Working Paper Series

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ECINEQ WP 2019 - 486

ECINEQ 2019 - 486
January 2019

www.ecineq.org

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Abstract

We provide an axiomatic approach to the measurement of individual resilience. Resilience has been an increasingly important topic in many social sciences but, as of now, there does not seem to be much literature on its theoretical foundations. This paper is intended to fill that gap. After an introduction to the notion of resilience and its possible determinants, we introduce a set of intuitively appealing properties that a resilience measure is required to possess. Our result is a characterization of the specific resilience ordering that satisfies these axioms.

*We thank Stefan Baumgärtner, Louise Grenier and seminar participants at the University of Montreal and the University of Oslo for comments and suggestions, and Anthony Lepinteur for research assistance. Asheim thanks CIREQ for facilitating his visits to Montreal. Financial support from the Fonds de Recherche sur la Société et la Culture of Québec and the Fonds National de la Recherche Luxembourg is gratefully acknowledged. The paper is also part of the research activities at the Centre for the Study of Equality, Social Organization, and Performance (ESOP) at the Department of Economics at the University of Oslo. ESOP was supported by the Research Council of Norway through its Centres of Excellence funding scheme, project number 179552.

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1 Introduction

Resilience has become a highly popular research topic over the last few decades in several disciplines. As Bonanno, Romero and Klein (2015) report, the frequency with which the term 'resilience' or one of its variants appear in the titles of articles published in social-sciences journals has quadrupled from 2000 to 2010, jumping to 800 occurrences. A similarly increasing trend is reported by Hodgson, McDonald and Hosken (2015) for the International Statistical Institute's Web of Science (ISI WoS) where its prevalence as a keyword in peer-reviewed papers in the ecology category has been rising steadily since the early 1970s. Particularly active contributors are psychologists and ecologists who routinely dedicate the first few pages of their writings to a discussion of the definition of the term and mention that it has taken on multiple meanings. The contributions of Ayed, Toner and Priebe (2018), Fletcher and Sarkar (2013), Bonanno (2012), Bonanno, Romero and Klein (2015), among others, are examples within the psychology literature; Hodgson, McDonald and Hosken (2015) or Standish, Hobbs, Mayfield, Bestelmeyer, Suding, Battaglia, Eviner, Hawkes, Temperton, Cramer, Harris, Funk and Thomas (2014), for example, can be consulted in the context of ecology.

The etymology of the term 'resilience' has its roots in the Latin verb *resilire,* meaning 'to jump back' or 'to recoil' and it is defined in the Merriam-Webster dictionary as "the capability of a strained body to recover its size and shape after deformation caused especially by compressive stress" or "an ability to recover from or adjust easily to misfortune or change." The first definition relates to the use of the term in physics, whereas the second describes it in relation to the social sciences. Both definitions help in visualizing the subject matter of our contribution: resilience captures the response in terms of the functioning of an individual when 'squeezed' by the occurrence of an adverse event such as the death of a spouse, a divorce, a job loss, a terrorist attack, a natural disaster or a severe injury. A resilient individual, once squeezed, is able to go back to the pre-event functioning level quickly. The variable that is mostly used in psychology to capture the functioning of an individual is his or her self-reported health status. Similar observations apply to macro settings, such as an ecosystem whose equilibrium is perturbed by human or natural activities.

An additional distinction in the psychological literature exists depending on the level of functioning reached at the end of the process. Resilience is often associated with a full recovery from the adverse event; the term *thriving* is applied when the person is better off after overcoming adversity as compared to before the event occurred; see, among others, Carver (1998). The latter phenomenon is also known as *growth following adversity* (Linley and Joseph, 2004) or *post-traumatic growth* (Tedeschi and Calhoun, 2004) which can be attributed to newly developed individual skills and a psychological sense of mastery following the negative event.

The confusion with the uses (and abuses, as Bonanno, 2012, one of the leading resilience researchers within psychology, puts it) of the term resilience is rooted in the fact that several contributors attempt to capture the characteristics of a resilient individual or system, rather than focusing on the process described above. In other words, instead of measuring the functioning process following an event (the 'squeeze'), they focus on the predictors of resilient outcomes affecting the process; these predictors may be personal, social, or a notion of system resources. This duality in the approaches to resilience is documented in the systematic review of the mental health literature by Ayed, Toner and Priebe (2018). They identify two broad categories of approaches to resilience, namely, *processes* and *characteristics.* Bonanno, Romero and Klein (2015) offer an integrative framework of this duality by discussing how the process of response to an event is influenced by characteristics, with the process being the subject matter of resilience rather than the characteristics of individuals or societies.

The same duality of approaches to resilience is present in ecology. The term has been used with different interpretations, leading to the "confusion of resilience" (Hodgson, McDonald and Hosken, 2015, p. 503). In the ecology literature, the majority of contributors follow Holling (1973) in defining resilience as a measure of the ability of ecosystems to absorb disturbances without changing identity. As Scheffer, Carpenter, Foley, Folke and Walker (2001) put it, resilience "corresponds to the maximum perturbation that can be taken without causing a shift to an alternative stable state" (p. 591). The alternative-process approach is proposed by Pimm (1984) according to which resilience indicates "how fast the variables return towards their equilibrium following a perturbation" (p. 322). The ecological literature defines Holling's interpretation of resilience as *ecological* resilience and Pimm's as *engineering* resilience (Gunderson, Allen and Holling, 2009), and some contributors (see Standish et al., 2014) propose to relabel Holling's definition by referring to it as 'resilience' and to name Pimm's definition 'recovery' to reduce the confusion about these two important concepts. We note that, in the field of ecology, there seems to be a preference for the characteristics approach.

An important first step in the measurement of resilience is to find an answer to the crucial question "resilient to what?" posed by Bonanno, Romero and Klein (2015) and the references therein in the context of psychological approaches. Hodgson, McDonald and Hosken (2015) address this issue in the ecological setting. Adverse events differ in terms of their intensity and the duration of their impact. Some events may last for a considerable amount of time, such as poverty, political violence, physical or sexual abuse; some other events are more transitory in nature, such as an accident, a terrorist attack or the death of a loved one.

There are numerous contributions by economists that address resilience in ecology by developing deterministic and stochastic models with regime shifts and estimating the underlying system properties. These models have been used to describe resource-management problems such as those pertaining to coral reefs, lakes, ocean-climate systems, woodland preservation, among others. For an excellent review see Li, Crépin and Folke (2018). However, there does not seem to be much of a literature within economics when it comes to the measurement of individual resilience. This is somewhat surprising because there appears to be a clear link between resilience and individual and social well-being—and the high economic costs associated with mental illness. An exception is the contribution of Etilé, Frijters, Johnston and Shields (2017) who propose an empirical measure of resilience estimating a dynamic finite-mixture model for the Australian population. These authors derive individual-specific values of the parameters that govern individual heterogeneity in the psychological response to ten major adverse events and identify three classes of individuals that differ in their responses to the events. We note that their approach is firmly based on empirical issues.

Measures of individual resilience have been proposed by psychologists in the form of measurement scales, such as the Connor-Davidson Resilience Scale (CD-RISC); for detailed reviews of resilience measurement scales see, for instance, Windle, Bennett and Noyes (2011) and Salisu and Hashim (2017). The CD-RISC scale is based on 25 items evaluated on a five-point Likert scale ranging from 0 (never) to 4 (nearly all of the time). These ratings are added across the 25 categories to arrive at a total score between 0 and 100, and higher scores indicate higher resilience. The scores themselves are obtained from individual answers to questions on the ability to adapt to change, the availability of close and secure relationships, the preference to take the lead in problem solving, and similar attributes.

In this paper, we provide an axiomatic approach to the measurement of resilience, thereby complementing the empirically oriented contributions alluded to above with a thorough theoretical analysis. We are not aware of any earlier work that addresses the theoretical foundations of measuring resilience and we hope that our observations provide a substantial step towards filling this gap. Thus, our work plays a role similar to that of Esteban and Ray's (1994) seminal paper on the measurement of polarization. The notion of polarization had been discussed in the literature prior to the publication of their article but Esteban and Ray's (1994) is the first contribution that provides a systematic theoretical examination of the phenomenon. Analogously, Bossert and D'Ambrosio (2013) represents the first axiomatic analysis of economic insecurity, a term that had been used with increasing frequency in the previous literature but had not been subjected to a thorough theoretical treatment.

Our starting point is the notion of a *health stream,* that is, a stream of values of individual health variables over time. These streams could be obtained by means of the mental health component of the Short-Form 12 Health Survey (SF-12), for instance, but our results are applicable to more general methods. In the illustrations of our measure, the health streams we use consist of self-assessed health status and satisfaction with own health. The objective is to establish an ordering defined on these streams that ranks them with respect to their relative resilience. Thus, we employ an ordinal interpretation of the notion of resilience. We propose a set of intuitively appealing properties of a resilience ordering and it turns out that there is a single specific ordering that satisfies all of them. Although our proposal is ordinal in nature (and, thus, statements regarding arithmetic means or similar statistics are not well-defined in our setting), there are numerous aggregation methods that can be employed, including those involving the ranks of individuals or any quantiles that permit us to draw conclusions regarding aggregate resilience. We note that our ordinal measure is attractive because it can be applied in empirical studies due to the availability of numerous datasets that are suited to our approach, such as the German Socio-Economic Panel Study (SOEP). Ordinal approaches to social index numbers are rather common in the theory of social index numbers. For instance, an ordinal approach to poverty measurement is presented by Sen (1976), and Blackorby and Donaldson (1984) and Ebert (1987) discuss ordinal inequality indices.

In the following section, we provide a detailed discussion and several examples that illustrate the information health streams are intended to convey in the context of assessing individual resilience. We then identify the notion of a *down spell* in Section 3, interpreted as a set of time periods during which an adverse event occurs and a subsequent (partial or full) recovery may or may not occur. These down spells form the foundation of our resilience measure. The formal definition of resilience orderings in general and of our specific proposal is given in Section 4. The axioms (properties) that we impose on a resilience measure are introduced and discussed in Section 5, and Section 6 contains our result—a characterization of the resilience ordering that possesses all of our properties. We conclude in Section 7 and establish the independence of our axioms in an appendix.

2 Preliminary observations and examples

As alluded to in the introduction, resilience is a phenomenon that is of increasing importance in many areas of research. While our approach and our results are applicable to a variety of settings, we focus on psychological aspects in order to work with a concrete example.

Our starting point is the choice of a socio-economic variable that we want to assess with respect to the notion of resilience. We assume that the data for which the requisite comparisons are to be performed consist of observed health values for a number of consecutive time periods. The term resilience is intended to capture the ability of an individual to recover quickly from adverse events. In order to exclude degenerate cases, we restrict attention to streams of health variables that cover at least three time periods. The finite length of a stream (and, thus, the number of observations) is denoted by T so that $T \in \mathbf{T}$, where $\mathbf{T} = \mathbb{N} \setminus \{1, 2\}$ is the set of positive integers excluding the numbers 1 and 2. For a possible stream length $T \in \mathbf{T}$, a health stream $x = (x_1, \ldots, x_T)$ is composed of T observations, one for each period from 1 to T. We assume that the observed health values are non-negative so that x is an element of $\mathbb{R}^T_+$, the set of all T-dimensional vectors with non-negative components. The length of a stream may vary so that the set from which x is chosen is the union $\cup_{T \in \mathbf{T}} \mathbb{R}^T_+$. We will introduce some restrictions on the set of possible streams that are in line with our interpretation and our objective of establishing a resilience ranking.

An important observation is that the variables we consider have to be interpreted in a way so that the resilience ordering to be established is invariant with respect to *increasing affine* transformations of the health variable but not with respect to *arbitrary* (not necessarily affine) increasing transformations. This assumption is needed to ensure that some of our axioms are meaningful. We note that this is a common and largely uncontroversial requirement that is (at least implicitly) made in most approaches to social index numbers, such as inequality or poverty orderings that are based on individual incomes.

