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Abstract

This paper provides a normative framework for distributional comparisons within a rank-dependent and bidimensional setting where individuals' well-being is characterized by a monetary and a non-monetary dimension (income and needs for instance) and when the focus is on inequality at the bottom as well as at the top of distribution in both dimensions. To this end, we develop third order inverse stochastic dominance conditions for classes of social welfare functions satisfying: i) Threshold Dependent Positional Transfer Sensitivity with respect to the monetary dimension (TDPT); ii) TDPT combined with downside inequality aversion with respect to the non-monetary dimension; iii) TDPT combined with upside inequality aversion with respect to the non-monetary dimension. Our results emerge along with the existing one, formulated by Zoli (2000) supporting downside inequality aversion both with respect to income and needs and Aaberge (2009) supporting upside inequality aversion with respect to income.

Keyword: sequential stochastic dominance; social welfare; inequality

JEL Cassification: D31, D63, I31

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Abstract

This paper provides a normative framework for distributional comparisons within a rank-dependent and bidimensional setting where individuals' well-being is characterized by a monetary and a non-monetary dimension and when the focus is on inequality at the bottom as well as at the top of distribution in both dimensions. To this end, we develop third order inverse stochastic dominance conditions for classes of social welfare functions satisfying: i) Threshold Dependent Positional Transfer Sensitivity with respect to the monetary dimension (TDPT); ii) TDPT combined with downside inequality aversion with respect to the non-monetary dimension; iii) TDPT combined with upside inequality aversion with respect to the non-monetary dimension. Our results emerge along with the existing one supporting downside inequality aversion both with respect to income and needs and upside inequality aversion with respect to income.

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1 Introduction

In recent years a growing effort has been devoted to expand the classical notion of income inequality towards a multidimensional space of achievements. Building upon a finding due to Kolm (1977), who noted the possibility of extending the results deriving from choice under uncertainty to the field of multidimensional evaluation,[1] Atkinson and Bourguignon (1982) developed a set of dominance conditions to rank bivariate distributions according to given classes of social welfare functions (SWF). They consider a notion of welfare dominance based on an additively separable social welfare function of the utilitarian type and investigate the implications of different assumptions about the form of this social welfare function, consisting in the imposition of certain sign restrictions on the utility functions. Such sign restrictions allow to introduce a degree of substitutability between attributes, expressing aversion to increasing correlation. The more sign-restrictions are imposed, the more complete the ordering becomes, but this comes at the price of making stricter assumptions on the shape of the SWF, which conveys a harder interpretability to the results. In their work, Atkinson and Bourguignon show that it is possible to assess the dominance according to each one of their SWFs by means of easily implementable statistical tests borrowed from portfolio theory and corresponding to first and second order stochastic dominance techniques extended to a bidimensional setting. It is worth noticing that this formulation encompasses a symmetric treatment of the attributes considered in the analysis, contrary to what happens in a subsequent work by the same authors where the attributes play an asymmetric role (Atkinson and Bourguignon, 1987). In particular, in this last work they develop a framework for ranking bivariate distributions of household income and 'needs', where needs reflect some non-income equity-relevant characteristics, such as the household's composition. Due to the transferability typical of income, it is used to compensate for deficiencies concerning the other dimension. They divide the populations to be compared into subgroups, each one of them being characterized by a different level of needs, and impose certain sign restrictions on the utility functions reflecting some desirable properties, requiring essentially that the needier is a household the higher is her marginal evaluation of income at any income level, and that such marginal evaluation decreases as income increases. In this way, it is possible to take into account, in the social evaluation, the interrelations existing between welfare components and, consequently, to

[1] The existing literature until that moment had exploited the same parallel with choice under uncertainty only in the unidimensional case (see e.g. Atkinson, 1970; Shorrocks, 1983; Dardadoni and Lambert, 1988).

