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Abstract

The question of who benefits from economic growth is most commonly assessed by using anonymous data from comparable cross sections to calculate changes in income inequality. An alternative is to utilize longitudinal data and assess the pattern of panel income changes. In this paper, we derive precise theoretical conditions reconciling various measures of rising/falling inequality together with various measures of convergent/divergent panel income changes. We have four main findings: i) for a large number of inequality indices, as well as for Lorenz curves, we derive precise conditions for rising inequality and convergent panel income changes to coexist, ii) we demonstrate that in order to observe both rising inequality and panel convergence, income changes in the panel have to be "large" (and in the right direction), where the meaning of "large" varies depending on the particular regression under analysis, iii) for a large number of inequality indices, as well as for Lorenz curves, we show that it is impossible to have both falling inequality together with divergent panel income changes in shares or in proportions, iv) we find a precise relationship between convergence/divergence of panel income changes in dollars on the one hand and the coefficient of variation, the correlation coefficient be- tween initial and final incomes, and the aggregate economic growth rate on the other.

Keywords: Income Inequality; Economic Mobility; Panel Income Changes.

JEL Classification: J31, D63..

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1 Introduction

Who benefits and who is hurt how much in income terms when an economy grows or contracts?[1] The more traditional way of answering this question is to compare data from two or more anonymous cross sections and gauge changing income inequality among individuals or households. Calculations of cross-sectional inequality measures such as Gini coefficients, income shares of particular quantiles of the income distribution, and comparisons of Lorenz curves have a long and distinguished history. A more recent technique within the anonymous tradition is to calculate Growth Incidence Curves (GICs) which, by design, compare the growth of incomes among anonymous quantiles of the income distribution (Ravallion and Chen, 2003).

An alternative way of gauging who benefits and who is hurt is to utilize data on a panel of people and assess the pattern of panel income changes, recognizing that some people change quantiles. Often called income mobility analysis, the assessment of panel income changes usually is carried out by means of regressions capturing income dynamics (e.g. Atkinson *et al.*, 1992), or by constructing what are called mobility profiles (e.g. Grimm, 2007; Van Kerm, 2009) or, synonymously, non-anonymous Growth Incidence Curves (Bourguignon, 2011).

[1]Throughout this paper, "income" will be used as shorthand for whichever magnitude is under examination.

The fundamental difference between the cross-sectional data and panel data approaches is this. When working with comparable cross-sections and looking at income inequality using such familiar tools as Lorenz curves and inequality indices, the analyst looks at the income of whoever is in the p'th position in each distribution (initial and final) regardless of whether that is the same person in one distribution as in the other.[2] By contrast, when looking at panel income changes, the analyst first identifies which individual is in the p'th position in the initial distribution and follows that person over time, even if that person is in a different position later on.

Thus, a statement about the persons in a particular group g, say, the richest 1% or poorest 10%, means different things in the two approaches. The standard inequality analysis permits statements of the type "the *anonymous* richest 1% got richer while the *anonymous* poorest 10% got poorer" while the panel data analysis makes a different type of statement: "those who *started* in the richest 1% experienced income changes of such and such amount while those who *started* in the poorest 10% experienced income changes of a different amount." To the extent that people move around within the income distribution, the two approaches provide different information.

Is it possible to have convergent panel income changes- that is, panel income changes decrease as initial income increases- and simultaneously to have rising income inequality? Is it possible to have divergent panel income changes along with falling income inequality? Are the possibilities in times of economic growth different from those in times of economic decline? Under what conditions do these different possibilities arise?

The purpose of this paper is to derive precise theoretical conditions reconciling various measures of rising/falling inequality together with various measures of convergent/divergent panel income changes; and identifying when certain impossibilities arise.

Empirically, the anonymous and the panel approach often give different answers to the question of who benefits from economic growth. A literature review turns up many instances of convergent panel income changes, many of them in times of rising inequality. Examples include cases of either i) convergent panel income changes in dollars/pesos/rupiah/etc.,[3] or ii) in per-

[2]The inequality literature terms this "anonymity".

[3]Among those studies see Dragoset and Fields (2008) for the United States, Fields *et al.* (2003a) and Fields *et al.* (2003b) for Indonesia, South Africa, Spain, and Venezuela, Cichello *et al.* (2005) for South Africa, and Fields *et al.* (2015) for Argentina, Mexico, and Venezuela.

centages,[4] coexisting with rising relative inequality. Many of these studies also have data for multiple periods, some of which exhibit constant or falling inequality, yet convergent panel income changes are found throughout.[5] An important case where divergent panel income changes in percentages and yuans were found together with rising inequality is China in the early 2000s (Yin *et al.*, 2006; Fields *et al.*, 2016).[6]

Our paper is not the first one trying to reconcile theoretically changes in inequality with convergence and divergence in panel data.

The idea that a pattern of panel changes whereby those initially at the bottom gain more than those initially at the top necessarily results in falling inequality was first raised by Francis Galton in 1886 in the context of the distribution of heights among parents and children. Later scholars demonstrated that no such implication holds, and Galton's assertion has come to be dubbed "Galton's fallacy" (see, for example, Bliss, 1999).

The literature also offers a claim regarding the opposite set of circumstances. Consider a panel of countries with per capita incomes in comparable currency units - Purchasing Power Parity-adjusted dollars, for example. Define β-divergence (convergence) as arising when a regression of final log-income on initial log-income produces a regression coefficient greater than (less than) one. Define σ-convergence (divergence) as arising when the variance of log-incomes falls (rises) from the initial year to the final year. It is proven in the literature that β-divergence measured in this way and σ-convergence measured in this way cannot arise simultaneously - more specifically, σ-convergence implies β-convergence, but β-convergence does not imply σ-convergence (Furceri, 2005; Wodon and Yitzhaki, 2006). Also, Jenkins and Van Kerm (2006) decompose changes in Generalized Gini indices into

[4]Among those studies see Gottschalk and Huynh (2010) for the United States, Fields *et al.* (2003a) for Indonesia, South Africa, Spain, and Venezuela, Quinn and Teal (2008) for Tanzania, Yin *et al.* (2006), Khor and Pencavel (2006), Fields *et al.* (2016) and Bárcena-Martin *et al.* (2019) for China.

[5]In Duval-Hernández *et al.* (2017), we present an analysis of changes in inequality and convergence in pesos, shares, and percentages over many periods in Mexico.

[6]There is also a number of studies where panel income changes are analyzed, but either i) inequality was not rising, or ii) the change in inequality was not reported. Among those studies we can list those for Indonesia and Peru (Grimm, 2007); Greece, Ireland, Italy, and Portugal (Van Kerm, 2009); Great Britain (Jenkins and Van Kerm, 2016); Italy (Palmisano and Peragine, 2015; Palmisano and Van de gaer, 2016). See also Bourguignon (2011) on growth of mean incomes for countries in different deciles of the world per capita income distribution.

a term reflecting share convergence and a term reflecting re-ranking, while Nissanov and Silber (2009) propose an alternative reconciliation of β- and σ-convergence, as defined above.[7]

Our contribution to this literature is that unlike the studies just cited, our reconciliation of changes in inequality and panel income changes is made using very general and widely used measures of both phenomena. In particular, our analysis of inequality changes is made first through commonly used inequality indices like the coefficient of variation, the variance of log-incomes, the Gini, the Atkinson, and the Generalized Entropy indices. Then, we provide results for cases of Lorenz-curve dominance.[8] Similarly, for the analysis of panel income changes, we estimate linear regressions between initial and final incomes, as traditionally used in studies of intra and inter-generational income mobility (e.g. Atkinson *et al.*, 1992; Solon, 1999, respectively), and the macro literature on absolute convergence of mean per capita incomes across countries (e.g. Barro, 1991; Sala-i-Martin, 1996).[9] By offering a reconciliation of widely used measures of inequality and panel income changes, we thus provide a framework that can be used by these several literatures.[10]

Overall, the results derived in this paper indicate that for each of the aforementioned inequality indices there is a corresponding panel income-change regression such that whenever inequality falls, it is impossible to have divergent panel income changes according to that regression. Another way to express the above is that divergence of panel income changes implies rising inequality, for suitably chosen pairs of relative inequality measures/panel income change regressions. These results capture the idea that whenever the anonymous distribution of income becomes less dispersed, the incomes of the initially poor and the initially rich converge to one another. This structure also applies when making inequality comparisons using Lorenz curves.

[7] O'Neill and Van Kerm (2008) apply the framework of Jenkins and Van Kerm (2006) to study σ and β convergence. In addition, Dhongde and Silber (2016) and Bárcena-Martin *et al.* (2019) provide a framework to analyze distributional change based on Gini-related measures that can serve to capture σ and β convergence.

[8] Furthermore, in cases when the Lorenz curves cross, we additionally analyze in Section 3.4 changes in inequality using the family of transfer-sensitive inequality indices, whenever one distribution third-order stochastically dominates another one.

[9] In the macroeconomics literature the term "absolute convergence" is used when the only explanatory variable in the regression is initial income.

[10] In a previous working paper (Duval Hernández *et al.*, 2015), we presented results pertaining to variance-based inequality indices and to Lorenz curve comparisons. Since then, we have derived many new results which we bring together in this paper.

However, the above conclusion crucially hinges on the matching of particular measures of inequality with specific income-change regressions. More precisely, if we leave unrestricted the choice of inequality measure/income-change regression pair, it is possible to find income processes where inequality falls, yet where there is divergence in panel income changes.[11]

Furthermore, we establish that in order to observe both rising inequality and panel convergence (a result often found in empirical studies), income changes of identified individuals in the panel have to be "large" (and in the right direction), where the meaning of "large" varies depending on the particular regression under analysis.

Dollar-change regressions are special because a uniform proportional increase in dollars makes the dollar-gains of the initially rich larger than those of the initially poor. Because of this property, during periods of strong positive economic growth there can be falling relative inequality together with divergent panel dollar changes, something that doesn't occur with regressions of share changes or of proportional changes.

Our paper goes beyond the previous literature in deriving precise conditions under which the four possibilities of rising/falling inequality can coexist with convergent/divergent panel income changes, and in identifying the pairings of inequality measures and panel income-change regressions for which falling inequality implies convergence in panel incomes. These conditions are derived in Section 3 and summarized in Section 4. First, we define our terms precisely.

2 Concepts and Definitions

The two key variables in this research are income inequality and panel income changes. "Income" is the term used for the economic variable of interest, which could be total income, labor earnings, consumption, or something else. The income recipient will be called a "person", but the results apply equally to households, workers, per capitas, adult equivalents, or country means.

