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New Perspectives on the Gini and Bonferroni Indices of Inequality

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Abstract

This paper rigorously demonstrates that for any unequal income distribution, the well-known Gini index of inequality is bounded above by the recently revived Bonferroni inequality index. The bound is exactly attained if and only if out of n incomes in the society, $(n-1)$ poor incomes are identical. The boundedness theorem is shown to possess a duality-type inequality implication. Reinterpreting a property of the absolute Gini index, noted by Weymark (1981), we propose a new postulate, 'additive monotonicity', for inequality indices and analyse its sensitivity to the absolute and relative Bonferroni, and the relative Gini indices. Finally, we look at the pattern of the income distribution when a society wishes to guarantee a minimum income for the worst off person and fixes the inequality levels, as measured by the Gini and the Bonferroni indices, at some specific values.

Keywords: Gini and Bonferroni indices; boundedness; additive monotonicity; maximin rule and lexicographic extension.

JEL Classification: C43, D31, D63, O15.

1 Introduction

The Gini index, the most popular index of inequality, is a relative index. A relative inequality index is one that remains invariant under equi-proportionate changes in all incomes. The Gini index has several advantages: it is easy to compute, can accommodate non-positive incomes, is bounded between zero and one for non-negative incomes, has a nice graphical interpretation, can be regarded as a measure of relative deprivation and possesses a nice compromise property – when multiplied by the positive mean it becomes an absolute index that remains unvarying for equal absolute changes in all incomes (see Blackorby and Donaldson, 1980 and Weymark, 1981). It as well satisfies the Daltonian population principle – an income by income replication of the population keeps inequality unchanged so that it becomes suitable for cross-population comparisons of inequality. Donaldson and Weymark (1980) employed this postulate to characterize the S-Gini family of inequality indices that contains the Gini index as a special case.

All these advantages of the Gini index, except the population principle, are shared by the less known Bonferroni index, which relies on the comparison of the partial means and the general mean of an income distribution. Only in recent years, some authors attempted to analyse this indicator of inequality from different perspectives. (See, among others, Nygard and Sandstrom (1981), Giorgi (1984), Tarsitano (1990), Giorgi and Mondani (1995), Aaberge (2000, 2007), Giorgi and Crescenzi (2001) and Chakravarty (2007).)

While the Gini index of an income distribution is double the area enclosed between the line of equality of the Lorenz curve of the distribution and the Lorenz curve itself, for the Bonferroni index the relevant device is the Bonferroni curve. Assuming that incomes are non-decreasingly ordered, Barcena-Martin and Silber (2013) defined the Bonferroni curve as the plot of the ratios between cumulative income shares and cumulative population proportions against the cumulative population proportions. The Bonferroni index turns out to the area between the Bonferroni curve and the horizontal line at height 1. The Bonferroni curve is just an alternative name of 'scaled conditional mean curve' considered by Aaberge (2007) who provided an excellent discussion on different properties of the index. In particular, he showed that the index may not satisfy the principle of diminishing transfer sensitivity (Kolm, 1976 and Shhorrocks and Foster, 1987), a stronger form of the principle of transfer that requires inequality to reduce under an income transfer from a person to anyone who has a lower income, unambiguously. However, it unambiguously fulfills the positional transfer sensitivity condition, a positional variant of the diminishing transfer sensitivity principle (see Mehran, 1976 and Kakwani, 1980). Barcena-Martin and Silber (2013) also devised an algorithm that simplifies the decomposition of the Bonferroni index with respect to income sources, income classes and population subgroups.

In Section 2 of the paper where we present the basics and preliminaries, we make a simple yet an extremely important observation which claims that the dominance relation generated by the non-intersecting Lorenz curves of two income distributions of a given total over a given population size is identical to that produced by the non-intersecting Bonferroni curves of the distributions. Given this, and many common attractive properties of the Gini and Bonferroni indices, it will certainly be worthwhile to inquire into the relationship between the two indices analytically. In Section 3 of the paper we rigorously establish that for any unequal income distribution the Gini index is bounded above by the Bonferroni index and the bound is exactly attained if and only if in any n-person society the $(n-1)$ poor incomes are equal. Exact equality case includes as well the possibility that the incomes are perfectly unequally distributed; a situation where the richest person monopolizes the entire income and the others receive the minimal income. This result has a highly interesting implication. Given a value within the common range of the two indices, there exists a two-person income distribution for which the Gini and Bonfronni indices take on this common value. We may regard this observation as a duality problem in the theory inequality measurement. While the primal question deals with the determination of the level of inequality for a given distribution, the dual situation's concern is to identify a distribution when the level of inequality is given.

Weymark (1981) noted that if the rank order of incomes across all sources of income is the same, while the absolute Gini inequality evaluator for the aggregated distribution is simply the sum of source-wise absolute Ginis, in the relative case, the mean-weighted sum of source-wise relative Ginis generates the aggregated relative Gini evaluator. If we treat the distribution produced by one source as the original income distribution of the society and the sum of the distributions generated by other sources as an incremental distribution, then Weymark's observation claims that the absolute Gini metric for the aggregated distribution increases. Taking cue from Weymark, in Section 4 of the article we introduce a new postulate, 'additive monotnicity', for inequality standards. According to additive monotonicity, inequality increases for unequal disproportionate increments across incomes, given that both the original and incremental distributions are non-decreasingly ordered. (Unequal disproportionate increments ensure that both the relative and absolute inequality levels for the incremental distribution are positive.) For instance, if in a 3-person society the income distribution $(1, 2, 6)$ expands to $(2, 4, 9)$, then while the first two incomes grow up by 100%, the third income escalates by 50%. Then additive monotonicity demands that the expanded distribution $(2, 4, 9)$ should be more unequal than its original sister $(1, 2, 6)$. Given non-decreasingness of the distributions of the original incomes as well as that of the increments, we explicitly demonstrate that for both the Gini and the Bonferroni indices, the original distribution does not have higher inequality than the aggregated one if and only if the incremental distribution is at least as unequal as its original counterpart. One reason for this paradoxical result is that higher mean is likely to lead to reduction in the relative distances between the mean and the incomes. Relative additions in low incomes outweigh the corresponding additions in high incomes. Another likely reason is non-decreasingness of incremental incomes. When we use the Gini index of inequality, for unordered distributions a sufficient, but not necessary, condition for the new distribution not to have higher inequality than the original distribution is that the inequality of the incremental distribution is less than that of its original twin. The section concludes with a discussion on a somewhat related axiom 'the principle of monotonicity in distance', introduced by Cowell and Flachaire (2017, 2018).

In Section 5 of the paper we make an unfamiliar observation on the two indices. If the incomes in a society follow a generalised arithmetic progression, defined in an unambiguous way, then as the population size increases substantially, the values of the Bonferroni and Gini indices approach respectively to two third and one half. In the case of simple arithmetic progression, the limiting values turn into one half and one third respectively. These results hold independently of the initial income and the common difference of the progression. One way to argue in favor of this bizarre observation is that a policy maker in a society with a large population size thinks that given a minimum guaranteed income, say the country's poverty threshold, for the poorest person, how should incomes be distributed such that the level of inequality, as measured by the Gini index, comes to be one third? This policy echoes the Rawlsian (1971) maximin rule, a criterion that prioritizes the welfare of the worst off individual of the society. For a finite population size incomes following either of the two progressions, both the Bonferroni and the Gini indices can be reduced by increasing the minimum income and the next higher income by the same absolute amount. This policy for reducing inequality may be interpreted as one-step lexicographic extension of the maximin criterion. In the case when incomes follow a geometric progression the limiting values of the two indices coincide at one, the common limiting value of the two metrics when the income distribution is perfectly unequal. However, the distribution following the geometric progression is not perfectly unequal and the result materializes independently of the common ratio. With a finite population size, for a pre-defined value of the minimum income, a reduction in the value of the common ratio over the relevant range decreases the proportionate gaps between consecutive incomes leading to a shrinkage in the level of overall inequality.

2 Preliminaries

Let $\mathbb{R}$ be the set of real numbers.

Consider a fixed homogeneous population $N = \{1, 2, \ldots, n\}$ of individuals. An income distribution in this population is represented by a non-decreasingly ordered vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ whose total is positive, that is, $x_1 \leq x_2 \leq \cdots \leq x_n$ and $\sum_{i=1}^{n} x_i > 0$. Thus, we allow the possibility that some of the incomes can be negative. By D_n we denote the set of all income distributions in the society with n individuals. Evidently, inequality is a vacuous concept for $n = 1$. We, therefore, assume at the outset that $n \geq 2$. Since many of our results hold on D_n, unless specified explicitly, we will assume that D_n is the set of all income distributions.

