a different (but, of course, closely related) subfield of statistical analysis.

Our central result is a characterization of what we refer to as the head-count ratio. It is a simple and intuitively appealing measure that conveys the basic principles of rank mobility in a transparent manner. The index calculates the rank-mobility value by counting the number of individuals who change position in the move from period zero to period one and divides the resulting number by the size of the population under consideration. The axioms employed in our characterization are plausible and not difficult to justify. We then proceed to a decomposition of the head-count ratio into an upwards head-count ratio and a downwards head-count ratio. As their labels suggest, these measures are obtained by dividing the number of those who move up (resp. down) by the population size. The benefit of such a decomposition into two opposing measures is the ability to compare the number of individuals who move down with the number of those who move up (at the expense of the former). The characterizations of these directional mobility measures are achieved by formulating suitable adaptations of our overall mobility axioms to the respective directional case. Again, the resulting properties are easily justifiable and intuitively appealing.

The next section introduces the basic notion of rank mobility. Our new rank-mobility measure—the head-count ratio—is presented, discussed and characterized in Section 3. A natural decomposition into upwards rank mobility and downwards rank mobility, along with the requisite characterization results, follows in Section 4. Section 4 also discusses some conceptual difficulties when considering weighted means of upwards and downwards rank mobility. Section 5 concludes with a discussion of our results.

2 Rank mobility

A society is represented by a finite set N of individuals. For our first result, we require that N has at least six members, that is, $n = |N| \geq 6$; for the remaining observations, it is sufficient to assume that $n \geq 2$. We will discuss these cardinality requirements regarding the set N in more detail once we reach the relevant parts of the paper. A strict ordering on N is a complete, transitive and antisymmetric binary relation $R \subseteq N \times N$. The set of all strict orderings on N is denoted by $\mathcal{R}^N$. For convenience, we sometimes express strict orderings in line notation, that is, we list the elements of N in decreasing order of rank. For instance, if $N = \{a, b, c\}$, $R = abc$ is the strict ordering that ranks a above b and b above c. For $a \in N$ and $R \in \mathcal{R}^N$, the rank of individual a in R is denoted by $r(a, R)$, that is,

$$r(a, R) = |\{b \in N : b\, R\, a\}|.$$

Thus, for $R = abc$, we have $r(a, R) = |\{a\}| = 1$, $r(b, R) = |\{a, b\}| = 2$ and $r(c, R) = |\{a, b, c\}| = 3$.

A rank-mobility measure uses as the only relevant information a pair of rankings of the individuals in N—one ranking for the previous period, one for the current period. Thus, we write such a measure as a function

$$M : \mathcal{R}^N \times \mathcal{R}^N \to [0, 1]$$

where $M(R^0, R^1)$ is the mobility associated with a move from the ranking R^0 in period zero (the previous period) to the ranking R^1 in period one (the current period). That the range of this measure is given by the interval $[0, 1]$ does not involve any loss of generality.

For a pair $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$, we denote the set of agents whose ranking improved when moving from period zero to period one by

$$R^0 \bigtriangleup R^1 = \{a \in N : r(a, R^0) > r(a, R^1)\}.$$

Analogously, the set of agents who dropped in the move from period zero to period one is denoted by

$$R^0 \bigtriangledown R^1 = \{a \in N : r(a, R^0) < r(a, R^1)\}.$$

The union of these two sets gives us the set of all agents whose ranking changed and we denote it by

$$R^0 \diamond R^1 = \{a \in N : r(a, R^0) \neq r(a, R^1)\}.$$

It follows immediately that, for any $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$,

$$R^0 \bigtriangleup R^1 = R^1 \bigtriangledown R^0 \quad \text{and} \quad R^0 \diamond R^1 = R^1 \diamond R^0.$$

That is, $R^0 \bigtriangleup R^1$ and $R^1 \bigtriangledown R^0$ are in a dual relationship, whereas the set $R^0 \diamond R^1$ has a symmetry property.