[11] In Section OA.6 in the Online Appendix we present income transitions illustrating that all combinations of rising/falling relative inequality can coexist with various measures of convergence and divergence in panel income changes, both in times of economic growth and in times of economic decline.

2.1 Notation

Consider an economy with n individuals observed over two time periods, initial (or 0), and final (or 1).[12]

Denote by d_{it} the income of individual i in period t measured in constant monetary units (e.g., real dollars). We drop the individual subindex i to denote vectors, e.g., $d_t = (d_{1t}, d_{2t}, \ldots, d_{nt})'$.

The basic building block of panel data analysis is the panel data matrix $\mathbf{D} = [d_0, d_1]$. If each column of $\mathbf{D}$ is divided by its respective mean, μ_t, we obtain the resulting matrix of shares $\mathbf{S} = [s_0, s_1]$.[13]

In addition to income shares, we will also deal with other strictly monotonic transformations of income, like log-incomes, denoted by $\log d_t$. More generically, we will denote by $y_t = f(d_t)$ a variable of income in dollars transformed by a specified strictly monotonically increasing function $f(\cdot)$.

A crucial feature of the panel data matrix $\mathbf{D}$ is that it involves pairs of incomes for each individual, which implies that if the i-th element of d_0 is moved to another row, the i-th element of d_1 must also be moved to the same row. In other words, in panel data analyses we are allowed to permute entire rows of $\mathbf{D}$, a property we call *Multi-period Anonymity*. This contrasts with the property of *Single-period Anonymity* (or simply *Anonymity*) commonly used in the analysis of cross-sectional inequality. Under single-period anonymity, we are allowed to separately permute a given column of $\mathbf{D}$ without necessarily permuting the elements in other columns of the data matrix. In mobility studies then, the assumption of single-period anonymity is replaced by multi-period anonymity, where the income *trajectories* matter without having to look at the names of the particular individuals experiencing such trajectories.

For the most part, income vectors and their transformations are sorted in ascending order of individuals' *initial-period* incomes.[14] An exception to this is when the final incomes are sorted in ascending order of *final-period* income; the resulting vector will be used in the Lorenz curve calculations

[12]While our analysis is set in a framework of income changes between an initial period and a final one, in Section 3.5 we discuss how this framework can be used to incorporate multi-period panel information.

[13]The inequality literature usually works with income expressed as a share of total income. In order to make an easier link with the regressions involving share changes, we will work throughout with shares of mean income. It is obvious that inequality comparisons are the same for shares of total income as for shares of mean income.

[14]This sorting is required for some inequality measures and immaterial for the convergence regressions.

below.

Definition 1 ***Vector of Final Shares in Ascending Order.***
Let $P(\cdot)$ be a permutation operator. Then, define $s_c = (s_{1c}, \ldots, s_{nc})$ ("c" for "counterfactual") as the final income-share vector when final incomes are sorted in ascending order of final *income, i.e.*

$$s_c \equiv P(s_1) \quad \text{such that} \quad s_{ic} \leq s_{jc} \;\; \forall\, i \leq j. \tag{1}$$

It is useful to illustrate the relationship between $\mathbf{D}$, $\mathbf{S}$, and $\mathbf{s_c}$ with a simple example. In particular, we display next a particular panel data matrix $\mathbf{D}$, together with its corresponding matrices $\mathbf{S}$, and $\mathbf{s_c}$.

$$\mathbf{D} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 10 & 9 \end{bmatrix}; \; \mathbf{S} = \begin{bmatrix} 0.23 & 0.69 \\ 0.46 & 0.23 \\ 2.31 & 2.08 \end{bmatrix}; \; \mathbf{s_c} = \begin{bmatrix} 0.23 \\ 0.69 \\ 2.08 \end{bmatrix}$$

Let r_{it} denote the population-normalized rank of individual i in period t, when the distribution in period t is sorted in ascending order of income in *that same period.* In other words, if R_{it} is the rank of individual i when the distribution is sorted in ascending order of income in period t, the normalized rank equals $r_{it} = R_{it}/n$.

Throughout this paper, the notation $[y_{10}, y_{20}, \ldots, y_{n0}] \to [y_{11}, y_{21}, \ldots, y_{n1}]$ will be used to denote the change in an income variable y for panel people $1, 2, \ldots, n$ from time 0 to time 1.

Another concept that we will need is that of a Rank-Preserving Transfer, defined next.

Definition 2 ***Equalizing Rank-Preserving Transfer***
A rank-preserving equalizing transfer $h > 0$ is a transfer of income between two individuals with ranks i and j and with dollar incomes $d_{j0} > d_{i0}$, such that:

$$\begin{aligned}
& d_{k0} = d_{k1} && \text{for } k \neq i, j, \\
& d_{j1} = d_{j0} - h, && \\
& d_{i1} = d_{i0} + h, && \text{where:} \\
& \text{if } j = i + 1, && h < (d_{j0} - d_{i0})/2; \\
& \text{if } j > i + 1, && h < \min[(d_{i+1,0} - d_{i0}), (d_{j0} - d_{j-1,0})].
\end{aligned}$$

A rank-preserving disequalizing transfer is defined similarly.[15] Equalizing transfers are sometimes called "progressive transfers", while disequalizing transfers are sometimes called "regressive transfers".

With this notation we can now specify how we will measure inequality and convergence/divergence of panel income changes.

2.2 Income Inequality

When is income inequality rising or falling? Income inequality and the change in income inequality are conceptualized and measured in a number of ways. "Relative inequality" is concerned with income comparisons measured in terms of ratios, "absolute inequality" with income comparisons measured in terms of dollar differences. In this paper we focus on relative inequality exclusively.

The way we measure inequality change in this paper is completely standard (e.g. Sen, 1997; Foster and Sen, 1997; Cowell, 2011), namely, we use the Lorenz functional (defined next) or a suitable inequality index to represent the inequality at two points in time and then to compare them.

Definition 3 *Lorenz Curve*
Let s_{jt} be the income-share of the individual in position j in period t, when shares are sorted in ascending order of income in that period. The Lorenz Curve of income in period t, LC_t, is the continuous piecewise linear function connecting the points

$$(F_i, L_i) = (i/n, \sum_{j=1}^{i} s_{jt}/n)$$

where $(F_0, L_0) = (0, 0)$.

A powerful and widely-used criterion for determining which of two income distributions is relatively more equal than another is the three-part Lorenz criterion, which states i) if Lorenz curve A lies somewhere above and never below Lorenz curve B, A is more equal than B, ii) if Lorenz curves A and B coincide, then A and B are equally unequal, and iii) if the Lorenz curves

[15] In this case, the final income of the poorer individual will be $d_{i1} = d_{i0} - h$, the final income of the richer individual will be $d_{j1} = d_{j0} + h$, and the last two conditions are replaced by $h < \min[d_{i,0} - d_{i-1,0}, d_{j+1,0} - d_{j,0}]$.

of A and B cross, the relative inequalities of A and B cannot be compared using the Lorenz criterion alone. To formally express the above criterion we need the following definition.

Definition 4 ***Lorenz Dominance***
Let s_{j0} be the initial income-share of the individual in position j, when shares are sorted in ascending order of initial income. Let s_{jc} be the final income-share of the individual in position j, when shares are sorted in ascending order of final income. The final income distribution Lorenz-dominates the initial distribution whenever

$$\begin{aligned} &s_{1c} + s_{2c} + \ldots + s_{jc} \geq s_{10} + s_{20} + \ldots + s_{j0} \;\; \text{for } j = 1, 2, \ldots, n-1 \text{ and} \\ &s_{1c} + s_{2c} + \ldots + s_{jc} > s_{10} + s_{20} + \ldots + s_{j0} \;\; \text{for some } j < n. \end{aligned} \tag{2}$$

Following standard notation, $LC_1 \succ LC_0$ means that the Lorenz curve in period 1 dominates that of period 0, namely, incomes in period 1 are more equally distributed than the ones in period 0 according to the Lorenz-criterion. This situation is sometimes also referred as a "Lorenz-improvement" when going from d_0 to d_1. The judgement is commonly made that when the final income distribution is more equal than the initial one, LC_1 is preferred to LC_0 by the Lorenz criterion. If the domination is weak we denote it as $LC_1 \succeq LC_0$, which means that incomes in period 1 are at least as equally distributed as those in period 0 by the Lorenz criterion. Similarly, if the previous inequalities are reversed we talk of a "Lorenz-worsening".

Judging a Lorenz-dominant distribution to be more equal than a Lorenz-dominated one is equivalent to making inequality comparisons on the basis of four commonly-accepted relative inequality axioms: anonymity, scale-independence, population-independence, and the transfer principle (Fields and Fei, 1978).

Yet, despite its appeal, the Lorenz criterion is not universally used for two reasons: it is ordinal, and it is incomplete. When the Lorenz criterion does render a verdict about which of two income distributions is more equal than another, it can only say that distribution A is more equal than distribution B but not how much more equal A is than B. And when Lorenz curves cross, the Lorenz criterion cannot render a verdict.

Those analysts who seek a complete cardinal comparison of the inequalities of two income distributions are led to use one or more inequality indices, $I(\cdot)$. For present purposes, these indices can be put into three categories:

1. Lorenz-consistent relative inequality indices: An inequality index is Lorenz-consistent if, when one Lorenz curve dominates another, the index registers the dominant distribution as (strictly) more equal (strong Lorenz-consistency) or equally unequal (weak Lorenz-consistency). A partial listing of strongly Lorenz-consistent relative inequality indices includes the Gini coefficient, Atkinson index, Theil index, the Generalized Entropy family of indices, and the coefficient of variation. Included among the weakly Lorenz-consistent inequality indices are the income share of the richest X%, income share of the poorest Y%, and the decile ratios (e.g. 90-10). For details, see Sen (1997), Foster and Sen (1997), and Cowell (2011).

2. Lorenz-inconsistent relative inequality indices: An inequality index is Lorenz-inconsistent if, when one Lorenz curve dominates another, it is ever the case that the index shows the Lorenz-dominant distribution to be less equal. One commonly-used relative inequality index is Lorenz-inconsistent: the variance of the logarithms of income. This index violates the transfer principle - that is, it is possible to make a rank-preserving transfer of income from a relatively rich person to a relative poorer person and yet the index can register an increase in relative inequality (Foster and Ok, 1999; Cowell, 2011).

3. Transfer-sensitive inequality indices: These indices are Lorenz-consistent, but in addition they can also unanimously rank distributions even in the presence of crossings in Lorenz-curves, as long as one distribution third-order stochastically dominates another. All members of the Atkinson index family, the Theil index, and more generally all of the Generalized Entropy measures with parameter smaller than 2 are "transfer-sensitive". The Gini index, however is not (Shorrocks and Foster, 1987).