To illustrate the basic information available, we now provide a few examples. Intuitively, our central hypothesis is that resilience is captured by assessing how quickly an individual recovers from an adverse event. In the context of psychological resilience, an adverse event is represented by a drop in self-reported health and the recovery (if any) is indicated by examining the extent to which and the speed with which the individual regains the pre-drop health level. Thus, we will focus on what we refer to as *down spells,* that is, the behavior of a stream x during a phase in which a sustained health level drops to a lower level and the subsequent (possibly partial) recovery that may or may not occur. (The symbols $a(\sigma^x)$ and $b(\sigma^x)$ in the following figures are meant to indicate the amplitude and the recovery delay in a given down spell but they can be safely ignored for the time being; they will be introduced and discussed in detail later on.)

Consider a stream of perceived health values as in the example of Figure 1. In the example, the starting point is a health score of 3, followed by a drop to 1 in the move from period 1 to period 2. There is a transition without a change from period 2 to period 3, and finally there is a recovery in the move from period 3 to period 4, at which point we are back at the initial health level of 3. The drop is interpreted as an adverse event, the return to the pre-drop level represents the recovery that occurs after remaining at the lower level for one time period. In this example, the individual recovers from the drop but the recovery does not occur with the highest possible speed.

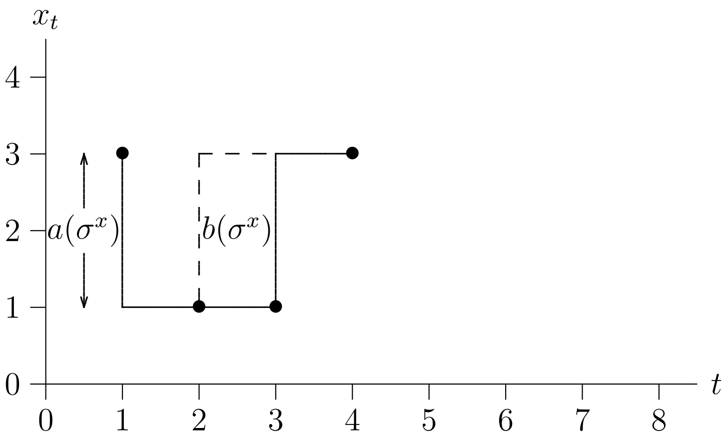

Figure 1: The health stream $x = (3, 1, 1, 3)$.

In Figure 2, the decline from 3 to 1 is slower. A drop from 3 to 2 occurs in period 1, followed by a drop from 2 to 1 in period 2. Recovery takes place as in Figure 1.

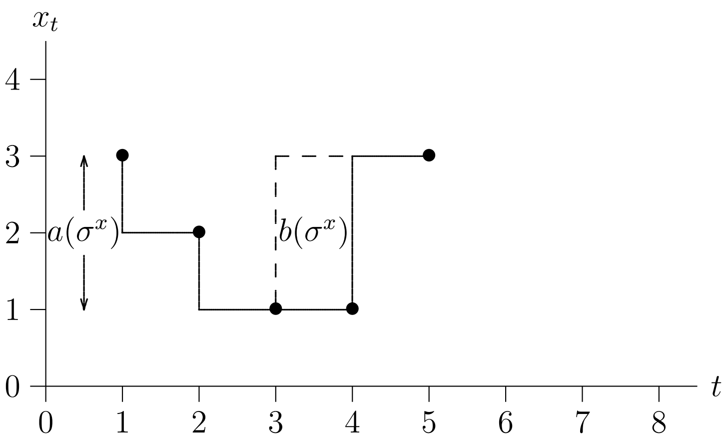

Figure 2: The health stream $x = (3, 2, 1, 1, 3)$.

In Figure 3, the speed of recovery is higher as compared to that in Figures 1 and 2—we return to the original level of 3 immediately after the completion of the downwards movement. A natural interpretation is to say that the individual of Figure 3 is more resilient than that of Figures 1 and 2 because recovery occurs at a higher speed.

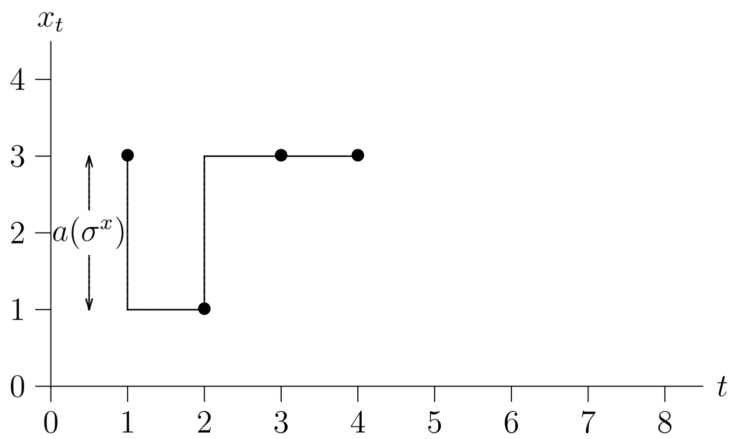

Figure 3: The health stream $x = (3, 1, 3, 3)$.

In Figure 4, there is an initial partial recovery from 2 to 3 in period 3 after a drop from 4 to 2 in the move from period 1 to period 2. This is followed in period 4 by a drop below the level of the initial downwards movement, after which a full recovery occurs in period 5. Because the recovery in the move from period 2 to period 3 is only partial, the down spell is not considered completed until we return to the pre-drop level of 4 two periods later. This contrasts with the example of Figure 7 below in which a full recovery occurs before there is a second down spell.

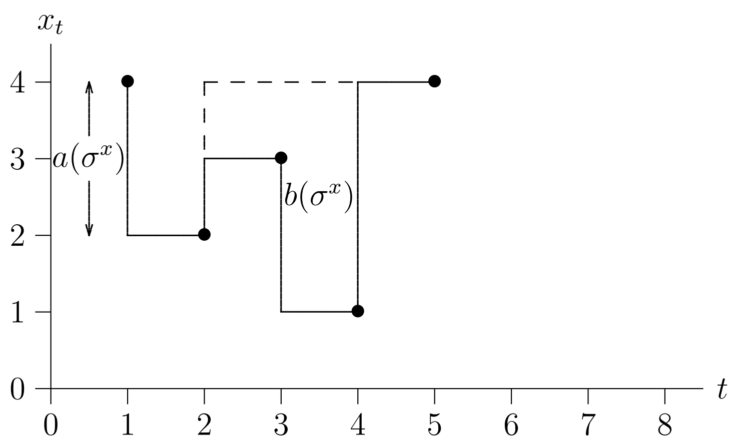

Figure 4: The health stream $x = (4, 2, 3, 1, 4)$.

In Figure 5, there is an excess recovery because the individual ends up at a level that exceeds the pre-drop level.

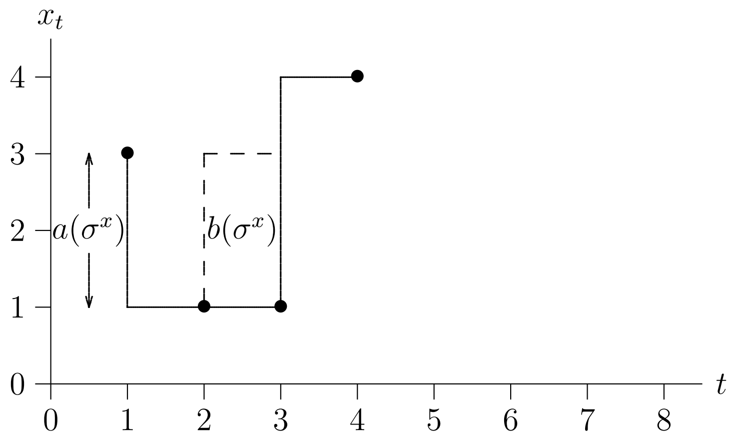

Figure 5: The health stream $x = (3, 1, 1, 4)$.

In Figure 6, there is no recovery. After the drop from 3 to 1, the score remains at 1 until the final period 4.

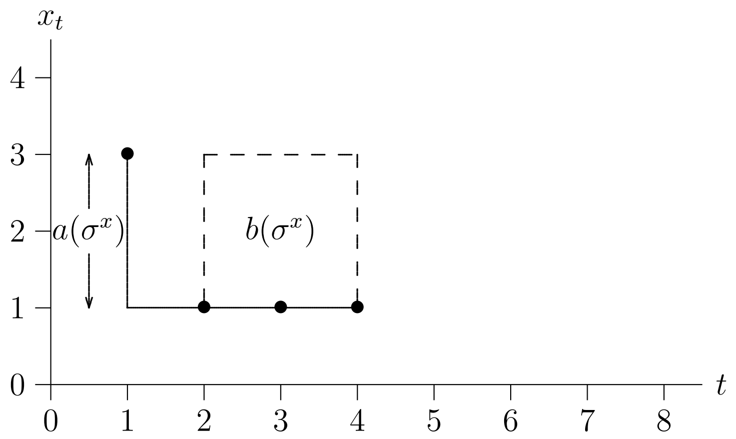

Figure 6: The health stream $x = (3, 1, 1, 1)$.

Figure 7 illustrates a stream with two down spells. The first spell involves a drop from 3 to 2 in period 1, followed by an immediate recovery. In the second spell, perceived health drops from 3 to 1 in period 4 and the full recovery occurs in period 8. As mentioned when discussing Figure 4, we now have a full recovery in period 3 and, therefore, a second down spell occurs two periods later when there is a drop from 3 to 1.

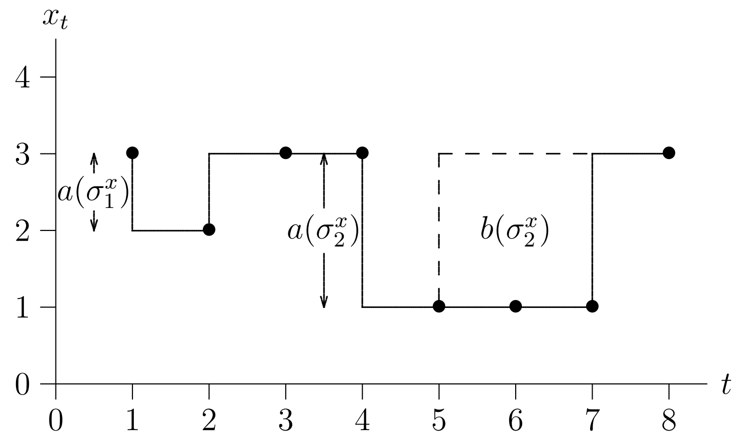

Figure 7: The health stream $x = (3, 2, 3, 3, 1, 1, 1, 3)$.

3 Down spells

Our approach to measuring resilience is ordinal in nature. That is, the resilience values we assign to the health-variable streams are only meant to make comparative statements such as 'stream x reflects more resilience than stream y' or 'the resilience of stream x and the resilience of stream y are equal.' In particular, the resilience values are not numerically significant so that statements such as 'x is associated with twice as much resilience as stream y' are not meaningful under such an ordinal interpretation.

As the examples in the previous section establish, health streams can be illustrated in a plausible and relatively straightforward manner. Although the precise definition of the down spells that form the foundation of our approach is somewhat more involved, it can be explained in terms of intuitively appealing concepts.

Consider a stream x of length T. To identify the down spells that are present in the stream x, we begin by partitioning the full set of time periods $\{1, \ldots, T\}$ into three sets. These three sets represent (i) the periods associated with sustained health; (ii) the time periods in which down spells occur; and (iii) the set of time periods in which (possibly partial) recoveries may or may not occur.