An inequality index I is a real valued function defined on the set of income distributions. Formally, $I : D_n \to \mathbb{R}$. The index I can be relative or absolute, where relativity means that for all $\mathbf{x} \in D_n$, $I(c\mathbf{x}) = I(\mathbf{x})$, $c > 0$ being any scalar. In contrast, absoluteness refers to the condition $I(\mathbf{x} + c\mathbf{1}) = I(\mathbf{x})$ where c is a scalar such that $\mathbf{x} + c\mathbf{1} \in D_n$, where $\mathbf{1}$ is the n-coordinated vector of ones. These two notions of inequality invariance reflect two different value judgment principles and each has its own merits and demerits (see Kolm, 1976).

An inequality index I is assumed to satisfy anonymity, that is, any reordering of incomes should keep inequality unchanged. Since I has been defined directly on ordered distributions, it fulfills anonymity. For any $\mathbf{x}, \mathbf{y} \in D_n$, $\mathbf{x}$ is said to be obtained from $\mathbf{y}$ by a progressive transfer, if for some pair (i, j), $y_j > 0$ and $c > 0$, $x_j = y_j - c$, $x_i = y_i + c \leq x_j$ and $x_k = y_k$ for all $k \neq i, j$. That is, $\mathbf{x}$ is deduced from $\mathbf{y}$ by a transfer of some positive amount of income for person j to a poorer person i, such that in the post-transfer distribution i does not become richer than j. The index I is said to satisfy the Pigou-Dalton transfer principle (transfer principle, for short) if $I(\mathbf{x}) < I(\mathbf{y})$, that is, inequality reduces under a progressive transfer. Under anonymity only rank preserving transfers are allowed. Anonymity and the transfer principle are regarded as minimal postulates for an inequality index.

Let $\mathbf{x} \in D_n$ be arbitrary.

For $i = 1, \ldots, n$, define

$$\begin{aligned} s_i &= x_1 + \cdots + x_i, \\ \mu_i &= s_i / i. \end{aligned}$$

So, $s_1, \ldots, s_n$ are the partial sums and $\mu_1, \ldots, \mu_n$ are the partial means. Note that since the components of $\mathbf{x}$ are non-decreasing, the partial sums and the partial means are also non-decreasing. *Since s_n is positive, both x_n and μ_n are also positive.*

Given $\mathbf{x} \in D_n$, the Gini index of $\mathbf{x}$ is defined to be $G(\mathbf{x})$, where $G(\mathbf{x})$ is given by the following expression.

$$G(\mathbf{x}) = \frac{1}{2n^2\mu_n} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|. \tag{1}$$

Since the components of $\mathbf{x}$ are assumed to be arranged in non-decreasing order, we have

$$G(\mathbf{x}) = \frac{1}{n^2\mu_n} \sum_{1 \leq i < j \leq n} (x_j - x_i). \tag{2}$$

The absolute Gini index for a distribution $\mathbf{x}$ will be denoted as $\mathsf{AG}(\mathbf{x})$ and is defined as follows.

$$\mathsf{AG}(\mathbf{x}) = \mu_n G(\mathbf{x}) = \frac{1}{n^2} \left(\sum_{1 \leq i < j \leq n} (x_j - x_i) \right). \tag{3}$$

Given $\mathbf{x} \in D_n$, the Bonferroni index of $\mathbf{x}$ is defined to be $B(\mathbf{x})$, where $B(\mathbf{x})$ is given by the following expression.

$$B(\mathbf{x}) = \frac{1}{\mu_n}\left(\mu_n - \frac{1}{n}\sum_{i=1}^{n}\mu_i\right).$$

This can equivalently be written in the following form.

$$B(\mathbf{x}) = \frac{1}{n\mu_n}\left(\sum_{i=1}^{n}(x_i - \mu_i)\right). \tag{4}$$

The absolute Bonferroni index for a distribution $\mathbf{x}$ will be denoted as $\mathsf{AB}(\mathbf{x})$ and is defined as follows.

$$\mathsf{AB}(\mathbf{x}) = \mu_n B(\mathbf{x}) = \frac{1}{n}\left(\sum_{i=1}^{n}(x_i - \mu_i)\right). \tag{5}$$

The indices $G(\mathbf{x})$ and $B(\mathbf{x})$ are sometimes referred to as relative Gini and Bonferroni indices respectively to distinguish them from the absolute Gini and Bonferroni indices $\mathsf{AG}(\mathbf{x})$ and $\mathsf{AB}(\mathbf{x})$ respectively.

Bonferroni curve: The Bonferroni curve of $\mathbf{x} \in D_n^+$ is a piecewise linear plot on the plane joining the point $(0,0)$ and the points $(i/n, \mu_i/\mu_n)$, for $i = 1,\ldots,n$, where D_n^+ is that subset of D_n in which all incomes are non-negative. So, the entire plot lies within the square $[0,1]^2$. Let $\mathbf{x}, \mathbf{x}' \in D_n^+$ be such that $s_n = s'_n$ (and so, $\mu_n = \mu'_n$). The distribution $\mathbf{x}$ Lorenz dominates the distribution $\mathbf{x}'$ if $s_i \geq s'_i$ for $i = 1,\ldots,n-1$, with '$>$' for at least one i in $\{1,\ldots,n-1\}$. We have the following simple characterisation of Lorenz domination.

Proposition 1 *Let $\mathbf{x}$ and $\mathbf{x}'$ be two distributions in D_n^+ with $\mu_n = \mu'_n = \mu > 0$. Then $\mathbf{x}$ Lorenz dominates $\mathbf{x}'$ if and only if the Bonferroni curve of $\mathbf{x}$ dominates the Bonferroni curve of $\mathbf{x}'$.*

Proof: Let μ denote the overall means of the two distributions. The Bonferroni curve of $\mathbf{x}$ dominates the Bonferroni curve of $\mathbf{x}'$ if and only if $\mu_i/\mu \geq \mu'_i/\mu$ for $i = 1,\ldots,n$, with '$>$' for at least one i in $\{1,\ldots,n-1\}$. The last condition holds (using the fact that μ is positive) if and only if $s_i \geq s'_i$ for $i = 1,\ldots,n$ with $>$ for at least one i. This last condition is the well-known Lorenz domination of $\mathbf{x}$ over $\mathbf{x}'$. □

In view of the Dasgupta-Sen-Starrett (1973) theorem, the Bonferroni superiority of $\mathbf{x}$ over $\mathbf{x}'$ is equivalent to the stipulation that $\mathbf{x}$ can be obtained from $\mathbf{x}'$ by a sequence of rank preserving progressive transfers. Social welfare equivalence of this condition is that $\mathbf{x}$ is regarded as socially better than $\mathbf{x}'$ by all strictly S-concave social welfare functions[1] This as well is equivalent to the specification that $\mathbf{x}$ is better than $\mathbf{x}'$ by the symmetric utilitarian rule, where the identical individual utility function is strictly concave.

[1] A social welfare function is called S-concave if for all $\mathbf{x} \in D_n^+$, $W(\mathbf{x}B) \geq W(\mathbf{x})$, where B is any $n \times n$ bistochastic matrix, a non-negative square matrix of order n each of whose of rows and columns sum to 1. W is called strictly S-concave, if the weak inequality is replaced by a strict inequality whenever $\mathbf{x}B$ is not a permutation of $\mathbf{x}$. All S-concave functions are symmetric.

3 Bonferroni Dominates Gini

Given any non-decreasing distribution $\mathbf{x}$ we prove that the Bonferroni index of $\mathbf{x}$ is at least as large as the Gini index of $\mathbf{x}$.

Theorem 1 (Bonferroni dominates Gini) *Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a non-decreasing distribution with mean $\mu_n > 0$. Then $B(\mathbf{x}) \geq G(\mathbf{x})$.*

Further, equality holds if and only if $x_1 = \cdots = x_{n-1} = x$. In this case,

$$B(\mathbf{x}) = G(\mathbf{x}) \quad = \quad \left(1 - \frac{1}{n}\right)\left(\frac{1 - x/x_n}{1 + (n-1)x/x_n}\right). \tag{6}$$

In words, the Bonferroni index of $\mathbf{x}$ is at least as large as the Gini index of $\mathbf{x}$ with equality holding if and only if the first $(n-1)$ components of $\mathbf{x}$ have the same value. In the later case, the equal value of the Bonferroni and Gini indices is given by (6).