In our work below, we emphasize Lorenz curve comparisons and Lorenz-consistent inequality indices. However, we give attention to the variance of log-incomes despite its Lorenz-inconsistency, because of its widespread use in the literature.

2.3 Divergent and Convergent Panel Income Changes

By definition, income mobility analysis entails looking at the joint distribution of individuals' incomes at two or more points in time. This entails an analysis of panel income changes since we follow particular individuals over time.

The income mobility literature distinguishes six mobility concepts: time-independence, positional movement, share movement, directional income movement, non-directional income movement, and mobility as an equalizer of longer-term incomes relative to initial (Fields, 2008). For purposes of characterizing the pattern of panel income changes in this paper, the relevant concept is directional income movement among panel people - that is, who gains or loses how much, from an initial date to a final one.

Panel income changes are said to be divergent when the income recipients who started ahead on average get ahead faster than those who started behind. It is convergent when those who started ahead on average get ahead more slowly (or fall behind more) than those who started behind. It is neutral when neither is the case.

What it means to get ahead at a faster, slower, or same rate itself requires careful specification. In the macroeconomics literature, the object of interest is nearly always the growth rate in percentages, often approximated by changes in log-income (see, for example, Barro, 1991; Sala-i-Martin, 1996). On the other hand, the literature on panel income changes among individuals or households presents a more varied picture; some studies use income changes in dollars, while others use changes in log-dollars, exact percentage changes, changes in income shares, or changes in income quantiles such as deciles or centiles (see Jäntti and Jenkins, 2015, for a recent and comprehensive review of the literature).

Much of the literature assesses divergence or convergence by a running a linear regression of final income on initial income or income change on initial income; such a regression is descriptive, not causal. In this paper, we follow this approach as well. Accordingly, we gauge divergence or convergence as follows.

Definition 5 ***Convergence and Divergence***
For a generic income variable y, which can be measured in dollars or by a strictly monotonically increasing function of dollars $y = f(d)$, define the

final-on-initial regression

$$y_1 = \alpha_y + \beta_y y_0 + u_y \tag{3.1}$$

and the change-on-initial regression

$$\Delta y \equiv y_1 - y_0 = \gamma_y + \delta_y y_0 + u_y. \tag{3.2}$$

The two regressions are linked by the relationship $\delta_y = \beta_y - 1$. Divergence of panel changes in y arises when $\beta_y > 1$, or equivalently, when $\delta_y > 0$, and convergence when $\beta_y < 1$, or equivalently, when $\delta_y < 0$. Otherwise, the panel changes in y are deemed neutral.

An alternative way of estimating convergence in shares is through the share-change on initial-y regressions

$$\Delta s = \kappa_y + \lambda_y y_0 + e_y, \tag{4}$$

in which case there will be "share-change-on-y" convergence whenever $\lambda_y < 0$, divergence if $\lambda_y > 0$; otherwise the share changes are deemed neutral. To emphasize, different regressions (and thus λ_y parameters) will arise depending on the specific transformation $y_0 = f(d_0)$ selected.

Finally, define the regression of the exact proportional changes in dollars on initial dollars

$$\text{pch d} \equiv (d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}. \tag{5}$$

Divergence of exact proportional changes arises when $\theta > 0$, convergence when $\theta < 0$. Otherwise, the exact proportional change patterns are deemed neutral.

Since relative inequality analysis is concerned with the distribution of income shares, it is natural to compare it to a regression also expressed in shares. Hence, the usefulness of conducting analysis using regressions (3.1) and (3.2) for $y = s = d/\mu$, as well as with equation (4). In this case, we subscript the parameters of the generic regressions (3.1) and (3.2) with the letter "s".

In spite of this natural connection between relative inequality and a share-change regression, often when someone is interested in finding out whether "the rich got richer and the poor, poorer" the reference is to changes in dollars and not merely in shares. For this reason we will also study changes in dollars. In this case, we subscript the parameters of the generic regressions (3.1) and (3.2) with "d".

Finally, in many applications, economists have been interested in studying whether proportional income changes are convergent or divergent. In particular they have studied whether on average initially richer individuals had proportional income changes larger than those of initially poorer individuals.

We can approximate proportional changes using a log-log regression or we can measure them exactly. In the first case we will use generic equations (3.1) and (3.2) for the transformation $y = \log d$, and we subscript their parameters with "log". For the analysis of exact proportional panel changes we use equation (5).

3 Mathematical Results

In this section we analytically develop a set of results that establish the connection between changes in relative inequality and our several panel income change concepts.

In what follows we will derive our results under two maintained assumptions. First, we will assume that in the initial period the income distribution is not completely equal. That is, we assume $V(y_0) > 0$. If by contrast, we were to allow cases where $V(y_0) = 0$, all the slope coefficients in our regressions would be undefined. The second assumption we will maintain is that between the initial and final periods, there is a change in relative inequality. In Section 3.6 we will briefly discuss the case when inequality remains unchanged.

In everything that follows we consider regressions done on population and abstract from all issues of inference. The proofs of some selected results are included in the Appendix, while all remaining proofs are included in an Online Appendix accompanying this paper.

3.1 Inequality Measures and Panel Changes in Dollars

Many analysts are especially concerned with the relationship between panel changes in dollars and changes in relative inequality. The study of panel dollar changes is relevant because when economic growth takes place, it might be too easy to find convergent proportional gains, for the simple reason that the poor start from much lower income levels. Percentage changes can converge even if the dollar gains of the initially poor individuals are smaller than those of the initially rich ones, and so studying the patterns of dollar changes rather than percentage changes in the panel provides a stronger test of whether incomes are converging or diverging when the economy is growing.

We can derive a connection between dollar regressions, either in final-on-initial form

$$d_1 = \alpha_d + \beta_d d_0 + u_d \tag{6}$$

or in change-on-initial form

$$\Delta d = \gamma_d + \delta_d d_0 + u_d \tag{7}$$

and the coefficient of variation. Unlike the variance of dollars (which is not scale-independent), the coefficient of variation has the advantages of being scale-independent and Lorenz-consistent.

Proposition 1 ***Changes in the Coefficient of Variation, Convergence in Dollars, and Economic Growth***

Let β_d be defined by the final-on-initial dollar regression (6), and denote the correlation coefficient from this regression by ρ_d. Let $CV(d_t)$ denote the coefficient of variation of income at period t, and let g denote the economy-wide growth rate in incomes between year 0 and year 1. Then there is divergence/convergence in dollars as follows:

$$\beta_d \gtreqless 1 \qquad (\text{i.e. } \delta_d \gtreqless 0) \iff \rho_d \frac{CV(d_1)}{CV(d_0)}(1+g) \gtreqless 1. \tag{8}$$

Proof: See Appendix

A look at equation (8) shows that in order to make a rising coefficient of variation compatible with convergent dollar changes ($\beta_d < 1$), we must either have a sufficiently strong economic decline ($g < 0$) or a sufficiently low intertemporal correlation ρ_d.

Consider an economy in which economic growth has taken place (i.e., $g > 0$) and income inequality as measured by the coefficient of variation has risen ($CV(d_1) > CV(d_0)$). If initial and final incomes were perfectly positively correlated - that is, if ρ_d were equal to $+1$ - then applying equation (8), we would know that panel income changes in dollars would necessarily be divergent (i.e., $\beta_d > 1$). However, if initial and final incomes are positively correlated but not perfectly so (i.e., $0 < \rho_d < 1$), room is left open for the possibility that a growing economy with rising income inequality might also have convergent dollar changes. Moreover, equation (8) also tells us that the smaller is ρ_d, the more room there is for positive economic growth, rising income inequality, and convergent dollar changes to coexist.

Some analysts may implicitly be supposing that income recipients who are high (low) to begin with will inevitably be high (low) at a later point in time. Whether or not this is the case is an empirical question. The answer should not, however, be assumed.

If during periods of economic decline, the dollar losses of the poor are larger than those of the rich, i.e., if there is divergence in dollars, then the income share of the rich will grow and so will inequality. Hence, in this case it is impossible to have a falling CV together with divergent dollar changes (this is apparent from equation (8)).

What if economic growth is positive and dollar changes are divergent? In that case the dollar gains of the initially poor can be smaller than those of the initially rich, yet the share gains of the anonymous poor can be higher than the share gains of the anonymous rich, leading to a fall in relative inequality. One such example occurs for transition $[5, 20] \rightarrow [7, 23]$, as it is the case that $\delta_d = 0.067$, yet the CV falls by 0.067.

In more precise terms, as with any relative inequality index, the coefficient of variation is independent of the measurement scale of income; yet the coefficient of a dollar-change regression is affected by proportional dollar-changes. Even when relative inequality is falling, if positive economic growth is strong enough, it can generate divergence in dollars by proportionally increasing incomes by $1 + g$.

3.2 Inequality Measures and Panel Changes in Shares and Proportions

Next we relate commonly used inequality measures to regressions of share and proportional panel changes. If the only change in incomes between the initial and final periods were a uniform rescaling by a given factor, then the income shares would remain unaltered, and the proportional changes would be constant for all individuals. In this case, our share and proportional change regressions would register coefficients equal to zero (i.e. neutral panel changes). Similarly, all relative inequality measures would remain unaltered. In other words, both relative inequality measures and these regressions have in common an invariance to economy-wide rescalings of income, and because of this, we will establish a connection between them under a common general structure.

As we will show, it is always the case that a fall in inequality, as gauged by a particular index, leads to convergence in a specified regression of share or proportional panel changes. Intuitively, this means that for each inequality index we can find a panel regression such that if the income shares of the *anonymous* rich and poor get closer together (i.e. if relative inequality falls), then either the income shares of the *initially* rich and poor also approach one another or the proportional income changes are larger for initially poorer individuals. The counterpart of this result is that if the individuals that started ahead experience larger share or proportional income gains as they go from one period to the next (i.e. if there is divergence), then inequality must rise, as judged by a specific relative inequality index.

A second property present in this general structure is that in order to observe rising relative inequality together with convergent panel changes, these panel changes need to be "large enough". As we will see, what constitutes "large" panel income changes depends on the index-regression pair under consideration.

The above properties can be expressed as a proposition schema, i.e. for any given relative inequality index I we have one resulting proposition.