We denote the set of periods in x with sustained health by $\mathbf{S}^x$. The idea is to include, starting from the first period, all time periods among those in $\{1, \ldots, T\}$ that are associated with maximal non-decreasing values in the health variable. Hence, if a decrease in the health level occurs, the individual no longer experiences sustained health. Formally, this set $\mathbf{S}^x$ is defined inductively as follows. The initial period (period 1) always belongs to this set so that we have $1 \in \mathbf{S}^x$. Now let $t \in \{2, \ldots, T\}$ and assume that we have examined each of periods 1 to $t-1$ to determine whether it is a member of $\mathbf{S}^x$. If $x_t \geq x_\tau$ for all $\tau \in \mathbf{S}^x \cap \{1, \ldots, t-1\}$, then $t \in \mathbf{S}^x$. Thus, if we reach period t and the level of health does not drop when considering those periods between 1 and $t-1$ that are already in the set $\mathbf{S}^x$, then period t is added to this set of periods with sustained health; if x_t is below one the values of $x_1, \ldots, x_{t-1}$ that have already been added to $\mathbf{S}^x$ in a previous step of the iteration, then t does not belong to the set of periods with sustained health.

To illustrate this construction, consider the health stream $x = (3, 2, 1, 1, 3) \in \mathbb{R}^5_+$ of Figure 2. According to the iterative procedure just defined, the initial period 1 is always in the set of periods with sustained health, that is, we have $1 \in \mathbf{S}^x$. To determine whether period $t = 2$ is in $\mathbf{S}^x$, we compare x_2 to the values that correspond to the periods that have already been added to $\mathbf{S}^x$. In this case, the only comparison is that involving x_1 because

$$\mathbf{S}^x \cap \{1, \ldots, t-1\} = \mathbf{S}^x \cap \{1\} = \{1\}.$$

We have $x_2 = 2 < 3 = x_1$ and, therefore, $2 \notin \mathbf{S}^x$. The same is true for periods 3 and 4. For $t = 3$, we have

$$\mathbf{S}^x \cap \{1, \ldots, t-1\} = \mathbf{S}^x \cap \{1, 2\} = \{1\}$$

and $x_3 = 1 < 3 = x_1$ so that, according to our definition, $3 \notin \mathbf{S}^x$. For $t = 4$, it follows that

$$\mathbf{S}^x \cap \{1, \ldots, t-1\} = \mathbf{S}^x \cap \{1, 2, 3\} = \{1\}$$

and $x_4 = 1 < 3 = x_1$ so that, again, $4 \notin \mathbf{S}^x$. The final candidate for membership in $\mathbf{S}^x$ is period $t = T = 5$. It follows that

$$\mathbf{S}^x \cap \{1, \ldots, t-1\} = \mathbf{S}^x \cap \{1, \ldots, 4\} = \{1\}$$

and $x_5 = 3 \geq 3 = x_1$ so that, by definition, $5 \in \mathbf{S}^x$ and hence $\mathbf{S}^x = \{1, 5\}$.

As another example, consider the stream $x = (3, 2, 3, 3, 1, 1, 1, 3) \in \mathbb{R}^T_+$; see Figure 7. Again, the starting point of the inductive procedure is to declare the initial period to be a member of the set of periods associated with sustained health, that is, $1 \in \mathbf{S}^x$. The only comparison required to determine whether period $t = 2$ belongs to $\mathbf{S}^x$ is that between x_2 and x_1 because

$$\mathbf{S}^x \cap \{1, \ldots, t-1\} = \mathbf{S}^x \cap \{1\} = \{1\}.$$

We have $x_2 = 2 < 3 = x_1$ so that $2 \notin \mathbf{S}^x$. For $t = 3$, we obtain

$$\mathbf{S}^x \cap \{1, \ldots, t-1\} = \mathbf{S}^x \cap \{1, 2\} = \{1\}$$

and, because $x_3 = 3 \geq 3 = x_1$, it follows that $3 \in \mathbf{S}^x$. Moving on to period $t = 4$, we have

$$\mathbf{S}^x \cap \{1, \ldots, t-1\} = \mathbf{S}^x \cap \{1, 2, 3\} = \{1, 3\}.$$

We obtain $x_4 = 3 \geq 3 = x_1 = x_3$ and, therefore, period 4 must be added to the set $\mathbf{S}^x$. Continuing the iteration, we obtain $5 \notin \mathbf{S}^x$ because $x_5 = 1$ is less than $x_1 = 3$ (and also less than x_3 and x_4) and, for the same reason, $6 \notin \mathbf{S}^x$ and $7 \notin \mathbf{S}^x$. Finally, for $t = T = 8$, we obtain

$$\mathbf{S}^x \cap \{1, \ldots, t-1\} = \mathbf{S}^x \cap \{1, \ldots, 7\} = \{1, 3, 4\}$$

and, because $x_8 = 3 \geq 3 = x_1 = x_3 = x_4$, it follows that $x_8 \in \mathbf{S}^x$ which leads us to the set $\mathbf{S}^x = \{1, 3, 4, 8\}$ of periods with sustained health in x.

Next, we define the set $\mathbf{D}^x$ of time periods in which down spells occur for a stream x. The construction of this set is very intuitive: a period t is part of a down spell if there is an earlier period τ in which there is sustained health such that the value of the health variable is in decline between τ and t. That is, for any $t \in \{1, \ldots, T\}$, $t \in \mathbf{D}^x$ if there exists a period $\tau \in \mathbf{S}^x \cap \{1, \ldots, t-1\}$ such that $x_\tau > \cdots > x_t$. It follows by definition that the set of periods that involve sustained health and the set of periods in which down spells occur must be disjoint. We use the streams of Figures 2 and 7 again to provide illustrations of this definition of $\mathbf{D}^x$.

In the stream $x = (3, 2, 1, 1, 3)$ depicted in Figure 2, we have $\mathbf{S}^x = \{1, 5\}$. Consider first the period $t = 2$. Because there exists a period $\tau \in \mathbf{S}^x \cap \{1, \ldots, t-1\} = \{1\}$ (namely, period $\tau = 1$) such that

$$x_\tau = x_1 = 3 > 2 = x_2,$$

it follows that period 2 is in $\mathbf{D}^x$. Moreover, because $1 \in \mathbf{S}^x \cap \{1, 2\}$ and

$$x_\tau = x_1 = 3 > 2 = x_2 > 1 = x_3,$$

period $t = 3$ must be a member of $\mathbf{D}^x$ as well. Because $x_3 = 1 \leq 1 = x_4$, the last inequality that defines membership in $\mathbf{D}^x$ is not satisfied for period 4 and, therefore, $4 \notin \mathbf{D}^x$. Thus, we obtain $\mathbf{D}^x = \{2, 3\}$ for this example.

For the stream $x = (3, 2, 3, 3, 1, 1, 1, 3)$ of Figure 7, we have $\mathbf{S}^x = \{1, 3, 4, 8\}$. Because

$$x_\tau = x_1 = 3 > 2 = x_2,$$

it follows that $t = 2$ belongs to the set $\mathbf{D}^x$. The next possible candidate for membership in $\mathbf{D}^x$ is period $t = 5$. There exists a period $\tau \in \mathbf{S}^x \cap \{1, \ldots, 4\} = \{1, 3, 4\}$ (namely, period $\tau = 4$) such that

$$x_\tau = x_4 = 3 > 1 = x_5$$

and it follows that period 5 is in $\mathbf{D}^x$. Because $x_5 = 1 \leq 1 = x_6 = x_7$, periods 6 and 7 are not associated with down spells and, therefore, $6 \notin \mathbf{D}^x$ and $7 \notin \mathbf{D}^x$. Therefore, we obtain $\mathbf{D}^x = \{2, 5\}$.

Finally, to complete the description of our partition, the set of periods $\mathbf{U}^x$ is defined as the complement of the union $\mathbf{S}^x \cup \mathbf{D}^x$ in $\{1, \ldots, T\}$, that is,

$$\mathbf{U}^x = \{1, \ldots, T\} \setminus (\mathbf{S}^x \cup \mathbf{D}^x).$$

We refer to the set $\mathbf{U}^x$ as the recovery phase, that is, the set of time periods during which recovery occurs. Note that $\mathbf{U}^x$ may be empty for some streams x. It is immediate that, in the example of Figure 2, we obtain

$$\mathbf{U}^x = \{1, \ldots, T\} \setminus (\mathbf{S}^x \cup \mathbf{D}^x) = \{1, \ldots, 5\} \setminus (\{1, 5\} \cup \{2, 3\}) = \{4\}$$

and, for the example illustrated in Figure 7, it follows that

$$\mathbf{U}^x = \{1, \ldots, T\} \setminus (\mathbf{S}^x \cup \mathbf{D}^x) = \{1, \ldots, 8\} \setminus (\{1, 3, 4, 8\} \cup \{2, 5\}) = \{6, 7\}.$$

With the partition $\{\mathbf{S}^x, \mathbf{D}^x, \mathbf{U}^x\}$ of $\{1, \ldots, T\}$ in hand, we can now proceed to a precise definition of a down spell. As seems natural, a down spell in stream x starts in a period in the set $\mathbf{S}^x$ of sustained health if the following period belongs to the set $\mathbf{D}^x$ in which down spells occur. Clearly, the number of down spells and their exact structure are stream-dependent. For the time being, we use the notation σ^x to indicate a generic down spell in x, without explicitly referring to the number of spells in a stream at this stage.

As hinted at above, a down spell σ^x in x starts in period $s(\sigma^x) \in \mathbf{S}^x$ if

$$(s(\sigma^x) + 1) \in \mathbf{D}^x.$$

For example, if $x = (3, 2, 1, 1, 3)$ as in Figure 2, it follows that $s(\sigma^x) = 1$ for the single spell σ^x in x because $(s(\sigma^x) + 1) = 2 \in \mathbf{D}^x$. Analogously, the stream $x = (3, 2, 3, 3, 1, 1, 1, 3)$ of Figure 7 has two down spells that start at $s(\sigma_1^x) = 1$ and at $s(\sigma_2^x) = 4$.

To identify the duration of a down spell σ^x, we use the following definition. If

$$\{s(\sigma^x) + 1, \ldots, d(\sigma^x)\} \subseteq \mathbf{D}^x \ \text{ and } \ (d(\sigma^x) + 1) \notin \mathbf{D}^x,$$

then the down spell σ^x ends at $d(\sigma^x)$. Note that this includes the possibility that $d(\sigma^x) = T$ if σ^x is the final down spell in the stream x. Thus, the down spell σ^x consists of the time periods in the set

$$D(\sigma^x) = \{s(\sigma^x) + 1, \ldots, d(\sigma^x)\}.$$

As is straightforward to verify, in the case of $x = (3, 2, 1, 1, 3)$, we obtain $d(\sigma^x) = 2$ and, for $x = (3, 2, 3, 3, 1, 1, 1, 3)$, it follows that $d(\sigma_1^x) = 2$ and $d(\sigma_2^x) = 5$.

If there exists $u(\sigma^x) \in \{d(\sigma^x), \ldots, T - 1\}$ such that

$$\mathbf{S}^x \cap \{d(\sigma^x) + 1, \ldots, u(\sigma^x)\} = \emptyset \ \text{ and } \ (u(\sigma^x) + 1) \in \mathbf{S}^x,$$

then a full recovery after the down spell σ^x occurs in time period $u(\sigma^x) + 1$. The set $U(\sigma^x)$ consists of the time periods after the down spell σ^x has finished and before a full recovery (if any) has occurred, that is,

$$U(\sigma^x) = \{d(\sigma^x) + 1, \ldots, u(\sigma^x)\},$$

where $u(\sigma^x) < T$ if a full recovery occurs and $u(\sigma^x) = T$ if a full recovery does not occur. In particular, $U(\sigma^x) = \emptyset$ if $d(\sigma^x) = u(\sigma^x)$ so that recovery is immediate when $u(\sigma^x) < T$ and no recovery is feasible when $u(\sigma^x) = T$ owing to the constraint imposed by reaching the final time period T. In the example of Figure 2, we obtain $u(\sigma^x) = 4$; for Figure 7, the requisite time periods are $u(\sigma_1^x) = 2$ and $u(\sigma_2^x) = 7$.

