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Abstract

This paper characterizes the curvature properties of applied social welfare functions. We choose as an example a general and non-abbreviated social welfare function introduced by Jorgenson and Slesnick (1983) that is frequently used in empirical work. We investigate the regularity of social preferences necessary to maximize social welfare with respect to prices and to guarantee that the social welfare function is well-behaved and suitable for theoretically plausible microsimulations. We use the notion of generalized convexity to define the necessary conditions for the social welfare function to be convex with respect to prices.

Keywords: Generalized convexity, social welfare function, inequality aversion.

JEL Classification: D30, D63, I31.

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1 Introduction

This paper characterizes the curvature properties of Social Welfare Functions (SWF) using generalized convexity results (Avriel 1972, Ben-Tal 1977, Caplin and Nalebuff 1991). This issue is important for ensuring the regularity of social preferences in order to maximize social welfare.

The curvature properties of a SWF are very important for applied welfare analysis. For instance, Atella et al. (2004) determine the set of prices, conditional on a given choice of degree of aversion to inequality, consistent with the maximization of the SWF. Identification of a society's aversion to inequality critically depends on the curvature properties of the social welfare function which is maximized to obtain a set of equilibrium prices. The authors empirically tested the curvature properties of the estimated SWF by finding reasonable global solutions, but they did not formally characterize the curvature properties of the SWF in general. This paper fills this knowledge gap.

As a working example, we select the Jorgenson and Slesnick (JS) SWF (Jorgenson and Slesnick 1983, 1984 a,b, 1987, 1990) because it is general and non-abbreviated, that is it is not summarized only by the mean income and a measure of dispersion of each income from the mean. The JS SWF was generally used for running microsimulations mainly estimating social cost of living indexes (Jorgenson and Slesnick 1983, 1984a, 1984b, 1987, 1990, Jorgenson 1990, 1997, Slesnick 1998, and Perali 2003). Jorgenson et al. (1992) use an intertemporal JS social welfare function to estimate the effect of a carbon tax on the reduction of carbon dioxide emissions. Jorgenson and Slesnick (2014) use their SWF for the measurement of social welfare maintaining consistency with US national accounts. Jorgenson and Schreyer (2017) also show how consumption-based measures of economic welfare at the individual and social level can be integrated into national accounts in general.

It should be remarked that the study of the curvature properties of the SWF is relevant not only in a maximization context but is also important to guarantee that the SWF is well-behaved and can therefore be used to implement theoretically plausible microsimulations. Normally, microsimulations are implemented in a partial equilibrium framework, studying for example the impact of macro policies separately for health, education, or housing, or examining consumption and labor supply. On the other hand, the social welfare function implements microsimulations in a general equilibrium setting jointly accounting for inter-sectoral effects provided that the curvature properties of the SWF are correctly imposed and tested at the econometric level.

The paper first sets notation and describes the properties of a general social welfare function formed by an efficiency and an equity term as originally described in Roberts (1980 b). Section 2 introduces the specification of the Roberts' SWF adopted by Jorgenson and Slesnick (1983, 1984a,b, 1987, 1990). The curvature properties of this function are characterized in Section 3. Conclusions summarize the contribution of the paper and describe the relevance of the results.

2 The applied Social Welfare Function

A social welfare functional assigns a social ordering defined on the set of social states X to each possible profile of individual utility functions in its domain. In our context, a social state X is described by the vector of quantities x consumed by K individuals. A social ordering R is a reflexive, complete and transitive binary relation that orders social states. The set $\mathcal{R}$ represents the set of all orderings defined on X. We define person k's utility function on the set of social states X as $u_k : X \to \mathbb{R}$, continuous and differentiable. The individual utility function describes the level of welfare for a given individual in each state. We also define the profile U formed by the vector of all real-valued individual utility functions as $U = (u_1, \dots, u_K) \in \mathcal{U}$, where $\mathcal{U}$ is the set of all possible profiles. For any $x \in X$, $U(x)$ denotes the vector $U(x) = (u_1(x), \dots, u_K(x)) \in \mathbb{R}^K$. To obtain a social preference ordering based on the individual utility functions, Sen (1970) defines a social welfare functional $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{R}$ where $\mathcal{D} \subseteq \mathcal{U}$ is the set of admissible profiles defining the domain of $\mathcal{F}$. The social welfare functional $\mathcal{F}$ maps the set of admissible utility profiles $\mathcal{D}$ to the set of all possible social orderings $\mathcal{R}$.[1] The social preference ordering that is obtained by applying $\mathcal{F}$ is denoted by $R_U = \mathcal{F}(U)$. The strict preference and the indifference relations corresponding to R_U are denoted by P_U and I_U respectively.

Some of the properties of social welfare functionals, traditionally formulated as axioms, that are considered desirables are the Unrestricted Domain (UD), the Independence of Irrelevant Alternatives (IR), and the Weak Pareto Principle (WP) (D'Aspremont and Gevers 1977, Roberts 1980b, Fleurbaey 2003, Bossert and Weymark 2004). Another condition that completes the set of Arrow-type properties relates to ordinal and interpersonal non-comparability. Following Roberts (1980b), the degree of comparability can be described in terms of the invariance class Φ being the set of invariance transformations ϕ such that $\forall U^1, U^2 \in \mathcal{D}$, if $\forall x \in X$, $U^2(x) = \phi(U^1(x))$, then $R_{U^1} = R_{U^2}$. The condition of Ordinal Non-Comparability can then be defined as

> **(ONC)** $\phi \in \mathit{\Phi}$ iff ϕ is a list of independent and strictly monotonically increasing transformations.

[1] Interestingly, Morreau and Weymark (2016) assume that the domain $\mathcal{D}$ is composed of profiles of utilities and profiles of utility scales or numerical grading scales.