For $k \in \{2, \ldots, n\}$, we say that a pair $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$ is of *degree* k if $|R^0 \diamond R^1| = k$. Analogously, for $k \in \{1, \ldots, n-1\}$, the pair $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$ is of *upwards* (resp. *downwards*) *degree* k if $|R^0 \bigtriangleup R^1| = k$ (resp. $|R^0 \bigtriangledown R^1| = k$).

3 The head-count ratio

We propose a simple and intuitive measure of rank mobility which is obtained by counting the number of individuals whose positions changed from period zero to period one and dividing the resulting number by the size of the population under consideration. This head-count ratio M^H is defined formally by letting, for all $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$,

$$M^H(R^0, R^1) = \frac{|R^0 \diamond R^1|}{n}.$$

The head-count ratio M^H can be characterized by means of four axioms.

Our first property is anonymity, a standard condition that requires a rank-mobility measure to treat all individuals impartially, paying no attention to their identities. Thus, we require that permuting the labels that we assign to the members of society does not change the value of a rank-mobility measure. Let $\pi\colon N \to N$ be a bijective function. For $R \in \mathcal{R}^N$, we define the relation R_π by letting, for all $a, b \in N$,

$$(\pi(a), \pi(b)) \in R_\pi \;\Leftrightarrow\; (a, b) \in R.$$

Thus, π permutes the labels of the individuals by assigning the label $\pi(a) \in N$ to the individual that was previously labeled $a \in N$.

Anonymity. For all $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$ and for all permutations $\pi\colon N \to N$,

$$M(R^0_\pi, R^1_\pi) = M(R^0, R^1).$$

Next, we normalize the maximal possible value of M to the number one.

Normalization. $\max\{M(R^0, R^1) : (R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N\} = 1$.

Normalization ensures that the rank-mobility measure is proportional in the sense that it allows for meaningful comparisons of societies with different population sizes. For instance, consider a society with a given number of members and a given pair of rankings. Suppose this society is to be compared with a society that is composed of twice as many individuals and there are two moves in the second group for each move in the first—that is, the proportions of those whose positions change are the same in both cases. It is natural to expect that the resulting mobility values be the same. Normalization guarantees that this proportionality property is satisfied. As a consequence, it is clear that the head-count ratio M^H is proportional in the above-described sense. In general, if an n-person society represented by a pair (R^0, R^1) is compared to a society with mn members (where $m \geq 2$) in a way such that the number of individuals who change positions in the original pair is multiplied by m, it follows immediately that the value of M^H calculated for the second society is

$$\frac{m|R^0 \diamond R^1|}{mn} = \frac{|R^0 \diamond R^1|}{n}$$

and hence the proportionality property is satisfied. See Bossert, Can and D'Ambrosio (2016) for a related discussion that concerns replication-invariant rank-mobility measures in a variable-population setting.

To reflect the feature that our measure only depends on those individuals who experience a change in their respective rank, we employ an invariance axiom. We require the rank-mobility values associated with two pairs of strict orderings $(R^0, R^1), (\overline{R}^0, \overline{R}^1) \in \mathcal{R}^N \times \mathcal{R}^N$ to coincide whenever the sets of individuals whose position changes are the same.

Change invariance. For all $(R^0, R^1), (\overline{R}^0, \overline{R}^1) \in \mathcal{R}^N \times \mathcal{R}^N$, if $R^0 \diamond R^1 = \overline{R}^0 \diamond \overline{R}^1$, then

$$M(R^0, R^1) = M(\overline{R}^0, \overline{R}^1).$$

Finally, we introduce an additivity property with an intuitive interpretation. Consider a situation in which the rank mobility associated with a move from period zero to period one is determined and, analogously, the rank mobility corresponding to a move from period one to period two is calculated (we can think of the time periods under consideration as years, for instance). Now suppose that we want to measure the mobility associated with the move from period zero to period two (that is, consider a biannual setting). If the set of individuals whose rank changed in the move from period zero to period one and the set of those whose rank changed in the move from period one to period two are disjoint, it seems natural to combine these distinct (annual) moves in an additive manner (as a biannual move). That is, if it so happens that the rank changes from period zero to period one and from period one to period two are distinct in the sense that they involve different individuals, the rank mobility associated with a move from period zero to period two is given by the sum of the rank mobility for the two moves from zero to one and from one to two. Thus, we impose the following property.

