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Manfred Krtscha

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Manfred Krtscha†
Karlsruher Institute of Technology, Germany

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In this paper we consider properties for inequality measurement. A property that should be demanded for every inequality measure we call "axiom" otherwise we call it only "desideratum". The most important new axiom (A6) and the desideratum "Cowell's Feature (CF)" are motivated carefully. (A6) is more restrictive than Zheng's (2007a) "unit consistency axiom" for partial inequality orderings, but it is not as restrictive as the overwhelmingly favoured "scale invariance" property. We will show that the combination of these two properties characterizes a type of differentiable inequality measure that the author had already introduced and characterized in 1994, but then with a stronger requirement. Since then, this measure has been widely employed in applied work because it has been perceived to possess some attractive properties. However, the aim of this paper is not only a better characterization of a single type of inequality measure, but also a numerical comparison of different "good" inequality measures that qualify under (A6). Our focus lies on the so-called intermediate measures, being a compromise between the scale invariant "relative inequality measures" and the translation invariant "absolute inequality measures", where equal absolute changes in all incomes do not affect the inequality value. We present three methods to construct strictly intermediate or centrist inequality measures, which are explained with the help of three examples. Then we undertake a comparison of how these illustrative measures satisfy our axioms. Finally we give a complete summary table showing all the properties of these inequality measures. A last relevant example has been taken from true life.

Keywords: unit consistency, ratio consistency, scale invariance, centrist inequality measures.

†**Contact details:** Karlsruher Institute of Technology, e-mail: krtscha@gmx.de.

Some Axiomatics of Inequality Measurement, With Specific Reference to Intermediate Indices

Manfred Krtscha

krtscha@gmx.de

Abstract: In this paper we consider properties for inequality measurement. A property that should be demanded for every inequality measure we call "axiom" otherwise we call it only "desideratum". The most important new axiom (A6) and the desideratum "Cowell's Feature (CF)" are motivated carefully. (A6) is more restrictive than Zheng's (2007a) "unit consistency axiom" for partial inequality orderings, but it is not as restrictive as the overwhelmingly favoured "scale invariance" property. We will show that the combination of these two properties characterizes a type of differentiable inequality measure that the author had already introduced and characterized in 1994, but then with a stronger requirement. Since then, this measure has been widely employed in applied work because it has been perceived to possess some attractive properties. However, the aim of this paper is not only a better characterization of a single type of inequality measure, but also a numerical comparison of different "good" inequality measures that qualify under (A6). Our focus lies on the so-called intermediate measures, being a compromise between the scale invariant "relative inequality measures" and the translation invariant "absolute inequality measures", where equal absolute changes in all incomes do not affect the inequality value. We present three methods to construct strictly intermediate or centrist inequality measures, which are explained with the help of three examples. Then we undertake a comparison of how these illustrative measures satisfy our axioms. Finally we give a complete summary table showing all the properties of these inequality measures. A last relevant example has been taken from true life.

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I. Introduction

Different forms of inequality measures are practically used. For a big number of individuals one normally considers classes x_1, x_2 ,..., x_r, where each class x_i is occupied by n_i individuals or with the probability p_i=n_i /n; n is the number of all individuals. Our focus always lies on the inequality of n individuals, even if these are divided into r classes. In fact we consider a *(statistical) inequality measure* as a sequence of functions $I^n : R^n_+ \rightarrow R_+$ which have to satisfy demands we call axioms if every inequality measure should satisfy them at any rate.

In Section II seven axioms for all inequality measures are presented.

In Section III a property, desired for all (intermediate) inequality measures and called "Cowell's Feature (CF)", is motivated by the "Compromise Concept (KC)" suggested in Krtscha (1994). This Feature is satisfied by all relative and all absolute inequality measures, also by many intermediate inequality measures being used in the past.

In Section IV we propose three methods for generating a "permissible" (i. e. all our axioms are satisfied) intermediate inequality measure for n>2. The methods of application are shown by different examples. The second method is based on Rao's (1984) fundamental view of inequality where the inequality measure for n>2 is founded on the case n=2. Therefore we prove our most important proposition that for n=2 the only type of differentiable inequality measures satisfying (A6) and (CF) is the type that was already characterized in Krtscha (1994) where for n>2 (KC) does not work, but for this case we show the possibility for a new characterization by means of a "decomposabiliy property".

In Section V by a summary table we compare the advantages and disadvantages of our three centrist measures.

In Section VI once more we compare these measures also from the "leftists' " and the "rightists' " point of view. By a last example we show that using an absolute or intermediate inequality measure may cause ambiguous judgments of the overall inequality. That is only possible, because the axiom (A6) is satisfied.

II. General Axioms for Inequality Measures

Axiom (A₁): *Pigou-Dalton Principle.*

$I^n(x_1,\dots, x_i+h, \dots, x_j-h, \dots, x_n) < I^n(\mathbf{x})$ *whenever a richer person* x_j *gives an amount* h *to a poorer person* x_i *on the condition* $h \in (0, (x_j - x_i)/2]$. *It is called "transfer principle" if we weaken the above strict inequality to "≤" .*

Axiom (A₂): *Symmetry.*

$I^n(P\mathbf{x}) = I^n(\mathbf{x})$, *where P is a permutation matrix.*

Axiom (A₃): *Minimality (Normalization).*

$I^n(\mathbf{x}) = 0$ *if and only if* $\mathbf{x}= \lambda\cdot\mathbf{1}$ *for all* $\lambda>0$.

Axiom (A₄): *Strict Monotonicity*

as to both extreme x_i *where* x_1 *is the smallest and* x_n *the greatest* $x_i \in R_+$.

$I^n(x_1, x_2,\dots, x_{n-1}, x_n) < I^n(x_1- h, x_2,\dots, x_{n-1},x_n + k)$ *if* $h>0$ *or* $k>0$ *and* $h, k, x_1 -h \geq 0$.

(see Krtscha (1996, p.16))

Axiom (A₅): *Replication Invariance.*

$I^n(\mathbf{x}) = I^{n.m}(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})$, *where* $(\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})$ *is a m-fold replication of* $\mathbf{x}$.

The next axiom is Zheng's (2007a) "unit consistency axiom" (A_6) which Zheng demanded for Lorenz dominance orderings. He demanded that changing the measure unit must not change the ordering of two distributions **x** and **y**.

