ECINEQ

Society for the Study of Economic Inequality

Working Paper Series

Prioritarian evaluation of well-being with an ordinal variable

Suman Seth
Gaston Yalonetzky

ECINEQ WP 2020 - 531

ECINEQ 2020 - 531
June 2020

www.ecineq.org

Prioritarian evaluation of well-being with an ordinal variable

Suman Seth
University of Leeds and OPHI

Gaston Yalonetzky
University of Leeds

Abstract

Additive social evaluation measures have been proposed and are commonly used to assess well-being with an ordinal variable. In this paper, we derive appropriate functional-form restrictions allowing additive social evaluation measures for ordinal variables to provide different degrees of priority to those relatively worse-off. To assess the robustness of societal well-being comparisons to alternative choices of distribution-sensitive measures, we propose tractable stochastic dominance conditions for different degrees of priority.

Keywords: Ordinal variables, measurement of well-being, Hammond transfer, inequality aversion, stochastic dominance, prioritarianism.

JEL Classification: I3, I31, D63.

1. Well-being evaluation with an ordinal variable

Non-monetary ordinal variables are frequently used for assessing societal well-being. Examples abound, ranging from different types of subjective well-being variables (e.g. OECD, 2013) to variables capturing access to basic services of varying quality (e.g. WHO and UNICEF, 2015). Unlike cardinally measurable variables, however, only ordered categories are observed for ordinal variables. Formally, suppose, well-being in a hypothetical society is assessed by some variable consisting of a fixed set of $S \geq 2$ ordered categories $c_1, \ldots, c_S$. Category c_s, for every s, reflects a strictly better-off situation than category c_{s-1} and is denoted by $c_s \succ_W c_{s-1}$, where $\succ_W$ is a binary and transitive relation. Thus, category c_S reflects the highest level of well-being; whereas, category c_1 reflects the lowest level. Each individual experiences only one category. The population proportion experiencing category c_s is denoted by p_s, where $p_s \geq 0$ for all s and $\sum_{s=1}^{S} p_s = 1$. We denote the distribution of population proportions (i.e. relative frequencies) across S categories by $\mathbf{p} = (p_1, \ldots, p_S)$ and the set of all such distributions over S categories by $\mathbb{P}$.

While evaluating societal well-being using an ordinal variable, especially in the subjective well-being literature, it is customary to assign some numerical valuation or scale (e.g. Cantril Ladder) to each category, respecting the categories' order, and to use an additive measure for evaluating overall social well-being (e.g. see Helliwell et al., 2019). We denote the numerical evaluation or scale of every category s by $w_s \in \mathbb{R}$ such that $w_s < w_t$ for all $s < t$ and summarise all S scales by vector $\mathbf{w} = (w_1, \ldots, w_S)$. We denote the class of additive societal well-being measures by:

$$W(\mathbf{p}) = \sum_{s=1}^{S} p_s w_s. \tag{1.1}$$

We denote the class of all well-being measures in Equation 1.1 by $\mathcal{W}$. Such measures have been axiomatically characterised by Gravel et al. (2011) and Apouey et al. (2019) and have been used by Gravel et al. (2020) to derive normative criteria for ranking distributions of an ordinal variable. All measures in $\mathcal{W}$ are *additively decomposable*; i.e. overall societal well-being can be expressed as a population-weighted average of the population subgroups' well-being, whenever that society's population is divided into mutually exclusive and collectively exhaustive subgroups.

Crucially, all measures in $\mathcal{W}$ satisfy *monotonicity*; i.e. they are sensitive to the order of the categories such that, all else unchanged, if an individual moves to a more affluent category, society's overall well-being evaluation should be higher. Though monotonicity is an important property, it does not ensure that well-being evaluations are robust across alternative scales (Allison and Foster, 2004). Moreover, as shown in the following illustration, monotonicity does not ensure that priority is given to relatively worse-off people. Consider the following four different population distributions across six ordered categories ($S = 6$):

$$\mathbf{p}^A = (0.1, 0.2, 0.2, 0.2, 0.2, 0.1), \quad \mathbf{p}^B = (0.05, 0.25, 0.2, 0.2, 0.2, 0.1),$$
$$\mathbf{p}^C = (0.1, 0.2, 0.2, 0.2, 0.15, 0.15), \quad \mathbf{p}^D = (0.1, 0.15, 0.2, 0.2, 0.2, 0.15).$$