Change additivity. For all $R^0, R^1, R^2 \in \mathcal{R}^N$, if $(R^0 \diamond R^1) \cap (R^1 \diamond R^2) = \emptyset$, then

$$M(R^0, R^2) = M(R^0, R^1) + M(R^1, R^2).$$

We can now state and prove our first characterization result. As will become clear once we go through the proof, the result is valid only for societies with at least six members.

Theorem 1. *Given any N with $n \geq 6$, a rank-mobility measure M satisfies anonymity, normalization, change invariance and change additivity if and only if $M = M^H$.*

Proof. It is straightforward to verify that M^H satisfies the axioms of the theorem statement. Now suppose that M is a rank-mobility measure that satisfies the axioms. Let $k \in \{2, \ldots, n\}$ and consider any pair $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$ that is of degree k, that is, $|R^0 \diamond R^1| = k$. Let $N' = R^0 \diamond R^1$ and define $\omega^k_{N'} = M(R^0, R^1)$. Now consider any $\overline{N} \subseteq N$ and $(\overline{R}^0, \overline{R}^1) \in \mathcal{R}^N \times \mathcal{R}^N$ such that

$$|\overline{R}^0 \diamond \overline{R}^1| = |\overline{N}| = |N'| = k.$$

If $\overline{N} = N'$, $M(\overline{R}^0, \overline{R}^1) = M(R^0, R^1) = \omega^k_{N'}$ follows from change invariance.

If $\overline{N} \neq N'$, there exists a permutation $\pi \colon N \to N$ such that

$$(R^0_\pi, R^1_\pi) = (\overline{R}^0, \overline{R}^1) \quad \text{and} \quad \overline{N} = \{\pi(a) : a \in N'\}.$$

By anonymity,

$$\omega^k_{\overline{N}} = M(\overline{R}^0, \overline{R}^1) = M(R^0_\pi, R^1_\pi) = M(R^0, R^1) = \omega^k_{N'}$$

and, because $\overline{N}$ can be any arbitrary subset of N with k members, it follows that $\omega^k_{\overline{N}} = \omega^k_{N'}$ for all pairs of degree k. Thus, $\omega^k_{N'}$ cannot depend on N' and we write it as ω^k. It therefore follows that $M(R^0, R^1) = \omega^k$ for all pairs (R^0, R^1) of degree k.

Next, we establish a relationship between the ω^k values. Let $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$ be of degree k. Without loss of generality, suppose that $R^0 \diamond R^1 = \{a_1, \ldots, a_k\}$.

Case 1: k is even. Let $(\overline{R}^0, \overline{R}^1) \in \mathcal{R}^N \times \mathcal{R}^N$ be such that $\overline{R}^0 \diamond \overline{R}^1 = R^0 \diamond R^1 = \{a_1, \ldots, a_k\}$ and

$$\begin{aligned} \overline{R}^0 &= (a_1 a_2)(a_3 a_4) \ldots (a_{k-1} a_k) \ldots, \\ \overline{R}^1 &= (a_2 a_1)(a_4 a_3) \ldots (a_k a_{k-1}) \ldots \end{aligned}$$

where we use the parentheses to emphasize the individuals who change positions when moving from $\overline{R}^0$ to $\overline{R}^1$. By change invariance, it follows that $M(R^0, R^1) = M(\overline{R}^0, \overline{R}^1) = \omega^k$ because $(\overline{R}^0, \overline{R}^1)$ is of degree k. The move from $\overline{R}^0$ to $\overline{R}^1$ can be decomposed into $k/2$ pairs of degree 2, where the sets of individuals whose positions change in each of these moves are given by $\{a_1, a_2\}, \ldots, \{a_{k-1}, a_k\}$. These sets are pairwise disjoint and change additivity can be applied repeatedly to conclude that

$$M(R^0, R^1) = M(\overline{R}^0, \overline{R}^1) = \omega^k = \frac{k}{2}\,\omega^2 \tag{1}$$

for all even k.