Axiom (A₆): *Unit Consistency Axiom.*

$I^n(\mathbf{x}) \leq I^n(\mathbf{y})$ *implies* $I^n(a\cdot\mathbf{x}) \leq I^n(a\cdot\mathbf{y})$ *and* $I^n(\mathbf{x}) \geq I^n(\mathbf{y})$ *implies* $I^n(a\cdot\mathbf{x}) \geq I^n(a\cdot\mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in R^n_+$ *and* $a>0$.

For $I^n(\mathbf{y})>0$ we can also demand: $R(\mathbf{x},\mathbf{y}):=I^n(\mathbf{x})/I^n(\mathbf{y})\leq 1$ *implies* $I^n(a\cdot\mathbf{x})/I^n(a\cdot\mathbf{y}) \leq 1$ *and* $R(\mathbf{x},\mathbf{y}):= I^n(\mathbf{x})/I^n(\mathbf{y}) \geq 1$ *implies* $I^n(a\cdot\mathbf{x})/I^n(a\cdot\mathbf{y}) \geq 1$ *for all* $a>0$.

(A consequence of this demand is that $R(\mathbf{x},\mathbf{y})=1$ implies $R(a\cdot\mathbf{x},a\cdot\mathbf{y})=1$ for all $a>0$.) Let us now demand the stronger axiom that this ratio $R(\mathbf{x},\mathbf{y})$ must not depend on a positive factor a, and we denote this demand (A6).

Axiom (A6): *Ratio Consistency Axiom.*

The ratio $R(\mathbf{x},\mathbf{y}):= I^n(\mathbf{x})/I^n(\mathbf{y})$ *does not depend on the measure unit, i. e. we have* $R(a\cdot\mathbf{x},a\cdot\mathbf{y})= R(\mathbf{x},\mathbf{y})$ *for all* $a >0$, *assuming* $I^n(\mathbf{y}) > 0$.

It is necessary to give a convincing motivation for the new demand (A6): The demand (A6) prevents a further statistical manipulation of the increasing inequality number of the incomes by an obscure inequality index $I^n_o(\mathbf{x})$.

Assuming a politician of party A states that by the wrong politics of party B the income inequality of the population shows an increase of exact 20%. That could be true if he had used the obscure inequality index $I^n{}_o(\mathbf{x})$ with $R_o(100\cdot\mathbf{x},100\cdot\mathbf{y}) = 100R_o(\mathbf{x},\mathbf{y})$. Being clever, he did not reveal that the income inequality ratio was measured by the use of cents instead of dollars knowing that the use of dollars had only indicated an increase of 0.2%. We see that (A_6) is satisfied, but not (A6).

Another example will show that (A6) is a stronger demand than (A_6): The standard Gini index is a relative inequality measure, thus it satisfies (A6). That axiom is also satisfied by the absolute inequality measure $Sp(\mathbf{x}):=\max\{x_i\}-\min\{x_i\}$ being linear homogeneous. However, the arithmetic or harmonic mean of both measures, suggested as simple and surely unknown compromises between a relative and an absolute inequality measure, do only satisfy our axioms (A_i), i =1,2,3,4,5,6 but not the axiom (A6), which on the other hand is satisfied by the geometric mean of these two measures. That will be shown by the following proposition.

Proposition 2.1. *Every continuous inequality measure $I^n(\mathbf{x})$ must have the property $I^n(a\mathbf{x})=a^\rho\cdot I^n(\mathbf{x})$ for all a>0 and fixed $\rho \geq 0$ because of (A6) and (A_4).*

Proof.

Because of (A6) we have

$I^n(\mathbf{x})/I^n(\boldsymbol{y}) = I^n(a\cdot\mathbf{x})/I^n(a\cdot\boldsymbol{y})$ or $I^n(a\cdot\mathbf{x})/I^n(\mathbf{x}) = I^n(a\cdot\mathbf{y})/I^n(\mathbf{y})$. (2.1)

For fixed $\mathbf{y}$ the right-hand side of the second equation is a function of a.

So we can write $I^n(a\cdot\mathbf{x})/I^n(\mathbf{x}) = f(a)$ or $I^n(a\cdot\mathbf{x}) = f(a)\cdot I^n(\mathbf{x})$, (2.2)

which implies $I^n(a\cdot(\mu\cdot\mathbf{x})) = f(a)\cdot I^n(\mu\cdot\mathbf{x})$. (2.3)

By (2.2) we have $I^n(\mu\cdot\mathbf{x}) = f(\mu)\cdot I^n(\mathbf{x})$ that is inserted in (2.3).

It causes $I^n(a\cdot(\mu\cdot\mathbf{x})) = f(a)\cdot I^n(\mu\cdot\mathbf{x}) = f(a)\cdot f(\mu)\cdot I^n(\mathbf{x})$. (2.4)

On the other hand by (2.2) we also have $I^n((a\cdot\mu)\cdot\mathbf{x}) = f(a\cdot\mu)\cdot I^n(\mathbf{x})$. (2.5)

The comparison of (2.4) and (2.5) entails the functional equation

$$f(a\cdot\mu) = f(a)\cdot f(\mu). \qquad (2.6)$$

This equation is a well-known Cauchy´s equation. As f is continuous and (partly) strictly increasing because of (A_4), there exists the only solution $f(a) = a^\rho$, $\rho \geq 0$ excluding the cases $I^n(\mathbf{x})=0$ and $\rho<0$.

(see a general solution in Eichhorn (1978, p.22)). Inserting it into (2.2) completes the proof. ■

This property is also mentioned and proven in Zheng (2007b) *Proposition 3* where inter alia the "decomposability" is assumed. For instance the absolute inequality measures $I^2_a(x,y) = \ln(1+|x-y|)$ or $I^2_b(x,y) = e^{|x-y|}-1$ satisfy the axioms (A_i), i =1,2,3,4,6, but Proposition 2.1 shows that they do not satisfy the axiom (A6).