If a social welfare functional $\mathcal{F}$ has the properties UD, IR, WP and ONC then by Arrow's impossibility theorem the only admissible social ordering is dictatorial.

Because the class of dictatorial social orderings is not appropriate for the evaluation of economic policies, it is crucial to weaken Arrow's assumptions to obtain a class of social welfare functions that can implement alternative ethical judgements. In line with Roberts (1980b), we first relax the Pareto principle and then we weaken the constraints on comparability. To make non-welfare characteristics play a role in determining a social ordering, Roberts (1980b) introduces the notions of Positive Association (PA)[2] ensuring that an increase in all levels of individual welfare must increase social welfare and of Non-Imposition (NI).[3]

Interestingly, as shown by Arrow (1963), under UD, IR, and ONC, the assumptions of PA and NI also imply WP. The objective is now to relax comparability in order to admit comparisons on a cardinal basis. As Sen (1970) showed, Arrow's dicatorial result is maintained even after replacing ONC with Cardinal Non-Comparable utility profiles

> **(CNC)** $\phi \in \Phi$ iff ϕ is a list of independent, strictly positive affine transformations, i.e. $\exists \alpha_k \in \mathbb{R}$ and $\exists \beta_k > 0$ such that $\phi_k(u_k) = \alpha_k + \beta_k u_k,\ k = 1, \ldots, K$.

By imposing that affine transformations have to preserve equality between units of different utility profiles, we obtain the property of Cardinal Unit Comparability

> **(CUC)** $\phi \in \Phi$ iff ϕ is a list of strictly positive affine transformations, i.e. $\exists \alpha_k \in \mathbb{R}$ and $\exists \beta > 0$ such that $\phi_k(u_k) = \alpha_k + \beta u_k,\ k = 1, \ldots, K$.

The set of social orderings is invariant to positive affine transformations with respect to the scaling parameter β that is the same for all individuals. CUC allows for interpersonal comparisons of welfare gains, but not for comparisons of welfare levels. Further, a class of non dictatorial social ordering can be obtained by extending the informational basis to CUC of the individual welfare functions. If a social welfare functional $\mathcal{F}$ satisfies UD, IR, WP and CUC, then there exists a continuous real-valued social welfare function W such that if $W\left(U\left(x^1\right)\right) > W\left(U\left(x^2\right)\right)$, then $x^1 P_U x^2$. As shown in Roberts (1980b), the social welfare function W can be represented as a weighted utilitarian

$$W\left(U\left(x\right)\right) = \sum_k a_k W(U(x)), \tag{1}$$

[2] Positive Association (PA). If $U^1(x^1) = U^2(x^1)$ and $U^1(x^2) > U^2(x^2)$, $\forall x^1 \in X \setminus \{x^2\}$, then $x^1 P_{U^2} x^2$ implies $x^1 P_{U^1} x^2$ and $x^2 P_{U^1} x^1$ implies $x^2 P_{U^2} x^1$.

[3] Non-Imposition (NI). $\forall x^1, x^2 \in X$, $\exists U^1, U^2 \in \mathcal{D}:\ x^1 P_{U^1} x^2,$ and $x^2 P_{U^2} x^1$.

where a_k is the weight of individual $k \in K$ belonging to the vector of weights $a \in \mathbb{R}^K_+$. Interestingly, if we introduce Anonymity (A) the names of the individuals are irrelevant and all individuals in society are given the same weight.[4] It is therefore possible to reverse the ranking of individual profiles and W reduces to a utilitarian social welfare function that must be symmetric with respect to the individual welfare functions u. The utilitarian representation of the SWF does not properly account for the distribution of welfare in the population and for the inequality of welfare levels.

To widen the range of admissible social orderings, the condition of Cardinal Full Comparability is necessary.

> **(CFC)** $\phi \in \Phi$ iff ϕ is a list of identical, strictly positive affine transformations, i.e. $\exists \alpha \in \mathbb{R}$ and $\exists \beta > 0$ such that $\phi_k(u_k) = \alpha + \beta u_k,\ k = 1, \ldots, K$.

CFC ensures that social orderings are invariant to any identical strictly positive affine transformation of the utility profile U, while ONC requires that social orderings are invariant with respect to strictly monotone increasing transformations that may not be the same across individuals.

Roberts (1980b, Theorem 4) demonstrates that if W satisfies UD, IR, WP and CFC, then there exists a function g, homogeneous of degree one, computed on the deviations of the levels of individual welfare from the mean level of welfare that defines the social welfare function as

$$W(U(x)) = \overline{W}(x) + g\left[U(x) - \overline{W}(x)\right],\ \text{for}\ \overline{W}(x) = \sum_k a_k u_k(x)\ \text{and}\ a \in \mathbb{R}^K_+. \tag{2}$$

The weights $a \in \mathbb{R}^K_+$ in Roberts (1980b) are all equal to $1/K$, because of the anonimity condition A. Such a class of admissible social judgments incorporates both an efficiency component given by the average individual welfare and an equity component measuring the inequality in the distribution of welfare. If dispersion increases, then social welfare decreases implying that g is a decreasing function. Then, for the SWF to be concave with respect to U, the function g must be concave in U.

As we have seen above, to incorporate non-welfare characteristics of social states, we need to replace WP with PA and NI. Maintaining UD, IR and CFC, Roberts (1980b) showed the existence of a social welfare function

$$W(U(x)) = F\left[\overline{W}(x) + g\left(U(x) - \overline{W}(x)\right), x\right] \tag{3}$$

[4]Anonimity (A). $\forall u \in \mathcal{D},\ uI^*u_\pi$, where u_π denotes a vector of permutation of the elements of u.

with $F : \mathcal{U} \times X \to \mathbb{R}$ and $\overline{W}(x) = \sum_k a_k(x)u_k(x)$. It incorporates non-welfare characteristics of social states through the weights $a_k(x)$, $g(x)$ and through F that depends directly on the social state x. This representation of social preferences suffers an identification problem analogous to the one encountered for individual preferences and the identification of equivalence scales (Perali 2003).