Proof: For $2 \leq j \leq n$, we have

$$\sum_{i=1}^{j-1}(x_j - x_i) \quad = \quad (j-1)(x_j - \mu_{j-1}).$$

So, using (2) we may write

$$G(\mathbf{x}) \quad = \quad \frac{1}{n^2\mu_n}\sum_{j=2}^{n}(j-1)(x_j - \mu_{j-1}).$$

Using (4), we have

$$B(\mathbf{x}) \quad = \quad \frac{1}{n\mu_n}\sum_{i=1}^{n}(x_i - \mu_i) = \frac{1}{n\mu_n}\sum_{j=2}^{n}(x_j - \mu_j).$$

So, $B(\mathbf{x}) \geq G(\mathbf{x})$ if and only if (using $\mu_n > 0$)

$$n\sum_{j=2}^{n}(x_j - \mu_j) \geq \sum_{j=2}^{n}(j-1)(x_j - \mu_{j-1}),$$

i.e., if and only if

$$\sum_{j=2}^{n}(n-j+1)x_j - \sum_{j=2}^{n}(n\mu_j - (j-1)\mu_{j-1}) \geq 0.$$

Let

$$S \quad = \quad \sum_{i=2}^{n}(n-i+1)x_i - \sum_{i=2}^{n}(n\mu_i - (i-1)\mu_{i-1}).$$

So, $B(\mathbf{x}) \geq G(\mathbf{x})$ if and only if $S \geq 0$. We have

$$\begin{aligned}
\sum_{i=2}^{n}(n\mu_i - (i-1)\mu_{i-1}) \quad &= \quad \sum_{i=2}^{n}n\mu_i - \sum_{i=2}^{n}(i-1)\mu_{i-1} \\
&= \quad n\mu_n + \sum_{i=2}^{n-1}n\mu_i - \sum_{i=2}^{n-1}i\mu_i - \mu_1 \\
&= \quad n\mu_n - \mu_1 + \sum_{i=2}^{n-1}(n-i)\mu_i.
\end{aligned}$$

Note that $\mu_1 = x_1$. Using the above simplification in the expression for S, we obtain

$$\begin{aligned} S &= \sum_{i=2}^{n}(n-i+1)x_i - n\mu_n + x_1 - \sum_{i=2}^{n-1}(n-i)\mu_i \\ &= -n\mu_n + x_1 + \sum_{i=2}^{n} x_i + \sum_{i=2}^{n}(n-i)x_i - \sum_{i=2}^{n-1}(n-i)\mu_i \\ &= -n\mu_n + \sum_{i=1}^{n} x_i + \sum_{i=2}^{n-1}(n-i)(x_i-\mu_i) \\ &= -n\mu_n + n\mu_n + \sum_{i=2}^{n-1}(n-i)(x_i-\mu_i) \\ &= \sum_{i=2}^{n-1}(n-i)(x_i-\mu_i). \end{aligned}$$

For $2 \leq i \leq (n-1)$, we have $(n-i) > 0$. Also, since μ_i is the mean of the numbers $x_1, \ldots, x_i$ with $x_1 \leq \cdots \leq x_i$, it follows that $x_i \geq \mu_i$. As a result, we have $S \geq 0$ and so $B(\mathbf{x}) \geq G(\mathbf{x})$.

Further, $B(\mathbf{x}) = G(\mathbf{x})$ if and only if $S = 0$, i.e., if and only if $x_i = \mu_i$ for $i = 2, \ldots, n-1$. The last condition is equivalent to $x_1 = x_2 = \cdots = x_{n-1}$. Taking this common value to be x provides the given expression for $G(\mathbf{x}) = B(\mathbf{x})$. Note that $\mu_n > 0$ implies $x_n \geq \mu_n > 0$ and so the division by x_n is possible.

□

We next prove an interesting consequence of Theorem 1, for which we assume that the set of income distributions is D_n^+. Let $\mathcal{C}$ be a class of distributions. The class $\mathcal{C}$ is complete for the Gini index if the values of the Gini index for the distributions in $\mathcal{C}$ span the entire interval $[0, 1-1/n]$, i.e., if

$$\{G(\mathbf{x}) : \mathbf{x} \in \mathcal{C}\} = [0, 1-1/n].$$

One may similarly define the completeness of $\mathcal{C}$ for the Bonferroni index.

Define the class of k-valued distributions to be the class $\mathcal{C}_k$ such that for any $\mathbf{x} \in \mathcal{C}_k$, the number of distinct values of the components of $\mathbf{x}$ is at most k. So, for any distribution $\mathbf{x}$ in $\mathcal{C}_1$, all components of $\mathbf{x}$ have the same value. Both the Gini and Bonferroni indices for any such $\mathbf{x}$ is 0.

The next class to consider is $\mathcal{C}_2$, i.e., the class of 2-valued distributions. We show that $\mathcal{C}_2$ is complete for both the Gini and the Bonferroni indices. It is clear from the definition of $\mathcal{C}_2$ that the following theorem may be regarded as a dual of Theorem 1.

Theorem 2 (Completeness of 2-valued distributions for the Gini and Bonferroni indices) *Let $\delta \in [0, 1-1/n]$. Then there is a distribution $\mathbf{x} \in \mathcal{C}_2$ such that $G(\mathbf{x}) = B(\mathbf{x}) = \delta$.*

In other words, given any value in the interval $[0, 1-1/n]$, it is possible to construct a distribution whose components take at most two distinct values, such that both the Bonferroni and Gini indices of this distribution is equal to the given value.

Proof: We consider distributions of the type $\mathbf{x} = (x, x, \ldots, x, y)$, i.e., the first $n-1$ components of $\mathbf{x}$ are equal to x and $x_n = y$, with $0 \leq x \leq y$ and $y > 0$. Clearly, any such $\mathbf{x}$ is in $\mathcal{C}_2$. and further, from Theorem 1, it follows that $G(\mathbf{x}) = B(\mathbf{x})$. Define

$$\lambda = \frac{n-1-n\delta}{(n-1)(1+n\delta)}. \tag{7}$$

and $x = \lambda y$ so that $x/y = x/x_n = \lambda$. Substituting the expression for $x/x_n = \lambda$ given in (7) into (6) and simplifying shows that $B(\mathbf{x}) = G(\mathbf{x}) = \delta$. □

From (7), we have $\delta = 0$ if and only if $\lambda = 0$, i.e., if and only if $x = x_n$. Also, $\delta = 1 - 1/n$ if and only if $\lambda = 0$, i.e., if and only if $x = 0$.

We observe that Theorem 2 provides a solution for the inverse problem for the Gini (resp. Bonferroni) index, i.e., given $\delta \in [0, 1 - 1/n]$, obtain a distribution whose Gini (resp. Bonferroni) index is equal to δ. This has an interesting practical consequence. Consider the Gini index. Suppose in a practical situation, the Gini index is computed and is found to be a value $\delta \in [0, 1 - 1/n]$. From this value of the Gini index what can be said about the distribution? Theorem 2 shows that there is a 2-valued distribution of the type $\mathbf{x} = (x, \ldots, x, x_n)$ whose Gini index is δ. So, simply looking at the Gini or the Bonferroni indices, one cannot rule out that the underlying distribution is such a degenerate type of distribution. This suggests that making any conclusion about the distribution based solely on its Gini or Bonferrroni indices may be misleading. It may also be noted that although Theorem 1 holds on the general domain D_n, its dual Theorem 2, holds on the domain $\mathcal{C}_2$.

4 Paradox of Additivity

The absolute Gini and Bonferroni indices satisfies an additivity condition as stated in the following result. For the absolute Gini index, this observation was already made by Weymark (1981) and the proof that we provide is only for the sake of completeness.

Theorem 3 *Let* $\mathbf{x}, \mathbf{d} \in D_n$ *and* $\mathbf{y} = \mathbf{x} + \mathbf{d}$. *Then*

$$\begin{aligned} \mathsf{AG}(\mathbf{y}) &= \mathsf{AG}(\mathbf{x}) + \mathsf{AB}(\mathbf{d}), & (8) \\ \mathsf{AB}(\mathbf{y}) &= \mathsf{AB}(\mathbf{x}) + \mathsf{AB}(\mathbf{d}). & (9) \end{aligned}$$

Proof: For $i = 1, \ldots, n$, let μ_i, δ_i and ν_i be the means of $(x_1, \ldots, x_i)$, $(d_1, \ldots, d_i)$ and $(y_1, \ldots, y_i)$ respectively. Note that $\nu_i = \mu_i + \delta_i$ for $i = 1, \ldots, n$.

For the result on the absolute Gini index, note that since both $\mathbf{x}$ and $\mathbf{d}$ are non-decreasing, so is $\mathbf{y}$. From (3) and (2), we have

$$\begin{aligned} \mathsf{AG}(\mathbf{y}) &= \nu_n G(\mathbf{y}) \\ &= \frac{1}{n^2}\left(\sum_{1\le i<j\le n}(y_j - y_i)\right) \\ &= \frac{1}{n^2}\left(\sum_{1\le i<j\le n}((x_j - x_i) + (d_j - d_i))\right) \\ &= \frac{1}{n^2}\left(\sum_{1\le i<j\le n}(x_j - x_i)\right) + \frac{1}{n^2}\left(\sum_{1\le i<j\le n}(d_j - d_i)\right) \\ &= \mu_n G(\mathbf{x}) + \delta_n G(\mathbf{d}) \\ &= \mathsf{AG}(\mathbf{x}) + \mathsf{AG}(\mathbf{d}). \end{aligned}$$

For the result on absolute Bonferroni index, we proceed using (5) and (4).

$$\begin{aligned}
\mathsf{AB}(\mathbf{y}) &= \nu_n B(\mathbf{y}) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}(y_i - \nu_i)\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}(x_i + d_i - \mu_i - \delta_i)\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}(x_i - \mu_i)\right) + \frac{1}{n}\left(\sum_{i=1}^{n}(d_i - \delta_i)\right) \\
&= \mu_n B(\mathbf{x}) + \delta_n B(\mathbf{d}) \\
&= \mathsf{AB}(\mathbf{x}) + \mathsf{AB}(\mathbf{d}).
\end{aligned}$$

□

From Theorem 3, we see that the inequality in $\mathbf{y}$, as measured by either the absolute Gini or the absolute Bonferroni index, is the sum total of the inequality in $\mathbf{x}$ and $\mathbf{d}$. So, the inequality in $\mathbf{y}$ is never lower than the inequality in $\mathbf{x}$.