Rao (1984) considered the population divided into r different classes $x_1, x_2,.., x_r$, where each class x_i is occupied by n_i individuals. For instance, you may think of r income classes for n officials. In this example we will shorten the inequality measure $I^n(\mathbf{x})$ of n officials, where n_i officials have the same income x_i:

$I^n(x_1, \dots, x_1, x_2, .., x_2, \dots\dots, x_r, .., x_r) = I^r(x_1, x_2, \dots, x_r, p_1, p_2, \dots, p_r)$,

where $p_i = n_i/n$ is the probability of the class x_i in the right side of this equation. Because of axiom (A_5) the number n does not affect the inequality number if all p_i are preserved. (One can simultaneously consider the classes x_i as individuals x_i if all p_i are equal. Then the p_i can be ignored.)

Rao preferred inequality measures belonging to the special class

$$I^r(x_1, x_2, .., x_r, p_1, p_2, .., p_r) = \sum_{i=1}^{r}\sum_{j=1}^{r} d(x_i, x_j)\, p_i\, p_j \qquad ®$$

with a symmetric distance function d: $R^2_+ \to R_+$, where the distances $d(x_i, x_j)$ need not satisfy the triangle inequality $d(x_i, x_j) \le d(x_i, x_k)+d(x_k, x_j)$. He could give the following statistical interpretation: $I^r(x_1, .., x_r, p_1, .., p_r)$ is the expected value of the inequality between two individuals x_i and x_j (taken from the classes x_i and x_j which need not be different) being arbitrarily singled out of the distribution **x**. This was Rao's fundamental idea for measuring the inequality $I^n(\mathbf{x})$ of any distribution $\mathbf{x} \in R^n_+$ if n>2. We will label this class by ® reminding us of Rao.

Because of $I^2(x_1,x_2,1/2,1/2)=d(x_1,x_2)$ we specially demand that the distance function $d(x_1,x_2)$ being $I^2(x_1,x_2)$ must be a "permissible" inequality measure. That means in detail: all axioms specially (A6) except for (A_5) must be fulfilled.
For that reason the already presented absolute inequality measure $I^2_a(x_i,x_j) = \ln(1+|x_i-x_j|)$ is excluded, but not the following example.

Example 1: *For* $\mathbf{x} \in R^2_{++}$ *the function* $I^2(x_1,x_2)=(x_1+x_2)\ln[(x_1+x_2)^2/4x_1x_2]$ *is a permissible inequality measure.*

Accepting Rao's point of view we have only to define a permissible inequality measure $I^2(x_1,x_2)$ in order to get a permissible inequality measure $I^n(\mathbf{x})$ for n>2.

III. Cowell's Feature

In 1994 Amiel and Cowell[1] discovered that not only every relative and absolute inequality measure, but also the suggested intermediate inequality measures of Pfingsten (1986) and Krtscha (1994) have a property that we call Cowell's Feature, and they conjectured that this property would be a general property of all inequality measures (before 1994). In order to motivate Cowell's Feature we will explain the "Compromise Concept (KC)" for constructing intermediate inequality measures proposed in Krtscha (1994) in a geometric way:

We consider $\mathbf{x}=(x_1, x_2 ,., x_n)$ lying on the hyperplane E: $x_1+x_2+...+x_n= c$ and **x**+ **s** lying on the hyperplane E*: $x_1+x_2+...+ x_n=c+s$, s>0. Both distributions **x** and **x**+**s** should have the same inequality number if **x**+**s** is a *fixed weighted arithmetic mean* of the two points $\mathbf{r}=\lambda\mathbf{x} \in E^*$ and $\mathbf{a}=\mathbf{x}+\mu\mathbf{1} \in E^*$. The fictive **r** is calculated as if the additional amount s>0 should be distributed with the relative point of view $(s_i/s_j=x_i/x_j)$, and the fictive **a** is calculated as if s should be distributed with the absolute point of view $(s_i=s_j=s/n)$.

Now we consider only points **x** having the same mean $m(\mathbf{x})=c/n$, and we can prove that the straight lines $(x_1+ts_1, x_2+ts_2, ., x_n+ts_n)$, $t\in R$, hit the same point $\mathbf{p}= (p, p,..., p)$ with $p \le 0$ if the $\mathbf{s}=(s_1, s_2,..., s_n)$ verifies Krtscha's Compromise Concept. For the fixed weighted arithmetic mean $\alpha\mathbf{a} +(1-\alpha)\mathbf{r}$ where α measures the degree of intermediateness we obtain $p= m(\mathbf{x})\cdot(\alpha-1)/\alpha$ by a longer elementary calculation. That hidden property leads to the following Feature which demands Krtscha's Compromise Concept only for points **x** with the same mean $m(\mathbf{x})$. We could call it an axiom for n=2 and only a desideratum for n>2, because we will not exclude "good inequality measures" which do not have this property for n>2.

1 The author got a personally sent unpublished telex (8.8.1994).

***(CF) Cowell's Feature*.**

For income distributions $\mathbf{x}=(x_1, x_2, ..., x_n)$ *with arithmetic mean* $m(\mathbf{x})$ *there exists a number* $t\in[0, 1]$ *depending only on* $m(\mathbf{x})$*, so that the following equation is valid:* $I^n(x_1, x_2,\ldots.., x_n) = I^n(x_1 + s_1, x_2 + s_2,\ldots.., x_n + s_n)$ *with* $s_i:= s\cdot(t\cdot x_i+1-t)/n\cdot(t\cdot m(\mathbf{x})+1-t)$*, where* $s>0$ *is an additional amount which is distributed into* s_i $(i=1, 2, \ldots., n)$ *parts given to each* x_i*.*

It is necessary to interpret (CF) also in a geometric way: The vector $\mathbf{s} = (s_1, s_2,..., s_n)$ shows the path from $\mathbf{x}^0$ to $\mathbf{x}^0+\mathbf{s}$ that must be chosen, if the inequality number has to be preserved. As the proportions s_i/s_j do not depend on s if m(**x**) is fixed, this path is a straight line $s\cdot\mathbf{s}$ ($s \geq 0$) starting at the point $\mathbf{p}= p\cdot\mathbf{1}$ with $p= 1-1/t(m)$ (assumed $t\neq 0$). A simple calculation shows that this **p** is the same point for all points **x** lying on the hyperplane $x_1+x_2+..+ x_n = x_1^o+x_2^0+ ...+ x_n^0$ through $\mathbf{x}^0$. The vector **s** lies on the plane which is spanned by the straight line through **O** and $\mathbf{x}^0$ (meaning the relative point of view) and the straight line $\mathbf{x} = \mathbf{x}^0 + \lambda\cdot\mathbf{1}$ with $\lambda >0$ (meaning the absolute point of view). If we specially choose $t(m)=1/[1-m\cdot(\alpha-1)/\alpha]$ we will obtain the $\mathbf{p}=(m(x)\cdot(\alpha-1)/\alpha)\cdot\mathbf{1}$. That point we had already calculated by Krtscha's Compromise Concept.