embed aversion to cumulative deprivation. Even in this case, the welfare dominance for the class of SWFs introduced is shown to be equivalent to readily implementable tests, namely first and second order sequential stochastic dominance conditions, suitable for a framework characterized by different subgroups of the population and consisting essentially in checking the dominance by sequential aggregation of the subgroups. In particular, conditions of first degree dominance are associated to restrictions imposed on the sign of the cross-derivative, reflecting the way in which the marginal valuation of income varies with needs, whereas conditions of second degree dominance are obtained by making assumptions about higher order derivatives. This approach, which represents an alternative to the use of equivalence scales to compare household incomes, has the advantage of allowing to rank family groups in terms of needs without, however, specifying how much a group is needier than another.

Later on, these techniques have been extensively adopted in the field of multidimensional inequality measurement. Atkinson (1992) extended the previous results within the context of poverty measurement, by developing dominance conditions to compare income distributions of individuals with different needs, when poverty is measured by means of the class of additively decomposable poverty indices. Also Jenkins and Lambert (1993) and Chambaz and Maurin (1998) contribute to the literature by providing respectively welfare and poverty rankings that extend previous results to situations where the compared populations show different marginal distributions of needs. While the tools developed until that moment were referred to first and second order sequential stochastic dominance conditions, Lambert and Ramos (2002) enriched the framework by developing third order sequential stochastic dominance conditions for social welfare functions satisfying Kolm's principle of diminishing transfers. All the works listed so far are based on a utilitarian ground, insofar as the evaluation is made through the utilitarian social welfare function. Conversely, in the rank-dependent framework, Zoli (2000) develops inverse sequential dominance conditions of first, second and third order, for classes of social welfare functions satisfying, in the last case, a positional version of the transfer sensitivity principle. The last two mentioned works, dealing with third order sequential dominance, despite being based respectively on a utilitarian and rank-dependent ground, share the common feature of being concerned about differences occurring in the lower part of the distribution, thus endorsing what has been called in Aaberge et al. (2013) downside inequality aversion.

However, recent developments in the literature have shown that it is not only inequality in the bottom of the distribution that matters and different arguments can be made to corroborate this statement.

The profile of inequality matters as inequality experienced at different parts of the distribution can play a different role in the economy. Top and bottom inequality can have different implications when it comes to evaluate the effect of inequality on the future prospects of growth of a society. High levels of top inequality often reflect the phenomenon of 'social separatism' and anti-social behaviours might arise as a consequence, especially when income inequality is reflected by political polarization. This is a situation in which the rich get involved in lobbying activities in order to force the introduction of policies that benefit themselves but that result into hampering the growth opportunities of the poor, for instance, preventing the implementation of pro-poor and other productive polices, like spending on human capital or infrastructure, appropriate the country's resources and subvert the legal and political institutions by rent-seeking and corruption (see Easterly 2001; Glaeser et al. 2003). This thesis is supported by the empirical evidence showing that it is mostly top inequality that is holding back growth at the bottom (Van der Weide and Milanovic 2018). At the same time, the evolution of top incomes is acquiring a central role in the academic and public debate (see Atkinson and Piketty, 2007; 2010; Piketty 2013, Piketty 2020).

Inequality at the top is relevant when observed from the perspective of relative concerns theory, according to which people have social preferences, that is, their utility also depends on the consumption or utility levels of others. Some theories of relative concerns predict negative welfare effects when friends and neighbours become better-off. Models of 'envy' assume that any improvement benefited by richer individuals acts as a negative externality on own utility (Friedman and Ostrov 2008), by contrast models of 'compassion' assume that a welfare improvement experienced by poorer individuals has a positive effect on own utility (Bolton and Ockenfels 2000). In more referenced models, envy and compassion coexist but they are combined such that the negative effect of an income increase of a richer individual more than outweigh the positive effect of an income increase of a poor individual (Fehr and Schmidt 1999).