As noted by Morreau and Weymark (2016), the SWF approach fails to distinguish changes in individual wellbeings from changes due to different measurement scales. They propose a scale dependent approach that pairs each utility profile with a profile of measurement scales.[5]

This class of welfare functionals is exempted from the criticism of welfarism, but requires an informational basis often too large to make welfare judgments operational. What is crucial, as for individual utility profiles, is to contract the informational basis while maintaining the identifiability of social preference orderings. The informational constraint imposed on F being independent of x is similar to the one imposed at the individual level to permit inter-household comparability. In order to obtain an operational social welfare function it is necessary to specify a functional form for g and for the individual welfare functions u_k compatible with the CFC requirements, as we now illustrate.

Jorgenson and Slesnick (Jorgenson and Slesnick 1983, 1984a,b, 1987, 1990) define the social welfare function on the vector V of the logarithms of individual indirect utility functions V_k, that is $V = (\ln V_1, \ldots, \ln V_K)$, belonging to the indirect utility possibility set $\mathcal{V}$. The JS SWF takes the form analogous to (3)

$$W(V, p|\rho) = \ln \overline{V}(V, p) + g(V, p|\rho), \tag{4}$$

where $\ln \overline{V}(V, p) = \frac{\sum_k m_o(p,d_k) \ln V_k}{\sum_k m_o(p,d_k)}$ and $g(V, p|\rho) = -\gamma(p)M(V, p)$, with

$$\gamma(p) = \left\{ \frac{\sum_{k\neq j} m_o(p, d_k)}{\sum_k m_o(p, d_k)} \left[1 + \left(\frac{\sum_{k\neq j} m_o(p, d_k)}{m_o(p, d_j)} \right)^{-(\rho+1)} \right] \right\}^{\frac{1}{\rho}} \quad \text{and} \tag{5}$$

$$M(V, p) = \left[\frac{\sum_k m_o(p, d_k)|\ln V_k - \ln \overline{V}(V, p)|^{-\rho}}{\sum_k m_o(p, d_k)} \right]^{-\frac{1}{\rho}}. \tag{6}$$

The notation in equation (4) makes the dependence on prices p explicit. The first term is a weighted average of individual welfare levels. The second term is a mean value function of degree ρ of the deviations of household welfare from the average.[6] The constant ρ determines

[5] Similarly, Bosmans *et al.* (2018) introduce the concept of reference set welfarism, based on the aggregation of reference money metric utilities used to represent the social order. This is done in order to avoid the criticism of Blackorby and Donaldson (1988) about the use of the sum of money metric utilities as a SWF, because it could be in general the sum of not concave functions (see also Khan and Schlee 2017).

[6] Using the definition of mean value function (Hardy, Littlewood and Pólya 1934) the function g can be

the curvature of the SWF and measures the degree of aversion to inequality in the distribution of welfare levels. The function g is homogenous of degree one being a mean value function of order ρ. It is a negative function that reaches the value of zero in the perfectly equal case where $\ln V_k = \ln \overline{V}(V,p), \forall k = 1,\dots,K$.

Note that $m_o(p,d_k)$ is an indicator of the size of consuming units depending on prices p and on the vector of attributes d_k used to construct equivalent total expenditure $y = y_k/m_o$, where y_k is the total expenditure of household k. The scale for the reference household is $m_o(p,d_j) = \min_k m_o(p,d_k)$. Jorgenson ans Slesnick (1983) incorporated a notion of horizontal equity that treats different individuals differently by introducing a weak form of anonimity requiring that all individuals with same characteristics receive the same weight. The SWF therefore maintains symmetry in V for identical individuals. Both the weights and the measure of inequality given by the function g must be the same for individuals with identical characteristics.

The SWF is equity-regarding in the sense that it obeys Dalton's principle of transfers requiring that a transfer from a richer to a poorer individual, that does not reverse their relative positions, must increase the level of social welfare. As a consequence, the weights associated to the individual welfare function must be $a_k(p) = m_o(p,d_k)/\sum_k m_o(p,d_k)$, with $\sum_k a_k(p) = 1,\ 0 < a_k(p) < 1$. For analytical convenience, the dependence on the demographic characteristics is dropped.

The SWF reaches a maximum when $\gamma(p)$ is equal to zero and it is positive only if $\gamma(p) < \ln \overline{V}/M(V,p)$. In order to simplify the notation, in equation (5) we substitute $a_j(p) = \min_k a_k(p)$ and

$$\frac{\sum_{k\neq j} m_o(p,d_k)}{m_o(p,d_j)} = \frac{1-a_j(p)}{a_j(p)}, \tag{7}$$

so that $\gamma(p)$ can be written as

$$\gamma(p) = \left\{(1-a_j(p))\left[1+(1/a_j(p)-1)^{-(\rho+1)}\right]\right\}^{1/\rho}. \tag{8}$$

Note that when $a_j(p) \to 1$, as if there were only one individual in the society, then $\gamma(p) \to +\infty$. While if $a_j(p) \to 0$, as usually happens when there is a high number of observations,[7] then $\gamma(p) \to 0$, except for the case $\rho = -1$ for which $\gamma(p) \to 1/2$. With $a_j(p) = 1/2$ then $\gamma(p) = 1$, independently by the value of ρ. Notice that with $0 < a_j(p) < 1/2$, then $0 < \gamma(p) < 1$. In particular, $a_j(p) \to 0$, and hence $\gamma(p) \to 0$, implies that the last individual

generalized as $g(x) = \phi^{-1}\{\sum_k \phi_k(f(V_k))\}$ where $\phi(f(V_k))$ is a continuous and strictly monotonic function of the form $f(V_k)^\rho$.