Proposition 2 ***Changes in Relative Inequality Indices and Convergence in Shares and Proportions***
For any given row in Table 1, let I denote the inequality index listed in column [1], and let π denote the corresponding coefficient of initial income in the regression in column [2] of such table, i.e. π will be one of $\{\delta, \lambda, \theta\}$, depending on the row. Then:

i) Falling inequality and convergent panel changes
If $\Delta I < 0$, then the income change regression in the corresponding row of column [2] is convergent, i.e. $\pi < 0$.

ii) Rising inequality and convergent panel changes
If $\Delta I > 0$ and if the condition in the corresponding row of column [3] of Table 1 is satisfied, then the respective income change regression is convergent, i.e., $\pi < 0$.

iii) Rising inequality and divergent panel changes
If $\Delta I > 0$ and if the condition in the corresponding row of column [3]

of Table 1 is reversed, then the respective income change regression is weakly divergent, i.e., $\pi \geq 0$.

iv) Falling inequality and divergent panel changes
In any row of Table 1, it is impossible to simultaneously have falling inequality, i.e., $\Delta I < 0$, and weakly divergent panel changes, i.e., $\pi \geq 0$.

Proof: See Online Appendix

We can also establish a corollary as the contrapositive of part i) of the above Proposition.[16]

Corollary 1 *If a regression in column [2] of Table 1 is weakly divergent, i.e., if $\pi \geq 0$, for $\pi \in \{\delta, \lambda, \theta\}$, then the inequality index in the correspoding row of the same table must rise.*

Proposition 2 and its corollary show that a falling inequality measure I implies convergent panel changes, as measured by a specific $\pi \in \{\delta, \lambda, \theta\}$ in the corresponding regression listed in column [2] of Table 1. Alternatively, it shows that weakly divergent panel changes in such regressions imply a rising inequality, as measured by the corresponding index in column [1] of that same Table. However, convergence does not imply falling inequality: $\pi < 0 \nRightarrow \Delta I < 0$.

The above proposition schema establishes results for the most commonly used inequality indices in the literature. More specifically, it contains results applying to inequality measures related to variance conditions (row 1 of Table 1), the Gini (row 2), the Generalized Entropy family (rows 3-5), as well as the Atkinson index (rows 6 and 7).[17]

The proposition results applied to row 1 of Table 1 link the variance of any monotonically increasing function of income in dollars $y = f(d)$ (e.g. logarithms, shares) with the coefficient of a regression of the changes in this generic variable y on its initial level y_0. As previously mentioned, in the macro

[16] Strictly speaking the corollary is the contrapositive of part *i)* of the Proposition when the inequality measure I falls *or remains constant*, i.e. when $I \leq 0$. Following the arguments outlined in the proof of the Proposition, it can be shown that a constant I implies convergence, provided that Δs is not zero for at least one person. However, as we mentioned before, save for Section 3.6, we will exclusively deal with cases when inequality changes from one period to the next.

[17] It is worth remembering that Theil's First Measure coincides with the Generalized Entropy index when $\alpha = 1$. Thus, the proposition includes results that apply to Theil's First Measure.

and labor literatures, it is common to assess changes in relative inequality by focusing on the variance of log-incomes, in spite of its Lorenz-inconsistency. In addition, it is easy to show that the variance of shares is the square of the coefficient of variation (see Lemma OA.1 in the Online Appendix), which is a Lorenz-consistent inequality measure. Hence, our proposition applies to the variance of log-dollars and the CV, as particular cases.[18]

It is instructive to illustrate how this proposition schema works for a particular index. In particular, we will illustrate its statement and intuition for the Gini index. In that case the proposition schema becomes:

Proposition 3 ***Changes in the Gini and Convergence of Share-Changes-on-Ranks***

Let G denote the Gini index. Let λ_r be defined by the share-changes-on-rank regression

$$\Delta s = \kappa_r + \lambda_r r_0 + e_r$$

Then:

i) Falling Gini and convergent share-changes-on-ranks

If the Gini falls, then the regression of share-changes on initial ranks is convergent, i.e., $\lambda_r < 0$.

ii) Rising Gini and convergent share-changes-on-ranks

If the Gini rises and if $\Delta G < 2E(s_1 \Delta r)$, then the regression of share-changes on initial ranks is convergent, i.e., $\lambda_r < 0$.

iii) Rising Gini and divergent share-changes-on-ranks

If the Gini rises and if $\Delta G \geq 2E(s_1 \Delta r)$, then the regression of share-changes on initial ranks is weakly divergent, i.e., $\lambda_r \geq 0$.

iv) Falling Gini and divergent share-changes-on-ranks

It is impossible to simultaneously have a falling Gini, i.e., $\Delta G < 0$, and weakly divergent share-changes-on-ranks, i.e., $\lambda_r \geq 0$.

[18] To reiterate, the results applying to the first row in Table 1 pertain to any monotonically increasing function of income, as long as we use the same function $y = f(d)$ as dependent and independent variables, i.e. as long as we run share-changes on initial shares, log-dollar changes on initial log-dollars, etc. As previously mentioned, a particular case of this result for the variance of logs and the coefficient in a log-change regression, was derived independently by Furceri (2005) and Wodon and Yitzhaki (2006).

Proof: See Appendix

In particular let's further explore how this Proposition operates in a simple two-person example. Consider in particular an economy in which the anonymous distribution of income in dollars changes from (1,3) to (1,5). The underlying panel possibilities are:

$$\begin{aligned} \text{Case I}: &\quad [1,3] \rightarrow [1,5] \\ \text{Case II}: &\quad [1,3] \rightarrow [5,1]. \end{aligned}$$

In Case I, income shares are divergent, and the Gini rises, so we are in part *iii*) of the above proposition. In Case II, however, the increase in the Gini ΔG equals 0.083, which is smaller than the weighted sum of rank changes $2E(s_1 \Delta r) = 0.66$.[19]. This puts us in part *ii*) of the above proposition, where convergent share changes co-exist with a rising Gini.

The mathematical condition listed in part *ii*) of the proposition specifies in what sense panel income changes need to be "large", so as to have rising inequality together with convergence. More specifically, it establishes that if panel changes are large enough so that the increase in the Gini is smaller than a weighted sum of rank changes, then it is possible to reconcile rising inequality (as measured by the Gini), together with convergent income changes.

Similar conditions specifying in what sense panel income changes need to be "large", so as to have rising inequality together with convergence, are listed in column [3] of Table 1 for other inequality measures. For instance, in the case of the Generalized Entropy index with parameter $\alpha = 0$ (row 5 of Table 1), the corresponding condition expressing that panel income changes need to be large enough is that the average exact proportional share changes $E[(s_1 - s_0)/s_0]$ have the opposite sign to the average change in log shares, $E(\Delta \log s)$. Changes in logs are an approximation to exact proportional changes, provided that the latter are small. Hence, when these two entities have opposite signs, it follows that share changes in the panel are large.

To conclude this section on inequality indices, note that the impossibilities in the above Propositions occur only when we pair-up a given inequality index with the appropriate panel income change regression. Absent the appropriate pairing, falling inequality and divergent panel income changes can both arise.

[19]The term $E(s_1 \Delta r)$ is a weighted sum of rank changes, because it equals $\sum \omega \Delta r$ for weight $\omega = s_1/n$.

We now turn to results linking our income change regressions to changes in inequality under Lorenz dominance.

3.3 Lorenz Dominance and Panel Income Changes

In spite of the wide use of the indices analyzed in the previous section, the Lorenz Dominance criterion remains the most widely accepted way of judging whether relative inequality has risen or fallen. The reason for this is that whenever this criterion provides an ordering of the inequalities of two distributions, all Lorenz-consistent indices agree with that ordering. In other words, when for two income distributions A and B the Lorenz criterion deems $LC_A \succ LC_B$, it will be the case that all Lorenz-consistent inequality indices deem distribution B to be more unequal.

It turns out that we can find a set of useful results linking Lorenz Dominance to our previous regression methods. We present those results next. It then follows that all these results also apply to the family of Lorenz-consistent inequality indices whenever the Lorenz curves of distributions A and B do not cross.

3.3.1 Lorenz Dominance and Share Changes

In this section we derive a connection between the Lorenz Dominance criterion

$$\begin{array}{ll} s_{1c} + s_{2c} + \ldots + s_{jc} \geq s_{10} + s_{20} + \ldots + s_{j0} & \text{for j = 1, 2, \ldots, n - 1 and} \\ s_{1c} + s_{2c} + \ldots + s_{jc} > s_{10} + s_{20} + \ldots + s_{j0} & \text{for some j < n} \end{array} \tag{2}$$

and a share-change regression

$$\Delta s = \gamma_s + \delta_s s_0 + u_s. \tag{9}$$

Equations (2) and (9) both involve initial and final income-shares. However, the final-period shares appear sorted differently in the two expressions. More specifically, in condition (2), final shares s_c are sorted in ascending order of *final* shares, while in equation (9) final shares s_1 preserve the order of *initial* shares.

It is easy to show that the sign of the coefficient δ_s in regression (9) is determined by the sign of the covariance

$$cov(\Delta s, s_0) = \frac{\sum_i (s_{i1} - s_{i0}) s_{i0}}{n},$$

since average share changes are zero by construction.

Using vector s_c as defined in (1), we can decompose this covariance as

$$cov(\Delta s, s_0) = \frac{\sum_i [(s_{i1} - s_{ic}) + (s_{ic} - s_{i0})] s_{i0}}{n}.$$

That is, whether share changes are convergent or divergent is determined by the sum of two terms, a structural mobility term and an exchange mobility term:

$$\begin{aligned} SM &= \frac{\sum_i (s_{ic} - s_{i0}) s_{i0}}{n} \\ XM &= \frac{\sum_i (s_{i1} - s_{ic}) s_{i0}}{n}. \end{aligned} \tag{10}$$

SM captures the component of the covariance associated with changes in the shape of the income distribution for anonymous people, and XM is the component of the covariance associated with positional change, under a fixed marginal distribution.[20]

We can derive the following two key Lemmas for these terms.

Lemma 1 *Let SM be given by equation (10), then:*

i) A Lorenz-improvement, $LC_1 \succ LC_0$, implies $SM < 0$.

ii) A Lorenz-worsening, $LC_1 \prec LC_0$, implies $SM > 0$.

Proof: See Appendix

In other words, in cases of Lorenz-dominance, the sign of SM fully reflects whether there has been a fall or a rise in inequality judged by the Lorenz-criterion.

As previously mentioned, when looking at income changes, we care not only about how the anonymous distribution of income evolves, but also about who moved to a different position across periods. This is reflected by the transition from s_c to s_1. In this transition, share changes will be convergent, since in the reranking of individuals there will always be a positive transfer of income shares from a relatively richer individual to a poorer one. This is expressed in the following Lemma.

Lemma 2

$$XM = \frac{\sum_i (s_{i1} - s_{ic}) s_{i0}}{n} \leq 0.$$

[20]This is so because if positions were to remain unchanged, i.e. $s_c = s_1$, the entire share change would be due to a change in the shape of the distribution, $s_c - s_0$.