Case 2: k is odd. Let $(\overline{R}^0, \overline{R}^1) \in \mathcal{R}^N \times \mathcal{R}^N$ be such that $\overline{R}^0 \diamond \overline{R}^1 = R^0 \diamond R^1 = \{a_1, \ldots, a_k\}$ and

$$\begin{aligned} \overline{R}^0 &= (a_1 a_2)(a_3 a_4) \ldots (a_{k-4} a_{k-3})(a_{k-2} a_{k-1} a_k) \ldots, \\ \overline{R}^1 &= (a_2 a_1)(a_4 a_3) \ldots (a_{k-3} a_{k-4})(a_k a_{k-2} a_{k-1}) \ldots. \end{aligned}$$

By change invariance, it follows that $M(R^0, R^1) = M(\overline{R}^0, \overline{R}^1) = \omega^k$ because $(\overline{R}^0, \overline{R}^1)$ is of degree k. The move from $\overline{R}^0$ to $\overline{R}^1$ can be decomposed into $(k-3)/2$ pairs of degree 2 and one pair of degree 3. The sets of individuals whose positions change in each of these moves are $\{a_1, a_2\}, \ldots, \{a_{k-4}, a_{k-3}\}$ and $\{a_{k-2}, a_{k-1}, a_k\}$. Again, these sets are pairwise disjoint and invoking change additivity repeatedly, it follows that

$$M(R^0, R^1) = M(\overline{R}^0, \overline{R}^1) = \omega^k = \frac{k-3}{2}\,\omega^2 + \omega^3 \tag{2}$$

for all odd k.

Because $n \geq 6$, there exist $(R^0, R^1), (\overline{R}^0, \overline{R}^1) \in \mathcal{R}^N \times \mathcal{R}^N$ such that

$$R^0 \diamond R^1 = \overline{R}^0 \diamond \overline{R}^1 = \{a_1, a_2, a_3, a_4, a_5, a_6\}$$

so that both of these pairs are of degree $k = 6$, and

$$\begin{aligned} R^0 &= (a_1 a_2)(a_3 a_4)(a_5 a_6) \ldots, \\ R^1 &= (a_2 a_1)(a_4 a_3)(a_6 a_5) \ldots, \\ \overline{R}^0 = R^0 &= (a_1 a_2 a_3)(a_4 a_5 a_6) \ldots, \\ \overline{R}^1 &= (a_2 a_3 a_1)(a_5 a_6 a_4) \ldots; \end{aligned}$$

again, parentheses are used to emphasize the positional changes. By change invariance, it follows that $M(R^0, R^1) = M(\overline{R}^0, \overline{R}^1) = \omega^6$. Using change additivity, we obtain

$$M(R^0, R^1) = \omega^6 = 3\,\omega^2$$

and

$$M(\overline{R}^0, \overline{R}^1) = \omega^6 = 2\,\omega^3.$$

Thus, $3\,\omega^2 = 2\,\omega^3$ and, solving for w^3 in Equation 2, it follows that

$$\omega^3 = \frac{3}{2}\,\omega^2.$$

Substituting into Equation 2, we obtain

$$\omega^k = \frac{k-3}{2}\,\omega^2 + \frac{3}{2}\,\omega^2 = \frac{k}{2}\,\omega^2$$

for all odd k. Thus, Equation 1 is true for all (odd and even) $k \in \{2, \ldots, n\}$.