[7]For instance, if there is a sample of 15,000 households of single persons and, in the reference household the equivalent adult is 1, then the value of $a_j(p) = m_o(p,d_j)/\sum_k m_o(p,d_k)$ is equal to 1/15,000.

is given a very small weight. While with $a_j(p) = 1/2$ there is no need of further ethic considerations, except for the dispersion among the individuals.

The values of $\gamma(p)$ also depend on the choice of ρ. The parameter ρ measures the society's constant degree of aversion to inequality. Within the admissible interval $(-\infty, -1]$ it affects the curvature of the social welfare function in the individual welfare space. Recall that $\gamma(p) \in (0, +\infty)$ and $\rho \in (-\infty, -1]$. The function $\gamma(p)$ is increasing with respect to ρ. This implies that the weight given to dispersion depends also on ρ. To illustrate the range of the function $\gamma(p)$, suppose that the household sample consists of 20,000 units so that $a_j(p) = 0.00005$. When $-2 < \rho < -1$, which is the empirically interesting case (Atella *et al.* 2004), then we have $0.007 < \gamma(p) < 0.5$, while for $-10 < \rho < -2$, then $0.00001 < \gamma(p) < 0.007$. Therefore, $\gamma(p)$ becomes increasingly more relevant as ρ approaches -1.

A further inspection of equation (8) reveals that when $\rho \to -\infty$ then we place the least possible weight upon equity as if all individuals had the same level of welfare and the social welfare function collapses to the weighted utilitarian case. If $\rho = -1$ then one recovers the egalitarian case giving maximum consideration to the inequality function $g(V, p|\rho)$. When the weights $a_k(p)$ take the same value for all k, then the potentially available level of welfare is maximum. This is Jorgenson and Slesnick's measure of efficiency. Note that a greater inequality aversion corresponds to a lower value of ρ. If ρ increases, then $\gamma(p)$ and $M(V, p)$ increases. This implies that the social planner is more willing to give up some welfare from the utilitarian position and s/he is less averse to inequality.

Jorgenson and Slesnick (1987, p. 311) state that "although the magnitude of money metric social welfare depends on the degree of aversion to inequality, we find that the qualitative features of comparisons among alternative policies for different values of this parameter are almost identical." However, this may be true only when alternative policies are ranked in relation to one society. If the same alternatives were compared across societies, then different degrees of society's aversion to inequality might explain the different social orderings. When the degree of aversion to inequality is estimated endogenously, ρ becomes a distinctive attribute of each society.

Atella *et al.* (2004) propose a scheme in which a benevolent social planner or ethical observer first chooses economic policies by maximizing W, specified *à la* JS, with respect to $\ln V_k$, with $k = 1, \dots, K$, for a set of prices at each given ρ. In the second part of the scheme, the households are asked to reveal the ρ that maximizes each household's welfare $\ln V_k$. Society is assumed to choose according to a majority rule (Black 1948). The mechanism critically depends on the choice of the set of prices p^* that minimizes society's welfare at each given ρ. The existence of a solution to this problem requires that the social welfare functional W be quasi-convex in prices p. The composition mapping W is strictly increasing in each function $\ln V_k$ and homogeneous of degree one in levels of the individual welfare. The logarithm transformation of the indirect utility function $\ln V_k(p, y_k)$ preserves its properties of (a) homogeneity of degree 0 in (p, y_k), (b) continuity at all strictly positive p, (c) non-increasing in p and non-decreasing in y_k, and (d) quasi-convexity in p.

A benevolent social planner elicits society's preferences towards inequality by gaining knowledge on the set of relative prices that corresponds to the maximization of $W(V, x|\rho)$ with respect to V at each $\rho \in (-\infty, -1]$. Then, s/he recovers the set of relative prices that maximizes the level of welfare of each household in the society, associated to a level of ρ. Here, V is a K-dimensional vector of individual welfare functions $\ln V_k(p, y_k)$ for $k = 1, .., K$. The welfare maximization is well-defined only if $W(V(\bar{p}, y), \bar{p}|\rho)$ is at least strictly quasiconcave with respect to V, for any fixed level of prices $\bar{p}$. A dual problem can be defined as showed in the next Section. It leads to the minimization of an indirect welfare function $\mathcal{W}(V^*(p, \bar{y}), p|\rho)$, with respect to p, where the level of income $\bar{y}_k$ is now given in the indirect utilities $V_k^*(p, \bar{y}_k)$.

We now investigate the curvature properties of a general SWF using well-known notions of concavity and convexity (Avriel 1972).

3 The Curvature of the SWF

To study the curvature properties of the JS SWF, we first recover the indirect social welfare function to be maximized with respect to prices. Consider the maximization of social welfare with respect to the vector of indirect utility functions V with exogenous prices that are predetermined at level $\bar{p}$ and a fixed level of aversion to inequality ρ

$$\max \left\{ W(V(\bar{p}, y),\ \bar{p}\ |\rho)\ :\ V \in \mathcal{V} \right\}, \tag{9}$$

where $\mathcal{V}$ is the indirect utility possibility set. The optimal value functions

$$V^*(p, \bar{y}) = (\ln V_1^*(p, \bar{y}_1), \ldots, \ln V_K^*(p, \bar{y}_K)),$$

solution of problem (9), depend on prices p and describe the maximum level of individual welfare attainable for a given level of equivalent total expenditure $\bar{y}$.