We should also observe that an increasing number of contributions in the literature adopt a rank-dependent and bidimensional approach to make comparisons between distributions of variables other than income and needs. A notable application can be found in the equality of opportunity literature, in which the downside and upside inequality aversion can be usefully implemented to define the attitude of a social planner with respect to individual efforts when an opportunity equalizing policy needs to be implemented (see, among others, Peragine 2002, Brunori et al. 2014). Alternative conceptions of inequality aversions are also present in the models developed along the rank-dependent approach to compare distributions of individual income growth and intertemporal poverty (see Aaberge and Peluso, 2019, Jenkins and Van Kerm 2016, Lo Bue and Palmisano 2020, Palmisano 2018).

Thus, it becomes compelling to consider alternatives to the standard attitude toward downside inequality aversion, which is encompassed by a social welfare function satisfying the principle of diminishing transfer sensitivity within the utilitarian approach and the principle of downside positional transfer sensitivity within the rank-dependent setting.

In this vein, Aaberge (2009) introduces the principle of upside positional transfer sensitivity and Aaberge et al. (2013) develop a theory for unambiguously ranking income distributions. In their paper, referred to a unidimensional context based on income only, they develop a third order inverse downward dominance condition, that places more emphasis on differences in the upper part of the distribution and holds for classes of social welfare functions satisfying the principle of upside positional transfer sensitivity. This condition endorses exactly the opposite view with respect to the standard dominance condition of third degree, renamed upward dominance, that places more emphasis on differences in the lower part of the distribution and holds for classes of social welfare functions satisfying the principle of downside positional transfer sensitivity.

In the present work, we draw together the arguments introduced above, namely the concern for the multidimensional evaluation of social welfare, the interest in inequalities affecting the top of the distribution and the application of the framework in other contexts. To this end, we develop third-degree inverse stochastic dominance conditions suitable for a rank-dependent and bidimensional framework and able to encompass different attitudes towards inequality aversion. In particular, a greater attention may be devoted to inequalities arising among the poorest as well as the richest individuals of the population, both in a monetary and a non-monetary sense. To be more precise, we introduce the principle of Withing-Group Threshold Dependent Positional Transfer Sensitivity (TDPT) in which the principle of upside positional transfer sensitivity coexists with a focus on the inequality at the top of the distribution. Different values of the thresholds allow to obtain different declinations of this property. In particular, one declination corresponds to the case in which the social planner endorses the standard downside positional transfer sensitivity; the other declination corresponds to the case in which the social planner endorses upside positional transfer sensitivity. We also introduce the principle of Between-Group Upside Inequality Aversion allowing to focus on inequality in the upper part of the distribution of the non-monetary dimension.

Hence, we develop third degree inverse dominance conditions for classes of social welfare functions satisfying:

i) TDPT respect to the monetary dimension;

ii) TDPT combined with downside inequality aversion with respect to the non-monetary dimension;

iii) TDPT combined with upside inequality aversion with respect to the non-monetary dimension.

We show that our results can be considered as a generalization of the existing one supporting downside inequality aversion both with respect to income and needs and upside inequality aversion with respect to income.

Notice that in all these cases the standard Pigou-Dalton transfer principle is preserved, what differs from one approach to the other is whether one puts more emphasis on differences among the richest or the poorest individuals, having in mind that, within our setting, each individual is characterized by a monetary and a non-monetary attribute.

A remark is in order at this point. The standard practice adopted in sequential dominance tests consists in dividing the population into subgroups depending on family size, age, type of housing. However, the method may also cover cases where this information is replaced by any other variable that is helpful for identifying and classifying individuals, such as, for instance, health, education, area of residence. Throughout the paper, we use the generic term ‘needs’, having in mind that it has a quite flexible and wide content.

This contribution represents an additional instrument in the researcher’s toolbox to help evaluating bidimensional distributions that can account for the whole profile of inequality and accommodate alternative declinations of the principle of inequality aversion.