Proof: See Appendix

With these two results we can proceed to analyze the connection between share mobility and changes in inequality as measured by Lorenz comparisons.

Proposition 4 ***Lorenz Dominance and Convergence in Shares***
Let δ_s be defined by the share-change regression

$$\Delta s = \gamma_s + \delta_s s_0 + u_s. \tag{9}$$

Then:

i) Falling inequality and convergent share changes
If there is a Lorenz-improvement, $LC_1 \succ LC_0$, then the regression of share changes on initial shares is convergent, i.e., $\delta_s < 0$.

ii) Rising inequality and convergent share changes
If there is a Lorenz-worsening, $LC_1 \prec LC_0$ and if $|XM| > SM$, then the regression of share changes on initial shares is convergent, i.e., $\delta_s < 0$.

iii) Rising inequality and divergent share changes
If there is a Lorenz-worsening, $LC_1 \prec LC_0$ and if $|XM| \leq SM$, then the regression of share changes on initial shares is weakly divergent, i.e., $\delta_s \geq 0$.

iv) Falling inequality and divergent share changes
It is impossible to simultaneously have a Lorenz-improvement, $LC_1 \succ LC_0$, and weakly divergent share changes, i.e., $\delta_s \geq 0$.

Proof: See Appendix

Corollary 2 *If the regression of share changes on initial shares is weakly divergent, i.e., $\delta_s \geq 0$, then either a weak Lorenz-worsening has taken place, $LC_1 \preceq LC_0$, or the Lorenz curves of incomes in periods 0 and 1 cross.*

The intuition (and proof) behind this proposition is related to a well-known result in the inequality literature stating that an equalization in the Lorenz sense can be achieved by a series of income transfers from richer to poorer individuals that keep unaltered the individual ranks between the initial and the final periods (see for instance Fields and Fei, 1978).

These progressive transfers generate by construction convergent share changes in the transition from s_0 to s_c (Lemma 1). However, when going from s_0 to s_1, we also need to consider the transition from s_c to s_1. In this last step the shape of the income distribution remains unchanged and pairs of individuals swap incomes and therefore positions. As we saw in Lemma 2, this positional rearrangement leads to convergent share changes always.

Hence, in the case of a Lorenz-improvement, both XM and SM go in the same direction, and share changes are convergent. However, in the case of a Lorenz-worsening, the two components will move in opposite directions, and depending on which force is dominant there will be convergence or divergence in shares as measured by δ_s in equation (9).

In contrast, if all individuals keep their same rank in the initial and final distributions (i.e. if there is zero positional mobility), vector s_c will equal the final share vector s_1, and the sign of δ_s is determined exclusively by SM. Given Lemma 1 and the connection between SM and δ_s, in the absence of positional changes, we have that a Lorenz-worsening leads to divergent share changes.

In other words, as long as we restrict ourselves to the case of no positional mobility and no crossings of Lorenz curves, share mobility and changes in inequality fully align, in the sense that rising inequality as gauged by Lorenz-worsening only occurs with divergent share-changes and falling inequality as gauged by Lorenz-improvement only occurs with convergent share-changes. If individuals swap positions from one period to the next, the direction of the inequality change and divergence/convergence need not align one-to-one.

As happened with the propositions in section 3.2, when panel income changes are large (and in the right direction), there can be convergence together with rising inequality. For Lorenz comparisons, the condition expressing "large" panel changes is that the exchange-mobility component $|XM|$ is larger than the structural-mobility one SM. This is Part *ii)* of Proposition 4.

Finally, Corollary 2 expresses the idea that when share changes are divergent, the income shares of the initially rich grow relative to others' shares (irrespective of whether there is positional change or not). This should lead to disequalization. Hence, the only possible way to register a fall in inequality in this instance is for Lorenz curves to cross.[21]

[21] As is well known, when Lorenz curves cross, a Lorenz-consistent measure can always be found showing rising inequality and another Lorenz-consistent measure can be found showing falling inequality.

3.3.2 Lorenz Dominance and Proportional Income Changes

We next explore the relationship between proportional changes in income and Lorenz-improvement/worsening.

Log-Income Approximation

The most common way to measure proportional convergence is by approximating proportional changes by differences in log-income and estimating a double-log regression

$$\Delta \log d = \gamma_{\log} + \delta_{\log} \log d_0 + u_{\log} \tag{11}$$

or its equivalent final-on-initial form $\log d_1 = \alpha_{\log} + \beta_{\log} \log d_0 + u_{\log}$. As we now show, these types of regressions have misleading properties.

Consider the following example:

$$[1, 1, 1, 1, 1, 1, 1, 1, 6, 9] \rightarrow [1, 1, 1, 1, 1, 1, 1, 1, 7, 8].$$

The richest person has transferred \$1 to the next richest person, which is a clear Lorenz-improvement. Inequality therefore falls by the Lorenz criterion and accordingly for any Lorenz-consistent inequality measure. Moreover, a rank-preserving transfer in dollars from the richest person to anyone lower down in the income distribution should be deemed convergent, as it brings convergence in dollars (in this case $\delta_d = -0.04$). However, if in this example, we regress the change in log-dollars on initial log-dollars, we obtain $\delta_{\log} = +0.00045$, and hence find divergence in log-dollars. Thus, in this example, a Lorenz-improvement has taken place and yet the regression of log-income changes on initial log-income registers divergence.[22]

The previous example illustrates a more general point: that log-incomes can be divergent if a progressive transfer occurs sufficiently high-up in the income distribution.

More precisely, we can show the following result for a single rank-preserving transfer that is sufficiently small:

Proposition 5 ***Lorenz Dominance and Log-income Panel Changes under a Single Rank-Preserving Transfer Sufficiently High Up in***

[22] In addition, the variance of log-incomes increases, which it must by Corollary 1 applied to logarithms and its variance.

the Income Distribution

Let δ_{log} be defined by the regression

$$\Delta \log d = \gamma_{log} + \delta_{log} \log d_0 + u_{log}. \tag{11}$$

Furthermore, let gm_0 denote the geometric mean of income in period 0, and note that $\exp(1) \approx 2.718$ *(Euler's number). Consider two individuals i and j such that $d_{i0} > d_{j0} > gm_0 * \exp(1)$. Let $h > 0$ be a small rank-preserving transfer between i and j. Then:*

a) If such a transfer h is equalizing, it produces a Lorenz-improvement $LC_1 \succ LC_0$ and a divergent regression coefficient, i.e. $\delta_{log} > 0$.

b) If such a transfer h is disequalizing, it produces a Lorenz-worsening $LC_1 \prec LC_0$ and a convergent regression coefficient, i.e. $\delta_{log} < 0$.

Proof: See Online Appendix

Proposition 5 suggests why it would be easy to misinterpret a log-change regression like (11). The log-change regression can indicate divergence as we define it, even when the income changes lead to a Lorenz-improvement. Rank-preserving equalizations which occur sufficiently high-up in the income distribution can lead to divergence in log-dollars. This is an unappealing property of the log-income change approximation to exact proportional change.[23]

As mentioned before, it is well known in the literature that the variance of log-incomes is not Lorenz-consistent. In fact Cowell (2011) shows that under a transfer similar to the one in Proposition 5, the variance of logarithms will move in the opposite direction to the Lorenz curve, and Foster and Ok (1999) show that the Lorenz-inconsistency of this variance can occur under even more general circumstances. Proposition 5 adds then further reasons to be cautious when using log regressions and variance of logs.

[23] A consequence of the above Proposition 5 is that we can find (see Table OA.2 in the Online Appendix) all possible combinations of Lorenz-worsening/improvement with convergent/divergent log-income changes. In particular, contrary to the share-change case, we can find examples that make compatible falling inequality as gauged by a Lorenz-improvement and divergent log-income changes.

Exact Proportional Changes

As previously mentioned, one alternative to the log-income changes regression (11) is to regress the exact proportional change in incomes on initial income as in equation (5). In this case, we can establish results and conditions linking Lorenz-improvements/worsenings with convergent/divergent exact proportional changes. In order to do this it is useful to define terms for *proportional* structural mobility (PSM) and *proportional* exchange mobility (PXM):

$$\begin{aligned} PSM &= \frac{1}{n}\sum_i \frac{s_{ic}-s_{i0}}{s_{i0}} \\ PXM &= \frac{1}{n}\sum_i \frac{s_{i1}-s_{ic}}{s_{i0}}. \end{aligned} \tag{12}$$

Similar to the analysis of share changes, PSM is a term capturing the average proportional share changes due to changes in the shape of the income distribution if positions remain unchanged. In turn, PXM reflects proportional share changes associated with positional rearrangements, under a fixed marginal distribution. We can establish the following lemmas for these two terms, which mirror Lemmas 1 and 2.

Lemma 3 *Let PSM be given by equation (12). Then:*

i) A Lorenz-improvement, $LC_1 \succ LC_0$, implies $PSM > 0$.

ii) A Lorenz-worsening, $LC_1 \prec LC_0$, implies $PSM < 0$.

Lemma 4

$$PXM = \frac{1}{n}\sum_i \frac{s_{i1}-s_{ic}}{s_{i0}} \geq 0.$$

Proof: See Online Appendix.

With these lemmas established, we can show the following results linking inequality changes and exact proportional changes.

Proposition 6 ***Lorenz Dominance and Convergence in Exact Proportional Changes***

Let θ be defined by the exact proportional change regression

$$\text{pch d} \equiv (d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}. \tag{5}$$

Then:

i) Falling inequality and convergent exact proportional changes

If there is a Lorenz-improvement, $LC_1 \succ LC_0$, then the exact proportional change regression is convergent, i.e., $\theta < 0$.

ii) Rising inequality and convergent exact proportional changes

If there is a Lorenz-worsening, $LC_1 \prec LC_0$, and if $PXM > |PSM|$, then the exact proportional change regression is convergent, i.e., $\theta < 0$.

iii) Rising inequality and divergent exact proportional changes

If there is a Lorenz-worsening, $LC_1 \prec LC_0$, and if $PXM \leq |PSM|$, then the exact proportional change regression is weakly divergent, i.e., $\theta \geq 0$.

iv) Falling inequality and divergent exact proportional changes

It is impossible to simultaneously have a Lorenz-improvement, $LC_1 \succ LC_0$, and weakly divergent exact proportional changes, i.e., $\theta \geq 0$.

Proof: See Online Appendix

Corollary 3 *If the exact proportional change regression is weakly divergent, i.e., $\theta \geq 0$, then either a weak Lorenz-worsening has taken place, $LC_1 \preceq LC_0$, or the Lorenz curves of incomes in periods 0 and 1 cross.*

The intuition of this Proposition is similar to the ones in Propositions 2 and 4: if income changes are large enough, and in a suitable pattern, we can have positional changes, rising inequality, and convergent proportional changes all taking place at the same time.