The indirect social welfare function $\mathcal{W}(V^*(p, \bar{y}),\ p\ |\rho)$ represents the maximum value of welfare, for any $p \in \mathcal{P}$. Problem (9) is equivalent to

$$\max \left\{ W(V(\bar{p}, y),\ \bar{p}\ |\rho) - \mathcal{W}(V^*(p, \bar{y}),\ p\ |\rho)\ :\ V \in \mathcal{V},\ p \in \mathcal{P} \right\}, \tag{10}$$

where $\mathcal{P} = \{p : \ 0 \leq p \leq 1, \ \text{with} \ \sum_i p_i = 1\}$ is the set of feasible normalized prices. Problem (10) is called the primal-dual problem in Silberberg and Suen (2001). It reaches a maximum at zero and solving it with respect to prices p is equivalent to solving the following problem

$$\min \left\{\mathcal{W}(V^*(p, \bar{y}), \ p \ |\rho) \ : \ p \in \mathcal{P}\right\}. \tag{11}$$

The indirect SWF $\mathcal{W}$ measures what the society is willing to give up to reach a given level of welfare. The decision variables are the prices, because they represent the direction along which to move to achieve the equilibrium.[8] Also notice that the properties of $\mathcal{W}$ are the same as those of an household indirect utility function: (a) homogeneity of degree 0 in (p, y), (b) continuity, (c) non-increasing in p and non-decreasing in y and (d) quasiconvexity in p.[9]

We now specialize on the JS SWF as a function of prices and provide the conditions that make these properties hold in Section 3.2. We first recall the basic notions of generalized concavity and convexity that will be used in the proofs.

3.1 Generalized Concavity: Basic Definitions

The analysis of generalized concavity requires the use of basic notions of quasiconcavity and quasimonotonicity.

Definition 1. A function $f : X \to \mathbb{R}$ defined on a convex subset $X \subseteq \mathbb{R}^n$ of a real vector space is quasiconcave iff for all $x^0, x^1 \in X$ and $\alpha \in [0, 1]$ we have

$$f\left(\alpha x^0 + (1 - \alpha)x^1\right) \geq \min\left\{f(x^0), f(x^1)\right\}. \tag{12}$$

A function $f : X \to \mathbb{R}$ is said quasiconvex if $-f$ is quasiconcave. An equivalent condition for functions that are differentiable at least once is the following.

Definition 2. Let f be differentiable on the open convex set $X \subseteq \mathbb{R}^n$. Then f is quasiconcave iff for every $x^0, x^1 \in X$ the following inequality is verified

$$f(x^0) \leq f(x^1) \Longrightarrow (x^1 - x^0)\nabla f(x^0) \geq 0 \tag{13}$$

[8] Problem (9) can be interpreted as a pre-transfer problem, where $\bar{p}$ is the vector of market prices, while, problem (11) can be interpreted as a post-transfer problem, where prices change. If, for instance, $p = \bar{p} + T$, with $T = (T_1, \ldots, T_n)$ being the vector of the amounts of the transfers, from condition $p \in \mathcal{P}$ we have that $\sum_i T_i = 0$. Note, however, that the pricing rule at the basis of the transfer principle can be more general (Bosmans et al. 2009) accounting for different transformations of prices.

[9] As noted in Bosmans *et al.* (2018), when implementing social welfare comparisons it is necessary to specify a reference price vector as commonly done with money metrics of individual utilities. Incidentally, the possibility to maximize the SWF $\mathcal{W}$ with respect to prices may provide an admissible set of reference prices corresponding to the optimal solution of the maximization of the social welfare function.

or vice versa

$$(x^1 - x^0)\nabla f(x^0) < 0 \Longrightarrow f(x^0) > f(x^1). \tag{14}$$

If a function is both quasiconcave and quasiconvex, then it is quasimonotone.

Definition 3. A function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, X convex, is quasimonotone if for every $x^0, x^1 \in X$ and for each $\alpha \in [0,1]$ it holds

$$\min\left\{f(x^0), f(x^1)\right\} \leq f\left(\alpha x^0 + (1-\alpha)x^1\right) \leq \max\left\{f(x^0), f(x^1)\right\}. \tag{15}$$

Monotonicity and quasi monotonicity are equivalent for univariate functions. In general, a function is both concave and convex if and only if it is affine. Quasimonotone functions are generalizations of affine functions in the case the concave and convex functions are replaced by quasiconcave or quasiconvex functions.

Consider the vector of functions $v = (v_1, .., v_K) \in \mathbb{R}^K_+$ and the vector of weights $\lambda \in \Lambda$ with $\Lambda = \{\lambda | 0 \leq \lambda_k \leq 1,\ \forall k = 1, .., K\}$. The mean value function $M_h(\lambda; v)$ on a K-dimensional space is defined as follows.

Definition 4. Mean value function in vector space. Let h be a continuous and strictly monotone real valued function,

$$M_h(\lambda; v) = h^{-1}\left(\sum_{k=1}^{K} \lambda_k h(v_k)\right), \tag{16}$$

where $\lambda \in \Lambda$ and h^{-1} is the inverse function of h .

Note that if h is the identity function $h(v) = I(v) = v$ then $M_h(.)$ is the arithmetic mean, while if $h(v) = \ln v$, then $M_h(.)$ is the geometric mean and if $h(v) = v^\rho$ then $M_h(.)$ is a power mean of order ρ. The same notation can be extended to the case of vector functions.

Definition 5. (h, G) concavity (Ben-Tal 1977). A real valued function f on $\mathbb{R}^K_+$ is (h, G)-concave if and only if $\forall \lambda \in \Lambda$ and $\forall v = (v_1, .., v_K) \in \mathbb{R}^K_+$:

$$f(M_h(\lambda; v) \geq M_G(\lambda; f(v_1), .., f(v_K)), \tag{17}$$

$$\text{that is,}\quad f\left(h^{-1}\left(\sum_{k=1}^{K} \lambda_k h(v_k)\right)\right) \geq G^{-1}\left(\sum_{k=1}^{K} \lambda_k G(f(v_k))\right). \tag{18}$$

The choice of $h(.)$ and $G(.)$ as identity functions $h(.) = G(.) = I(.)$ defines the family of concave functions. When only $h(.)$ is chosen as the identity function $h(.) = I(.)$ then the G-concave family of functions is generated. G-concave functions are concave functions transformable by a continuous increasing function over a range.