By normalization, there exists a pair $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$ such that M assumes its maximal value of 1 at (R^0, R^1). Because ω^k is increasing in k, this maximal value is obtained for $k = n$. Using Equation 1, it follows that $\omega^n = (n/2)\,\omega^2 = 1$ and hence $\omega^2 = 2/n$. Thus, for any $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$ of degree k, it follows that

$$M(R^0, R^1) = \omega^k = \frac{k}{2}\,\frac{2}{n} = \frac{|R^0 \diamond R^1|}{n} = M^H(R^0, R^1). \blacksquare$$

The assumption that N consist of at least six members is essential in the above proof. The ω^k values are determined by considering multiples of degree-two situations and of degree-three situations, and the combination of the two yields the desired values. But this method can only be applied if there are at least six individuals to begin with: six is the smallest number that can be expressed as a multiple of two and as a multiple of three. It is straightforward to see that if N has fewer than six members, other measures become available and, thus, this minimal-cardinality assumption cannot be dispensed with in the above theorem.

4 Upwards and downwards mobility ratios

Suppose now that we want to focus on upwards rank mobility, that is, we pay particular attention to individuals who move up in the ranking. A natural modification of the head-count ratio M^H is obtained if the number of those who change position is replaced by the number of those who move up in the ranking, again divided by the total number of agents. Thus, we define the upwards head-count ratio M^U by letting, for all $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$,

$$M^U(R^0, R^1) = \frac{|R^0 \bigtriangleup R^1|}{n}.$$

Note that M^U does not satisfy the normalization axiom employed in our characterization of M^H. This is because imposing a maximal value of one on a directional measure would generate a conflict with the proportionality principle. Indeed, if we were to assume that upwards rank mobility achieves a maximal value of one, the characterization below would yield the ratio

$$\frac{|R^0 \triangle R^1|}{n-1}. \tag{3}$$

This is the case because the minimal positive number of upwards changes is one and the maximal number of upwards changes is $n-1$: if someone moves up in the ranking, at least one other person must move down. It is immediate that the measure of Equation 3 is not proportional. An m-fold multiple of an n-person society leads to the ratio

$$\frac{m|R^0 \triangle R^1|}{mn-1}.$$

Clearly, this ratio is not equal to that of Equation 3. In contrast, the upwards head-count ratio M^U obviously is proportional. For this reason, the normalization axiom used in Theorem 1 needs to be modified in order to arrive at M^U. To ensure proportionality, the maximal value achieved by a measure of upwards rank mobility must be equal to $(n-1)/n$ rather than one. This alternative normalization is very plausible because upwards rank mobility cannot but yield merely a partial picture of overall mobility. Thus, we obtain the following directional variant of the normalization property.

Directional normalization. $\max\{M(R^0, R^1) : (R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N\} = (n-1)/n$.

Analogously, we can define a downwards head-count ratio M^D by concentrating on downwards moves. That is, for all $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$,

$$M^D(R^0, R^1) = \frac{|R^0 \bigtriangledown R^1|}{n}.$$

Again, we divide by n so that the resulting measure is proportional.

It is now immediate that the head-count ratio M^H can be decomposed naturally into its upwards and downwards constituent parts by adding M^U and M^D. That is, we have, for all $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$,

$$M^H(R^0, R^1) = M^U(R^0, R^1) + M^D(R^0, R^1).$$

We now provide characterizations of M^U and of M^D. To do so, the two axioms of change invariance and change additivity can be rephrased in a natural way to capture upwards and downwards rank mobility. Instead of focusing on the set of agents whose positions change in the move from R^0 to R^1, we restrict attention to those whose positions improve (resp. worsen) so that we obtain the following properties.