The severity of the down spell σ^x is measured by

$$a(\sigma^x) = x_{s(\sigma^x)} - x_{d(\sigma^x)}$$

where the letter a is associated with the 'a' in *a*mplitude of the down spell. Note that the length of a decline does not matter, only the amplitude. The delay in recovery after the down spell σ^x is measured by

$$b(\sigma^x) = \sum\nolimits_{t \in U(\sigma^x)} \left(x_{s(\sigma^x)} - x_t \right)$$

where the letter b represents the 'b' in coming *b*ack.

We restrict attention to streams with at least one down spell for which (partial) recovery is not made infeasible by the time constraint—that is, after a drop from a sustained level of health, there is at least one time period left before the final period T is reached. Let $T \in \mathbf{T}$ and $x \in \mathbb{R}^T_+$, and consider all down spells σ^x in x for which recovery is not made infeasible so that $d(\sigma^x) < T$. To exclude trivial cases, we only consider streams that contain at least one such down spell.

Now denote the number of such down spells by m^x and the i^{th} of these spells by σ^x_i. Define

$$\Sigma(x) = \{\sigma^x_1, \ldots, \sigma^x_{m^x}\}$$

as the set of all down spells for which recovery is not made infeasible by the time constraint. By assumption, this set is non-empty. To simplify our exposition, we write σ^x instead of σ^x_1 if $m^x = 1$, that is, if there is only one permissible down spell in the stream x; this does not create any ambiguity. The set H_T defined by

$$H_T = \{x \in \mathbb{R}^T_+ \mid \Sigma(x) \neq \emptyset\}$$

contains all streams of length $T \in \mathbf{T}$ for which the notion of resilience is well-defined in the sense that recovery is not excluded by reaching the end of the sampling period T. Because T may be any integer greater than or equal to three, the set of such streams of any length is given by

$$\Omega = \bigcup_{T \in \mathbf{T}} H_T.$$

Thus, the set Ω constitutes the set of streams that we want to be able to compare by means of what we refer to as a *resilience ordering.*

4 Resilience orderings

A resilience ordering is a complete and transitive binary relation $\succsim$ defined on Ω with the interpretation 'at-least-as-resilient-as.' Thus, for any two streams x and y in Ω, the statement 'x is at least as resilient as y' is expressed by the relational statement $x \succsim y$. The relation $\succsim$ is complete if any two streams x and y in Ω can be compared, that is, if

$$x \succsim y \text{ or } y \succsim x$$

for all $x, y \in \Omega$. Transitivity requires that if $x \succsim y$ and $y \succsim z$ for any three streams $x, y, z \in \Omega$, it must also be true that $x \succsim z$. The relation $\succsim$ can be partitioned into a 'more-resilient-than' relation $\succ$ and an 'as-resilient-as' relation $\sim$, defined by letting, for all $x, y \in \Omega$,

$$x \succ y \text{ if } [x \succsim y \text{ and not } y \succsim x]$$

and

$$x \sim y \text{ if } [x \succsim y \text{ and } y \succsim x].$$

The specific resilience ordering $\succsim^r$ that we propose and characterize in this paper is based on comparing the values of a resilience measure $r\colon \Omega \to (0,1]$ that is defined in terms of the amplitudes $a(\sigma_i^x)$ and the recovery delays $b(\sigma_i^x)$ associated with the spells that are present in a stream $x \in \Omega$. This resilience measure is defined by letting, for all $x \in \Omega$,

$$r(x) = \frac{\sum_{i=1}^{m^x} a(\sigma_i^x)}{\sum_{i=1}^{m^x} a(\sigma_i^x) + \sum_{i=1}^{m^x} b(\sigma_i^x)}.$$

The measure reflects how quickly an individual recovers from adverse events. An adverse occurrence is represented by a drop in the value of the health variable and recovery is interpreted as a subsequent (not necessarily immediate) return to the pre-drop level. Thus, r increases with the amplitude of a spell and decreases with the recovery delay: a *ceteris-paribus* (partial) recovery from a more severe drop is associated with higher resilience, and a *ceteris-paribus* longer recovery delay means that resilience is lower. Clearly, $r(x)$ takes on values that are greater than zero and smaller than or equal to one because $a(\sigma_i^x)$ is positive and $b(\sigma_i^x)$ is non-negative. Furthermore, $r(x) = 1$ if recovery is always immediate, that is, if $u(\sigma_i^x) = d(\sigma_i^x)$ for all $i \in \{1, \ldots, m^x\}$.

Our resilience ordering $\succsim^r$ is now defined by declaring $x \in \Omega$ to be at least as resilient as $y \in \Omega$ if the value of the resilience measure r at x is greater than or equal to the value of r at y. That is, for all $x, y \in \Omega$,

$$x \succsim^r y \;\Leftrightarrow\; r(x) \geq r(y).$$

We reiterate that the resilience measure r does *not* have any numerical significance—no comparisons other than relative resilience rankings are permissible in the ordinal setting considered throughout this paper.

The notion of *vulnerability* may be defined as the inverse of resilience, that is, as the value of a function $v\colon \Omega \to [1, \infty)$ defined by

$$v(x) = \frac{1}{r(x)}$$

for all $x \in \Omega$. Thus, for our particular measure, we obtain

$$v(x) = \frac{\sum_{i=1}^{m^x} a(\sigma_i^x) + \sum_{i=1}^{m^x} b(\sigma_i^x)}{\sum_{i=1}^{m^x} a(\sigma_i^x)} = \sum_{i=1}^{m^x} \left(\frac{a(\sigma_i^x)}{\sum_{j=1}^{m^x} a(\sigma_j^x)} \cdot \frac{a(\sigma_i^x) + b(\sigma_i^x)}{a(\sigma_i^x)} \right)$$

for all $x \in \Omega$, where $a(\sigma_i^x)/(\sum_{j=1}^{m^x} a(\sigma_j^x))$ is the endogenous weight given to down spell σ_i^x, and $(a(\sigma_i^x) + b(\sigma_i^x))/a(\sigma_i^x)$ is the vulnerability exhibited in down spell σ_i^x. Thus, each spell is weighted according to its fraction of the total amplitude—the sum of the amplitudes over all spells. The vulnerability ordering associated with our resilience ordering $\succsim^r$ is simply its reverse ordering, that is, $x \in \Omega$ is at least as vulnerable as $y \in \Omega$ if $y \succsim^r x$.

The resilience measure r and its inverse v have an intuitive geometric interpretation in the case of a single spell σ^x. The distance $a(\sigma^x)$—the amplitude—is an indicator of the severity of the shock represented by the drop, and the area is $b(\sigma^x)$ the recovery delay. Thus, it seems natural to use the expression

$$\frac{a(\sigma^x)}{a(\sigma^x) + b(\sigma^x)}$$

as the value of our measure of resilience. Analogously, its inverse

$$\frac{a(\sigma^x) + b(\sigma^x)}{a(\sigma^x)}$$

is the associated index of vulnerability.

Now we apply the resilience and vulnerability measures to our examples. Consider first Figure 1. The stream of health values is given by $x = (3, 1, 1, 3)$. We have $\mathbf{S}^x = \{1, 4\}$, $\mathbf{D}^x = \{2\}$ and $\mathbf{U}^x = \{3\}$. There is a singleton set $\Sigma(x) = \{\sigma^x\}$ of down spells for which recovery is not made impossible by the end-of-sample constraint, with $s(\sigma^x) = 1$, $d(\sigma^x) = 2$ and $u(\sigma^x) = 3$. Furthermore, we have $a(\sigma^x) = 2$ and $b(\sigma^x) = 2$ so that

$$r(x) = \frac{2}{2+2} = \frac{1}{2} \text{ and } v(x) = \frac{1}{r(x)} = 2.$$

In Figure 2, we have $x = (3, 2, 1, 1, 3)$. It follows that $\mathbf{S}^x = \{1, 5\}$, $\mathbf{D}^x = \{2, 3\}$ and $\mathbf{U}^x = \{4\}$. There is a singleton set $\Sigma(x) = \{\sigma^x\}$ of down spells with $s(\sigma^x) = 1$, $d(\sigma^x) = 3$ and $u(\sigma^x) = 4$. We obtain $a(\sigma^x) = 2$ and $b(\sigma^x) = 2$ so that, as in the example of Figure 1,

$$r(x) = \frac{2}{2+2} = \frac{1}{2} \text{ and } v(x) = \frac{1}{r(x)} = 2.$$

In the example of Figure 3, the requisite stream is $x = (3, 1, 3, 3)$. We obtain $\mathbf{S}^x = \{1, 3, 4\}$, $\mathbf{D}^x = \{2\}$ and $\mathbf{U}^x = \emptyset$. There is a singleton set $\Sigma(x) = \{\sigma^x\}$ of down spells with $s(\sigma^x) = 1$, $d(\sigma^x) = 2$ and $u(\sigma^x) = 2$. Furthermore, we have $a(\sigma^x) = 2$ and $b(\sigma^x) = 0$ so that

$$r(x) = \frac{2}{2+0} = 1 \text{ and } v(x) = \frac{1}{r(x)} = 1.$$

In Figure 4, we have $x = (4, 2, 3, 1, 4)$. It follows that $\mathbf{S}^x = \{1, 5\}$, $\mathbf{D}^x = \{2\}$ and $\mathbf{U}^x = \{3, 4\}$. There is a singleton set $\Sigma(x) = \{\sigma^x\}$ of down spells with $s(\sigma^x) = 1$, $d(\sigma^x) = 2$ and $u(\sigma^x) = 4$. We obtain $a(\sigma^x) = 2$ and $b(\sigma^x) = 1 + 3 = 4$ so that

$$r(x) = \frac{2}{2+4} = \frac{1}{3} \text{ and } v(x) = \frac{1}{r(x)} = 3.$$

In the example of Figure 5, the requisite stream is $x = (3, 1, 1, 4)$. We obtain $\mathbf{S}^x = \{1, 4\}$, $\mathbf{D}^x = \{2\}$ and $\mathbf{U}^x = \{3\}$. There is a singleton set $\Sigma(x) = \{\sigma^x\}$ of down spells with $s(\sigma^x) = 1$, $d(\sigma^x) = 2$ and $u(\sigma^x) = 3$. Furthermore, we have $a(\sigma^x) = 2$ and $b(\sigma^x) = 2$ so that

$$r(x) = \frac{2}{2+2} = \frac{1}{2} \text{ and } v(x) = \frac{1}{r(x)} = 2.$$

There is no recovery in Figure 6. The stream of health values is $x = (3, 1, 1, 1)$. We have $\mathbf{S}^x = \{1\}$, $\mathbf{D}^x = \{2\}$ and $\mathbf{U}^x = \{3, 4\}$. There is a singleton set $\Sigma(x) = \{\sigma^x\}$ of down spells with $s(\sigma^x) = 1$, $d(\sigma^x) = 2$ and $u(\sigma^x) = 4$. It follows that $a(\sigma^x) = 2$ and $b(\sigma^x) = 2 + 2 = 4$ so that

$$r(x) = \frac{2}{2+4} = \frac{1}{3} \text{ and } v(x) = \frac{1}{r(x)} = 3.$$

In the case of two down spells as depicted in Figure 7, the stream of health values is $x = (3, 2, 3, 3, 1, 1, 1, 3)$. We have $\mathbf{S}^x = \{1, 3, 4, 8\}$, $\mathbf{D}^x = \{2, 5\}$ and $\mathbf{U}^x = \{6, 7\}$. There is a set of two down spells $\Sigma(x) = \{\sigma_1^x, \sigma_2^x\}$. For the first spell, we obtain $s(\sigma_1^x) = 1$, $d(\sigma_1^x) = 2$ and $u(\sigma_1^x) = 2$. It follows that $a(\sigma_1^x) = 1$ and $b(\sigma_1^x) = 0$. The requisite numbers for the second spell are $s(\sigma_2^x) = 4$, $d(\sigma_2^x) = 5$, $u(\sigma_2^x) = 7$, $a(\sigma_2^x) = 2$ and $b(\sigma_2^x) = 2 + 2 = 4$. Thus, we obtain

$$r(x) = \frac{1+2}{1+2+0+4} = \frac{3}{7} \text{ and } v(x) = \frac{1}{r(x)} = \frac{7}{3}.$$

Thus, according to our resilience ordering $\succsim^r$, the most resilient stream is that of Figure 3 (with a resilience value of 1), followed by those in Figures 1,2 and 5 (with a resilience of 1/2). Next, we have Figure 7 with a resilience of 3/7 and, at the bottom, the least resilient (and thus most vulnerable) ones in Figures 4 and 6 with a resilience level of 1/3.