Motivated by Theorem 3, we put forward the following postulate.

Postulate of Additive Monotonicity: *An inequality index $I : D_n \to \mathbb{R}$ is said to satisfy the postulate of additive monotonicity, if for all $\mathbf{x} \in D_n$, $I(\mathbf{x}+\mathbf{d}) > I(\mathbf{d})$, where $\mathbf{d} \in D_n$ is not of the form $c\mathbf{x}$ or $c\mathbf{1}$, where $c > 0$ is any scalar.*

Theorem 3 shows that both the absolute Gini and the absolute Bonferroni indices unambiguously satisfy the postulate of additive monotonicity. Next we consider the question of whether the (relative) Gini and Bonferroni indices satisfy this postulate. Towards this end, we need to obtain some results on the additive properties of the Gini and Bonferroni indices.

Let $\mathbf{x}$ and $\mathbf{d}$ be two distributions and let $\mathbf{y} = \mathbf{x} + \mathbf{d}$. The following results relate the Gini and Bonferroni indices of $\mathbf{y}$ with those of $\mathbf{x}$ and $\mathbf{d}$. For the Gini index this property was noted by Weymark (1981). We present the proof here in order to make the paper self contained.

Theorem 4 (Additivity of Gini, Weymark (1981)) *Let $\mathbf{x}, \mathbf{d} \in D_n$ be two distributions with means $\mu > 0$ and $\delta > 0$ respectively. Let $\mathbf{y} = \mathbf{x} + \mathbf{d}$. Let $\alpha = \mu/(\mu + \delta)$. Then*

$$G(\mathbf{y}) = \alpha G(\mathbf{x}) + (1-\alpha)G(\mathbf{d})$$

Further,

$$\min(G(\mathbf{x}), G(\mathbf{d})) \leq G(\mathbf{y}) \leq \max(G(\mathbf{x}), G(\mathbf{d}))$$

with equality if and only if $G(\mathbf{x}) = G(\mathbf{d})$.

Proof: Note that the mean of $\mathbf{y}$ is $\mu+\delta$. Since $\mathbf{x}$ and $\mathbf{d}$ are non-decreasing, so is $\mathbf{y}$. For $1 \leq i < j \leq n$, we have $y_j - y_i = x_j - x_i + d_j - d_i$. So, from (2), we have

$$\begin{aligned} G(\mathbf{y}) &= \frac{1}{n^2(\mu+\delta)} \sum_{1\leq i<j\leq n} (y_j - y_i) \\ &= \frac{1}{n^2(\mu+\delta)} \sum_{1\leq i<j\leq n} ((x_j - x_i) + (d_j - d_i)) \\ &= \frac{1}{n^2(\mu+\delta)} \left(\sum_{1\leq i<j\leq n} (x_j - x_i) + \sum_{1\leq i<j\leq n} (d_j - d_i) \right) \\ &= \frac{1}{n^2(\mu+\delta)} \left(n^2\mu G(\mathbf{x}) + n^2\delta G(\mathbf{d}) \right) \\ &= \alpha G(\mathbf{x}) + (1-\alpha) G(\mathbf{d}). \end{aligned}$$

Given that both μ and δ are positive, $0 < \alpha, 1-\alpha < 1$. Using this along with the expression for $G(\mathbf{y})$ in terms of $G(\mathbf{x})$ and $G(\mathbf{d})$, the inequalities follow. □

Theorem 4 requires both $\mathbf{x}$ and $\mathbf{d}$ to be non-decreasing. We consider a generalisation of Theorem 4, where these conditions are not required to hold. The generalisation, however, comes at a cost of obtaining a less sharper result. Note that if a distribution is not necessarily non-decreasing, the Gini index cannot be computed using (2) and the formulation (1) has to be used.

Theorem 5 (Weak Additivity of Gini) *Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{R}^n$ be two distributions which are not necessarily non-decreasing and having means $\mu > 0$ and $\delta > 0$ respectively. Let $\mathbf{y} = \mathbf{x} + \mathbf{d}$ and $\alpha = \mu/(\mu+\delta)$. Then*

$$G(\mathbf{y}) \leq \alpha G(\mathbf{x}) + (1-\alpha) G(\mathbf{d}) \leq \max(G(\mathbf{x}), G(\mathbf{d})).$$

Consequently, if $G(\mathbf{d}) < G(\mathbf{x})$, then $G(\mathbf{y}) \leq G(\mathbf{x})$.

Proof: Note that the mean of $\mathbf{y}$ is $\mu+\delta$. Since $\mathbf{x}$ and $\mathbf{d}$ are not necessarily non-decreasing, $\mathbf{y}$ is also not necessarily non-decreasing. So, we cannot use (2) to compute the Gini indices of $\mathbf{x}$, $\mathbf{d}$ and $\mathbf{y}$. Instead, we need to use (1). For $1 \leq i < j \leq n$, we have $|y_i - y_j| = |x_i - x_j + d_i - d_j| \leq |x_i - x_j| + |d_i - d_j|$. So, from (1), we have

$$\begin{aligned} G(\mathbf{y}) &= \frac{1}{2n^2(\mu+\delta)} \sum_{i=1}^{n}\sum_{j=1}^{n} |y_i - y_j| \\ &\leq \frac{1}{2n^2(\mu+\delta)} \sum_{i=1}^{n}\sum_{j=1}^{n} (|x_i - x_j| + |d_i - d_j|) \\ &= \frac{1}{2n^2(\mu+\delta)} \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - x_j| + \sum_{i=1}^{n}\sum_{j=1}^{n} |d_i - d_j| \right) \\ &= \frac{1}{2n^2(\mu+\delta)} \left(2n^2\mu G(\mathbf{x}) + 2n^2\delta G(\mathbf{d}) \right) \\ &= \alpha G(\mathbf{x}) + (1-\alpha) G(\mathbf{d}). \end{aligned}$$

Since both μ and δ are positive, $0 < \alpha, 1-\alpha < 1$. This shows the last inequality. □

In Theorem 5, the condition $G(\mathbf{d}) < G(\mathbf{x})$ is sufficient for the inequality $G(\mathbf{y}) \leq G(\mathbf{x})$ to hold. Consider for example, $\mathbf{x} = (1,3,5,7,9)$ and $\mathbf{d} = (2,1,1,2,1)$ so that $\mathbf{y} = (3,4,6,9,10)$. In this case, we have $G(\mathbf{d}) = 0.17 < 0.32 = G(\mathbf{x})$ and consequently, we obtain $G(\mathbf{y}) = 0.24 < 0.32 = G(\mathbf{x})$. On the other hand, the condition $G(\mathbf{d}) < G(\mathbf{x})$ is not necessary for the inequality $G(\mathbf{y}) \leq G(\mathbf{x})$ to hold. It is possible that $G(\mathbf{d}) > G(\mathbf{x})$ and yet $G(\mathbf{y}) \leq G(\mathbf{x})$. As an example, consider $\mathbf{x} = (1,3,5,7,9)$ and $\mathbf{d} = (0,1,0,1,1)$ with $\mathbf{y} = (1,4,5,8,10)$. In this case, $G(\mathbf{d}) = 0.4 > 0.32 = G(\mathbf{x})$ and yet we have $G(\mathbf{y}) = 0.31 < 0.32 = G(\mathbf{x})$.

We prove a result on additivity of the Bonferroni index.

Theorem 6 (Additivity of Bonferroni) *Let $\mathbf{x}, \mathbf{d} \in D_n$ be two distributions with means $\mu > 0$ and $\delta > 0$ respectively. Let $\mathbf{y} = \mathbf{x} + \mathbf{d}$. Let $\alpha = \mu/(\mu + \delta)$. Then*

$$B(\mathbf{y}) = \alpha B(\mathbf{x}) + (1-\alpha) B(\mathbf{d}).$$

Further,

$$\min(B(\mathbf{x}), B(\mathbf{d})) \leq B(\mathbf{y}) \leq \max(B(\mathbf{x}), B(\mathbf{d}))$$

with equality if and only if $B(\mathbf{x}) = B(\mathbf{d})$.

Proof: For $1 \leq i \leq n$, let μ_i (resp. δ_i) be the mean of $x_1, \ldots, x_i$ (resp. $d_1, \ldots, d_i$). Similarly, for $1 \leq i \leq n$, let ν_i be the mean of $y_1, \ldots, y_i$. Then $\nu_i = \mu_i + \delta_i$, for $i = 1, \ldots, n$. We have $\mu_n = \mu$ and $\delta_n = \delta$ and $\nu_n = \mu_n + \delta_n = \mu + \delta \neq 0$.