Upwards invariance. For all $(R^0, R^1), (\overline{R}^0, \overline{R}^1) \in \mathcal{R}^N \times \mathcal{R}^N$, if $R^0 \triangle R^1 = \overline{R}^0 \triangle \overline{R}^1$, then

$$M(R^0, R^1) = M(\overline{R}^0, \overline{R}^1).$$

Downwards invariance. For all $(R^0, R^1), (\overline{R}^0, \overline{R}^1) \in \mathcal{R}^N \times \mathcal{R}^N$, if $R^0 \bigtriangledown R^1 = \overline{R}^0 \bigtriangledown \overline{R}^1$, then

$$M(R^0, R^1) = M(\overline{R}^0, \overline{R}^1).$$

Upwards additivity. For all $R^0, R^1, R^2 \in \mathcal{R}^N$, if $(R^0 \bigtriangleup R^1) \cap (R^1 \bigtriangleup R^2) = \emptyset$, then

$$M(R^0, R^2) = M(R^0, R^1) + M(R^1, R^2).$$

Downwards additivity. For all $R^0, R^1, R^2 \in \mathcal{R}^N$, if $(R^0 \bigtriangledown R^1) \cap (R^1 \bigtriangledown R^2) = \emptyset$, then

$$M(R^0, R^2) = M(R^0, R^1) + M(R^1, R^2).$$

The property of anonymity need not be changed; it is just as natural and plausible in the directional case. We can now state and prove two theorems that parallel Theorem 1. However, their proofs differ somewhat from that of our first theorem. In particular, it is now no longer necessary to require that there be at least six agents—any society with at least two members is covered by these results.

Theorem 2. *Given any N with $n \geq 2$, a rank-mobility measure M satisfies anonymity, directional normalization, upwards invariance and upwards additivity if and only if $M = M^U$.*

Proof. Again, it is straightforward to verify that M^U satisfies all of the axioms. Now suppose that M is a rank-mobility measure that satisfies the four axioms. Let $k \in \{1, \ldots, n-1\}$ and consider any pair $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$ that is of upwards degree k, that is, $|R^0 \bigtriangleup R^1| = k$. Replacing change invariance with upwards invariance and $R^0 \diamond R^1$ with $R^0 \bigtriangleup R^1$ in the corresponding part of the proof of Theorem 1, we conclude that there exists ω^k such that $M(R^0, R^1) = \omega^k$ for all pairs $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$ that are of upwards degree k.

Now consider the following three rankings.

$$\begin{aligned} R^0 &= a_{k+1}a_1a_2a_3 \ldots a_{k-2}a_{k-1}a_k \ldots, \\ R^1 &= a_1a_2a_3 \ldots a_{k-2}a_{k-1}a_{k+1}a_k \ldots, \\ R^2 &= a_1a_2a_3 \ldots a_{k-2}a_{k-1}a_ka_{k+1} \ldots. \end{aligned}$$

Note that $R^0 \bigtriangleup R^1 = \{a_1, \ldots, a_{k-1}\}$ and $R^1 \bigtriangleup R^2 = \{a_k\}$. Therefore, (R^0, R^1) is of degree $k-1$ and (R^1, R^2) is of degree 1. Moreover, the two sets are disjoint and we can employ upwards additivity to obtain

$$\omega^k = M(R^0, R^2) = M(R^0, R^1) + M(R^1, R^2) = \omega^{k-1} + \omega^1$$

and a simple iteration argument yields

$$\omega^k = k\,\omega^1$$

for all $k \in \{1, \ldots, n-1\}$. Because ω^k is increasing in k, it follows that its maximal value is achieved at pairs of degree $n-1$. Thus, by normalization, $\omega^{n-1} = (n-1)\,\omega^1 = (n-1)/n$ and hence

$$\omega^1 = \frac{1}{n}.$$

Substituting back, it follows that

$$M(R^0, R^1) = \frac{|R^0 \triangle R^1|}{n} = M^U(R^0, R^1). \blacksquare$$

Clearly, the above proof works in the same way if upwards invariance and upwards additivity are replaced with downwards invariance and downwards additivity. Thus, we immediately obtain the following characterization of our measure of downwards rank mobility M^D.