5 Properties of a resilience ordering

In our characterization, we focus on the restriction of the resilience ordering to streams with a single down spell. Thus, we define the sets

$$H_T^1 = \{x \in H_T \mid |\Sigma(x)| = 1\}$$

for all $T \in \mathbf{T}$, and the domain considered in our main result is given by

$$\Omega^1 = \{x \in \Omega \mid |\Sigma(x)| = 1\}.$$

The restriction of a resilience ordering to streams with a single down spell is referred to as a single-spell resilience ordering.

For some of our properties, it is convenient to employ the following definition and notation. Two streams $x, y \in H_T^1$ *have the same timing structure* if $\mathbf{S}^x = \mathbf{S}^y$, $\mathbf{D}^x = \mathbf{D}^y$ and $\mathbf{U}^x = \mathbf{U}^y$. If $x, y \in H_T^1$ have the same timing structure, we write $s := s(\sigma^x) = s(\sigma^y)$, $d := d(\sigma^x) = d(\sigma^y)$, $u := u(\sigma^x) = u(\sigma^y)$, and $\mathbf{U} := \mathbf{U}^x = \mathbf{U}^y$. Note that, in this case, $\mathbf{U} = \{d + 1, \ldots, u\}$ and $|\mathbf{U}| = u - d$.

5.1 Recovery neutrality

We begin with a property that ensures that all periods in the recovery phase are treated equally by our measure. This implies, in particular, that no discounting can be employed. Thus, if the order of the health-variable values that occur during recovery is changed, this is a matter of equal resilience. In other words, the property ensures that our measure treats all time periods in which recovery occurs equally, paying no attention to the order in which the requisite health-variable values appear in a stream.

Recovery neutrality. For all $T \in \mathbf{T}$ and for all $x, y \in H_T^1$ with the same timing structure, if $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$ and $(y_\tau)_{\tau \in \mathbf{U}}$ is a permutation of $(x_\tau)_{\tau \in \mathbf{U}}$, then

$$x \sim y.$$

To illustrate this property, consider the streams $x = (5, 1, 2, 4, 3, 5)$ and $y = (5, 1, 3, 2, 4, 5)$. The two streams have the same timing structure with $\mathbf{U} = \{3, 4, 5\}$ and, because $y_3 = 3 = x_5$, $y_4 = 2 = x_3$ and $y_5 = 4 = x_4$, it follows that (y_3, y_4, y_5) is obtained from permuting (x_3, x_4, x_5). Thus, recovery neutrality requires that

$$(5, 1, 2, 4, 3, 5) = x \sim y = (5, 1, 3, 2, 4, 5).$$

5.2 Recovery translation invariance

Translation invariance is a commonly-imposed condition in the design of social index numbers; for example, absolute measures of inequality such as those of Kolm (1976) or Blackorby and Donaldson (1980) are translation invariant. In our setting, the property is defined for pairs of streams with the same timing structure. Although the label translation invariance typically refers to situations in which the same value is added to all components, our version goes beyond that by allowing these additions to be specific to each time period. Nevertheless, we use the term translation invariance because it captures the motivation underlying the axiom. Note that the property is analogous to the axiom of *independence of income source* employed by Weymark (1981) in the context of social welfare functions and inequality measures defined on income distributions. See also Blackorby, Bossert and Donaldson (2005, p. 118) who use a related axiom that they label *incremental equity* in a characterization of utilitarianism.

Recovery translation invariance as defined below demands that adding or subtracting the same vector of health values to the recovery phase of two streams without changing the common timing structure does not affect the relative ranking of the two streams.

Recovery translation invariance. For all $T \in \mathbf{T}$, for all $x, y \in H_T^1$ with the same timing structure and for all $z \in \mathbb{R}^T$ such that $x_\tau = y_\tau$ and $z_\tau = 0$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$, if $(x + z), (y + z) \in H_T^1$ and $\mathbf{U}^{(x+z)} = \mathbf{U}^{(y+z)} = \mathbf{U}$, then

$$(x + z) \succsim (y + z) \;\Leftrightarrow\; x \succsim y.$$

Again, we employ an example to illustrate this axiom. Let $x = (5, 1, 2, 4, 3, 5)$ and $y = (5, 1, 3, 3, 3, 5)$. The two streams have the same timing structure with $\mathbf{U} = \{3, 4, 5\}$. Defining $z = (0, 0, -1, -1, 1, 0) \in \mathbb{R}^6$, it follows that $(x + z), (y + z) \in H_T^1$ and $\mathbf{U}^{(x+z)} = \mathbf{U}^{(y+z)} = \mathbf{U}$. Therefore, recovery translation invariance requires that

$$(5, 1, 1, 3, 4, 5) = (x + z) \succsim (y + z) = (5, 1, 2, 2, 4, 5)$$

if and only if

$$(5, 1, 2, 4, 3, 5) = x \succsim y = (5, 1, 3, 3, 3, 5).$$

5.3 Recovery monotonicity

We assume that our resilience ordering possesses a plausible monotonicity property with respect to the health-variable values experienced in the recovery phase. In particular, if all values in the recovery phase increase (while all other health-variable values remain unchanged), resilience increases.

Recovery monotonicity. For all $T \in \mathbf{T}$ and for all $x, y \in H_T^1$ with the same timing structure such that $\mathbf{U} \neq \emptyset$, if $x_\tau > y_\tau$ for all $\tau \in \mathbf{U}$ and $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$, then

$$x \succ y.$$

For example, if $x = (5, 2, 3, 5, 4, 6)$ and $y = (5, 1, 2, 4, 3, 6)$, recovery monotonicity requires that $(5, 2, 3, 5, 4, 6) = x \succ y = (5, 1, 2, 4, 3, 6)$ because $x_\tau > y_\tau$ for all $\tau \in \mathbf{U} = \{3, 4, 5\}$ and $x_\tau = y_\tau$ for all $\tau \in \{1, 2, 6\}$.

5.4 Amplitude and recovery consistency

The following axiom requires that certain movements along a stream leave the value of vulnerability unchanged; in particular, only the amplitude and the recovery delay are of importance. The duration of a downwards movement is irrelevant—all that matters is the amplitude of the drop. Furthermore, we do not distinguish between full recovery and excess recovery; any recovery that takes us beyond the pre-drop level is treated in the same way as a recovery to the pre-drop level. Finally, anything that happens prior to the down spell plays no role.

Amplitude and recovery consistency. For all $T, T' \in \mathbf{T}$, for all $x \in H_T^1$ and for all $y \in H_{T'}^1$ such that $|\mathbf{U}^x| = |\mathbf{U}^y|$, if there exists $t \in \mathbb{Z}$ such that $x_{s(\sigma^x)} - x_{d(\sigma^x)} = y_{s(\sigma^y)} - y_{d(\sigma^y)}$ and $x_{s(\sigma^x)} - x_\tau = y_{s(\sigma^y)} - y_{t+\tau}$ for all $\tau \in \mathbf{U}^x$, then

$$x \sim y.$$

Let $x = (3, 1, 2)$ and $y = (4, 4, 2, 3)$. We have $s(\sigma^x) = 1$, $s(\sigma^y) = 2$, $d(\sigma^x) = 2$, $d(\sigma^y) = 3$, $\mathbf{U}^x = \{3\}$ and $\mathbf{U}^y = \{4\}$. Because

$$x_1 - x_2 = y_2 - y_3 = 2 \;\text{ and }\; x_1 - x_3 = y_2 - y_4 = 1,$$

amplitude and recovery consistency requires that $x \sim y$.

5.5 Continuity

We employ a mild continuity property that ensures that small changes in the values of the health variables do not lead to large changes in vulnerability provided that the time period in which down spell ends does not change. This is another well-established condition that is employed throughout the literature concerned with the design of social index numbers.

Continuity. For all $T \in \mathbf{T}$, for all sequences $\langle x^k \rangle_{k \in \mathbb{N}}$, with $x^k \in H^1_T$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} x^k = x \in H^1_T$, and for all $y \in H^1_T$,

$$\left[x^k \succsim y \text{ for all } k \in \mathbb{N} \text{ and } \lim_{k\to\infty} d(\sigma^{x^k}) = d(\sigma^x)\right] \Rightarrow x \succsim y$$

and

$$\left[y \succsim x^k \text{ for all } k \in \mathbb{N} \text{ and } \lim_{k\to\infty} d(\sigma^{x^k}) = d(\sigma^x)\right] \Rightarrow y \succsim x.$$

Consider the sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ defined by letting $x^k = (5, 1, 2, 4, 3 - 1/k, 5)$ for all $k \in \mathbb{N}$. Furthermore, let $x = (4, 1, 3, 2, 3, 4)$. It follows that

$$\lim_{k\to\infty} x^k = (5, 1, 2, 4, 3, 5).$$

Continuity demands that if $x^k \succsim (4, 1, 3, 2, 3, 4)$ for all $k \in \mathbb{N}$, then

$$\lim_{k\to\infty} x^k = (5, 1, 2, 4, 3, 5) \succsim (4, 1, 3, 2, 3, 4)$$

and, likewise, if $(4, 1, 3, 2, 3, 4) \succsim x^k$ for all $k \in \mathbb{N}$, then

$$(4, 1, 3, 2, 3, 4) \succsim (5, 1, 2, 4, 3, 5) = \lim_{k\to\infty} x^k.$$

5.6 Homogeneity of degree zero

Our final axiom is homogeneity of degree zero. This well-established requirement demands that resilience is invariant with respect to the multiplication of all health-variable values by the same positive constant. The homogeneity property defined below is frequently employed to ensure that a measure is relative. Again, prominent examples can be found in the literature on inequality measurement. For instance, the relative indices of Atkinson (1970), Kolm (1969) and Sen (1973) are homogeneous of degree zero; see also Blackorby and Donaldson (1978).

Homogeneity of degree zero. For all $T \in \mathbf{T}$, for all $x \in H^1_T$ and for all $\lambda \in \mathbb{R}_{++}$,

$$\lambda \cdot x \sim x.$$

For $x = (5, 1, 2, 4, 3, 5)$ and $\lambda = 1/2$, homogeneity of degree zero requires that

$$(5/2, 1/2, 1, 2, 3/2, 5/2) = \lambda \cdot (5, 1, 2, 4, 3, 5) \sim (5, 1, 2, 4, 3, 5).$$

6 A characterization

Our main result is the following characterization of the resilience ordering $\succsim^r$.

Theorem 1. *A single-spell resilience ordering $\succsim$ satisfies recovery neutrality, recovery translation invariance, recovery monotonicity, amplitude and recovery consistency, continuity and homogeneity of degree zero if and only if $\succsim = \succsim^r$.*

Proof. If. To show that $\succsim^r$ satisfies recovery neutrality, assume that $T \in \mathbf{T}$ and $x, y \in H_T^1$ have the same timing structure. If $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$ and $(y_\tau)_{\tau \in \mathbf{U}}$ is a permutation of $(x_\tau)_{\tau \in \mathbf{U}}$, it follows immediately that $r(x) = r(y)$ and hence $x \sim^r y$.