However, that special choice is not necessary. First we choose t= constant.

1.) t=0 for all $\mathbf{x} \in R^n_+$: An absolute inequality measure is generated satisfying the axiom (A6). All s_i are s/n for i=1,..,n, and the point **p** does not exist.

2.) t=1 for all $\mathbf{x} \in R^n_+$: A relative scale invariant measure is generated, and $\mathbf{p}= \mathbf{0}$.

3.) $t=\mu$ with fixed $\mu \in(0, 1)$ for all $\mathbf{x} \in R^n_+$: Pfingsten's (1986) strict intermediate measure is obtained. This measure does not fulfil (A_6) and (A6). The point **p** is fixed and depends on μ. If we choose $t= \mu = ½$ we get $\mathbf{p} = \mathbf{-1}$ for all **x**.

The parameter t depends on *m*(**x**): Verifying Krtscha's proposal the point **p** cannot be fixed and the additional amount s>0 must be so small that we will stay in a small neighborhood of **x**. You shall better say the vector **s** defined by (CF) shows the direction of a path through **x** where the inequality number remains constant.

We will now consider the special case *n=2* where (CF) becomes a demand for the direction of the equiinequality curve through **x**. Krtscha demanded that the direction of the equiinequality curve in every point $\mathbf{x}\in R^2_+$ is defined by the direction indicated by a weighted arithmetic mean (with the fixed parameter $\alpha \in [0 ,1]$) of the points **r** and **a** as already explained. If α is once chosen, it must be preserved in each point $\mathbf{x} \in R^2_+$. Therefore Krtscha's demand is more restrictive than Cowell's Feature where t or α must only be preserved in all points **x** having the same mean $m(\mathbf{x})=(x_1+x_2)/2$. On the other hand (CF) allows to change the parameter t abruptly at a certain value m which could be defined by $m:= \beta m_0$ where m_0 are the mean costs of living. Then it would be reasonable that we demand t=1 for m(**x**)<m meaning the relative point of view and t=0 for m(**x**) ≥ m meaning the absolute point of view. (In this case an equiinequality curve is a broken straight line determined by the parameter m.)

The idea of abrupt changing t was suggested by Azpitarte and Alonso-Villar (2014) according to the ”ray invariance concept “of Seidl and Pfingsten (1997) similar to the papers of Del Rio and Ruiz-Castillo (2000), Del Rio and Alonso-Villar (2010) and Yoshida 2005) that contain further intermediate invariance concepts. In the articles of Zoli (1999) and (2012) the inequality perception goes from the relative point of view to the absolute point of view when incomes increase, and that flexibility is achieved by two parameters.

However, not only Krtscha's demand, but also Cowell's demand for an equally consequent treatment of situations with the same mean could be considered as too restrictive, and changing α or t also in points on the straight line $x_1+x_2=2c$ in order to approach the relative point of view when going away from (c,c) has already been proposed indirectly. For instance our Example 1 has this property. It is taken from Zheng (2007b) where the third example in Zheng's *Proposition 4* does not satisfy (CF) for $\mathbf{x}\in R^2_{++}$ although it has many good properties. However, considering any $x_i\rightarrow 0$ it reacts as problematically as a relative inequality measure (α=0) where both limit-situations $x_1= 0$, $x_2 =1$ and $x_1= 0$, $x_2 = 100$ are equally judged. (Because of this well-known problem the variable α should not have the limit α=0. For that reason the property (CF) which prevents this limit could be another **reasonable axiom for n=2** .)

In the next Section IV for n=2 we will show that the combination of (CF) and (A6), being a strong but reasonable demand, implies that it is not allowed to change the degree of intermediateness after α has been chosen once.

IV. Generating Intermediate Inequality Measures

In Subramanian (2014) it was clearly explained how the economic inequality of a distribution could be judged by three appropriate measures. An inequality measure will be labeled "strictly intermediate" or "centrist", if it possesses the compromise property (CP) defined in Bossert and Pfingsten (1990).

Definition (CP): *For all* $n\geq 2$, $x\in R_+$ *such that* $\mathbf{x} \neq\lambda\cdot\mathbf{1}$ $(\lambda>0)$:

a) $I^n(\mathbf{x}) < I^n(\alpha\cdot\mathbf{x})$ *for all* $\alpha>1$, *b)* $I^n(\mathbf{x}) > I^n(\mathbf{x}+\lambda\cdot\mathbf{1})$ *for all* $\lambda>0$.

The property a) means that the multiplication of **x** with a number α>1 increases the value $I^n(\mathbf{x})$, whereas a relative inequality measure, preferred by the "rightists", would not change $I^n(\mathbf{x})$. The property b) means that the addition of the same positive value λ to every x_i, i=1, 2,....., n decreases the value $I^n(\mathbf{x})$, whereas an absolute inequality measure, preferred by the "leftists", would not change $I^n(\mathbf{x})$. If we abstain from the word "strictly" in both inequalities a) and b) we will have to accept relative and absolute inequality measures as extreme intermediate inequality measures. However, we are not interested in these extreme measures and show how we can generate any centrist inequality measure $I^n(\mathbf{x})$ for n>2 by the following three methods.

Method 1: We take an absolute inequality measure $I^n_a(\mathbf{x})$ being linear homogeneous and the corresponding relative inequality measure $I^n_r(\mathbf{x})$, fulfilling the axioms (A_i) i=1, 2,.., 6. Then we take a weighted geometric mean of both measures with a fixed parameter α∈(0,1) defining the degree of intermediateness so that (CP) is satisfied. (That could also be done by the product $I^n_a(x))^{\alpha}\cdot(I^n_r(x))^{\beta})$. Later it will be clear that (CF) is satisfied.