3.3.3 Lorenz Dominance and Changes in Dollars

While the previous subsections established a clear connection between change in inequality as gauged by the Lorenz criterion and share and proportional panel changes, some evaluators may be interested specifically in panel income changes in dollars. To close this section we discuss the relationship between changes in inequality under Lorenz-dominance and a dollar-change regression.[24]

[24] In separate work we establish a connection between panel changes in dollars and the so-called Absolute Lorenz Curves, as defined by Moyes (1999).

The connection between Lorenz-dominance and a dollar-change regression can be established through equation (8), derived previously in Propostion 1,

$$\beta_d \gtreqless 1 \qquad \text{(i.e. } \delta_d \gtreqless 0) \iff \rho_d \frac{CV(d_1)}{CV(d_0)}(1+g) \gtreqless 1. \tag{8}$$

Given that the coefficient of variation is Lorenz-consistent, a Lorenz-improvement will make the ratio $CV(d_1)/CV(d_0)$ smaller than 1. Depending on the movement of the other components ρ_d and g, we can end up having convergence ($\beta_d < 1$) or divergence ($\beta_d > 1$) in dollars. The intuition presented in section 3.1 holds in this case, and will not be repeated here.

3.4 Extensions to Cases Involving Single Lorenz Crossings

As previously noted, all the results in section 3.3 were derived by analyzing rising or falling inequality as judged by Lorenz-worsenings or improvements. Of course, it is possible for the Lorenz curves of two distributions to cross, which often happens in practice.[25] How far can we go when Lorenz curves cross? In addition to our results for specific inequality indices, which still apply, it is also possible to establish more general results for the class of "Transfer-Sensitive inequality indices" (Shorrocks and Foster, 1987). These indices allow certain pairs of distributions to be ranked in the presence of Lorenz-crossings by giving greater weight to transfers that occur in the lower part of the income distribution. More specifically this family of indices can rank two distributions when one distribution third-order stochastically dominates the other one.[26]

In section OA.3 in the Online Appendix file we present a result linking the class of Transfer Sensitive indices with the share change regression (9).

3.5 Extension to Multiple Periods

So far all the results presented relate changes in inequality to panel convergence over two periods, initial and final. However, the above framework can

[25]See Atkinson (1973, 2008) for a classic discussion of the available evidence on Lorenz-crossings using real data in a cross-country setting.

[26]Shorrocks and Foster (1987) show that the Atkinson family and the Generalized Entropy class with $\alpha < 2$ satisfy the transfer-sensitive property, but the Gini coefficient does not.

be used to analyze information arising from income vectors covering more than two periods.

One particular such extension occurs when in the above analysis we replace the vector of final incomes, y_1, with a vector of individual average incomes, y_a, where the average is taken separately for each individual over all the periods for which there is information available. This y_a can be considered a measure of "longer-term" incomes.

In this case, instead of comparing changes in inequality between two points in time, we compare the inequality in "longer-term" incomes relative to the initial inequality. This comparison has attracted the attention of economists because it indicates to what extent income mobility equalizes longer-term incomes relative to initial (see for instance, Fields, 2010).

In this setting, the interpretation of convergence in panel change regressions will be different as well. More specifically, convergence will now mean that individual income trajectories over time are such that initially poorer individuals will have the most positive (or less negative) differences between their "longer-term" incomes and their initial incomes. As usual, the precise income concept will depend on the regression under consideration. For a detailed exposition we refer the reader to Section OA.4 in the Online Appendix.

Finally, another way in which multi-period information can be incorporated within the framework of our paper, is if instead of working with vectors of initial and final incomes, we replace them with corresponding vectors averaging income across various years around the initial period, and across many years around the last period observed. For instance, in a dataset spanning 15 years of income information one could average the income in the first five and the last five years, and use such averages instead of y_0 and y_1 above. By averaging over various years, this exercise could help in reducing the impact of measurement error and of reversion to the mean due to transitory income shocks.

In summary, the two-period framework exploited throughout the paper is not as limiting as it might first seem. Instead, it can be used to analyze many economically meaningful questions in a straightforward manner while using richer multi-period datasets.

3.6 Special Cases

To conclude this section we briefly discuss three special types of changes in the income distribution vector to gain additional understanding of how different patterns of Lorenz curve changes and divergent/convergent panel income changes can arise.

First, we consider the case where the anonymous income distribution vector does not change, but where individuals swap positions. Since there is no change in the anonymous distribution, the Lorenz curves and all inequality indices will remain unchanged. However, the positional swaps that occur will lead to convergence in our panel regressions no matter how we measure it.

The second special case we consider is one in which all incomes change proportionally, i.e. all incomes are scaled-up or down by a constant multiplicative factor κ. In this case, all relative inequality measures will remain the same, also the panel income changes are recorded as neutral, i.e. neither convergent nor divergent, by three of our panel regressions. The one exception arises in the case of the dollar change regression, because a uniform proportional increase(decrease) in dollars makes the dollar gains(losses) of the initially rich larger than those of the initially poor.

Finally, we consider the case when all individuals keep their same positions and yet there is a Lorenz-worsening. This case is of interest because, as we saw in the previous sections, it is not possible to rule out the existence of convergent panel income changes when inequality rises, due to the fact that there might be crossings among panel people as we go from one period to the other. It is then interesting to see whether in the absence of positional changes, rising relative inequality is or is not a sufficient condition for divergent panel changes. Furthermore, this scenario of a Lorenz-worsening with no positional change, seems to be what many people have in mind when they think of increases in inequality.

In this third special case, the regressions of shares changes and exact proportional changes record divergence. However, to get divergence in dollar changes, we need to assume the additional condition that economic growth has been non-negative.[27] Also, in the case of the log-change regression, we have an ambiguous result depending on where in the distribution the disequalizing income changes are taking place (see Proposition 5).[28]

The results for the three special cases are summarized and established in

[27]Under negative growth we could find convergent, neutral or divergent dollar changes depending on the magnitude of the growth rate of income, g.

[28]This conclusion does not extend to cases of rising relative inequality, without Lorenz-worsening. For instance, in the case of the transition $[1, 2.99, 3] \to [2, 2.001, 5]$, there is a rise in inequality (the CV goes from 0.40 to 0.47), the Lorenz curves cross, there is no positional change, yet all of our regressions register convergence, rather than divergence.

section OA.5 of the Online Appendix.

This concludes our derivation of results. We turn now to a summary of the results and a concluding discussion.

4 Summary of Results and Concluding Observations

This paper has explored mathematically the relationship between changing relative income inequality in the cross section and panel income changes. In spite of all four combinations - rising inequality and convergent panel income changes, rising inequality and divergent panel income changes, falling inequality and convergent panel income changes, and falling inequality and divergent panel income changes - being possible under scenarios of positive and negative aggregate income growth, we find that for each way of assessing change in inequality, there is a corresponding panel income change regression, such that two properties hold: i) if the income distribution becomes less unequal relatively then there is convergence in panel incomes, and ii) if panel income changes are divergent, then relative inequality rises. Intuitively, the first property says that if the anonymous rich and the anonymous poor get closer together relatively when going from period 0 to period 1, then the identified rich and poor cannot be farther apart in relative terms. The second property means that if the identified initially rich and poor move relatively farther away from one another, then the anonymous income distribution must become relatively more unequal. In our paper, we show that these intuitive properties hold for arbitrary n-person economies and for a wide variety of ways of measuring inequality, provided that we pair each relative inequality measure with the appropriate panel income-change regression.

Dollar-change regressions are interesting, because they indicate how much additional purchasing power each recipient gains in the course of economic growth. During periods of strong positive economic growth, there can be falling relative inequality, convergent panel income changes in relative terms, and yet divergent dollar changes. In other words, the two intuitive properties in the preceding paragraph do not apply to dollar change regressions precisely because dollar changes are not invariant to proportional rescalings.

What may be problematical is the use of log-dollar change regressions, like the ones commonly used in the macro and labor literatures, because they have the unappealing property of deeming panel income changes divergent when rank-preserving equalizing transfers occur sufficiently high-up in the income distribution.

To conclude let us return to where we started; namely with the reconciliation between i) convergent panel income changes and rising inequality, and between ii) divergent panel income changes and falling inequality.

Convergence can occur in spite of rising inequality provided that panel income changes are large enough in absolute value as inequality increases. In particular, our paper shows how it is possible to have convergent dollar changes, together with rising relative inequality measures, in times of economic growth - a combination that is often observed in empirical data. Finally, whether divergent panel income changes and falling inequality can coexist or not depends crucially on the way inequality and divergence are measured.

The results derived in this paper open up additional questions as to the empirical nature of individual income changes. For instance, when rising inequality is observed together with convergent panel income changes in empirical work, is this finding driven by a few individuals experiencing large changes, by many individuals experiencing moderate changes, or are both important? Exploring the precise way in which these individual income changes occur is an important question for future research.

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Table 1: Inequality Indices and Panel Change Regressions

	Index [1]	Income-Change Regression [2]	Convergence Condition [3]
(1)	*Variance of y*	$\Delta y = \gamma_y + \delta_y y_0 + u_y$	$\Delta V(y) < V(\Delta y)$
(2)	*Gini G*	$\Delta s = \kappa_r + \lambda_r r_0 + e_r$	$\Delta G < 2E(s_1 \Delta r)$
Generalized Entropy GE			
(3)	$\alpha \neq 0, 1$		
	$GE = (\alpha(\alpha-1))^{-1} E\left[s_{it}^{\alpha} - 1\right]$	$\Delta s = \kappa_\alpha + \lambda_\alpha f^*(s_0, \alpha) + e_\alpha$	$\alpha \Delta GE < E(s_1 \Delta f^*(s, \alpha))$
(4)	$\alpha = 1$		
	$GE = E(s_{it} \log s_{it})$	$\Delta s = \kappa_{\log s} + \lambda_{\log s} \log s_0 + e_{\log s}$	$\Delta GE < E(s_1 \Delta \log s)$
(5)	$\alpha = 0$		
	$GE = E\left[\log(1/s_{it})\right]$	$(d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}$	$E(\Delta \log s) < 0 < E\left(\frac{s_1 - s_0}{s_0}\right)$
Atkinson A			
(6)	$\epsilon < 1, \epsilon \neq 0$		
	$A = 1 - [E(s_{it}^{\epsilon})]^{\frac{1}{\epsilon}}$	$\Delta s = \kappa_\epsilon + \lambda_\epsilon f^*(s_0, \epsilon) + e_\epsilon$	$\epsilon \Delta \tau(A) < E(s_1 \Delta f^*(s, \epsilon))$
(7)	$\epsilon = 0$		
	$A = 1 - \prod_{i=1}^{n} s_{it}^{1/n}$	$(d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}$	$E(\Delta \log s) < 0 < E\left(\frac{s_1 - s_0}{s_0}\right)$

y is an arbitrary monotonically increasing transformation of income in dollars d. s denotes income in dollars as a share of mean income, $\mu = E(d)$ i.e. $s = d/\mu$. r is the normalized income rank running from 1/n to 1. While $f^*(s, \nu)$ is the monotonic transformations of shares $f^*(s, \nu) = s^{\nu-1}/(\nu - 1)$; which in the case of the GE index we set to $\nu = \alpha$ and for the Atkinson family to $\nu = \epsilon$. Finally, $\tau(A) = (1-A)^{\epsilon}/(\epsilon(\epsilon - 1))$ is a monotonically increasing transformation of the Atkinson index A.