Note that $y_i - \nu_i = x_i - \mu_i + d_i - \delta_i$. From (4)

$$\begin{aligned}
B(\mathbf{y}) &= \frac{1}{n(\mu+\delta)} \sum_{i=1}^{n} (y_i - \nu_i) \\
&= \frac{1}{n(\mu+\delta)} \sum_{i=1}^{n} ((x_i - \mu_i) + (d_i - \delta_i)) \\
&= \frac{1}{n(\mu+\delta)} \left(\sum_{i=1}^{n} (x_i - \mu_i) + \sum_{i=1}^{n} (d_i - \delta_i) \right) \\
&= \frac{1}{n(\mu+\delta)} \left(n\mu B(\mathbf{x}) + n\delta B(\mathbf{d}) \right) \\
&= \alpha B(\mathbf{x}) + (1-\alpha) B(\mathbf{d}).
\end{aligned}$$

Since $0 < \alpha, 1 - \alpha < 1$, the inequalities follow.

□

Considering distributions in D_n, Theorem 4 (resp. Theorem 6) shows that a sufficient condition for the Gini (resp. Bonferroni) inequality metric of the original distribution not to exceed that of the aggregate distribution is that the incremental distribution is more unequal than its original twin. We make this statement precise in the following manner.

Theorem 7 (Additivity Paradox) *Let $\mathbf{x}, \mathbf{d} \in D_n$ and $\mathbf{y} = \mathbf{x} + \mathbf{d}$. Then*

1. *$G(\mathbf{y}) \leq G(\mathbf{x})$ if and only if $G(\mathbf{d}) \leq G(\mathbf{x})$.*
2. *$B(\mathbf{y}) \leq B(\mathbf{x})$ if and only if $B(\mathbf{d}) \leq B(\mathbf{x})$.*

Proof: For the first point, we use Theorem 4. If $G(\mathbf{d}) \leq G(\mathbf{x})$, then $\min(G(\mathbf{d}), G(\mathbf{x})) = G(\mathbf{d})$ and $\max(G(\mathbf{d}), G(\mathbf{x})) = G(\mathbf{x})$. So, from Theorem 4, we have $G(\mathbf{d}) \leq G(\mathbf{y}) \leq G(\mathbf{x})$. On the other hand, if $G(\mathbf{d}) > G(\mathbf{x})$, then $\min(G(\mathbf{d}), G(\mathbf{x})) = G(\mathbf{x})$ and $\max(G(\mathbf{d}), G(\mathbf{x})) = G(\mathbf{d})$. So, from Theorem 4, we have $G(\mathbf{x}) < G(\mathbf{y}) < G(\mathbf{d})$. This proves the first point.

The argument for the Bonferroni index is similar to the argument given above and is based on Theorem 6. □

From Theorem 7 it emerges that for both the Gini and the Bonferroni standards, the inequality in the aggregate distribution obtained by aggregating a non-decreasingly ordered original distribution and a non-decreasingly ordered incremental distribution is at most the inequality in the original distribution if and only if the inequality in the incremental distribution is at most the inequality in the original distribution.

The content of Theorem 7 is that by adding two non-decreasing distributions, it is possible to lower the inequality below one of them. If $\mathbf{x}$ is an initial non-decreasing wealth distribution and $\mathbf{d}$ is an incremental non-decreasing wealth distribution, then the wealth distribution given by $\mathbf{y} = \mathbf{x} + \mathbf{d}$ may have inequality lower than the inequality of $\mathbf{x}$. The paradoxical issue here is that the incremental wealth distribution $\mathbf{d}$ is also *non-decreasing*. This means that it is possible to decrease the inequality in wealth distribution while ensuring that the increments provided to the richer people are more than the increments provided to the poorer people. More compactly, it is possible to decrease inequality while ensuring that the rich get even richer. To bring out the paradoxical situation more clearly, we consider two practical applications of Theorem 7.

Salary increments: Suppose the non-decreasingly ordered distribution $\mathbf{x}$ represents the various salary levels in an organisation in a particular year. The next year, each salary level is increased by a non-negative amount. The increments are given by the distribution $\mathbf{d}$. The sums of the original salary levels and the increments are the new salary levels which are given by the distribution $\mathbf{y}$. A possible social welfare goal in determining salary levels is to decrease the inequality in the various salary levels. Supposing inequality in the salary levels is measured by the Gini or the Bonferroni indices. Theorem 7 tells us that it is possible to provide a non-decreasing sequence of increments and yet achieve an overall decrease in inequality. In other words, it is possible to make the better paid even more better paid and yet achieve an overall lower level of inequality.

Let us consider a concrete example to illustrate the above paradox. Suppose that the initial salary levels are given by $\mathbf{x} = (1, 3, 6, 10, 15)$ which are entries in thousand-dollar unit. Suppose the increments are $\mathbf{d} = (1, 2, 3, 4, 5)$ which are again in thousand-dollar unit. In other words, the salary at the lowest level is increased by a 1000 Dollars, at the next level by 2000 Dollars and so on, finally at the last level by 5000 Dollars. So, people who were already getting a higher salary, receives a higher amount of increment. Let $\mathbf{y} = \mathbf{x} + \mathbf{d} = (2, 5, 9, 14, 20)$ be the new salary levels. How does the inequality in $\mathbf{y}$ compare to the inequality in $\mathbf{x}$? Since the salary increments make the better paid even more better paid, one may expect that the inequality in $\mathbf{y}$ has increased compared to the inequality in $\mathbf{x}$. However, if one measures inequality using either the Gini or the Bonferroni indices, then it turns out that the inequality in $\mathbf{y}$ has actually decreased in comparison to the inequality in $\mathbf{x}$. We have, $G(\mathbf{y}) = 0.36 < 0.4 = G(\mathbf{x})$ which from Theorem 7 happens due to the fact that $G(\mathbf{d}) = 0.27 < 0.4 = G(\mathbf{x})$. Similarly, we have, $B(\mathbf{y}) = 0.43 < 0.47 = B(\mathbf{x})$ which from Theorem 7 happens due to the fact that $B(\mathbf{d}) = 0.33 < 0.47 = B(\mathbf{x})$.

Tax rebates: We consider an example in the scenario of providing income tax rebates. Suppose that the gross incomes are given by the non-decreasing distribution $\mathbf{w}$. Let $\mathbf{u}$ be the distribution of the present tax deductions, so that the net incomes are given by the distribution $\mathbf{x} = \mathbf{w} - \mathbf{u}$. Let us assume that the components of $\mathbf{x}$ are also arranged in non-decreasing order. Suppose the goverment announces a populist new tax scheme which leads to a tax distribution $\mathbf{v}$. Being populist, the taxes for all persons get reduced so that $v_i \leq u_i$, for $i = 1, \ldots, n$. So, the tax rebates are given by the distribution $\mathbf{d} = \mathbf{u} - \mathbf{v}$. Due to the new tax policy the net income distribution becomes $\mathbf{y} = \mathbf{w} - \mathbf{v} = \mathbf{x} + \mathbf{d}$. In other words, the new net incomes can be seen as the sum of the previous net incomes and the tax rebates. So effectively, the i-th net income gets increased by an amount d_i. Suppose now that the government decides to provide higher tax rebates to those with higher incomes, i.e., $d_i \leq d_{i+1}$ for $i = 1, \ldots, n-1$. This means that people who were already taking home a higher net pay now takes home an even higher net pay. Consequently, one may be led to believe that the government policy has lead to an increase in inequality. However, if inequality is measured using either the Gini or the Bonferroni indices, this is not necessarily true. If the inequality in $\mathbf{d}$ is less than the inequality in $\mathbf{x}$, then the inequality in the net incomes after the new taxation policy is actually less than the inequality in the net incomes prior to this taxation policy. In other words, it is possible to reduce inequality by allowing people with already higher net pay to take home an even higher net pay.

Let us consider a numerical example. Suppose that the gross income is $\mathbf{w} = (6, 8, 11, 15, 20)$ (with thousand-dollar unit). Suppose that the initial tax deductions are $\mathbf{u} = (5, 5, 5, 5, 5)$ which is the same for all levels. The net incomes are $\mathbf{x} = (1, 3, 6, 10, 15)$. Further, suppose that the new tax deductions are $\mathbf{v} = (4, 3, 2, 1, 0)$. So, people with higher incomes are required to pay less taxes and the resulting tax rebates are $\mathbf{d} = (1, 2, 3, 4, 5)$. The new net income distribution is $\mathbf{y} = (2, 5, 9, 14, 20)$. The distributions $\mathbf{x}$, $\mathbf{d}$ and $\mathbf{y}$ are the same as those considered in the previous example. So, we have both $G(\mathbf{x}) > G(\mathbf{y})$ and $B(\mathbf{x}) > B(\mathbf{y})$. Effectively, in this scenario requiring the people with higher incomes to pay lesser taxes has led to a reduction in the inequality, as measured by Gini and Bonferroni indices.