The remainder of the paper is organized as follows: Section 2 outlines the framework, Section 3 presents the theoretical results, Section 4 concludes.

2 Theoretical model

2.1 Notation

Let individuals s, belonging to population S, be described by an income level y and by an additional non-monetary dimension of well-being, that we indicate with the generic term 'needs'. Consider the population S as partitioned into groups S_i, $i = 1, ..., n$, characterized by a decreasing level of these needs, in such a way that we can consider group i to be needier and consequently poorer in a non-monetary sense than group $i+1$. We assume that the needs level of individuals gives an extra information about their identity to be read together with the traditional information about the monetary dimension, represented by income in this specific case. If $F_i(y)$ is the cumulative income distribution of group i and we denote by q_i^F the share of individuals belonging to group i, we have that the overall cumulative income distribution is $F(y) = \sum_{i=1}^{n} q_i^F F_i(y)$. Let $\mu(F)$ be the mean income of distribution F. Moreover, let $\mathcal{F}$ be the set of all such cumulative distributions and $F_i^{-1}(p) = \inf \{y : F_i(y) \geq p\}$ with $p \in [0,1]$ the left continuous inverse of $F_i(y)$ denoting the income y of an individual at the p^{th} percentile of the distribution of group i.

We are interested in a formulation of the rank-dependent social welfare function that is able to capture not only the extent of monetary wealth, but also its non-monetary side, proxied by the membership of individuals to different need-based groups. A Yaarri (1988) type SWF coherent with our framework can be defined as follows:

$$W(F) = \sum_{i=1}^{n} q_i^F \int_0^1 v_i(p) F_i^{-1}(p) dp \tag{1}$$

$v_i(p)$ is the weight attached to the income of an individual ranked at the p^{th} percentile in group i, which we assume to be a twice differentiable function. It is clear that the aggregation of incomes encompasses two different weighting procedures: on the one hand, within each group, a weighted average of the incomes of individuals is obtained on the basis of the their position in the income ranking; on the other hand, incomes of each group, weighted by the relevant population share, are aggregated according to weights that are specific for each group. Restrictions on weights will define different classes of social welfare functions, characterized by different normative implications.

The framework proposed is general enough to be applied to compare different kinds of distributions. For instance, letting the non-monetary variable be represented by time, the framework could be used for the assessment of intertemporal distributions of income. In this case, the weighting function would act as a discount rate. Alternatively, our model could be implemented for the normative evaluation of growth processes by letting the non-monetary variable referring to the position of individuals in the reference distribution (i.e. normalized rank either in the pre-growth income distribution or in the post-growth distribution or in both) and the monetary variable referring to individual income growth.[2] This framework is also suitable to make comparisons across distributions when the researcher adopts an opportunity egalitarian perspective. For instance the 'need group' would encompass all those individuals characterized by the same set of exogenous factors that affect their outcome.[3] Last, if we replace the monetary variable with the number of deprivation counts experienced by each individual within a group, we could obtain a model to evaluate multidimensional poverty.

2.2 Properties

In this section, we discuss different restrictions on the weight function. By imposing a greater number of such restrictions, although we make stricter assumptions about the properties of the social welfare function, we will be able to perform more comparisons between intersecting distribution functions.

Property 1. (Welfare Monotonicity)

$$v_i(p) \geq 0 \ for \ any \ i = 1, ..., n, \ for \ any \ p \in [0, 1]$$

In Property 1 we impose a standard monotonicity assumption, requiring that social welfare is a nondecreasing function of individuals' incomes. This amounts to specifying non-negative weights.

Property 2. (Between Groups Inequality Aversion)

$$v_i(p) \geq v_{i+1}(p) \ for \ any \ i = 1, ..., n-1, \ for \ any \ p \in [0, 1]$$

[2]Such framework would allow to evaluate individual income growth by giving different weights to the growth of different individuals according to their position in the reference distribution. See on this Lo Bue and Palmisano 2020, Palmisano 2018, Palmisano and Peragine 2015.