The first example for this method comes from the variance $V^n(\mathbf{x})$. So we denote it

$$\tilde{V}^n(\mathbf{x}) = (1/nm) \sum_{i=1}^{n} (x_i - m)^2 \qquad \text{with } m = (x_1+ x_2 +....+ x_n)/n. \tag{4.1}$$

The second example to method 1 comes from the Gini index $G^n(\mathbf{x})$. It is denoted

$$\check{G}^n(\mathbf{x}) = (1/2\cdot n^2 m^{1/2}) \sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j| \quad \text{with } m = (x_1+x_2+\ldots+x_n)/n. \tag{4.2}$$

Method 2: We start with a permissible (strict) intermediate inequality measure $I^2(x_i, x_j)$. This $I^2(x_i, x_j)$ must not depend on the global mean $m(\mathbf{x})$, $\mathbf{x}\in R^n$. Then we take the *arithmetic mean* of the inequality measures $I^2(x_i, x_j)$ of all possible pairs (x_i, x_j) so that $I^n(\mathbf{x})$ belongs to the class ®. However, we have to accept that the demand (CF) is not satisfied for n>2.

The following example is generated by this method.

$$F^n(\mathbf{x}) = (1/n^2) \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2/(x_i + x_j) \quad . \tag{4.3}$$

Method 3: The only difference to method 2 is that we take the *weighted arithmetic mean* of the inequality measures of all possible $I^2(x_i, x_j)$ where the weights $w(x_i, x_j)$ depend on the global mean $m(\mathbf{x})$ in order to value each $I^2(x_i, x_j)$ with regard to the global mean $m(\mathbf{x})$. A possible weight is $w(x_i, x_j)=(x_i+x_j)/m(\mathbf{x})$. (Later we will see that an inequality measure can simultaneously be generated by method 1 and 3.)

For obtaining permissible differentiable inequality measures $I^2(\mathbf{x})$ we will prove:

Proposition 4.1. *If $I^2(\mathbf{x})$: $R^2_+ \rightarrow R_+$ is differentiable and satisfies the axiom (A6) and the Feature (CF), then it has to satisfy the differential equation $dx_2/dx_1 = (\alpha\cdot x_1 + (2-\alpha)\cdot x_2)/((2-\alpha)\cdot x_1 + \alpha\cdot x_2)$, $\alpha \in [0, 1]$.*

Proof.

The Feature (CF) demands that in the ratio $s_2/s_1 = (tx_2+1-t)/(tx_1+1-t)$ the variable t has to depend only on the mean $m(\mathbf{x})$.

That means $s_2/s_1= (t(m)x_2+1-t(m)) / (t(m)x_1+1-t(m))$, (4.4)

and axiom (A6) demands the same ratio for any $\lambda\cdot\mathbf{x}$ ($\lambda > 0$) instead of $\mathbf{x}$, implying λm instead of m. From the equivalence of these ratios for all λ we have to satisfy the equation

$$(mt(\lambda m)\lambda x_2+1-t(\lambda m))/(t(\lambda m)\lambda x_1+1-t(\lambda m))=(t(m)x_2+1-t(m))/(t(m)x_1+1-t(m)). \tag{4.5}$$

Choosing $\lambda=1/m$ in (4.5) we obtain

$$(t(1)x_2/m +1- t(1))\cdot(t(m)x_1 +1-t(m)) = (t(1)x_1/m +1- t(1))\cdot(t(m)x_2 +1-t(m)).$$

By elementary calculation we get the solution $t(m) = t(1)/(m\cdot(1-t(1))+t(1))$.

Inserting it into (4.4), it follows that $s_2/s_1 = (x_2\cdot t(1)+m\cdot(1-t(1))/(x_1\cdot t(1)+m\cdot(1-t(1)) = (2x_2\cdot t(1)+(x_1+x_2)\cdot(1-t(1)) / (2x_1\cdot t(1)+(x_1+x_2)(1-t(1))$.

Substituting $\alpha:= 1-t(1)$ we finally obtain

$$s_2/s_1 = (\alpha\cdot x_1 + (2-\alpha)\cdot x_2) / ((2-\alpha)\cdot x_1 + \alpha\cdot x_2) , \alpha \in [0, 1]. \tag{4.6}$$

Assuming differentiability of $I^2(x_1 ,x_2)$ we get the differential equation

$$dx_2/dx_1= (\alpha\cdot x_1 +(2-\alpha)\cdot x_2) /((2-\alpha)\cdot x_1+\alpha\cdot x_2) , \alpha \in[0, 1]. \tag{4.7}$$

■

It is solved in Krtscha (1994) where the solution is

$(x_2-x_1)^{1/(1-\alpha)} = c(x_1+ x_2)$ for $c \geq 0$, $x_2 \geq x_1$, $0 \leq \alpha < 1$, and for the case $\alpha = 1$ the differential equation (4.7) implies $dx_2/dx_1 = 1$ being characteristic for an absolute inequality measure, whereas for the case $\alpha = 0$ the differential equation (4.7) implies $dx_2/dx_1 = x_2/x_1$ being characteristic for a relative inequality measure. Being only interested in centrist inequality measures we prefer the *"fair compromise"*, generated by $\alpha=1/2$. In this case we obtain $t(1)=1/2$ and $t(m)= 1/(m+1)$, and the equation $(x_2-x_1)^2 =c(x_1+x_2)$ describes points (x_1,x_2) of a parabola as an equiinequality curve of $I^2(x_1,x_2) = \phi((x_2-x_1)^2 /(x_2+x_1))$ where $\phi : R_+ \rightarrow R_+$ is any strictly increasing function with $\phi(0) = 0$ being differentiable in R_{++}. All these functions are permissible and differentiable "fair intermediate inequality measures".

With the special function $\phi_a(t) = t$ and the mean $m = (x_1+x_2)/2$ we obtain

$$I^2(x_1, x_2) =(x_2-x_1)^2 /(x_2 + x_1)= (1/2m) \sum_{i=1}^{2} (x_i - m)^2 \quad , \tag{4.1a}$$

where the natural generalization $\tilde{V}^n(\mathbf{x})$ in (4.1) was generated by method 1.

It satisfies (A6) and also (CF). That is geometrically clear for n=3, because the equiinequality surfaces $\tilde{V}^3(\mathbf{x})=c$ are generated by rotating the parabolas $(x_2-x_1)^2 = c(x_1+x_2)$ around the axis $\mathbf{x}= \lambda\cdot\mathbf{1}$.