From Theorem 7, we have that the Gini and the Bonferroni indices do not in general satisfy the postulate of additive monotonicity. They satisfy the weak form of the postulate (that is, $G(\mathbf{x} + \mathbf{d})$ is greater than or equal to $G(\mathbf{x})$ and $B(\mathbf{x}+\mathbf{d})$ is greater than or equal to $B(\mathbf{x})$) if and only if the inequality in $\mathbf{d}$ is at least as large as the inequality in $\mathbf{x}$. In the examples discussed above, the distribution $\mathbf{d}$ has been considered to be increments to the original distribution $\mathbf{x}$. So, the Gini and the Bonferroni indices satisfy the weak form of the postulate of additive monotonicity if and only if the inequality in the increments is at least as large as the inequality in the original distribution.

4.1 On a Postulate of Cowell and Flachaire

A postulate, called the principle of monotonicity in distance, has been introduced by Cowell and Flachaire (2017, 2018). The idea behind the postulate is the following. Suppose a distribution $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^+$ is changed to a distribution $\mathbf{x}' = (x'_1, \ldots, x'_n) \in D_n^+$, where the income of exactly one person (say the i-th person) is increased. In other words, $x'_j = x_j$ for $j = 1, \ldots, n$ and $j \neq i$; $x'_i = x_i + \delta$ where $\delta > 0$. Let μ and μ' be the means of $\mathbf{x}$ and $\mathbf{x}'$ respectively. Suppose that $|x_i - \mu| < |x'_i - \mu'|$. Then the increment to the i-th income has moved the i-th person further away from the mean. The principle of monotonicity in distance requires that the inequality in $\mathbf{x}$ to be lower than the inequality in $\mathbf{x}'$.

Cowell and Flachaire show that the Gini and the Theil indices do not satisfy the principle of monotonicity in distance. The example they provide has $\mathbf{x} = (1, 2, 3, 4, 5, 9, 10)$ and $\mathbf{x}' = (1, 2, 3, 4, 7, 9, 10)$ with $\mu = 4.857$ and $\mu' = 5.143$. The change from $\mathbf{x}$ to $\mathbf{x}'$ is to increase the income of only the fifth person from 5 to 7, i.e., $x_5 = 5$ and $x'_5 = 7$ and all other incomes remain the same. We have $|x_5 - \mu| = 5 - 4.857 = 0.143$ and $|x'_5 - \mu'| = 7 - 5.143 = 1.857$. So, compared to the distribution $\mathbf{x}$, in $\mathbf{x}'$, the income of the fifth person has moved further away from the mean and the principle of monotonicity in distance requires that $G(\mathbf{x}) < G(\mathbf{x}')$. However, we have $G(\mathbf{x}) = 0.361 > 0.357 = G(\mathbf{x}')$.

We take a closer look at the above example. We note that for $i = 1, \ldots, 4$, $x_i = x'_i$ and since $\mu < \mu'$, we have $|x_i - \mu| = \mu - x_i < \mu' - x_i = \mu' - x_i' = |x_i' - \mu|$. So, it is not only that the deviation from the mean of the fifth person's income has increased, in fact, the deviations from the mean of the incomes of all the first five persons have increased. On the other hand, let us now consider the deviations from the mean of the last two persons. Note that $|x_6 - \mu| = 9 - 4.857 = 4.143 > 3.857 = |x_6' - \mu'|$ and $|x_7 - \mu| = 10 - 4.857 = 5.143 > 4.857 = 10 - 5.143 = |x_7' - \mu'|$. This shows that due to the increase of the income of the fifth person, the deviations from the mean of the incomes of the first five persons have increased, while the deviation from the mean of the last two persons have decreased. Focusing only on the deviation from the mean of the fifth person's income does not provide a complete picture. Given the complete picture of how the various deviations behave, it does not appear to be intuitive to expect that the overall inequality in $\mathbf{x}'$ should be greater than that in $\mathbf{x}$. This is especially so since the deviations from the mean of people with higher incomes have actually gone down. Since the Gini index is an overall measure of inequality, it is perhaps not unexpected that the inequality in $\mathbf{x}$ is more than the inequality in $\mathbf{x}'$.

The point of the above discussion is to argue that the principle of monotonicity in distance needs discussion at greater length before it is accepted as a compelling postulate for an inequality index. The above discussion has been in the context of the example provided by Cowell and Falchaire. It is not too difficult to lift the essence of the discussion to a more general result.

Proposition 2 *Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a non-decreasing distribution with mean $\mu > 0$. Let i in $\{1, \ldots, n\}$ with $x_i \geq \mu$ and $\delta > 0$ be such that the distribution $\mathbf{x}' = (x_1', \ldots, x_n')$ defined as $x_j' = x_j$, for $j = 1, \ldots, n$ and $j \neq i$; $x_i' = x_i + \delta$, is also non-decreasing. Let μ' be the mean of $\mathbf{x}'$. Let $j \in \{1, \ldots, n\}$. Then*

$$\begin{array}{lcll} |x_j - \mu| & < & |x_j' - \mu'| & \text{if } j = i; \\ |x_j - \mu| & < & |x_j' - \mu'| & \text{if } j \neq i \text{ and } x_j \leq \mu; \\ |x_j - \mu| & > & |x_j' - \mu'| & \text{if } j \neq i \text{ and } x_j \geq \mu'. \end{array}$$

Remark: For $j \neq i$ such that $\mu < x_j < \mu'$, the relationship between $|x_j - \mu|$ and $|x_j' - \mu'| = |x_j - \mu'|$ cannot be determined in general.

Proof: Note that, $\mu' = \mu + \delta/n$.

For $j = i$, we have $|x_i' - \mu'| = |x_i + \delta - (\mu + \delta/n)| = |x_i - \mu| + |\delta - \delta/n| > |x_i - \mu|$.

For $j \neq i$, we have $x_j = x_j'$. If $x_j \leq \mu$, then $|x_j' - \mu'| = |x_j - (\mu + \delta/n)| = \mu + \delta/n - x_j = |x_j - \mu| + \delta/n > |x_j - \mu|$. On the other hand, if $x_j \geq \mu'$, we have $|x_j' - \mu'| = |x_j - \mu'| = x_j - \mu' = x_j - (\mu + \delta/n) = x_j - \mu - \delta/n < |x_j - \mu|$. □

Proposition 2 shows that if only the income of the i-th person is increased, then the deviations from the mean of the incomes of one set of persons increase while the devations from the mean of the incomes of another set of persons decrease. Consequently, it is a priori not reasonable to expect that the overall inequality should increase. In particular, one may note that the deviations from the means of the higher income persons decrease. So, a reduction of inequality cannot be wholly unexpected.

5 Benchmark Distributions

Suppose $G(\mathbf{x})$ is computed for a distribution $\mathbf{x}$. From this value, what can be inferred about the distribution $\mathbf{x}$? Note that since the entire information in the n-component distribution $\mathbf{x}$ is represented by a single value $G(\mathbf{x})$, any inference based on this value is tentative. Nevertheless, the purpose of computing the index is to be able to obtain some understanding of the inequality present in the distribution $\mathbf{x}$.

One way to obtain such an understanding is to compare $G(\mathbf{x})$ with $G(\mathbf{z})$, where $\mathbf{z}$ is some benchmark distribution which is well understood. Similar considerations hold for $B(\mathbf{x})$.

For a positive real number x, two simple benchmark distributions are the following.

$$\mathsf{eq} = (x, \dots, x), \qquad \mathsf{ineq} = (0, \dots, 0, x).$$

It is well known and easy to verify that $G(\mathsf{eq}) = 0 = B(\mathsf{eq})$ and $G(\mathsf{ineq}) = (1 - 1/n) = B(\mathsf{ineq})$. Given a distribution $\mathbf{x}$, if $G(\mathbf{x})$ (or, $B(\mathbf{x})$) turns out to be close to 0, then one interpret this by saying that the distribution $\mathbf{x}$ is more or less equitable. Similarly, if the value of $G(\mathbf{x})$ (or, $B(\mathbf{x})$) turns out to be close to $(1 - 1/n)$, then the interpretation would be that distribution $\mathbf{x}$ has a great amount of inequality.

So, knowing values of an inequality index for certain benchmark distributions help in interpreting the computed value of the index for a given distribution $\mathbf{x}$. To the best of our knowledge, the values of Gini and Bonferroni indices are not known for any benchmark distributions other than eq and ineq.

In this section, we derive values of the Gini and Bonferroni indices on two standard distributions, namely distributions following either the arithmetic or the geometric progression.

5.1 Arithmetic Progression

We assume at the outset that the set of income distributions is D_n^{++}, the strictly positive part of D_n^{+}. This domain restriction ensures that the minimum income x_1 is positive. Consequently, in order to maintain parity with the Rawlsian maximin policy, one can set x_1 at a pre-specified level, say at the country's poverty threshold, the income necessary to maintain the subsistence standard of living.