Note that distribution $\mathbf{p}^B$ is obtained from $\mathbf{p}^A$ by moving 5% of population from the lowest category to the second-lowest category; distribution $\mathbf{p}^C$ is obtained from $\mathbf{p}^A$ by moving 5% of population from the second-highest category to the highest category; and distribution $\mathbf{p}^D$ is obtained from $\mathbf{p}^A$ by moving 5% of population from the second-lowest category to the highest category. Combining a measure in $\mathcal{W}$ with a scale akin to a Cantril Ladder like $\mathbf{w} = (1, 2, 3, 4, 5, 6)$, we obtain the following social evaluation ranking:

$$W(\mathbf{p}^A) < W(\mathbf{p}^B) = W(\mathbf{p}^C) < W(\mathbf{p}^D). \tag{1.2}$$

Monotonicity explains and justifies why social evaluations are higher for $\mathbf{p}^B$, $\mathbf{p}^C$ and $\mathbf{p}^D$ than for $\mathbf{p}^A$. But the rankings among $\mathbf{p}^B$, $\mathbf{p}^C$ and $\mathbf{p}^D$ could be controversial. Unlike distributions $\mathbf{p}^C$ and $\mathbf{p}^D$, distribution $\mathbf{p}^B$ has been obtained from distribution $\mathbf{p}^A$ by improving the situation of 5% of the population experiencing the worst category. Hence, contrary to the ranking in expression 1.2, should the social evaluation of $\mathbf{p}^B$ be higher than that of both $\mathbf{p}^C$ and $\mathbf{p}^D$? The answer depends on the degree of priority the social planner is willing to give to those relatively worse-off. The next section presents different normative criteria for providing different *degrees of priorities* to the relatively worse-off.

2. Providing priority to the worse-off

Ethically-grounded appeals to render social welfare evaluations sensitivity to inequality (e.g. Atkinson, 1970; Foster and Sen, 1997; Kolm, 1998) together with the United Nations' renewed pledge to 'leaving no one behind' (United Nations, 2018), justify devising social evaluation measures that, effectively, prioritise improvements among those experiencing the worst categories. Recently, Gravel et al. (2020) and Apouey et al. (2019) have proposed ways to incorporate sensitivity to inequality in the additive social evaluation measures presented in Equation 1.1. Specifically, Gravel et al. (2020) pioneered the operationalisation of *Hammond transfers* (Hammond, 1976) in the case of ordinal variables, while Apouey et al. (2019) put forward the remarkably identical *equity principle*. In general, the Hammond transfer principle requires that a reduction in the gap between a poorer individual and a richer one, *ceteris paribus*, should improve social evaluation, irrespective of the size of the gain for the poorer individual and the size of the loss for the richer one, as long as their relative ranks remain unaltered.[1]

In this paper, instead of being unduly restrictive, we introduce an intuitive normative criterion for giving different degrees of *priority to the worse-off* when assessing well-being in the *ordinal-variable* framework, drawing from Parfit (1997) and Seth and Yalonetzky (2020). The degrees of priority vary between a minimum and a maximum. In one extreme, the *minimal priority* (PRI-MIN) criterion requires that, *ceteris paribus*, moving an ϵ fraction of worse-off people to an adjacent improved category should lead to a larger increase in social well-being than moving an ϵ fraction of a relatively better-off people to a respective adjacent improved category. In the other extreme, the *maximal priority* (PRI-MAX) criterion, which

[1] With continuous variables, the well-known Pigou-Dalton transfer principle is a particular case of the Hammond transfer principle, with the additional restriction that the sizes of the gains and losses should be equal.

is conceptually equivalent to the Hammond transfer principle for ordinal variables (Gravel et al., 2020), requires that, *ceteris paribus*, moving an ϵ fraction of worse-off people to an adjacent improved category should lead to a larger increase in well-being than moving an ϵ fraction of relatively better-off people to *any* improved category. With more than three categories, intermediate forms of priority between minimal and maximal are also feasible.