Now we establish recovery translation invariance. Let $T \in \mathbf{T}$, $x, y \in H_T^1$ and $z \in \mathbb{R}^T$ be such that x and y have the same timing structure, $x_\tau = y_\tau$ and $z_\tau = 0$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$, $(x + z), (y + z) \in H_T^1$ and $\mathbf{U}^{(x+z)} = \mathbf{U}^{(y+z)} = \mathbf{U}$. It follows that

$$\begin{aligned} a(\sigma^{x+z}) = a(\sigma^{y+z}) = a(\sigma^x) = a(\sigma^y), \\ b(\sigma^{x+z}) = b(\sigma^x) - \textstyle\sum_{t \in \mathbf{U}} z_t, \\ b(\sigma^{y+z}) = b(\sigma^y) - \textstyle\sum_{t \in \mathbf{U}} z_t. \end{aligned}$$

Therefore,

$$\begin{aligned} (x+z) \succsim^r (y+z) \quad &\Leftrightarrow \quad \frac{a(\sigma^{x+z})}{a(\sigma^{x+z}) + b(\sigma^{x+z})} \geq \frac{a(\sigma^{y+z})}{a(\sigma^{y+z}) + b(\sigma^{y+z})} \\ &\Leftrightarrow \quad \frac{a(\sigma^x)}{a(\sigma^x) + b(\sigma^x) - \sum_{t \in \mathbf{U}} z_t} \geq \frac{a(\sigma^x)}{a(\sigma^x) + b(\sigma^y) - \sum_{t \in \mathbf{U}} z_t} \\ &\Leftrightarrow \quad b(\sigma^x) \leq b(\sigma^y) \\ &\Leftrightarrow \quad x \succsim^r y. \end{aligned}$$

Next, we prove that $\succsim^r$ satisfies recovery monotonicity. Assume that $T \in \mathbf{T}$ and $x, y \in H_T^1$ have the same timing structure with $\mathbf{U} \neq \emptyset$. Furthermore, assume that $x_\tau > y_\tau$ for all $\tau \in \mathbf{U}$ and $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$. This immediately implies that $b(\sigma^x) < b(\sigma^y)$ and, because $\succsim^r$ decreases in the recovery delay, it follows that $x \succ^r y$.

Now consider amplitude and recovery consistency. Let $T, T' \in \mathbf{T}$, $x \in H_T^1$ and $y \in H_{T'}^1$ be such that $|\mathbf{U}^x| = |\mathbf{U}^y|$. Furthermore, assume that there exists $t \in \mathbb{Z}$ such that $x_{s(\sigma^x)} - x_{d(\sigma^x)} = y_{s(\sigma^y)} - y_{d(\sigma^y)}$ and $x_\tau - x_{d(\sigma^x)} = y_{t+\tau} - y_{d(\sigma^y)}$ for all $\tau \in \mathbf{U}^x$. By definition, this implies that

$$a(\sigma^x) = x_{s(\sigma^x)} - x_{d(\sigma^x)} = y_{s(\sigma^y)} - y_{d(\sigma^y)} = a(\sigma^y)$$

and

$$b(\sigma^x) = \sum_{\tau \in \mathbf{U}} \left(x_{s(\sigma^x)} - x_\tau\right) = \sum_{\tau \in \mathbf{U}} \left(y_{s(\sigma^y)} - y_{t+\tau}\right) = b(\sigma^y)$$

and hence $r(x) = r(y)$, which implies $x \sim^r y$.

That continuity is satisfied follows immediately from the continuity of the restriction of the function r to H_T^1 for all $T \in \mathbf{T}$, provided that we require that $\lim_{k \to \infty} x^k = x \in H_T^1$ and $\lim_{k \to \infty} d(\sigma^{x^k}) = d(\sigma^x)$.

Finally, we prove that $\succsim^r$ is homogeneous of degree zero. Let $T \in \mathbf{T}$, $x \in H_T^1$ and $\lambda \in \mathbb{R}_{++}$. It follows that

$$a(\sigma^{\lambda \cdot x}) = \lambda \cdot x_{s(\sigma^x)} - \lambda \cdot x_{d(\sigma^x)} = \lambda \cdot \left(x_{s(\sigma^x)} - x_{d(\sigma^x)}\right) = \lambda \cdot a(\sigma^x)$$

and

$$b(\sigma^{\lambda \cdot x}) = \sum_{t \in \mathbf{U^x}} \left(\lambda \cdot x_{s(\sigma^x)} - \lambda \cdot x_t\right) = \lambda \cdot \sum_{t \in \mathbf{U^x}} \left(x_{s(\sigma^x)} - x_t\right) = \lambda \cdot b(\sigma^x).$$

Therefore, $r(\lambda \cdot x) = r(x)$ and hence $\lambda \cdot x \sim^r x$. □

In the only-if part of the proof, we proceed in several steps to illustrate how adding one axiom at a time successively narrows down the set of possible orderings until we arrive at the desired conclusion.

We begin by showing that the conjunction of recovery neutrality and recovery translation invariance implies that the criterion is insensitive to the distribution of health values in the recovery phase provided that the sum of the health values in the recovery phase remains unchanged.

Lemma 1. *If a single-spell resilience ordering $\succsim$ satisfies recovery neutrality and recovery translation invariance, then, for all $T \in \mathbf{T}$ and for all $x, y \in H_T^1$ with the same timing structure, if $x_\tau = y_\tau$ for all $\tau \in \{1, \dots, T\} \setminus \mathbf{U}$ and $\sum_{\tau \in \mathbf{U}} x_\tau = \sum_{\tau \in \mathbf{U}} y_\tau$, then*

$$x \sim y.$$

Proof. Assume that $x, y \in H_T^1$ have the same timing structure, $\sum_{\tau \in \mathbf{U}} x_\tau = \sum_{\tau \in \mathbf{U}} y_\tau$ and $x_\tau = y_\tau$ for all $\tau \in \{1, \dots, T\} \setminus \mathbf{U}$.

It is trivially true that $x \sim y$ if $|\mathbf{U}| = u - d$ equals 0 or 1.

Now assume that $|\mathbf{U}| = u - d \geq 2$. Define the vectors $x^0, \dots, x^{u-d-1}$ by $x^0 = x$ and

$$x^k = x^{k-1} + w^k \quad \text{for all } k = 1, \dots, u - d - 1,$$

where, for all $k = 1, \dots, u - d - 1$, $w^k \in \mathbb{R}^T$ is given by $w^k_{u-k+1} = y_{u-k+1} - x^{k-1}_{u-k+1}$, $w^k_{u-k} = -w^k_{u-k+1}$ and $w^k_\tau = 0$ for $\{1, \dots, T\} \setminus \{u - k, u - k + 1\}$. Note that

$$x^{u-d-1}_{d+1} = x_{d+1} - y_{d+2} + x^{u-d-2}_{d+2} = \cdots = x_{d+1} + \sum_{\tau=d+2}^{u} (x_\tau - y_\tau) = y_{d+1}$$

since, by assumption, $\sum_{\tau=d+1}^{u} y_\tau = \sum_{\tau=d+1}^{u} x_\tau$, while, by construction, $x^{u-d-1}_\tau = y_\tau$ for all $\tau \in \{d + 2, \dots, u\}$. Hence, $x^{u-d-1} = y$.

It remains to be shown that $x^k \sim x^{k-1}$ for all $k \in \{1, \dots, u - d - 1\}$. Note that

$$x^{k-1}_{u-k} + x^{k-1}_{u-k+1} = x^k_{u-k} + x^k_{u-k+1}$$

since $w^k_{u-k} = -w^k_{u-k+1}$. Hence, we can define the scalars α^k and β^k as follows.

$$\begin{aligned}\alpha^k &= \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k} + x^{k-1}_{u-k+1}\right) = \tfrac{1}{2} \cdot \left(x^k_{u-k} + x^k_{u-k+1}\right),\\ \beta^k &= \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k} - x^k_{u-k}\right) = \tfrac{1}{2} \cdot \left(x^k_{u-k+1} - x^{k-1}_{u-k+1}\right).\end{aligned}$$

Let $z^k \in \mathbb{R}^T$ be given by

$$\begin{aligned}z^k_{u-k+1} &= \alpha^k - \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k+1} + x^k_{u-k+1}\right),\\ z^k_{u-k} &= \alpha^k - \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k} + x^k_{u-k}\right)\end{aligned}$$

and $z^k_\tau = 0$ for all $\tau \in \{1, \ldots, T\} \setminus \{u-k, u-k+1\}$. Then

$$\begin{aligned}x^k_{u-k+1} + z^k_{u-k+1} &= \alpha^k + \tfrac{1}{2} \cdot \left(x^k_{u-k+1} - x^{k-1}_{u-k+1}\right) = \alpha^k + \beta^k\\ &= \alpha^k + \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k} - x^k_{u-k}\right) = x^{k-1}_{u-k} + z^k_{u-k}\end{aligned}$$

and

$$\begin{aligned}x^{k-1}_{u-k+1} + z^k_{u-k+1} &= \alpha^k - \tfrac{1}{2} \cdot \left(x^k_{u-k+1} - x^{k-1}_{u-k+1}\right) = \alpha^k - \beta^k\\ &= \alpha^k - \tfrac{1}{2} \cdot \left(x^{k-1}_{u-k} - x^k_{u-k}\right) = x^k_{u-k} + z^k_{u-k}.\end{aligned}$$

By recovery neutrality and recovery translation invariance, it follows that $x^k \sim x^{k-1}$ for all $k \in \{1, \ldots, u-d-1\}$ and hence $x \sim y$ by transitivity. $\square$

Our next step consists of adding recovery monotonicity to the two axioms of the above lemma. As a consequence, it follows that an additive criterion must be used to compare any two streams with the same timing structure and with identical health-variable values in the periods prior to the recovery phase.

Lemma 2. *If a single-spell resilience ordering $\succsim$ satisfies recovery neutrality, recovery translation invariance and recovery monotonicity, then, for all $T \in \mathbf{T}$ and for all $x, y \in H^1_T$ with the same timing structure, if $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$, then*

$$x \succeq y \;\Leftrightarrow\; \sum_{\tau \in \mathbf{U}} x_\tau \geq \sum_{\tau \in \mathbf{U}} y_\tau.$$

Proof. Assume that $x, y \in H^1_T$ have the same timing structure and $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$. Note that the equivalence stated in the lemma is trivially true if $|\mathbf{U}| = u - d = 0$. Thus, we can without loss of generality assume that $|\mathbf{U}| = u - d > 0$. In view of Lemma 1, we only have to prove that, under the assumptions of the lemma statement, the inequality

$$\sum_{\tau \in \mathbf{U}} x_\tau > \sum_{\tau \in \mathbf{U}} y_\tau$$

implies $x \succ y$.

The implication follows directly from recovery monotonicity if $|\mathbf{U}| = u - d = 1$.