Definition 6. ρ-concavity (Hardy, Littlewood and Pólya 1936, Avriel 1972, Caplin and Nalebuff 1991). Consider $\rho > 0$, a non-negative function f, with convex support is called ρ-concave if and only if

$$f\left(\sum_{k=1}^{K} \lambda_k v_k\right) \geq \left[\sum_{k=1}^{K} \lambda_k f(v_k)^{\rho}\right]^{\frac{1}{\rho}}, \ \forall \lambda \in \Lambda. \tag{19}$$

The definition refers to f^{ρ} being concave for positive ρ. For negative ρ, $-f^{\rho}$ is concave as it is in the case considered here. The parameter ρ is a measure of the degree of concavity of the function. The definition of ρ-concavity is also obtained as a special case of (18) by letting $h(.)$ be the identity function and $G(.) = [f(v)]^{\rho}$. Note that the standard definition of concavity is obtained when $\rho = 1$ and the mean value function takes the form of an arithmetic mean. The case of $\rho = 0$ corresponds to log-concavity to which a geometric mean is associated.

Further, recall that the sum of concave functions is concave, but this property does not hold in general for quasiconcave functions. Consider the sum of f_1, strictly increasing convex function, and f_2, strictly decreasing convex function. This function is convex, but it is not in general quasiconcave, even if both f_1 and f_2 are quasiconcave. For differentiable functions, quasiconcavity is related to a property of monotonicity of the gradient, that is not guaranted by the sum of an increasing and decreasing function. Therefore, it is useful to define the following class of quasiconcave functions.

Definition 7. Uniform Quasiconcavity. Two functions f_1 and f_2 are said uniformly quasiconcave if and only if

$$\min\{f_i(x_1), f_i(x_2)\} = f_i(x_1) \text{ or } f_i(x_2), \ \forall i = 1, 2, \forall x_1, \ x_2 \in \mathbb{R}^n. \tag{20}$$

Note that the sum of uniformly quasiconcave functions is also quasiconcave, and it holds the same for the product.

Proposition 1. *(Prékopa et al. 2011) Given two functions f_1 and f_2 uniformly quasiconcave and non-negative, then their product is quasiconcave.*

In the next section, we focus on the curvature properties of the JS SWF.

3.2 The Curvature Properties of the JS Social Welfare Function

Reconsider the Jorgenson and Slesnick specialization for a SWF:

$$W(V, p|\rho) = \ln \overline{V}(V, p) - \gamma(p) \left\{\sum_k a_k(p) \ |\ln V_k(p) - \ln \overline{V}(V, p)|^{-\rho}\right\}^{-\frac{1}{\rho}}, \tag{21}$$

where $\ln \overline{V}(V,p) = \frac{\sum_k m_o(p,d_k) lnV_k(p)}{\sum_k m_o(p,d_k)} = \sum_k a_k(p) \ln V_k(p)$ with $a_k(p) \in [0,1], \sum_k a_k(p) = 1$, $\gamma(p)$ is given in equation (8), k is the number of households in the society, and $m_o(p,d_j) = \min_k m_o(p,d_k)$ is the scale for the reference household. In line with Jorgenson and Slesnick (1990), the SWF embeds the following properties.

The JS SWF is equity regarding, because at a given level of average welfare, social welfare declines as the distribution of welfare levels becomes more dispersed. It is also efficiency regarding because it is strictly increasing if an individual utility function increases, all other things equal.

Further, the increase in the average level of individual welfare $\ln \overline{V}(V,p)$ must be larger than $g(V,p|\rho)$ representing the dispersion in individual welfare levels, if the individual welfares are considered as "goods". The weight $\gamma(p)$ is therefore chosen as a function of the weight $a_j(p) = m_o(p,d_j)/\sum_k m_o(p,d_k)$ respecting this condition.

This "monotonicity" property is known as positive association (PA). It has been formalized by Arrow (1950, 1963) in order to generalize the Pareto principle. It imposes that if an alternative state rises, or does not fall, in the ordering of each individual and it was preferred before the change, then it is still preferred. Equity considerations represented by the function g are affected by the size of the population through γ which depends on $a_j(p)$.

The concavity with respect to V can be deduced studying the curvature of the components $\ln \overline{V}(V,p)$ and $g(V,p|\rho)$. The term $\ln \overline{V}(V,p) = \sum_k a_k(p) \ln V_k(p)$ is a weighted sum of concave functions. The term $g(V,p|\rho) = -\gamma(p)\left[\sum_k a_k(p)|\ln V_k - \ln \overline{V}(V,p)|^{-\rho}\right]^{-\frac{1}{\rho}}$ is concave, because $M(V,p)$ is convex, being a weighted $\ell^{-\rho}$ norm of non-negative elements.

Recalling the equivalence between problem (9) and (11), we now focus on the properties of the SWF with respect to changes in prices. To learn about the curvature properties of the JS social preferences with respect to prices p, it is also crucial to know the curvature properties of the weighting function $a_k(p)$ which, in turn, depends on the structure of the demographic function $m_o(p,d_k)$. The equivalence scale $m_o(p,d_k)$ assigns a weight to each household in proportion to its needs. This weight depends on prices and exogenous attributes d_k and represents the number of household equivalent members. Menon et al. (2016) show that for the household income $y_k = y\, m_o(p,d_k)$ to be a plausible expenditure function, the demographic function $m_o(p,d_k)$ must satisfy the conditions described below.