We refer to the general property as *priority to the worse-off of order* α (PRI-α), which requires that, *ceteris paribus*, moving an ϵ fraction of worse-off people to an adjacent improved category leads to a larger increase in social well-being than moving an ϵ fraction of better-off people up to an α (≥ 1) number of adjacent improved categories. A formal general statement of the property, which includes PRI-MIN (i.e. $\alpha = 1$) and PRI-MAX (i.e. $\alpha = S - 2$) as *limiting cases*, is the following:

Priority of order $\boldsymbol{\alpha}$ to the worse-off (PRI-$\boldsymbol{\alpha}$) For any $\mathbf{p}, \mathbf{p}', \mathbf{q}' \in \mathbb{P}$, for some $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq S - 2$, for some $s < t < S$ and for some $\epsilon \in (0, 1)$, if (i) $\mathbf{p}'$ is obtained from $\mathbf{p}$ such that $p'_s = p_s - \epsilon$ while $p'_u = p_u \; \forall u \neq \{s, s+1\}$, and (ii) $\mathbf{q}'$ is obtained from $\mathbf{p}$ such that $q'_t = p_t - \epsilon$ while $q'_u = p_u \; \forall u \neq \{t, \min\{t + \alpha, S\}\}$, then $W(\mathbf{p}') > W(\mathbf{q}')$ for any $W \in \mathcal{W}$.

As the value of α increases, a social planner's prioritisation of the worse-off rises. In Theorem 2.1, we present the subclasses of measures $\mathcal{W}_\alpha$ that satisfy the PRI-α property:

Theorem 2.1 For any $S \geq 3$ and for some $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq S - 2$, a well-being measure $W \in \mathcal{W}$ satisfies property PRI-α if and only if

a. $2w_s > w_{s-1} + w_{s+\alpha} \; \forall s = 2, \ldots, S - \alpha$ and $2w_s > w_{s-1} + w_S \; \forall s = S - \alpha + 1, \ldots, S - 1$ whenever $\alpha \leq S - 3$.

b. $2w_s > w_{s-1} + w_S \; \forall s = 2, \ldots, S - 1$ whenever $\alpha = S - 2$.

Proof. The sufficiency part is straightforward. We prove the necessity part as follows. Suppose, $S \geq 3$ and $\alpha \in \mathbb{N}$ such that $1 \leq \alpha \leq S - 2$. Consider some $s' < t \leq S - 2$. Now, $\mathbf{p}'$ is obtained from $\mathbf{p}$, such that $p'_{s'} = p_{s'} - \epsilon$ while $p'_u = p_u \; \forall u \neq \{s', s' + 1\}$, and so $p'_{s'+1} = p_{s'+1} + \epsilon$. Likewise, $\mathbf{q}'$ is obtained from $\mathbf{p}$, such that $q'_t = p_t - \epsilon$ while $q'_u = p_u \; \forall u \neq \{t, t'\}$ for some $t' = \min\{t + \alpha, S\}$, and so $q'_{t'} = p_{t'} + \epsilon$. So, by property PRI-$\alpha$:

$$W(\mathbf{p}') > W(\mathbf{q}'). \tag{2.1}$$

Combining Equations 1.1 and 2.1 and substituting $t = s' + 1 = s$ for any $s = 2, \ldots, S - 1$, we obtain

$$w_s - w_{s-1} > w_{t'} - w_s. \tag{2.2}$$

First, suppose $t' = s + \alpha \leq S$ or $s \leq S - \alpha$. Then, $w_{t'} = w_{s+\alpha}$ and Equation 2.2 results in $2w_s - w_{s-1} > w_{s+\alpha}$ for all $s = 2, \ldots, S - \alpha$. Second, suppose $t' = \min\{s + \alpha, S\} = S$ or $s > S - \alpha$. Hence, Equation 2.2 results in $2w_s - w_{s-1} > w_S \; \forall s = S - \alpha + 1, \ldots, S - 1$. ■

In Corollaries 2.1 and 2.2, we present two limiting cases that satisfy properties PRI-MIN and PRI-MAX, respectively, which directly follow from Theorem 2.1:

Corollary 2.1 For any $S \geq 3$, a well-being measure $W \in \mathcal{W}$ satisfies property PRI-MIN (i.e., PRI-α for $\alpha = 1$) if and only if $2w_s > w_{s-1} + w_{s+1}$ $\forall s = 2, \dots, S-1$.[2]

Corollary 2.2 For any $S \geq 3$, a well-being measure $W \in \mathcal{W}$ satisfies property PRI-MAX (i.e., PRI-α for $\alpha = S-2$) if and only if $2w_s > w_{s-1} + w_S$ $\forall s = 2, \dots, S-1$.