Theorem 3. *Given any N with $n \geq 2$, a rank-mobility measure M satisfies anonymity, directional normalization, downwards invariance and downwards additivity if and only if $M = M^D$.*

In some applications, one may want to attach different weights to upwards and downwards moves, thus expressing overall rank mobility as a weighted mean of M^U and M^D. Letting $\alpha \in [0, 1]$ denote the weight attached to upwards mobility, the corresponding measure can be expressed as

$$M^\alpha(R^0, R^1) = \alpha M^U(R^0, R^1) + (1-\alpha) M^D(R^0, R^1) \tag{4}$$

for all $(R^0, R^1) \in \mathcal{R}^N \times \mathcal{R}^N$. The interpretation of Equation 4 it straightforward: the higher the value of α, the higher the relative importance of upwards movements. In general, the maximal value of M^α is

$$M^\alpha_{max} = \begin{cases} (1-\alpha) + \alpha/(n-1) & \text{if } \alpha \in [0, 1/2]; \\ \alpha + (1-\alpha)/(n-1) & \text{if } \alpha \in (1/2, 1]. \end{cases}$$

This observation raises an immediate concern when it comes to the choice of a normalization property. Because these maximal values are α-specific, it is far from obvious how a suitable normalization axiom may be formulated. Although seemingly appealing at first sight, there appears to be a conceptual problem with calculating weighted means of the directional measures. Because the measures themselves have a strong ordinal flavor, it is not an easy task to endow these weighted means with a solid interpretation. For example, does a weight of 1/4 accurately capture the idea that downwards mobility is three times as important as upwards mobility? Based on these considerations, it seems plausible to us that calculating weighted means may take us well beyond the ordinal nature of the measures studied here.

5 Discussion

The measures proposed in this contribution are simple and intuitive indices of rank mobility based on the count of individuals who change their position from one period to the next. However, our approach can serve as the basis of more complex aggregation procedures. For example, one may want to capture the intensity of period-to-period movements by incorporating information on the number of positions involved in each individual change. Another interesting aspect of mobility not explored here is the likelihood of expected changes and weighted counts may be performed, where different weights are given to changes occurring in different positions of the ranking.

An interesting statistic derived from our measures is the ratio of upward and downward mobility ratios, that is, the ratio

$$\frac{M^U(R^0, R^1)}{M^D(R^0, R^1)} = \frac{|R^0 \bigtriangleup R^1|}{|R^0 \bigtriangledown R^1|}$$

(provided that the number of those who move is not equal to zero). This ratio is a useful indicator of improvement relative to deterioration in the move from one period to the following.

From an applied perspective, note that the proposed measures are defined on strict rankings. But ties occur frequently in data such as that relating to household incomes, for instance. The measure of income which is usually attributed to individuals is equivalent household income and, thus, all household members have the same income level and hence the same rank. Of course, this issue can be dealt with by considering households to be the relevant units rather than individuals. An additional potential problem arises in self-reported income data: individuals tend to report rounded values such as \$3,500 rather than \$3,473.84, say. The resulting clustering gives rise to further ties in the ranking. One way in which our measures can be applied in such cases is to consider all possible ways of breaking ties in these rankings, then calculate the index values for each of the resulting strict rankings and, finally, perform an averaging operation.

Ties in rankings are unlikely to occur in many settings where the mobility of groups or countries is considered. The average income of a group (such as teachers, physicians etc.) can be used as a proxy of the social status of these groups in a society. Analogously, countries are frequently ranked according to GDP per capita or to indices beyond GDP such as the Human Development Index or the Better Life Index—and ties are extremely rare in these contexts. What matters in public debates on these issues is the relative ranking of countries derived from these indices rather than the values of the indicators *per se.* Thus, this is another area in which our results can be applied successfully.

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