Now assume that $\sum_{\tau\in\mathbf{U}} x_\tau > \sum_{\tau\in\mathbf{U}} y_\tau$ and $|\mathbf{U}| = u - d \geq 2$. Define $x', y' \in H^1_T$ as follows. Let $y'_{d+1} = y_{d+1}$ and choose $x'_{d+1} \in (y_{d+1}, x_s)$ such that $x'_{d+1} - y_{d+1} < \sum_{t\in\mathbf{U}}(x_t - y_t)$. Moreover, define

$$x'_\tau = \frac{1}{u-d-1} \cdot \left(\sum_{t\in\mathbf{U}} x_t - x'_{d+1} \right) \quad \text{and} \quad y'_\tau = \frac{1}{u-d-1} \cdot \left(\sum_{t\in\mathbf{U}} y_t - y'_{d+1} \right)$$

for all $\tau \in \{d+2, \ldots, u\}$, and let $x'_\tau = y'_\tau = x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$. By definition, x' and y' have the same timing structure as x and y and, moreover, we have

$$\sum_{\tau\in\mathbf{U}} x'_\tau = \sum_{\tau\in\mathbf{U}} x_\tau \quad \text{and} \quad \sum_{\tau\in\mathbf{U}} y'_\tau = \sum_{\tau\in\mathbf{U}} y_\tau$$

as well as $x'_\tau > y'_\tau$ for all $\tau \in \mathbf{U}$. By Lemma 1, $x' \sim x$ and $y' \sim y$. By recovery monotonicity, $x' \succ y'$ and hence $x \succ y$ by transitivity. □

If the axiom of amplitude and recovery consistency is employed in addition to the properties previously imposed, it follows that knowledge of the recovery delays $b(\sigma^x)$ and $b(\sigma^y)$ is sufficient to rank the streams x and y, provided that they are associated with recovery phases of the same length and share the same amplitudes.

Lemma 3. *If a single-spell resilience ordering $\succsim$ satisfies recovery neutrality, recovery translation invariance, recovery monotonicity and amplitude and recovery consistency, then, for all $T, T' \in \mathbf{T}$, for all $x \in H^1_T$ and for all $y \in H^1_{T'}$ such that $|\mathbf{U}^x| = |\mathbf{U}^y|$, if $x_{s(\sigma^x)} - x_{d(\sigma^x)} = y_{s(\sigma^y)} - y_{d(\sigma^y)}$, then*

$$x \succsim y \;\Leftrightarrow\; b(\sigma^x) \leq b(\sigma^y).$$

Proof. Assume that $T, T' \in \mathbf{T}$, $x \in H^1_T$ and $y \in H^1_{T'}$ are such that $|\mathbf{U}^x| = |\mathbf{U}^y|$ and $x_{s(\sigma^x)} - x_{d(\sigma^x)} = y_{s(\sigma^y)} - y_{d(\sigma^y)}$. Let $u = |\mathbf{U}^x| + 2 = |\mathbf{U}^y| + 2$ and define $x', y' \in H^1_u$ as follows. Let $x'_1 = y'_1 = \max\{x_{s(\sigma^x)}, y_{s(\sigma^y)}\}$, $x'_2 = y'_2 = \max\{x_{d(\sigma^x)}, y_{d(\sigma^y)}\}$, $x'_\tau = x_{d(\sigma^x)-2+\tau} + \max\{0, y_{s(\sigma^x)} - x_{s(\sigma^y)}\}$ and $y'_\tau = y_{d(\sigma^y)-2+\tau} + \max\{0, x_{s(\sigma^x)} - y_{s(\sigma^y)}\}$ for all $\tau \in \{3, \ldots, u\}$. Hence, $s(\sigma^{x'}) = s(\sigma^{y'}) = 1$, $d(\sigma^{x'}) = d(\sigma^{y'}) = 2$ and $\mathbf{U}^{x'} = \mathbf{U}^{y'} = \{3, \ldots, u\}$. Note that, by construction, $x'_\tau \geq x_\tau \geq 0$ and $y'_\tau \geq y_\tau \geq 0$ for all $\tau \in \{1, \ldots, u\}$. By amplitude and recovery consistency, $x' \sim x$ and $y' \sim y$ and, by Lemma 2,

$$x' \succsim y' \;\Leftrightarrow\; b(\sigma^{x'}) \;=\; (u-2) \cdot x'_1 - \sum_{\tau=3}^{u} x'_\tau \;\leq\; (u-2) \cdot y'_1 - \sum_{\tau=3}^{u} y'_\tau \;=\; b(\sigma^{y'})$$

because $x'_1 = y'_1$. The result follows since $b(\sigma^x) = b(\sigma^{x'})$ and $b(\sigma^y) = b(\sigma^{y'})$ and $\succsim$ is transitive. □

The next property to be added is continuity. This axiom allows us to extend the result of the previous lemma to any two streams x and y with the same amplitudes; the recovery phases associated with x and y may now differ in length.

Lemma 4. *If a single-spell resilience ordering $\succsim$ satisfies recovery neutrality, recovery translation invariance, recovery monotonicity, amplitude and recovery consistency and continuity, then, for all $T, T' \in \mathbf{T}$, for all $x \in H^1_T$ and for all $y \in H^1_{T'}$, if $x_{s(\sigma^x)} - x_{d(\sigma^x)} = y_{s(\sigma^y)} - y_{d(\sigma^y)}$, then*

$$x \succsim y \;\Leftrightarrow\; b(\sigma^x) \leq b(\sigma^y).$$

Proof. Assume that $T, T' \in \mathbf{T}$, $x \in H^1_T$ and $y \in H^1_{T'}$ are such that $x_{s(\sigma^x)} - x_{d(\sigma^x)} = y_{s(\sigma^y)} - y_{d(\sigma^y)}$.

If $|\mathbf{U}^x| = |\mathbf{U}^y|$, then the result follows from Lemma 3.

Now assume that $|\mathbf{U}^x| \neq |\mathbf{U}^y|$; without loss of generality, assume that $|\mathbf{U}^x| < |\mathbf{U}^y|$. Let $u = |\mathbf{U}^y| + 2$ and define $x', y' \in H^1_u$ as follows. Let $x'_1 = y'_1 = \max\{x_{s(\sigma^x)}, y_{s(\sigma^y)}\}$, $x'_2 = y'_2 = \max\{x_{d(\sigma^x)}, y_{d(\sigma^y)}\}$, $x'_\tau = x_{d(\sigma^x)-2+\tau} + \max\{0, y_{s(\sigma^x)} - x_{s(\sigma^y)}\}$ for all $\tau \in \{3, \ldots, |\mathbf{U}^x| + 2\}$, $x'_\tau = x'_1$ for all $\tau \in \{|\mathbf{U}^x| + 3, \ldots, u\}$ and $y'_\tau = y_{d(\sigma^y)-2+\tau} + \max\{0, x_{s(\sigma^x)} - y_{s(\sigma^y)}\}$ for all $\tau \in \{3, \ldots, u\}$. Hence, $s(\sigma^{x'}) = s(\sigma^{y'}) = 1$, $d(\sigma^{x'}) = d(\sigma^{y'}) = 2$, $\mathbf{U}^{x'} = \{3, \ldots, |\mathbf{U}^x| + 2\}$ and $\mathbf{U}^{y'} = \{3, \ldots, u\}$. Note that, by construction, $x'_\tau \geq x_\tau \geq 0$ and $y'_\tau \geq y_\tau \geq 0$ for all $\tau \in \{1, \ldots, u\}$. By amplitude and recovery consistency, $x' \sim x$ and $y' \sim y$.

If $|\mathbf{U}^x| = 0$ (and, by assumption, $|\mathbf{U}^y| > 0$), then $b(\sigma^{x'}) = 0 < b(\sigma^{y'})$. Construct the sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ by letting, for all $k \in \mathbb{N}$, x^k be defined by $x^k_1 = x'_1$, $x^k_2 = x'_2$ and $x^k_\tau = x'_1 - \varepsilon/k$ for all $\tau \in \{3, \ldots, u\}$, where $(u-2) \cdot \varepsilon < b(\sigma^{y'})$, so that $b(\sigma^{y'}) > b(\sigma^{x^1}) > \cdots > b(\sigma^{x^k}) > b(\sigma^{x^{k+1}}) > \cdots > b(\sigma^{x'}) = 0$. Note that $|\mathbf{U}^{x^k}| = |\mathbf{U}^y|$. By Lemma 3, $\sigma^{y'} \succ \sigma^{x^k}$ for all $k \in \mathbb{N}$ and $\sigma^{x^k} \succ \lim_{\kappa\to\infty} \sigma^{x^\kappa} = \sigma^{x'}$ for all $k \in \mathbb{N}$ by continuity because $\lim_{\kappa\to\infty} x^\kappa = x'$ and $b(\sigma^{x^k})$ is strictly decreasing in k. Hence,

$$x \sim x' \succ y' \sim y$$

because $\succsim$ is transitive.

Finally, assume that $|\mathbf{U}^x| > 0$. Construct the sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ by letting, for all $k \in \mathbb{N}$, x^k be defined by $x^k_1 = x'_1$, $x^k_2 = x'_2$, $x^k_3 = x'_3 + (u - |\mathbf{U}^x| - 2) \cdot \varepsilon/k$, $x^k_\tau = x'_\tau$ for all $\tau \in \{4, \ldots, |\mathbf{U}^x| + 2\}$ and $x^k_\tau = x'_1 - \varepsilon/k$ for all $\tau \in \{\mathbf{U}^x| + 3, \ldots, u\}$, where $(u - |\mathbf{U}^x| - 2) \cdot \varepsilon < x'_1 - x'_3$, so that $x^k_2 \leq x'_3 < x^k_3 < x^k_1$. It follows that $|\mathbf{U}^{x^k}| = |\mathbf{U}^y|$. By construction, $b(\sigma^{x^1}) = \cdots = b(\sigma^{x^k}) = b(\sigma^{x^{k+1}}) = \cdots = b(\sigma^{x'})$. Combined with Lemma 3, it follows that $\sigma^{x^k} \sim \sigma^{x'}$ for all $k \in \mathbb{N}$ by continuity; note that $\lim_{\kappa\to\infty} x^\kappa = x'$. Hence, by Lemma 3 and transitivity,

$$x' \sim x^k \succsim y' \;\Leftrightarrow\; b(\sigma^{x'}) = b(\sigma^{x^k}) \leq b(\sigma^{y'})$$

for all $k \in \mathbb{N}$ so that

$$x \sim x' \succsim y' \sim y \;\Leftrightarrow\; b(\sigma^x) = b(\sigma^{x'}) \leq b(\sigma^{y'}) = b(\sigma^y).$$

□

We are now ready to prove the only-if part of our axiomatization. Adding homogeneity of degree zero implies that we can divide by the amplitude of a down spell so that we arrive at the ordering $\succsim^r$ represented by the resilience measure r.

Proof. Only if. Assume that $\succeq$ is an ordering that satisfies the axioms of the theorem statement. Let $T, T' \in \mathbf{T}$, $x \in H_T^1$ and $y \in H_{T'}^1$. Define

$$\lambda^x = \frac{1}{x_{s(\sigma^x)} - x_{d(\sigma^x)}} = \frac{1}{a(\sigma^x)} \quad \text{and} \quad \lambda^y = \frac{1}{y_{s(\sigma^y)} - y_{d(\sigma^y)}} = \frac{1}{a(\sigma^y)}.$$

Let $x' \in H_T^1$ and $y' \in H_{T'}^1$ be defined by $x' = \lambda^x \cdot x$ and $y' = \lambda^y \cdot y$. By homogeneity of degree zero,

$$x \sim x' \quad \text{and} \quad y \sim y'.$$

We have that

$$x'_{s(\sigma^x)} - x'_{d(\sigma^x)} = \frac{x_{s(\sigma^x)} - x_{d(\sigma^x)}}{a(\sigma^x)} = 1 = \frac{y_{s(\sigma^y)} - y_{d(\sigma^y)}}{a(\sigma^y)} = y'_{s(\sigma^y)} - y'_{d(\sigma^y)},$$

thus, by Lemma 4,

$$\begin{aligned}
x' \succsim y' \quad &\Leftrightarrow \quad \frac{b(\sigma^x)}{a(\sigma^x)} = b(\sigma^{x'}) \leq b(\sigma^{y'}) = \frac{b(\sigma^y)}{a(\sigma^y)} \\
&\Leftrightarrow \quad 1 + \frac{b(\sigma^x)}{a(\sigma^x)} \leq 1 + \frac{b(\sigma^y)}{a(\sigma^y)} \\
&\Leftrightarrow \quad \frac{a(\sigma^x) + b(\sigma^x)}{a(\sigma^x)} \leq \frac{a(\sigma^y) + b(\sigma^y)}{a(\sigma^y)} \\
&\Leftrightarrow \quad \frac{a(\sigma^x)}{a(\sigma^x) + b(\sigma^x)} \geq \frac{a(\sigma^y)}{a(\sigma^y) + b(\sigma^y)}.
\end{aligned}$$

As established above, we also have $x \sim x'$ and $y \sim y'$ so that we obtain

$$x \sim x' \succsim y' \sim y \ \Leftrightarrow \ x \succsim^r y,$$

using the definition of $\succsim^r$. □

That our axioms are independent is established in the appendix.