Appendix

Proofs of Selected Results

Proof of Proposition 1. By definition

$$\rho_d = \frac{cov(d_1, d_0)}{\sqrt{V(d_1)}\sqrt{V(d_0)}}$$

and

$$\beta_d = \rho_d \frac{\sqrt{V(d_1)}}{\sqrt{V(d_0)}}.$$

However,

$$\frac{\sqrt{V(d_1)}}{\sqrt{V(d_0)}} = \frac{\sqrt{V(d_1)}/\mu_1}{\sqrt{V(d_0)}/\mu_0}\,\frac{\mu_1}{\mu_0} = \frac{CV(d_1)}{CV(d_0)}\,\frac{\mu_1}{\mu_0}.$$

Moreover,

$$\mu_1 = (1+g)\mu_0$$

where g is the economy-wide income growth rate. Combining these equations together we obtain equation (8). □

Proof of Proposition 3. We can express the Gini index at time t as

$$G_t = -\frac{n+1}{n} + \frac{2}{n}\sum_{i=1}^{n} r_{it}s_{it}.$$

Hence, the change in Ginis can be expressed as

$$\begin{aligned}
G_1 - G_0 &= \frac{2}{n}\sum_{i=1}^{n}(r_{i1}s_{i1} - r_{i0}s_{i0}) \\
&= \frac{2}{n}\sum_{i=1}^{n}(r_{i1}s_{i1} - r_{i0}s_{i1} + r_{i0}s_{i1} - r_{i0}s_{i0}) \\
&= \frac{2}{n}\sum_{i=1}^{n}[(r_{i1} - r_{i0})s_{i1} + r_{i0}(s_{i1} - s_{i0})].
\end{aligned}$$

In other words, we arrive at the following decomposition

$$\Delta G = \frac{2}{n}\sum_{i=1}^{n}(s_{i1}\Delta r_i + r_{i0}\Delta s_i)$$
$$= 2[E(s_1\Delta r) + E(r_0\Delta s)] \tag{A.1}$$

To establish the results in this proposition we will first show that $E(s_1\Delta r)$ is always non-negative. Then we will show that that the sign of λ_r is given by the sign of the second term in (A.1), i.e., $E(r_0\Delta s)$.

The following Lemma establishes the first result.

Lemma

$$\sum_{i=1}^{n} s_{i1}\Delta r_i \geq 0.$$

$\triangleleft$*Proof of Lemma*

Order the individuals in ascending order of initial income shares, and create an $n \times n$ matrix A, whose rows and columns identify the individuals in that order- same order for rows and columns.

Let the entries of the A matrix be filled as follows: if individual p overtakes individual q (i.e., p's rank is less than q's initially, but greater in final shares), then $A(p, q) = 1$ and $A(q, p) = -1$. That is, $A(p, q) = 1$ if p's share of final income exceeds that of q when p's share of initial income was less than q's. If two individuals do not overtake one another then $A(q, p) = 0$. By constructing the matrix A in this manner we ensure that for all $i, j \in \{1, \ldots, n\}$ it is the case that $A(i, j) = -A(j, i)$.

From this construction, it follows that we can express the change in ranks for any given individual as the column sum for a given row of matrix A, i.e.,

$$\Delta r_i = \sum_{q=1}^{n} A(i, q).$$

Hence we can write,

$$\sum_{i=1}^{n} s_{i1}\Delta r_i = \sum_{i=1}^{n} s_{i1}\left(\sum_{q=1}^{n} A(i, q)\right).$$

This sum aggregates terms of the form $s_{i1}A(i, j)$ for three types of pairs (i, j):

a) $Z = \{i, j \in (1, \dots, n) | A(i, j) = 0\}$,

b) $Pos = \{i, j \in (1, \dots, n) | A(i, j) = 1\}$,

c) $Neg = \{i, j \in (1, \dots, n) | A(i, j) = -1\}$.

For each element in the set Pos there is a corresponding element in the set Neg (since $A(i, j) = -A(j, i)$), so we can now sum over the pairs

$$\begin{aligned}\sum_{i=1}^{n} s_{i1} \Delta r_i &= \sum_{(p,q) \notin Z} [s_{p1} A(p, q) + s_{q1} A(q, p)] \\ &= \sum_{(p,q) \notin Z} (s_{p1} - s_{q1})\end{aligned}$$

yet we know that $s_{p1} > s_{q1}$ since p overtook q going up, hence we have a sum of positives (or zeroes if there was no positional change).

This completes the proof of the Lemma.▷

The intuition behind this result is that for any upward rank change there will be one or more downward rank changes such that the overall sum of the upward and downward rank changes is zero. The upward rank change is multiplied by a larger *final* income share than are the downward rank changes. This is true for all upward rank changes, individually and together.

As a consequence of this last Lemma and equation (A.1) we can establish that if $\Delta G < 0$, it must be that the second term

$$\frac{2}{n} \sum_{i=1}^{n} r_{i0} \Delta s_i = 2E(r_0 \Delta s)$$

is negative.

This term however, is a (rescaled) covariance between share changes and initial ranks. In particular,

$$\begin{aligned}cov(\Delta s, r_0) &= E(r_0 \Delta s) - E(r_0)E(\Delta s) \\ &= E(r_0 \Delta s),\end{aligned}$$

as $E(\Delta s) = 0$, by construction. If this term is negative (as it is when $\Delta G < 0$) then $\lambda_r < 0$, since by definition

$$\lambda_r = \frac{cov(\Delta s, r_0)}{V(r_0)}.$$

This proves part i).

To prove ii) notice that by virtue of the decomposition (A.1), if $0 \leq \Delta G < 2E(s_1 \Delta r)$, then it follows that $E(r_0 \Delta s) = cov(\Delta s, r_0) < 0$, and hence $\lambda_r < 0$.

By a similar logic, if $\Delta G \geq 2E(s_1 \Delta r)$, then by (A.1) it follows that $cov(\Delta s, r_0) \geq 0$, and hence $\lambda_r \geq 0$. Furthermore, the above inequalities automatically guarantee that $\Delta G \geq 0$, as we know from the previous Lemma that $E(s_1 \Delta r) \geq 0$. Together this establishes part iii) of the Proposition.

Finally, the impossibility in part iv) follows immediately from part i). $\square$

Proof of Lemma 1. Proof of part ii)

Let s_0 be the initial vector of shares and let s_c be defined as in (1). Namely, let s_c be the vector of final-period shares sorted in ascending order of final income.

Theorem 2.1 in Fields and Fei (1978) implies that if the distribution of s_0 Lorenz-dominates that of s_c, i.e. if $LC_0 \succ LC_c$, then it is possible to go from s_0 to s_c by means of a sequence of rank-preserving disequalizing transfers.

One convenient way of representing such transfers is by indexing them as $h(i, j)$ where the first argument, i, indicates which individual is making a transfer and the second one, j, which one is receiving it.

Since the transfers are disequalizing, and no one makes a transfer to himself, they satisfy the following conditions:

$$\begin{aligned} &h(i,j) = 0 \quad \text{for } d_{i0} \geq d_{j0} \\ &h(i,j) \geq 0 \quad \text{for } d_{i0} < d_{j0} \quad \text{with strict inequality for some pair } \{i,j\}. \end{aligned}$$

The total transfers made by individual i will be the sum over the second index j, namely

$$h(i, \cdot) = \sum_{j=1}^{n} h(i, j).$$

Similarly, the total transfers received by this same individual will be the sum over the first index, namely

$$h(\cdot, i) = \sum_{j=1}^{n} h(j, i).$$

Hence, the change in this person's income share can be expressed as the difference between the two previous quantities, i.e.

$$s_{ic} - s_{i0} = h(\cdot, i) - h(i, \cdot) = \sum_{j=1}^{n} h(j, i) - \sum_{j=1}^{n} h(i, j).$$

By construction, the sum of the share changes over all individuals is zero, hence each person's share loss is somebody else's share gain, and also each share gain is somebody else's loss. In other words, any given transfer $h(i, j)$ appears with a positive sign in the share change of individual j, and with a negative sign in the share change of individual i. Furthermore, at any given stage of the sequence of transfers, the sender i is always poorer than the receiver j, since the transfer is disequalizing. Hence, for each transfer $h(i, j)$ we have

$$h(i, j)(\tilde{s}_j - \tilde{s}_i) \geq 0,$$

where $\tilde{s}_i$ and $\tilde{s}_j$ are the shares of individuals i and j, respectively, at the given stage of the sequence of transfers where $h(i, j)$ takes place.

Since each of these transfers are rank-preserving, it follows that at any given stage of the sequence of transfers, $\tilde{s}_j - \tilde{s}_i \geq 0$ implies

$$h(i, j)(s_{j0} - s_{i0}) \geq 0.$$

Notice however, that SM in equation (10) can be rewritten as

$$\begin{aligned} SM &= n^{-1} \sum_i (s_{ic} - s_{i0}) s_{i0} \\ &= n^{-1} \sum_i \left(\sum_{j=1}^{n} h(j, i) - \sum_{j=1}^{n} h(i, j) \right) s_{i0}. \end{aligned}$$

That is, SM will be the average of terms $h(i, j)(s_{j0} - s_{i0})$ for all the transfers $h(i, j)$. Since these terms are non-negative, and some will be strictly positive, then SM will be positive.

In other words, we have shown that $LC_0 \succ LC_c$ implies $SM > 0$. However, by construction, the Lorenz curve of the vector s_c is the same as that of the final income vector s_1 (i.e. $LC_c = LC_1$), so we have that a Lorenz-worsening $LC_0 \succ LC_1$ implies $SM > 0$.