Let $d_i = x_{i+1} - x_i$ for $1 \leq i \leq n-1$. Since we have assumed that the components of $\mathbf{x}$ are non-decreasing, it follows that $d_i \geq 0$ for $i = 1, \dots, n-1$. If $\mathbf{x}$ is an AP, then $d_1 = \dots = d_{n-1}$. We consider a more general situation, where the d_i's themselves form an AP, i.e., there is a common difference d, such that $d_i = d_1 + (i-1)d$. In other words, the d_i's grow linearly with i. Note that we do not insist that $d \geq 0$. A negative value for d indicates that the difference between two successive components decreases with i. On the other hand, we do have the constraint of non-decreasing behaviour on $\mathbf{x}$. This implies that $d_{n-1} \geq 0$. So, d must satisfy the condition $d \geq -d_1/(n-1)$.

Theorem 8 (Gini and Bonferroni for generalised AP) *Let* $\mathbf{x} = (x_1, \dots, x_n) \in D_n^{++}$. *Let* $d_i = x_{i+1} - x_i$ *for* $i = 1, \dots, n-1$ *and* $d \geq -d_1/(n-1)$ *be such that* $d_i = d_1 + (i-1)d$. *Then*

$$\begin{aligned} G(\mathbf{x}) &= \left(\frac{n^2-1}{2n}\right)\left(\frac{2d_1 + d(n-2)}{6x_1 + 3d_1(n-1) + d(n-1)(n-2)}\right) \\ B(\mathbf{x}) &= \left(\frac{n-1}{6}\right)\left(\frac{9d_1 + 4d(n-2)}{6x_1 + 3d_1(n-1) + d(n-1)(n-2)}\right). \end{aligned}$$

Proof: For $2 \le j \le n$ and $1 \le i \le j-1$, we have $x_j - x_i = d_i + \cdots + d_{j-1}$. So, $\sum_{i=1}^{j-1}(x_j - x_i) = \sum_{i=1}^{j-1} i d_i$ and therefore

$$\begin{aligned}
&\sum_{j=2}^{n}\sum_{i=1}^{j-1}(x_j - x_i) \\
&= \sum_{j=2}^{n}\sum_{i=1}^{j-1} i d_i = \sum_{i=1}^{n-1}(n-i) i d_i = \sum_{i=1}^{n-1}(n-i) i (d_1 + (i-1)d) \\
&= d_1\left(n\sum_{i=1}^{n-1} i - \sum_{i=1}^{n-1} i^2\right) + d\left((n+1)\sum_{i=1}^{n-1} i^2 - n\sum_{i=1}^{n} i - \sum_{i=1}^{n-1} i^3\right) \\
&= d_1\left(\frac{n^2(n-1)}{2} - \frac{n(n-1)(2n-1)}{2}\right) \\
&\quad + d\left(\frac{(n+1)n(n-1)(2n-1)}{6} - \frac{n^2(n-1)}{2} - \left(\frac{n(n-1)}{2}\right)^2\right) \\
&= \left(\frac{n(n^2-1)}{12}\right)(2d_1 + d(n-2)).
\end{aligned}$$

Let $s_i = x_1 + \cdots + x_i$ and $\mu_i = s_i/i$. Then

$$\begin{aligned}
s_i &= i x_1 + \sum_{k=1}^{i-1}(i-k) d_k \\
&= i x_1 + \sum_{k=1}^{i-1}(i-k)(d_1 + (k-1)d).
\end{aligned}$$

The last expression after simplification leads to

$$s_i = \frac{i}{6}\left(6x_1 + 3d_1(i-1) + d(i-1)(i-2)\right).$$

Using (2), we have

$$\begin{aligned}
G(\mathbf{x}) &= \frac{1}{n s_n}\sum_{1 \le i < j \le n}(x_j - x_i) \\
&= \left(\frac{n^2-1}{2n}\right)\left(\frac{2d_1 + d(n-2)}{6x_1 + 3d_1(n-1) + d(n-1)(n-2)}\right).
\end{aligned}$$

This shows the expression for the Gini index.

We have

$$\begin{aligned}
x_i &= x_1 + d_1 + \cdots + d_{i-1} \\
&= x_1 + \left(\frac{i-1}{2}\right)(2d_1 + (i-2)d).
\end{aligned}$$

From this we obtain

$$\sum_{i=1}^{n}(x_i - \mu_i) = \frac{1}{6}\sum_{i=1}^{n}\left(3d_1(i-1) + 2d(i-1)(i-2)\right).$$

After simplification, we obtain

$$\sum_{i=1}^{n}(x_i - \mu_i) = \frac{n(n-1)}{36}\left(9d_1 + 4d(n-2)\right).$$

Using (4)

$$\begin{aligned} B(\mathbf{x}) &= \frac{1}{n\mu_n}\sum_{i=1}^{n}(x_i - \mu_i) \\ &= \left(\frac{n-1}{6}\right)\left(\frac{9d_1 + 4d(n-2)}{6x_1 + 3d_1(n-1) + d(n-1)(n-2)}\right). \end{aligned}$$

This shows the expression for the Bonferroni index. □

For incomes following a simple arithmetic progression we have the following theorem.

Theorem 9 (Gini and Bonferroni for AP) *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^{++}$ *be a distribution such that* $x_i = x_1 + (i-1)d_1$ *for* $i = 1, \ldots, n$, *where* d_1 *is a non-negative real number. Then*

$$G(\mathbf{x}) = \frac{d_1(n^2-1)}{3n(2x_1 + (n-1)d_1)} \quad \textit{and} \quad B(\mathbf{x}) = \frac{d_1(n-1)}{2(2x_1 + (n-1)d_1)}.$$

Proof: The result follows from Theorem 8 by setting $d = 0$. □

Theorems 8 and 9 indicate that by increasing the incomes of the first and second worst off persons by the same absolute quantity, we can reduce the values of the both the Gini and Bonferroni inequality metrics when incomes follow the generalized or simple arithmetic progression. This is simply one-step lexicographic spreading out of the maximin principle.

We make a clear distinction here between Sen's (1970) lexicographic extension of the maximin criterion and our one-step lexicographic spreading out. According to Sen's criterion of two distributions over a given population size n; maximize the welfare of the worst-off individual; for equal welfare of the worst-off individual, maximize the welfare of the second worst-off individual;...; for equal welfare of the worst-off individual, equal welfare of the second worst-off individual,..., equal welfare of the $(n-1)$th worst-off individual, maximize the welfare of the best-off (nth) individual. Our one-step lexicographic extension requires that (possibly different) incomes of the worst-off and the second worst-off individuals should be increased by the same rank-preserving absolute amount with the objective of making the resulting distribution more equitable.

Corollary 1 (Asymptotic behaviour of Gini and Bonferroni on AP) *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^{++}$ *be a distribution such that* $x_i = x_1 + (i-1)d_1$ *for* $i = 1, \ldots, n$, *where* d_1 *is a non-negative real number. Then,* $G(\mathbf{x}) \to 1/3$ *and* $B(\mathbf{x}) \to 1/2$ *as* $n \to \infty$.

Proof: Given $x_1 > 0$, using the scale invariance property of both the Gini and the Bonferroni indices, we can scale down all the entries of the distribution $\mathbf{x}$ by x_1 without changing the values of $G(\mathbf{x})$ and $B(\mathbf{x})$. Such scaling down transforms the AP given by $\mathbf{x}$ to an AP whose first term is 1 and common difference is d_1/x_1. Now, using Theorem 9 we obtain

$$\begin{aligned} G(\mathbf{x}) &= \frac{(d_1/x_1)(n^2-1)}{3n(2 + (n-1)(d_1/x_1))} = \left(\frac{1}{3} + \frac{1}{3n}\right)\left(\frac{(d_1/x_1)}{2/(n-1) + (d_1/x_1)}\right), \\ B(\mathbf{x}) &= \frac{(d_1/x_1)(n-1)}{2(2 + (n-1)(d_1/x_1))} = \frac{(d_1/x_1)}{2\left(2/(n-1) + d_1/x_1\right)}. \end{aligned}$$

As n increases to infinity, both $1/(3n)$ and $2/(n-1)$ go to zero. Consequently, $G(\mathbf{x})$ goes to 1/3 and $B(\mathbf{x})$ goes to 1/2. □

From Corollary 1, the asymptotic behaviour of $G(\mathbf{x})$ and $B(\mathbf{x})$ do not depend on either the value of the first term x_1 or the value of the common difference d_1. Consequently, if the population size is large enough, then the asymptotic behaviour indicated in Corollary 1 may be assumed to hold for the given distribution.

Equitable distribution of wealth is an ideal and is unlikely to be achieved in practice. Perhaps, the next acceptable distribution would be that of an AP. For a large enough value of n, suppose the Gini index of a distribution $\mathbf{x}$ comes out to be close to 1/3. This may be interpreted as being an acceptable inequality. Similarly, if the value of the Bonferroni index of $\mathbf{x}$ comes out to be close to 1/2, then the inequality in $\mathbf{x}$ may be considered acceptable.