With the other special function $\phi_b(t) = t^{1/2}$ and $m = (x_1+x_2)/2$ we obtain

$$I^2(x_1, x_2) = (1/2\cdot 2^2 m^{1/2}) \sum_{i=1}^{2}\sum_{j=1}^{2} |x_i - x_j| \quad , \tag{4.2b}$$

where the natural generalization $\check{G}^n(\mathbf{x})$ in (4.2) was generated by method 1, too.

It also satisfies (CF) that is later verified by an example for n=3 where its equiinequality curves on every plane E: $x_1+x_2+x_3=3c$ are regular hexagons H(c) with center $c\cdot\mathbf{1}$. Its equiinequality curves in the plane spanned by the straight line through $\mathbf{O}$ and $\mathbf{x} \in H(c)$ and the straight line $\mathbf{x} + t\cdot\mathbf{1}$ are parabolas through any $\mathbf{x}$ with constant $m(\mathbf{x})=c$, and the parabola-tangents in $\mathbf{x}$ hit $s\cdot\mathbf{1}$ in the same point. It is also a possible "fair compromise measure", and without an additional axiom - Krtscha (1994) suggested one - we do not get a characterization of $\tilde{V}^n(\mathbf{x})$.

Meanwhile a different characterization of $\tilde{V}^n(\mathbf{x})$ is possible by a paper of Chakravarty (2000). He characterized the variance $V^n(\mathbf{x})$ as the only absolute inequality measure which is "additive subgroup decomposable" where the weights of the "within-group components" are the population-shares. (See also Subramarian (2011)) where the weight $w(\mathbf{x}^j)$ attached to subgroup j could depend on the population share $\pi(\mathbf{x}^j)$ or on the income share $\sigma(\mathbf{x}^j)$ or both.) Using this result and defining the weights $w(\mathbf{x}^j)$ of the within-group components by the *income shares* $\sigma(\mathbf{x}^j)$, in Nov. 2014 Subramarian and D. Jayaray noticed that $\tilde{V}^n(\mathbf{x})$ belongs to the general family of intermediate inequality measures being excellent, because it satisfies the unit consistency and the special subgroup decomposability. They called it "Krtscha measure $K(\mathbf{x})$", and they used it in many papers, for example in their paper (2015). We will now describe this result in their notation.

Let $\mathbf{x}$ be divided into s exhaustive and exclusive subgroups $\mathbf{x}^j =(x^j_1, x^j_2 ,.. , x^j_{nj})$, j=1, 2, ... , s. Then for $K^n(\mathbf{x})=K^n(\mathbf{x}^1, \mathbf{x}^2,.., \mathbf{x}^s)$ we obtain two components.

The first component is called the

within-group component $K_w(\mathbf{x})=\sigma(\mathbf{x}^1)\cdot K^{n1}(\mathbf{x}^1)+\sigma(\mathbf{x}^2)\cdot K^{n2}(\mathbf{x}^2)+....+\sigma(\mathbf{x}^s)\cdot K^{ns}(\mathbf{x}^s)$,

the second component is called the

between-group component $K^n_b(\mathbf{x}) = K^n(\mathbf{x}^1_0, \mathbf{x}^2_0, ..., \mathbf{x}^s_0)$

obtained by replacing all incomes within each subgroup j by the relevant subgroup mean $\mathbf{x}^j_0$.

Therefore the inequality measure $K^n(\mathbf{x})$ satisfies the

subgroup decomposability $K^n(\mathbf{x}) = K_w(\mathbf{x}) + K^n_b(\mathbf{x})$.

The advantage of a decomposable inequality measure is called "very appealing" by Shorrocks (1980 and 1987) and "extremely convenient" by Subramanian (2011). It is also used by Foster (1983) in order to characterize the Theil measure. The only reason, because we do not add this property and the Feature (CF) to our axioms is that they are too restrictive, and we will not exclude good measures based on Rao's fundamental inequality idea, which do not have these properties and do not satisfy the following demand being a special consequence of the subgroup decomposability.

Limit when Merging (LM)*: Assuming equal means $m(\mathbf{x})$ and $m(\mathbf{y})$ of the distributions* $\mathbf{x}$ *and* $\mathbf{y}$ *we want the inequality* $I^{n+m}(\mathbf{x},\mathbf{y}) \leq \max(I^n(\mathbf{x}), I^m(\mathbf{y}))$ *to be satisfied.*

This desideratum (LM) is a minor restriction. It is only characteristic for all inequality measures depending on the distances $d(x_i, m(\mathbf{x}))$ and not on $d(x_i, x_j)$.

Now we use the second possibility to extend the fair compromise measure $I^2(x_1,x_2)=(x_2-x_1)^2/(x_2+x_1)$ for n>2 by *method 2*. Following Rao's conception ® the function $I^2(x_1,x_2)$ in (4.1a) can be interpreted as a "distance" $d_{i,j}=(x_i-x_j)^2/(x_i+x_j)$ between x_i and x_j. Then we consider $F^n(\mathbf{x})$ as the arithmetic mean of all $d_{i,j}$:

$$F^n(\mathbf{x}) = (1/n^2) \sum_{i=1}^{n}\sum_{j=1}^{n} (x_i - x_j)^2/(x_i + x_j) \quad . \tag{4.3}$$

This measure does not satisfy Cowell's Feature (CF) and the desideratum (LM). That will be shown by the next Example 2, but it has a special property: We consider the distribution $(x_1, x_2, .., x_n)$ where a poor individual x_i is listed in the left tail where his neighbors are also poor. For all poor people the difference $\Delta= |x_{i+1}-x_i|$ to the neighbor is felt more perceptible than the same difference Δ within the right tail where the rich people are listed. We suppose that there could be a general agreement that the differences $\Delta= |x_i -x_j|$ and also $\Delta^2=(x_i-x_j)^2$ being a component of the distances $d_{i,j}=d(x_i, x_j)$ should have a smaller weight for the same Δ if x_i and x_j are bigger. By accumulating these weighted distances we get the inequality measure defined by (4.3) being an aggregation of these individually felt inequalities.

Therefore we will call an inequality measure depending on the distances d_{ij} "*sensitive"* if it has the described property[2] for n≥3.