7 Concluding remarks

In this paper, we propose and axiomatize a measure of resilience based solely on the properties of the health streams—the fundamental determinants of our notion of resilience. More specifically, our approach treats down spells as the crucial experiences that reflect an individual's ability to recover. Implicit in our definition is the assumption that, in a down spell, the severity of the down movement matters but not its duration. Likewise, our properties imply a specific way of identifying the dividing line between a downwards movement and the recovery phase. These features represent modeling choices that we consider attractive in the measurement of resilience. Of course, there are alternative methods of defining the amplitude of a down spell and the transition from a drop to the period in which recovery can occur, and it may be useful to explore some of these in future work. A more general approach is to enrich the framework by taking into consideration information concerning the influencing forces that precipitate a down spell. However, if we were to adopt such a procedure we would have to rely on enriched data which might not be as easily available as the health streams on which our resilience ordering is based.

Our measure of resilience has a drawback at the level of the individual of not being continuous when a sequence of health streams converges to a health stream for which the down spell ends in a different period. To exemplify, consider Figure 4 and the let the health value in period 3 (instead of having the value 3) approach the health value 2 from below. To be specific, consider the sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ of streams with $x^k = (4, 2, 2 - \frac{1}{k+1}, 1, 4)$, so that $x = \lim_{k \to \infty} x_k$ is given by $x = (4, 2, 2, 1, 4)$. For each element of this sequence, $d(\sigma^{x^k}) = 4$, meaning that the down spell ends in period 4 and our measure of resilience $r(x^k)$ equals 1, since full recovery is immediate. However, in the limit, $d(\sigma^x) = 2$ and $r(x) = \frac{2}{7}$, since full recovery now takes two periods.

In practical applications, health is represented by a finite number of discrete values. In particular, when applying a Short Form Health Survey such as SF-36 (or the shorter version SF-12), health might be assigned integer values on a scale of 0 to 100 where a score of 0 indicates the worst possible perceived health and a score of 100 is equivalent to full health.

By not relying on information at the individual level concerning the forces that precipitate the individual down spells, our measure of resilience can be used for measuring the resilience of large populations of subjects. At such an aggregate level, the issue of whether the measure is continuous at the individual level loses much of its importance. Thus, the measure of resilience that we propose might be well-posed to identify the effects of interventions designed to improve mental-health resilience. This can be accomplished by studying populations of subjects some of whom are treated and some of whom remain untreated.

A possible concern is that our ordinal measure of resilience may not allow for a sufficient degree of differentiation across individuals. Nevertheless, this may not pose a serious problem, as demonstrated by the results when applying our measure to the German Socio-Economic Panel. The SOEP is an ongoing panel survey with yearly re-interviews (see http://www.diw.de/gsoep). It is a representative longitudinal micro-level study providing a wide range of demographic and socio-economic information on private households and all household members. The first data was collected in 1984 from a sample of randomly-selected adult respondents in the Federal Republic of Germany. Since then, the same individuals have been surveyed annually. In 1990 the survey was expanded to include the states of the former German Democratic Republic. New samples were included later on to collect information on specific population groups or to boost the sample size. Every year since 1994, individuals are asked to rate their health by responding to the question "How would you describe your current health?" with possible answers on a five-point scale, ranging from "bad" (1) to "very good" (5). They are also asked "How satisfied are you with your health?" where responses are given on an 11-point scale from 0 ("completely dissatisfied") to 10 ("completely satisfied"). We analyzed the years from 1994 to 2016 and restricted the sample to respondents for whom we have at least six consecutive observations, leaving us with 15,015 individuals. A histogram representing the relative frequencies of the attained resilience levels is provided in Figure 8 for self-assessed health status and in Figure 9 for health satisfaction. As is evident from these diagrams, there is considerable variation in both cases.

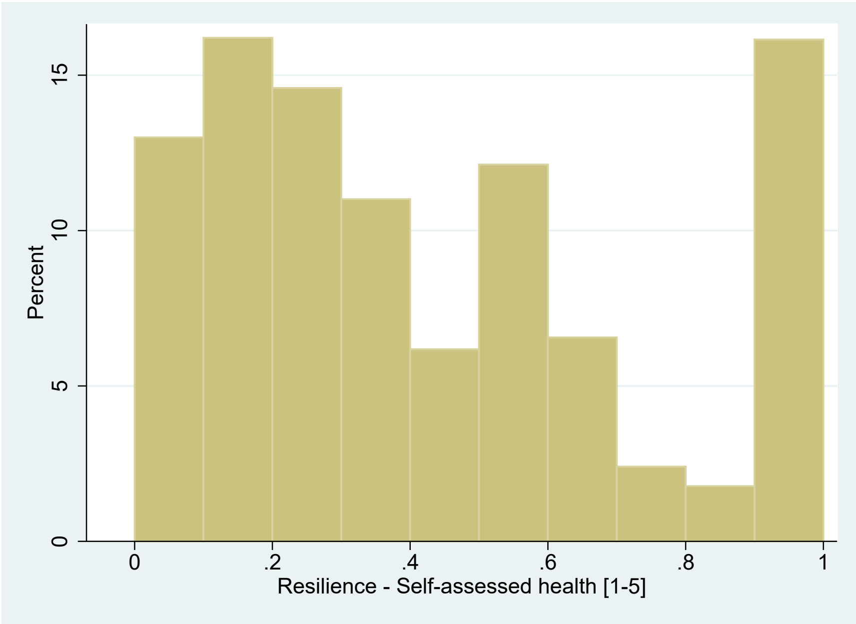

Figure 8: Self-assessed health

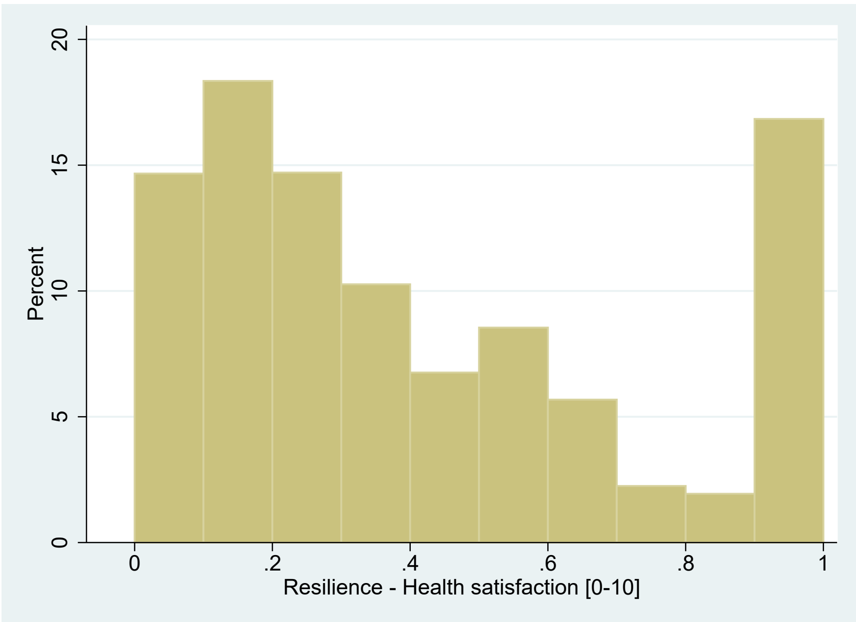

Figure 9: Health satisfaction

We conclude by noting that our approach is general enough to accommodate the assessment of resilience in the context of other variables. These could be both at an individual level, such as equivalent household income, and aggregate variables of economic performance, such as unemployment rates and GDP growth rates of countries.

A Appendix: Independence of the axioms

For each of the six axioms employed in our characterization, we provide an example that violates the axiom and satisfies the remaining properties. We note that the five examples that satisfy recovery monotonicity also satisfy the following stronger property.

Strong recovery monotonicity. For all $T \in \mathbf{T}$ and for all $x, y \in H_T^1$ with the same timing structure such that $\mathbf{U} \neq \emptyset$, if $x_\tau \geq y_\tau$ for all $\tau \in \mathbf{U}$ with at least one strict inequality and $x_\tau = y_\tau$ for all $\tau \in \{1, \ldots, T\} \setminus \mathbf{U}$, then

$$x \succ y.$$

Thus, the examples also show that strengthening the monotonicity axiom in this way does not affect the independence of the axioms. Note that our ordering $\succsim^r$ possess this stronger property.

A.1 Recovery neutrality

Let $\delta \in (0,1)$ and define, for all $x \in \Omega^1$,

$$r^1(x) = \frac{a(\sigma^x)}{a(\sigma^x) + \sum_{t \in U(\sigma^x)} \delta^{t-d(\sigma^x)} \cdot (x_{s(\sigma^x)} - x_t)}$$

and, for all $x, y \in \Omega^1$, $x \succsim^1 y$ if and only if $r^1(x) \geq r^1(y)$. The ordering $\succsim^1$ satisfies all our axioms except for recovery neutrality.

A.2 Recovery translation invariance

Let $\delta \in (0,1)$ and define, for all $x \in \Omega^1$,

$$r^2(x) = \frac{a(\sigma^x)}{a(\sigma^x) + \sum_{t \in U(\sigma^x)} \delta^{t-d(\sigma^x)} \cdot (x_{s(\sigma^x)} - x_{\pi(t)})},$$

where $\pi \colon U(\sigma^x) \to U(\sigma^x)$ is a bijection satisfying $x_{\pi(t)} \leq x_{\pi(t+1)}$ for all $t \in U(\sigma^x) \setminus \{u(\sigma^x)\}$, and, for all $x, y \in \Omega^1$, $x \succsim^2 y$ if and only if $r^2(x) \geq r^2(y)$. The ordering $\succsim^2$ satisfies all our axioms except for recovery translation invariance.

A.3 Recovery monotonicity

Let $\succsim^3$ be the universal indifference relation, that is, for all $x, y \in \Omega^1$, $x \sim^3 y$. The ordering $\succsim^3$ satisfies all our axioms except for recovery monotonicity.

A.4 Amplitude and recovery consistency

Define, for all $x \in \Omega^1$,

$$r^4(x) = \frac{a(\sigma^x) \cdot (d(\sigma^x) - s(\sigma^x))}{a(\sigma^x) \cdot (d(\sigma^x) - s(\sigma^x)) + b(\sigma^x)}$$

and, for all $x, y \in \Omega^1$, $x \succsim^4 y$ if and only if $r^4(x) \geq r^4(y)$. The ordering $\succsim^4$ satisfies all our axioms except for amplitude and recovery consistency.

A.5 Continuity

Define, for all $x \in \Omega^1$,

$$r^5(x) = \frac{a(\sigma^x)}{a(\sigma^x) + |\mathbf{U}^x| \cdot b(\sigma^x)}$$

and, for all $x, y \in \Omega^1$, $x \succsim^5 y$ if and only if $r^5(x) \geq r^5(y)$. The ordering $\succsim^5$ satisfies all our axioms except for continuity.

A.6 Homogeneity of degree zero

Define, for all $x, y \in \Omega^1$, $x \succsim^6 y$ if and only if $b(\sigma^x) \leq b(\sigma^y)$. The ordering $\succsim^6$ satisfies all our axioms except for homogeneity of degree zero.

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