Corollary 2 (Asymptotic behaviour of Gini and Bonferroni on generalised AP) *Let $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^{++}$ be arbitrary. Let $d_i = x_{i+1} - x_i$ for $i = 1, \ldots, n-1$ and $d > 0$ be such that $d_i = d_1 + (i-1)d$. Then $G(\mathbf{x}) \to 1/2$ and $B(\mathbf{x}) \to 2/3$ as $n \to \infty$.*

Proof: Given $x_1 > 0$, we scale down all the components of $\mathbf{x}$ by x_1 without changing the values of $G(\mathbf{x})$ and $B(\mathbf{x})$. Scaling down by x_1 results in d_1 becoming d_1/x_1 and d becoming d/x_1. So, using Theorem 8, we have

$$\begin{aligned} G(\mathbf{x}) &= \frac{1}{2}\left(1 - \frac{1}{n^2}\right)\left(\frac{2d_1/(x_1 n) + d/x_1(1-2/n)}{6/n^2 + 3d_1/x_1(1/n - 1/n^2) + d/x_1(1-1/n)(1-2/n)}\right) \\ B(\mathbf{x}) &= \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(\frac{9d_1/(x_1 n) + 4d/x_1(1-2/n)}{6 + 3d_1/x_1(1/n - 1/n^2) + d/x_1(1-1/n)(1-2/n)}\right). \end{aligned}$$

Again, we have $G(\mathbf{x}) \to 1/2$ and $B(\mathbf{x}) \to 2/3$ as $n \to \infty$. □

Consider the distribution $\mathbf{x} = (x_1, \ldots, x_n)$ with $d_i = x_{i+1} - x_i$ and d_i satisfying $d_i = d_1 + (i-1)d$. If $d = 0$, then $\mathbf{x}$ is an AP and Corollary 1 shows that the Gini and Bonferroni indices asymptotically go to 1/3 and 1/2 respectively. On the other hand, if $d > 0$ (and d_1 is independent of n), then Corollary 2 shows that the Gini and Bonferroni indices asymptotically go to 1/2 and 2/3 respectively. The result does not depend on the actual value of d. So, the asymptotic values of the Gini and the Bonferroni indices increase sharply if d increases from 0 to a very small positive value.

5.2 Geometric Progression

Theorem 10 (Gini and Bonferroni on GP) *Let $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^{++}$ be a distribution such that $x_i = x_1 r^{i-1}$ for $i = 1, \ldots, n$, where $r > 1$ is a real number. Then*

$$G(\mathbf{x}) = \left(1 - \frac{1}{n}\right)\left(\frac{r^n + 1}{r^n - 1} - \left(\frac{2r}{(n-1)(r-1)}\right)\left(\frac{r^{n-1} - 1}{r^n - 1}\right)\right) \quad \text{and} \quad B(\mathbf{x}) = 1 - \frac{P_n}{r^n - 1} + \frac{H_n}{r^n - 1}.$$

In the above, H_n is the n-th harmonic number defined as $H_n = \sum_{i=1}^n (1/n)$ and $P_n = \sum_{i=1}^n (r^i/i)$.

Proof: Using scale invariance of the Gini and Bonferroni indices, we may assume that $x_1 = 1$.

For $i = 1, \ldots, n$, we have

$$\mu_i = \frac{1 + r + r^2 + \cdots + r^{i-1}}{i} = \frac{r^i - 1}{i(r-1)}.$$

In particular, $n\mu_n = (r^n - 1)/(r-1)$.

We first consider the case of Gini index. For $2 \le j \le n$,

$$\begin{aligned}\sum_{i=1}^{j-1}(x_j - x_i) &= \sum_{i=1}^{j-1}\left(r^{j-1} - r^{i-1}\right)\\ &= (j-1)r^{j-1} - \left(1 + r + \cdots + r^{j-2}\right)\\ &= (j-1)r^{j-1} - \frac{r^{j-1}-1}{r-1}.\end{aligned}$$

From (2), we obtain

$$\begin{aligned}G(\mathbf{x}) &= \frac{1}{n^2\mu_n}\sum_{j=2}^{n}\left((j-1)r^{j-1} - \frac{r^{j-1}-1}{r-1}\right)\\ &= \frac{1}{n^2\mu_n}\left(\sum_{j=1}^{n-1} jr^j - \frac{1}{r-1}\left(\sum_{j=1}^{n-1} r^j - (n-1)\right)\right)\\ &= \frac{1}{n^2\mu_n}\left(\frac{(n-1)r^n}{r-1} - \frac{r(r^{n-1}-1)}{(r-1)^2} - \frac{1}{r-1}\left(\frac{r(r^{n-1}-1)}{r-1} - (n-1)\right)\right)\\ &= \frac{1}{n^2\mu_n}\left(\frac{n-1}{r-1}(r^n+1) - \frac{2r}{(r-1)^2}(r^{n-1}-1)\right)\\ &= \frac{1}{n^2\mu_n}\left(\frac{(n-1)(r^n+1)(r-1) - 2r(r^{n-1}-1)}{n(r-1)(r^n-1)}\right)\\ &= \left(1 - \frac{1}{n}\right)\left(\frac{r^n+1}{r^n-1} - \left(\frac{2r}{(n-1)(r-1)}\right)\left(\frac{r^{n-1}-1}{r^n-1}\right)\right).\end{aligned}$$

Next we consider Bonferroni index. Using (4), we have

$$\begin{aligned}B(\mathbf{x}) &= \frac{1}{n\mu_n}\sum_{i=1}^{n}(x_i - \mu_i)\\ &= \frac{1}{n\mu_n}\sum_{i=1}^{n}\left(r^{i-1} - \frac{r^i-1}{i(r-1)}\right)\\ &= \frac{r-1}{r^n-1}\left(\frac{r^n-1}{r-1} - \frac{P_n}{r-1} - \frac{H_n}{r-1}\right)\\ &= 1 - \frac{P_n}{r^n-1} + \frac{H_n}{r^n-1}.\end{aligned}$$

□

The expression for $G(\mathbf{x})$ in Theorem 10 can be written as

$$G(\mathbf{x}) = \left(1-\frac{1}{n}\right)\left(\left(\frac{1+1/r^n}{1-1/r^n}\right) - \left(\frac{2r}{(n-1)(r-1)}\right)\left(\frac{1/r - 1/r^n}{1-1/r^n}\right)\right). \tag{10}$$

Since $r > 1$, as $n \to \infty$, $1/r^n \to 0$. Also, $1/n \to 0$ as $n \to \infty$. So, from (10), the expression for $G(\mathbf{x})$ in Theorem 10 goes to 1 as n goes to infinity.

A GP with common ratio greater than 1 has all the components in the distribution $\mathbf{x}$ to be unequal. So, from Theorem 1, we have $B(\mathbf{x}) > G(\mathbf{x})$ for such a GP distribution $\mathbf{x}$. Since we already know that

$G(\mathbf{x})$ goes to 1 as n goes to infinity, it follows that $B(\mathbf{x})$ also goes to 1 as n goes to infinity. So, when the distribution is a GP with common ratio greater than 1, then both the Gini and the Bonferroni indices go to 1 as n goes to infinity. This asymptotic behaviour does not depend on the actual value of the common ratio. As long as it is greater than 1, as n grows, eventually the inequality in the distribution will be the maximum possible.

It is well known that a GP grows much faster compared to an AP. Correspondingly, one may expect that the inequality in a GP will be greater than that in an AP. From Corollary 1, asymptotically, the Gini index of an AP goes to 1/3 and the Bonferroni index of a GP goes to 1/2. In contrast, the above discussion shows that both the Gini and the Bonferroni indices of a GP go to 1. This difference in the asymptotic behaviours of the Gini and Bonferroni indices on distributions growing as a GP compared to distributions growing as an AP, quantifies the faster growth of inequality in a GP compared to that in an AP.

Given a pre-specified value of $x_1 > 0$, say the poverty line, a reduction in the value of r over the interval $(1, \infty)$ reduces the proportionate gaps between incomes leading to a reduction in the level of inequality.

6 Concluding Remarks

This paper has explored some relationships between the Gini and Bonferroni inequality standards. Some unexplained behaviours of the two indices have been analyzed as well. Following a characteristic of the absolute Gini index, remarked by Weymark (1981), we introduced a new axiom, 'additive monotonicity,' for indicators of inequality. While the absolute Bonferroni index, like the absolute Gini, unambiguously fulfills this postulate, necessary and sufficient conditions for its satisfaction by the relative counterparts of the two indices, are identified. It now remains to be examined further implications of this postulate, particularly, how the members of the generalised entropy measures behave with respect to the postulate. Its role in characterisation of inequality indicators can as well be explored. Another line of investigation that seems worthwhile is to consider a multidimensional version of the axiom and examine its implication on multidimensional inequality standards, say, on the Gajods-Weymark (2005) multidimensional generalized Gini's. We leave these as future research programs.

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