By way of illustration, Table 1 compares well-being evaluation rankings of the four distribution from the previous section ($\mathbf{p}^A$, $\mathbf{p}^B$, $\mathbf{p}^C$ and $\mathbf{p}^D$) using scales satisfying different degrees of priority ($\alpha = 1, 2, 3, 4$). Clearly, for any degree of priority, $\mathbf{p}^B$ yields higher level of social well-being than $\mathbf{p}^C$. However, with relatively low priority ($\alpha = 1, 2$), $\mathbf{p}^D$ has higher level of social well-being than $\mathbf{p}^B$. By contrast, when maximum priority is provided to people relatively worse-off (i.e. $\alpha = 4$), then social well-being becomes higher in $\mathbf{p}^B$ than even in $\mathbf{p}^D$.

Table 1: Degree of priority and social well-being ranking

α	Scale-vector ($\mathbf{w}$)	Well-being rankings
1	$(1, 2.2, 3.3, 4.3, 5.2, 6)$	$W(\mathbf{p}^A) < W(\mathbf{p}^C) < W(\mathbf{p}^B) < W(\mathbf{p}^D)$
2	$(1, 3.2, 4.5, 5.3, 5.7, 6)$	$W(\mathbf{p}^A) < W(\mathbf{p}^C) < W(\mathbf{p}^B) < W(\mathbf{p}^D)$
3	$(1, 3.5, 4.8, 5.5, 5.8, 6)$	$W(\mathbf{p}^A) < W(\mathbf{p}^C) < W(\mathbf{p}^B) = W(\mathbf{p}^D)$
4	$(1, 3.6, 4.9, 5.5, 5.8, 6)$	$W(\mathbf{p}^A) < W(\mathbf{p}^C) < W(\mathbf{p}^D) < W(\mathbf{p}^B)$

3. Priority dominance conditions

In this section, we present tractable dominance conditions to test the robustness of a well-being ranking of distributions to alternative reasonable comparison criteria for all well-being measures in subclass $\mathcal{W}_\alpha$ for some $\alpha \in \{1, 2, ..., S-2\}$. We refer to dominance for a particular value of α as PRI-α dominance. In order to conclude dominance between two distributions for some α, we are required to compare the weighted sums of population proportions of these two distributions for $S-1$ distinct sets of weights. We denote the r^{th} set of weights by $\boldsymbol{\omega}^r = (\omega_1^r, \dots, \omega_S^r)$ for all $r = 1, \dots, S-1$ and derive the explicit values of these weights (ω_s^r $\forall s = 1, \dots, S$ and $\forall r = 1, \dots, S-1$) in Theorem 3.1 drawing from Seth and Yalonetzky (2020, Theorem 4).

[2]The restriction in Corollary 2.1 is identical to that obtained by Apouey et al. (2019, Proposition 2). However, we should point out that the result, as claimed by Apouey et al. (2019), does not follow from their own equity principle property. Consider two distributions $\mathbf{p}' = (\frac{1}{3}, 0, 0, \frac{2}{3})$ and $\mathbf{q}' = (0, \frac{2}{3}, 0, \frac{1}{3})$ as suggested by Apouey et al. (2019) and a scale-vector $\mathbf{w} = (0, \frac{1}{8}, \frac{6}{25}, \frac{1}{3})$. Clearly, $W(\mathbf{p}') > W(\mathbf{q}')$, contradicting their claim.