The proof of part i) follows by reproducing the previous steps, now with rank-preserving equalizing transfers. □

Proof of Lemma 2. Recall s_c is a permutation of s_1. In particular, s_c is sorted in ascending order of s_1, and we will assume that both s_0 and s_1 are sorted in ascending order of s_0. Since vectors s_1 and s_c have the same elements, the only differences between them are the ones due to positional changes. If nobody changes positions $s_c = s_1$, and $XM = 0$, trivially. Hence, we will assume from now on that $\exists\, i \leq n$ such that $s_{i1} \neq s_{ic}$.

Denote the difference between s_{i1} and s_{ic} by

$$\eta_i \equiv s_{i1} - s_{ic}.$$

In other words, η_i is the difference between the final-period share of the individual ranked i in the initial distribution and the final-period share of the individual ranked i in the final distribution, when each of these distributions is sorted in ascending order.

Also, denote the (ordered) set of individual indices by $I = \{1, 2, \ldots, n\}$.

Since we want to establish that

$$\sum_{i \in I} (s_{i1} - s_{ic}) s_{i0} \leq 0,$$

we need only include in the sum those individuals who changed position, since $\eta_i = 0$ for those who did not change position.

The (ordered) set of indices for individuals with non-zero positional change is denoted by $\tilde{I} = \{a(1), a(2), \ldots, a(m)\}$, for $m \leq n$. That is, there are m individuals who changed positions and $a(1), \ldots, a(m)$ are their indices in the original set I.

The next claim will be useful in what follows.

Claim *For all $a(j) \in \tilde{I}$, let $a(m) \geq a(j)$. Then,*

$$\sum_{i=0}^{k} \eta_{a(m-i)} \leq 0 \quad \forall k < m.$$

$\triangleleft$ *Proof of Claim*

Start with $\eta_{a(m)} = s_{a(m)1} - s_{a(m)c}$. The share $s_{a(m)c}$ is the highest final-period share among those individuals who changed positions (since s_c is sorted in ascending order of final shares and $s_{a(m)c}$ is its last element). In contrast, $s_{a(m)1}$ is the final-period share of the initially richest individual who changed positions (since we assumed s_1 to be sorted in ascending order of initial income). Since the person initially in the position $a(m)$ must have moved lower in the distribution, it follows that $s_{a(m)1} < s_{a(m)c}$, and thus $\eta_{a(m)} < 0$.

Now consider the sum:

$$\eta_{a(m)} + \eta_{a(m-1)} = (s_{a(m)1} + s_{(m-1),1}) - (s_{a(m)c} + s_{a(m-1),c}).$$

In this expression, the terms in s_c are the two largest shares (among those who changed positions) because s_c is ordered in ascending order of s_1. In contrast, the terms $s_{a(m)1}$ and $s_{a(m-1),1}$ may or may not be the largest, hence

$$\eta_{a(m)} + \eta_{a(m-1)} \leq 0.$$

Now continue to the top three, top four, etc. The same logic as before yields

$$\sum_{i=0}^{k} \eta_{a(m-i)} = \sum_{i=0}^{k} s_{a(m-i),1} - \sum_{i=0}^{k} s_{a(m-i),c}.$$

Again, note that the elements in the s_c sum are the largest k+1 final shares among those who changed positions, while the elements in the s_1 sum need not be the largest k+1 final shares.

This establishes the claim.▷

Since $\forall a(i) \in \tilde{I}$, $\eta_{a(i)} \neq 0$, we can partition the index set $\tilde{I}$ into alternating subsets of contiguous indices for individuals with positive and negative positional changes. That is, we can express

$$\tilde{I} = \{M_1, M_2, \ldots, M_h\} \quad h \leq m,$$

where the partition subsets M_k have the following properties:

i) For all $a(i) \in M_k$, it is either true that $\eta_{a(i)} > 0$ or $\eta_{a(i)} < 0$.

ii) For any sets M_k and M_l, with $k < l$, and for all $a(i) \in M_k$ and $a(j) \in M_l$, we have $a(i) < a(j)$.

Note that in the partition $\tilde{I} = \{M_1, M_2, \ldots, M_h\}$, the first subset, M_1, contains observations with positive η's. This is because $\eta_{a(1)} \in M_1$ and this term is strictly positive, since for all individuals who changed positions it is the case that $\eta_{a(j)} \neq 0$ and $s_{a(1)c}$ is the smallest final-period share among all position-changers. A similar logic establishes that the last subset in the partition, M_h, contains elements with negative η's.

To simplify notation we will denote the subsets with positive elements by P_j and the ones with negative changes by N_j. Hence, we can reexpress our partition as

$$\tilde{I} = \{P_1, N_1, \ldots, P_g, N_g\} \quad g < h,$$

Furthermore, for each of these P_j subsets denote their maximum as $\hat{p}_j = \max P_j$.

Next, define the following sums over such subsets:

$$SP_j = \sum_{i \in P_j} \eta_i; \quad SN_j = \sum_{i \in N_j} \eta_i; \quad S_j = SP_j + SN_j;$$
$$XP_j = \sum_{i \in P_j} \eta_i s_{i0}; \quad XN_j = \sum_{i \in N_j} \eta_i s_{i0}; \quad X_j = XP_j + XN_j.$$

For any X_j as defined above it is the case that:

$$\begin{aligned}
X_j &= \sum_{i \in N_j} \eta_i s_{i0} + \sum_{i \in P_j} \eta_i s_{i0} \\
&\leq \sum_{i \in N_j} \eta_i s_{i0} + s_{\hat{p}_j 0} \sum_{i \in P_j} \eta_i \quad (\text{since } \hat{p}_j = \max P_j,\ \eta_i > 0\ \ \forall i \in P_j, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and } s_0 \text{ is sorted in ascending order}) \\
&= \sum_{i \in N_j} \eta_i s_{i0} + s_{\hat{p}_j 0}(S_j - SN_j) \quad (\text{by definition of } S_j) \\
&= \sum_{i \in N_j} \eta_i (s_{i0} - s_{\hat{p}_j 0}) + s_{\hat{p}_j 0} S_j \quad (\text{since } SN_j = \sum_{i \in N_j} \eta_i).
\end{aligned}$$

Observe that $\forall i \in N_j$, it is the case that $\eta_i < 0$, and also $s_{i0} - s_{\hat{p}_j 0} \geq 0$ (by the fact that s_0 is sorted in ascending order and for any subset j, if $i \in N_j$ and $l \in P_j$, then $l < i$). Therefore the following inequality holds

$$\sum_{i \in N_j} \eta_i (s_{i0} - s_{\hat{p}_j 0}) \leq 0,$$

always.

Now consider the other term in the expression of X_j, namely, $s_{\hat{p}_j 0} S_j$. Our goal is to prove that the summation of these terms across all subsets is also non-positive. To establish this result we will work with the partial sum of such terms, starting from the highest index g all the way down to an arbitrary $k < g$, that is

$$\sum_{h=0}^{k} s_{\hat{p}_{g-h}0} S_{g-h}.$$

In particular, it is the case that

$$0 \geq s_{\hat{p}_{g-k}0} \sum_{h=0}^{k} S_{g-h} \quad (\text{since } \sum_{h=0}^{k} S_{g-h} \leq 0 \text{ by the above Claim that}$$

$$\sum_{i=0}^{k} \eta_{a(m-i)} \leq 0 \quad \forall k < m)$$

$$= s_{\hat{p}_{g-k}0} \sum_{h=0}^{k-1} S_{g-h} + s_{\hat{p}_{g-k}0} S_{g-k}$$

$$\geq s_{\hat{p}_{g-k+1}0} \sum_{h=0}^{k-1} S_{g-h} + s_{\hat{p}_{g-k}0} S_{g-k} \ \ (\text{as } \sum_{h=0}^{k-1} S_{g-h} \leq 0 \text{ and } s_{\hat{p}_{g-k}0} < s_{\hat{p}_{g-k+1}0})$$

$$= s_{\hat{p}_{g-k+1}0} \sum_{h=0}^{k-2} S_{g-h} + s_{\hat{p}_{g-k+1}0} S_{g-k+1} + s_{\hat{p}_{g-k}0} S_{g-k}.$$

Continuing these steps $k-2$ more times establishes that

$$\sum_{h=0}^{k} s_{\hat{p}_{g-h}0} S_{g-h} \leq 0,$$

as desired.

In other words we have shown that for any j

$$\sum_{i \in N_j} \eta_i (s_{i0} - s_{\hat{p}_j 0}) \leq 0,$$

and for any $k \leq g$,

$$\sum_{h=0}^{k} s_{\hat{p}_{g-h}0} S_{g-h} \leq 0.$$

Hence, summing across all partitions we obtain

$$\sum_{j=1}^{g} X_j \leq \sum_{j=1}^{g} \left[\sum_{i \in N_j} \eta_i (s_{i0} - s_{\hat{p}_j 0}) + s_{\hat{p}_j 0} S_j \right] \leq 0,$$

but $n^{-1}\sum_{j=1}^{g} X_j = XM$, so this completes the proof of the Lemma. □

Proof of Proposition 4. Consider the share-change regression

$$\Delta s \equiv s_1 - s_0 = \gamma_s + \delta_s s_0 + u_s.$$

The coefficient δ_s equals

$$\delta_s = \frac{cov(\Delta s, s_0)}{V(s_0)}.$$

Hence, its sign will be determined by the sign of the covariance

$$\begin{aligned}
cov(\Delta s, s_0) &= n^{-1}\sum_i (s_{i1} - s_{i0})s_{i0} - \overline{\Delta s}\cdot\overline{s_0} \\
&= n^{-1}\sum_i (s_{i1} - s_{i0})s_{i0} \qquad (\text{since } \overline{\Delta s} = 0) \\
&= n^{-1}\sum_i [(s_{i1} - s_{ic}) + (s_{ic} - s_{i0})]s_{i0} \\
&= XM + SM
\end{aligned}$$

for XM and SM defined in (10). Hence,

$$sgn(\delta_s) = sgn(XM + SM).$$

By Lemma 1.i), a Lorenz-improvement $LC_1 \succ LC_0 \implies SM < 0$. By Lemma 2, $XM \leq 0$ always. Hence, if $LC_1 \succ LC_0$ then $XM + SM < 0$, and therefore $\delta_s < 0$. This proves part i) of the Proposition.

Part ii) follows again from the equation $sgn(\delta_s) = sgn(XM + SM)$. Namely, if there is a Lorenz-worsening, $LC_1 \prec LC_0$, then by Lemma 1, $SM > 0$. However, if $|XM| > SM$, it will still be the case that $\delta_s < 0$.

If in contrast, $|XM| \leq SM$, then $SM + XM \geq 0$, and hence $\delta_s \geq 0$. This establishes part iii). Finally, part iv) immediately follows also from part i).

This completes the proof of the Proposition. □