Properties of the Household Equivalence Scales. The equivalence scale $m_o(p,d_k)$ is positive and non-decreasing in p_i, homogeneous of degree zero in p and quasi-concave.

Now, consider the normalized scale $a_k(p) = m_o(p,d_k)/\sum_k m_o(p,d_k)$. It weights the household equivalence scales relative to the sum of all types in the sample. It ranges in the

$[0, 1]$ interval and can be therefore interpreted as a relative frequency of a certain type in the population. An increase in p results in an increase in the cost of the needs of an individual. The increase can be more or less than proportional, depending on the compensation effect of the economies of scale. Notice also that the scale $a_k(p)$ has the same properties of $m_o(p, d_k)$, because the sum of equivalent incomes $y_k = y\, m_o(p, d_k)$ gives

$$\sum_k y_k = \sum_k y\, m_o(p, d_k) \quad \text{and} \quad \frac{\sum_k y_k}{\sum_k m_o(p, d_k)} = \frac{\sum_k y\, m_o(p, d_k)}{\sum_k m_o(p, d_k)}. \tag{22}$$

Then, we have

$$\frac{Y}{\sum_k m_o(p, d_k)} = \frac{\sum_k y\, m_o(p, d_k)}{\sum_k m_o(p, d_k)} = \sum_k a_k(p) y = y, \tag{23}$$

where Y represents the total income in the society and $a_k(p)y$ measures the total income per equivalent household. The weight $a_k(p)$ scales income y and therefore it satisfies the same properties of $m_o(p, d_k)$.

The following result states that the weights $a_k(p)$ are non decreasing in prices, but in a way that does not dominate the decrease of $\ln V_k(p)$, as shown below.

Proposition 2. *The change in the average of individual welfares* $\ln \overline{V}(V, p)$*, due to an increase in prices is such that*

$$\frac{\partial \ln \overline{V}(V, p)}{\partial p_i} = \sum_k \left[\frac{\partial a_k(p)}{\partial p_i} \ln V_k(p) \, (1 + \varepsilon_k) \right] < 0, \;\; i = 1, \ldots, n, \tag{24}$$

where $\varepsilon_k = \frac{\partial \ln V_k(p)/\partial p_i}{\partial a_k(p)/\partial p_i} \frac{a_k(p)}{\ln V_k(p)}$.

Proof. The property of Positive Association ensures that $\ln \overline{V}(V, p)$ must be increasing in each $V_k(p)$ and decreasing in each p_i. Hence,

$$\frac{\partial \ln \overline{V}(V, p)}{\partial p_i} = \sum_k \left[\frac{\partial a_k(p)}{\partial p_i} \ln V_k(p) + \frac{\partial \ln V_k(p)}{\partial p_i} a_k(p) \right] < 0. \tag{25}$$

Grouping terms in equation (25) we can see that the size of the change in the average of individual welfares depends on the relative change in $\ln V_k$ with respect to the change in the weight $a_k(p)$ through a change in price p_i

$$\sum_k \left[\frac{\partial a_k(p)}{\partial p_i} \ln V_k(p) \, (1 + \varepsilon_k) \right] \leq 0, \;\text{ with } \varepsilon_k = \frac{\partial \ln V_k(p)/\partial p_i}{\partial a_k(p)/\partial p_i} \frac{a_k(p)}{\ln V_k(p)} \leq 0.$$

This implies that the monotonicity of $\ln \overline{V}(V, p)$ depends on the values of ε_k, that is the elasticity of the function $\ln V_k(p)$ with respect to $a_k(p)$. $\square$

Note that the sign of each partial derivative of $\ln \overline{V}$ depends on the elasticity ε_k between $\ln V_k$ and a_k, with $k = 1, \ldots, K$. Under anonimity, $a_k = 1/K$, $\forall k$ and $\partial \ln \overline{V}(V,p)/\partial p_i = (1/K)\sum_k \partial \ln V_k(p)/\partial p_i$ is negative because each $\partial \ln V_k(p)/\partial p_i$ is negative.

We now show the generalized convexity of $\ln \overline{V}$.

Proposition 3. *The function* $\ln \overline{V}$ *is quasiconvex with respect to* p.

Proof. From Positive Association and Proposition 2, each function $a_k(p) \ln V_k(p)$ is decreasing and quasiconvex by Definition 2. Then $\ln \overline{V}(V,p)$ is the sum of uniformly quasiconvex functions (see Definition 7) and hence it is quasiconvex. $\square$

In order to to examine the properties of the real value function $g(V,p|\rho) = -\gamma(p)M(V,p)$, we introduce the following property of Monotonicity of the Deviations of the individual welfare functions from the mean with respect to p_i, $i = 1, \ldots, n$.

Property of Monotonicity of the Deviations (MD). Define $\delta_k(V,p)$ as the deviation function $\delta_k(V,p) = \ln V_k(p) - \ln \overline{V}(V,p)$ such that $\partial \ln \delta_k(V,p)/\partial p_i \geq 0$, $\forall k = 1, \ldots, K$ and $\forall i = 1, \ldots, n$.

Note that the property requires that if $\delta_k(V,p) > 0$, then $\partial \delta_k(V,p)/\partial p_i \geq 0$. Therefore, if $\ln V_k(p) > \ln \overline{V}(V,p)$, then $\partial \ln V_k(p)/\partial p_i \geq \partial \ln \overline{V}(V,p)/\partial p_i$ and the difference between $\ln V_k(p)$ and $\ln \overline{V}(V,p)$ is not decreasing. While if $\delta_k(V,p) < 0$, then $\partial \delta_k(V,p)/\partial p_i \leq 0$, and the difference between $\ln V_k(p)$ and $\ln \overline{V}(V,p)$ is not increasing, or equivalently, the difference between $\ln \overline{V}(V,p)$ and $\ln V_k(p)$ is not decreasing.