[3]See on this Peragine 2002.

In Property 2, we are interested in a welfare representation that gives priority to individuals exhibiting higher needs, that is, if two individuals belonging to different groups are ranked at the same percentile in the income distribution of their respective group, the social planner should give higher weight to the income of the needier individual. An interesting application of this property is in the context of equality of opportunity, as it captures the essence of this theory, that is, inequalities due to exogenous factors (i.e. the non-monetary variable) are inequitable and needs to be compensated by society.

A class of social welfare functions satisfying both Properties 1 and 2 displays weights such that:

$$v_i(p) \geq v_{i+1}(p) \ \geq 0 \ for \ any \ i = 1, ..., n-1, \ for \ any \ p \in [0,1]$$

Property 3. (Within Groups Inequality Aversion)

$$v_i'(p) \leq 0 \ for \ any \ i = 1, ..., n, \ for \ any \ p \in [0,1]$$

This property corresponds to the standard principle of transfers applied over homogeneous populations. In particular, we restrict our attention to individuals belonging to the same group, i.e. characterized by the same level of needs. The social planner should display inequality aversion, thus evaluating more a distribution where the incomes of individuals of the same group are more equal. In fact, imposing negative first derivatives on the weight function $v_i(p)$ is equivalent to impose that a progressive income transfer from individuals belonging to group i and ranked at the $(p+\pi)^{th}$ percentile of the income distribution to individuals of the same group i but ranked at the p^{th} income percentile, with $\pi \geq 0$, is welfare enhancing (see Mehran, 1976 and Yaari, 1987, 1988).

Property 4. (Between Groups Diminishing Inequality Aversion)

$$v_i'(p) \leq v_{i+1}'(p) \ for \ any \ i = 1, ..., n-1, \ for \ any \ p \in [0,1]$$

This property states that the same progressive transfer of income is evaluated differently if it takes place in different need groups. Namely, W increases more the higher is the needs level of the group within which the progressive transfer takes place. Following Zoli (2000), we consider two identical progressive transfers $\delta > 0$ from individuals ranked at the $(p+\pi)^{th}$ percentile of the income distribution to individuals belonging to the same group, but ranked at the p^{th} income percentile, with $\pi \geq 0$. These transfers take place within two different groups: the former is within group i, while the latter is within group $i+1$, that is less needy. W obeys Between Groups Diminishing Inequality Aversion if the impact on social welfare of the transfer within group i is greater than the impact of the same transfer applied to group $i+1$, that is:

$$\Delta_i W(p, \pi, \delta) \geq \Delta_{i+1} W(p, \pi, \delta) \qquad for\ any\ p, \pi, \delta, i$$

which corresponds to:

$$-\delta v_i(p+\pi) + \delta v_i(p) \geq -\delta v_{i+1}(p+\pi) + \delta v_{i+1}(p) \qquad for\ any\ p, \pi, \delta, i$$

Simplifying for δ and for small π it becomes:

$$v_i'(p) \leq v_{i+1}'(p) \qquad for\ any\ p, i$$

Properties 3 and 4, taken together, require that:

$$v_i'(p) \leq v_{i+1}'(p) \leq 0\ for\ any\ i = 1, ..., n-1,\ for\ any\ p \in [0, 1]$$

In other words, the social concern for within group inequality decreases with the level of needs. We are introducing a diminishing sensitivity to horizontal equity concerns, by prescribing that the lower is the level of inequality in the individual income within a group, the higher is social welfare, conditionally on individuals' need. Property 3 and 4 are conditions linking vertical and horizontal equity, saying that horizontal equity is more important within worst-off groups.