2 That is not the level-sensitivity in Subramanian (2011) and the transfer-sensitivity in Zoli (2002).

V. Examples for Strict Intermediate Inequality Measures

In this chapter we will compare three centrist inequality measures with each other. All can be called “fair compromise measures”. The first measure $F^n(\mathbf{x})$, defined in (4.3), is not generated by *method 1,* but by *method 2* as it belongs to the class ® being the simple arithmetic mean of the distances $d_{i,j}=(x_i-x_j)^2/(x_i+x_j)$. That is a permissible centrist inequality measure. It is strictly convex because all $d_{i,j}$ are strictly convex functions.. For the measure defined in (4.3) we have introduced the notation $F^n(\mathbf{x})$ that shall remind us of the “felt” smaller importance of the “distances” with the same differences $|x_i - x_j|$ but with higher (x_i+x_j). This “sensitive measure” can also be interpreted by Rao as

$$F^n(x_1, x_2,., x_n, p_1, p_2,.., p_n)= \sum_{i=1}^{n}\sum_{j=1}^{n}((x_i-x_j)^2/(x_i+x_j))\,p_i\,p_j \quad \text{with } p_i = p_j=1/n.$$

The second strict intermediate inequality measure $\tilde{V}^n(\mathbf{x})$ is also strictly convex. By *method 1* it is derived from the well-known variance $V^n(\mathbf{x})$ in probability.

It can also be generated by *method 3* where we extend Rao's point of view by externally weighting each distance $d_{i,j}$: *S*tarting with the measure $I^2(x_i, x_j)=(x_i-x_j)^2/(x_i+x_j)$ as it was also done by $F^n(\boldsymbol{x})$, we will then use the external weights $(x_i+x_j)/m(\mathbf{x})$ for a weighted arithmetic mean of all n^2 possible $I^2(x_i, x_j)$; so we will obtain (4.1). (In order to show this equivalence one has to use a well-known equation for the variance divided by m.) The motivation for these weights could be given by an economist who wishes weights for the felt inequality $I^2(x_i, x_j)$ that represent the economic importance of (x_i+x_j) compared with the total level $m(\mathbf{x})$. Thereby the “sensitivity” of $F^n(\mathbf{x})$ is eliminated, and in (4.1) there is no more sensitivity for fixed mean m if n>2. (However, an economist does it not deplore.)

For the third measure we take the simple geometric mean of the Gini index $G^n(x)$ and the corresponding absolute Gini measure. For that reason we denoted it as

$$\check{G}^n(\mathbf{x}) = (1/2n^2m^{1/2}) \sum_{i=1}^{n}\sum_{j=1}^{n}|x_i-x_j| \quad \text{with } m= (x_1+ x_2 +....+ x_n)/n.$$

It can also be generated by *method 3* if we start with the permissible measure $I^2(x_i ,x_j)=|x_i -x_j|/(x_i+x_j)^{1/2}$ satisfying (CF). Then we choose the root of the weights that we have taken when we generated (4.1) so that the weighted arithmetic mean leads to $\check{G}^n(\mathbf{x})$. It is also convex, but not strictly convex.

At last we will give a summary table with the properties satisfied or not by the proposed three indices which satisfy the axioms (A_1) - (A_6), (A6) and (CP).

	$\tilde{V}^n(\mathbf{x})$	$\check{G}^n(\mathbf{x})$	$F^n(\mathbf{x})$
Rao's class ®	no	no	yes
(CF)	yes	yes	no
(LM)	yes	no	no
Decomposability	yes	no	no
Sensitivity	no	no	yes
Differentiability	yes	no	yes

The proofs of the statements are elementary or consequences of well-known facts: $\check{G}^n(\mathbf{x})$ satisfies the Pigou-Dalton principle (A_1), because that property is satisfied by the Gini coefficient. $F^n(\mathbf{x})$ satisfies (A_1) because of the strict convexity implying the Schur-convexity and so on.

Testing (CF) we take the following example.

Example 2: The points $\mathbf{a}$=(2,4,6) and $\mathbf{b}$=(2,0,10) have the same mean, but they have different inequality numbers.

The tangent plane to the equiinequality surface of $F^3(\mathbf{x}) = F^3(\mathbf{a})$ in $\mathbf{a}$ hits $s\cdot\mathbf{1}$ in $-3.732\cdot\mathbf{1}$, whereas the tangent plane to the equiinequality surface of $F^3(\mathbf{x}) = F^3(\mathbf{b})$ in $\mathbf{b}$ intersects $s\cdot\mathbf{1}$ in $-3.545\cdot\mathbf{1}$. For the equiinequality paths $\mathbf{s}$ in $\mathbf{a}$ and $\mathbf{b}$ it is therefore not possible to start from the same point on $s\cdot\mathbf{1}$. That contradicts (CF).

For $\check{G}^3(\mathbf{x})=\check{G}^3(\mathbf{a})$ the tangent plane in $\mathbf{a}$ hits $s\cdot\mathbf{1}$ in the point $-4\cdot\mathbf{1}= -m(\mathbf{x})\cdot\mathbf{1}$. For $\check{G}^3(\mathbf{x})=\check{G}^3(\mathbf{b})$ the tangent plane in $\mathbf{b}$ hits $s\cdot\mathbf{1}$ in the same point as expected by (CF).

Testing (LM) we take the following example.

Example 3: *The distributions* $\mathbf{x}$*=(0, 3, 5) and* $\mathbf{y}$*=(0.2, 2.015, 5.785) have the same mean* $m(\mathbf{x})=m(\mathbf{y})=8/3$ *and we obtain* $F^3(\mathbf{x})=1.888$, $F^3(\mathbf{y})=1.89356$ *and* $\check{G}^3(\mathbf{x})=0.680$, $\check{G}^3(\mathbf{y})=0.6375$.

Then $F^6(\mathbf{x},\mathbf{y}) =1.90586 > \max (F^3(\mathbf{x}), F^3(\mathbf{y}))$ contradicts (LM) as well $\check{G}^6(\mathbf{x},\mathbf{y}) = 0.723 > \max(G^3(\mathbf{x}), G^3(\mathbf{x}))$.

Moreover, this very exceptional example shows that even the strict convexity of $F^n(\mathbf{x})$ does not imply (LM) which is a stronger demand.