We can describe $\partial \delta_k(V,p)/\partial p_i$ as a variation of the difference in the welfare of household k from the mean. Given a change in p_i, the change in the household welfare $|\partial \ln V_k(p, y_k)/\partial p_i|$ is lower than the change in the mean $|\partial \ln \overline{V}(V,p)/\partial p_i|$, for a richer household. The opposite holds for a poorer household. Further, MD maintains the ranking of individual k after a price change.

Proposition 4. *The function*

$$\begin{aligned} g(V,p|\rho) &= -\gamma(p)M(V,p) = \\ & -\left\{(1-a_j(p))\left[1+\left(\frac{1-a_j(p)}{a_j(p)}\right)^{-(\rho+1)}\right]\right\}^{\frac{1}{\rho}}\left[\sum_k a_k(p)|\ln V_k(p) - \ln \overline{V}(V,p)|^{-\rho}\right]^{-\frac{1}{\rho}}, \end{aligned} \tag{26}$$

is non-increasing and quasiconvex with respect to p.

Proof. Note that $\partial g(V,p)/\partial p_i = -\left(M(V,p)\partial\gamma(p)/\partial p_i + \gamma(p)\partial M(V,p)/\partial p_i\right)$, with

$$\begin{aligned}\frac{\partial\gamma(p)}{\partial p_i} &= \frac{1}{\rho}\left\{(1-a_j(p))\left[1+\left(\frac{1-a_j(p)}{a_j(p)}\right)^{-(\rho+1)}\right]\right\}^{\frac{1}{\rho}-1}\cdot \\ &\cdot\left(-\frac{\partial a_j(p)}{\partial p_i}\right)\left[1+\left(\frac{1-a_j(p)}{a_j(p)}\right)^{-(\rho+1)}\left(1-\frac{\rho+1}{a_j(p)}\right)\right]\geq 0 \end{aligned}\tag{27}$$

because $a_j(p)$ is non-decreasing. Then the sign of the partial derivative of $g(V,p|\rho)$ depends also on the partial derivative of $M(V,p)$, which is $\frac{\partial M(V,p)}{\partial p_i} = \mu_i^1(V,p)\mu_i^2(V,p)$, with $\mu_i^1(V,p) = -\frac{1}{\rho}\left[\sum_k a_k(p)|\ln V_k(p) - \ln\overline{V}(V,p)|^{-\rho}\right]^{-\frac{1}{\rho}-1} \geq 0$ and

$$\mu_i^2(V,p) = \sum_k\left[\frac{\partial a_k}{\partial p_i}|\ln V_k(p) - \ln\overline{V}(V,p)|^{-\rho}\right]\left[1-\frac{\rho}{\frac{\partial \ln a_k}{\partial p_i}}\frac{\partial\ln\left(\ln V_k - \ln\overline{V}(V,p)\right)}{\partial p_i}\right].\tag{28}$$

Because $\partial a_k(p)/\partial p_i \geq 0$, then $\frac{\partial a_k(p)}{\partial p_i}|\ln V_k(p) - \ln\overline{V}(V,p)|^{-\rho} \geq 0$ and $-\rho\Big/\frac{\partial\ln a_k(p)}{\partial p_i} \geq 0$. Finally, $\partial\ln\left(\ln V_k(p) - \ln\overline{V}(V,p)\right)/\partial p_i \geq 0$, because of the MD Property. Hence, we have $\mu_i^2(V,p)\geq 0$ and $\partial g(V,p)/\partial p_i \geq 0$, $\forall i = 1,\ldots,n$. Then, γ is quasiconcave because it is an increasing transformation of the quasiconcave function $a_j(p)$. The function M is also quasiconcave in prices because it is an increasing transformation of the sum of uniformly quasiconcave functions. In fact, $M(V,p) = \left[\sum_k a_k(p)|\ln V_k - \ln\overline{V}(V,p)|^{-\rho}\right]^{-\frac{1}{\rho}} = \left[\sum_k f_k(V,p)\right]^{-\frac{1}{\rho}}$, where $f_k(V,p) = a_k(p)|\ln V_k - \ln\overline{V}(V,p)|^{-\rho}$ is non-decreasing and quasiconcave in prices, because of the MD Property. □

We now use these results to formalize the curvature property of the function $W(V,p|\rho)$.

Proposition 5. *The function $W(V,p|\rho)$ is decreasing and quasiconvex with respect to p.*

Proof. Proposition 4 states that the product $g(V,p) = -\gamma(p)M(V,p)$ is non-increasing and quasiconvex. Propositions 2 and 3 ensure that $\ln\overline{V}(V,p)$ is decreasing and quasiconvex. Consequently, $\ln\overline{V}(V,p)$ and $g(V,p)$ are uniformly quasiconvex and their sum is decreasing and quasiconvex with respect to p. □

4 Conclusions

The main contribution of this study is the definition of the curvature properties of each object composing the SWF and of the social functional in its aggregate. Only a regular SWF is suitable both for welfare maximization and theoretically plausible microsimulations.

A SWF is well-behaved when the regularity properties described in the study are empirically respected.

By completing the set of requirements with the characterization of the curvature properties, the Jorgenson and Slesnick SWF is ready for a more general use. The concavity results obtained here can be extended to other functional forms by using analogous lines of proof. Further, for example, the knowledge of the effects of price variations on the SWF permits analyzing the conditions necessary for optimal welfare-improving price subsidies and taxation.

For future research, the efficiency and equity considerations involved in the JS SWF should also be extended to the exact aggregation process summing up the individual welfare of each family member to a household welfare function (Chavas *et al.* 2018) and then to a SWF. This research endeavor requires that our knowledge of inter-household comparisons be extended to the realm of inter-personal/intra-family comparisons in order to construct household welfare functions that can be plausibly aggregated both at the household and social level.

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