Theorem 3.1 For any $\mathbf{p}, \mathbf{q} \in \mathbb{P}$, for some $S \geq 3$, and for some $\alpha \in [1, S-2] \subseteq \mathbb{N}$, $W(\mathbf{p}) > W(\mathbf{q})$ for all $W \in \mathcal{W}_\alpha$ if and only if, with at least one strict inequality, $\sum_{s=1}^{S} \omega_s^r (p_s - q_s) \leq 0 \; \forall r = 1, \ldots, S-1$ such that

$$\omega_s^r = \begin{cases} 0 & \text{for } s > r \text{ and } r = 1, \ldots, S-1 \\ 1 & \text{for } s = 1 \text{ and } r = 1, \ldots, S-1 \\ 2^{1-s} & \text{for } s = 2, \ldots, r \text{ and } r = 2, \ldots, \alpha+1 \\ \dfrac{2^{r-s}}{\sum_{j=0}^{\overline{r}} (-1)^j \binom{r-\alpha j-1}{j} 2^{r-(\alpha+1)j-1}} & \text{for } s = r-\alpha, \ldots, r \text{ and } r = \alpha+2, \ldots, S-1 \\ \dfrac{\sum_{j=0}^{\overline{s}} (-1)^j \binom{r-s-\alpha j}{j} 2^{r-s-(\alpha+1)j}}{\sum_{j=0}^{\overline{r}} (-1)^j \binom{r-\alpha j-1}{j} 2^{r-(\alpha+1)j-1}} & \text{for } s = 2, \ldots, r-\alpha-1 \text{ and } r = \alpha+3, \ldots, S-1 \end{cases},$$

where $\overline{r} = \left\lfloor \dfrac{r-1}{\alpha+1} \right\rfloor$ and $\overline{s} = \left\lfloor \dfrac{r-s}{\alpha+1} \right\rfloor$.[3]

Proof. By our definition above, we know that $w_S - w_1 > 0$. Consider the following variable transformation $\omega_s = (w_S - w_s)/(w_S - w_1)$ for all s. Clearly, $1 = \omega_1 > \omega_2 > \cdots > \omega_S = 0$. For $\mathbf{p}, \mathbf{q} \in \mathbb{P}$, define $P(\mathbf{p}) = \sum_{s=1}^{S} \omega_s p_s$. Note that $P(\mathbf{p})$ is the additively decomposable poverty measure for ordinal variables introduced by Seth and Yalonetzky (2020, Theorem 1) when the poverty threshold category is fixed at c_{S-1} (i.e. there is only one non-deprivation category, namely c_S).[4]

Then we can easily establish that $W(\mathbf{p}) - W(\mathbf{q}) > 0$ if and only if $P(\mathbf{p}) - P(\mathbf{q}) = \sum_{s=1}^{S} \omega_s (p_s - q_s) < 0$, since $P(\mathbf{p}) = (w_S - W(\mathbf{p}))/(w_S - w_1)$ and so $W(\mathbf{p}) - W(\mathbf{q}) = -(w_S - w_1)[P(\mathbf{p}) - P(\mathbf{q})]$. Moreover, if scales in $\mathbf{w}$ satisfy the restrictions defining the class $\mathcal{W}_\alpha$ (Theorem 2.1), then it can be easily verified that the weights in $\boldsymbol{\omega} = (\omega_1, ..., \omega_S)$ satisfy the restrictions defining the class of poverty measures $\mathcal{P}_\alpha$ presented in Theorem 2 of Seth and Yalonetzky (2020) with poverty threshold category c_{S-1}.[5]

Hence, the dominance conditions presented in Theorem 4 of Seth and Yalonetzky (2020) for poverty threshold category c_{S-1} are applicable to ascertain whether $W(\mathbf{p}) - W(\mathbf{q}) > 0$ for all $W \in \mathcal{W}_\alpha$ because Theorem 4 states that $\sum_{s=1}^{S} \omega_s^r (p_s - q_s) \leq 0 \; \forall r = 1, \ldots, S-1$ (with at least one strict inequality and the ω_s^r as defined in Theorem 3.1) if and only if $P(\mathbf{p}) - P(\mathbf{q}) < 0$ for all $P \in \mathcal{P}_\alpha$. This completes our proof. ■

Theorem 3.1 states that, for a given $\alpha \in [1, 2, ..., S-2]$, well-being in distribution $\mathbf{p}$ is

[3] By $\lfloor b \rfloor$ for any $b \in \mathbb{R}_{++}$, we denote the largest possible non-negative integer that is not greater than b.