VI. Conclusion and a Last Example

In a weather report we sometimes can read two different degrees for the same day. The first degree is the normally measured temperature, while the second temperature is the "subjectively felt temperature" which sometimes is lower because of a cold wind. We can also compare the measurement of the inequality with a weather report. In the common statistics the inequality of the incomes is normally measured by the Gini coefficient which does not depend on the monetary unit. The Gini coefficient would also remain unchanged if all incomes increase for instance by 10% (in reality). However, in the opinion of the "leftists", represented by the trade unions and socialistic politicians, the inequality of the incomes will increase in this case. This increasing inequality would become visible by the use of an absolute or strictly intermediate inequality measure, for instance by one of the three measures we have presented. For our assumed example where all incomes increase by 10% the inequality ratio $R(\mathbf{x},\mathbf{y})=I^n(\mathbf{x})/I^n(\mathbf{y})$ would also indicate an increase of β=10% using the linear homogeneous centrist inequality measures $\tilde{V}^n(\mathbf{x})$ and $F^n(\mathbf{x})$, but $\check{G}^n(\mathbf{x})$ indicates the value 4.88%. We think that the "rightists" [3]would prefer $\check{G}^n(\mathbf{x})$, but the "leftists" would rather prefer the value of 10%, because it is the same value that an absolute inequality measure will indicate if it is linear homogeneous. However, let us assume another scenery: all salaries x_i in $\mathbf{x}$ increase to $x_i+m(\mathbf{x})/10$. Then $\tilde{V}^n(\mathbf{x})$ and $F^n(\mathbf{x})$ are reduced to 90.9% of the former values, whereas $\check{G}^n(\mathbf{x})$ is only reduced to 95.3% of the former value. This reduction would be accepted more easily by the "leftists",

3 S. Kolm introduced the name"rightists" for people who in the presence of income-growth insist on the use of relative inequality measures and "leftists" who insist on the use of absolute inequality measures.

because they refuse any reduction of the inequality number in this scenery. So they would have a problem to prefer one of these intermediate inequality measures.

If we are to recommend the best of these three measures, we will remind the reader that $\tilde{V}^n(\mathbf{x})$ is the only intermediate measure which satisfies all our axioms and desiderata except for the property we called "sensitivity".
As $F^n(\mathbf{x})$ possesses this property we should first calculate the inequality number by this measure in order to get the "felt inequality".
Then at any rate we should compare $F^n(\mathbf{x})$ with the best measure $\tilde{V}^n(\mathbf{x})$ or with $\check{G}^n(\mathbf{x})$, because the latter measure is closer to the Gini index which is mostly used and preferred for the comparison of the inequality numbers of *different countries at the same time* not depending on the unit of the money. However, for the comparison of different inequality numbers concerning the *same country at different times* we recommend a centrist inequality measure where the comparison of the inequality ratios is significant. Because of (A6) it does not depend on the unit of the money, too. (For example you need no exchange rate of dollars and rupees if you compare the growth of the inequality of the consumption expenditure from 1983 to 2010 in India with the corresponding growth in the USA.) Therefore we will finish our paper with a realistic test of the inequality ratios of our recommended three measures considering two situations: The big subgroup $B \subset X$ of all employees X of the German railway company had got lower salaries than the officials in subgroup A, and we tested how our three inequality measures judge the inequality of $X = A \cup B$ before and after the lower payment was removed. Example 4 describes that realistic but very simplified situation:

Example 4: *The employees X of the railway company in Germany can be divided into two groups A and B. In group A there are the officials. In group B which is about n=10 times bigger than A there are the non-officials. Each group has engine-drivers and the same number of other personnel. In group A the monthly salary of an engine-driver is € 3500, whereas another official of group A gets € 2500. In group B the engine-drivers and the other personnel only get 90% of the officials' salaries meaning a reduction with the factor γ = 0.9.*

Not all people agreed with the repeated strikes of group B, but all people agreed that the different payment was unfair, because one could not find reasons for a different payment apart from the fact that the officials are not allowed to strike. It was also generally thought that the same (higher) payment of officials and non-officials would certainly cause a smaller inequality number of the incomes within all employees of the railway company. However, that general belief may be wrong! That can be seen by the calculation of the inequality numbers and ratios using our three centrist measures:

Because of the replication axiom we need not know the absolute numbers of the employees and get $\tilde{V}^n(\mathbf{x}) = \tilde{V}^n(3500, 2500, 3150, .,3150, 2250,.., 2250) = 500\tilde{V}^n(7, 5, 6.3, ., 6.3, 4.5, ., 4.5) = \mathbf{78.56}$
(3500 and 2500 represent the officials, whereas 3150 and also 2250 are taken 10 times.)
If all employees have the same salaries as the officials we will get the inequality number $\tilde{V}^n(\mathbf{y}) = \tilde{V}^n(3500, \ldots, 3500, 2500, \ldots, 2500) = 500\tilde{V}^n(7, 5) = \mathbf{83.3}$.

(3500 and also 2500 are taken 11 times.)

Then we get the inequality ratio $R(\mathbf{y},\mathbf{x})$ = **1.060** that shows an increase of the overall inequality number $\tilde{V}^n(\mathbf{x})$. The reason for this interesting effect is the increase of the within-component of the non-officials that is higher than the decrease of the between component of $\tilde{V}^n(\mathbf{x})$.

However, choosing the inequality measure $F^n(\mathbf{x})$ we get $R(\mathbf{y},\mathbf{x})$ = **1.0042**, and choosing $\check{G}^n(\mathbf{x})$ we obtain $R(\mathbf{y},\mathbf{x})$ = **0.9944.**

These values indicate a nearly unchanged overall inequality. That means a significant difference between these centrist inequality measures. Generally speaking, only the use of an absolute or an intermediate inequality measure $I^n(\mathbf{x})$ can show the effect that a multiplied income-growth of the incomes of a subgroup increases the overall inequality of the incomes because of the inequality-increase of the subgroup (not depending on the decomposability of $I^n(\mathbf{x})$). As to our three examples we have twice seen that by a multiplied income-growth this increase is linear and once that it is sub-linear. In addition, it is shown that in extreme situations there is no simple correlation between unfairness and inequality. This example is therefore a further motivation for the use of axiom (A6).

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