[4] Note that Seth and Yalonetzky (2020) denote the poverty measure for a threshold category c_k for some $k \in \{1, 2, \ldots, S-1\}$ by $P(\mathbf{p}, c_k)$. For a fixed poverty threshold category c_{S-1}, however, we simply use the notation $P(\mathbf{p})$.

[5] For instance, consider the case $\alpha = S-2$ in Theorem 2.1. Subtracting $2w_S$ from both sides, dividing by $(w_S - w_1)$ and rearranging, yields: $2\omega_s < \omega_{s-1}$ for all $s = 2, ..., S-1$, which is essentially the restriction defining the class of poverty measures $\mathcal{P}_{S-2}$ in Seth and Yalonetzky (2020).

higher than that in distribution $\mathbf{q}$ for all $W \in \mathcal{W}_\alpha$ (i.e., $\mathbf{p}$ PRI-α dominates $\mathbf{q}$) if and only if $\sum_{s=1}^{S} \omega_s^r p_s \leq \sum_{s=1}^{S} \omega_s^r q_s$ for all $r = 1, \ldots, S-1$ with at least one strict inequality.

4. Concluding remarks

Building on previous attempts in the literature on well-being measurement with ordinal variables, we have gone fruitfully further in the direction of operationalising different concepts of 'priority to the less advantaged', which ensures that the policy-maker has an incentive to assist those relatively worse off. This proposal echoes the notions of precedence to poorer people among the poor put forward by Seth and Yalonetzky (2020). We have shown that it is possible to devise reasonable social evaluation measures prioritising well-being improvements among the less advantaged when variables are ordinal. We have axiomatically characterised subclasses of ordinal-variable welfare measures based on different degrees of priority to the less advantaged. Each subclass is defined by a restriction on the admissible scale-vectors.

References

Allison, R. A. and J. E. Foster (2004). Measuring health inequality using qualitative data. *Journal of Health Economics 23*, 505–24.

Apouey, B., J. Silber, and Y. Xu (2019). On inequality-sensitive and additive achievement measures based on ordinal data. *Review of Income and Wealth*. DOI: 10.1111/roiw.12427.

Atkinson, A. (1970). On the measurement of economic inequality. *Journal of Economic Theory 2*(3), 244–63.

Foster, J. E. and A. K. Sen (1997). *On Economic Inequality*. Oxford University Press.

Gravel, N., B. Magdalou, and P. Moyes (2020). Ranking distributions of an ordinal variable. *Economic Theory*. 10.1007/s00199-019-01241-4.

Gravel, N., T. Marchant, and A. Sen (2011). Comparing societies with different numbers of individuals on the basis of their average advantage. In M. Fleurbaey, M. Salles, and J. Weymark (Eds.), *Social Ethics and Normative Economics*, pp. 261–277. Springer.

Hammond, P. J. (1976). Equity, Arrow's conditions, and Rawls' difference principle. *Econometrica 44*(4), 793–804.

Helliwell, J., R. Layard, and J. Sachs (2019). *World Happiness Report 2019*. New York: Sustainable Development Solutions Network.

Kolm, S.-C. (1998). *Justice and equity*. MIT Press.

OECD (2013). *OECD Guidelines on Measuring Subjective Well-Being*. OECD. OECD Publishing.

Parfit, D. (1997). Equality and priority. *Ratio 10*(3), 202–221.

Seth, S. and G. Yalonetzky (2020). Assessing deprivation with ordinal variables: theory and application to water sanitation in bangladesh. *World Bank Economic Review*. DOI: 10.1093/wber/lhz051.

United Nations (2018). Leaving no one behind. Excerpt from Committee for Development Policy, Report on the twentieth, See Official Records of the Economic and Social Council, 2018, Supplement No. 13 (E/2018/33).

WHO and UNICEF (2015). Wash post-2015. proposed indicators for drinking water, sanitation and hygiene. Technical report, WHO / UNICEF Joint Monitoring Programme for Water Supply and Sanitation.