Property 5. (Within Group Downside Positional Transfer Sensitivity)

$$v_i''(p) \geq 0\ for\ any\ i = 1, ..., n,\ for\ any\ p \in [0, 1]$$

We confine our attention to intra-group comparisons and make use of the results, due to Zoli (1999), referred to homogeneous populations. The principle we are going to use, defined in Zoli (1999) as positional transfer sensitivity, has been redefined by Aaberg et al. (2013) as downside positional transfer sensitivity (DPTS), in order to distinguish such principle from the different upside version of it.

To better understand the meaning of the DPTS, consider a fixed progressive transfer δ taking place between individuals belonging to group i and showing equal difference in ranks, but located in different positions within their group's distribution. We require the transfer taking place at lower ranks to be more equalizing, and thus more welfare improving, than the transfer taking place at higher ranks. In particular, we consider two progressive transfers - within group i - of the same amount, δ, one from individuals at rank $q+\pi$ to individuals at rank q, and another from rank $p+\pi$ to rank p, with $q \leq p$ and π expressing the equal difference in ranks, and denote the change in social welfare associated with them, respectively, by $\Delta_i W(q,\pi,\delta)$ and $\Delta_i W(p,\pi,\delta)$. W satisfies the DPTS if and only if $\Delta_i W(q,\pi,\delta) \geq \Delta_i W(p,\pi,\delta)$ *for all* $q \leq p$, where $p = q + \varepsilon$. In formal terms, this means that:

$$-\delta v_i(q+\pi) + \delta v_i(q) + \delta v_i(p+\pi) - \delta v_i(p) \geq 0 \qquad for\ any\ p,\pi,\delta,i$$

Hence, we may interpret this condition as the requirement that a combination of a progressive and a regressive transfer of the same amount δ but taking place respectively from the $(q+\pi)^{th}$ to the q^{th} percentile and from the p^{th} to the $(p+\pi)^{th}$ percentile of the income distribution, with $q \leq p$, does not lead to a welfare loss. Such combination has been called in Zoli (1999) an 'Elementary Favorable Composite Positional Transfer' (EFCPT). In this context, it will be useful to refer to is as an 'Elementary Downside Favorable Composite Positional Transfer' (EDFCPT), in order to distinguish it form its Upside counterpart.

We now proceed by refining the above expression in order to obtain a condition in terms of the weight function. Simplifying for δ and for π small enough we get that:

$$v_i'(p) - v_i'(q) \geq 0 \qquad for\ any\ p,i$$

Having in mind that $p = q + \varepsilon$ and for small ε this could be written as:

$$v_i^{''}(q) \geq 0 \qquad for\ any\ p,i$$

Therefore, in terms of the weight function, the principle of within-group downside positional transfer sensitivity is equivalent to impose positive second derivatives on the social weights. Following Aaberge et al. (2013), an inequality averse social planner who supports the principle of DPTS is said to exhibit downside (positional) inequality aversion. Since every time we adopt the intra-group perspective, within our needs-based framework, individuals differ only in the extent of the income component, this means that our social planner could be said to exhibit downside inequality aversion with respect to income.

Property 6. (Within Group Upside Positional Transfer Sensitivity)

$$v_i''(p) \leq 0 \; for \; any \; i = 1, ..., n, \; for \; any \; p \in [0, 1]$$

We confine again our attention to intra-group comparisons and exploit the results, due to Aaberge et al. (2013), referred to homogeneous populations. The principle of upside positional transfer sensitivity (UPTS) states that the same progressive transfer from a richer to a poorer individual, with a fixed number of people between the donor and the receiver, is valued more if it occurs at higher income levels. Clearly, an inequality averse social planner who supports the principle of UPTS can be said to exhibit upside (positional) inequality aversion with respect to income.

To better understand the meaning of the UPTS, consider a fixed progressive transfer δ taking place between individuals with equal difference in ranks. We require the transfer taking place at lower ranks to be less equalizing, and thus less welfare improving, than the transfer taking place at higher ranks.

We consider two progressive transfers - within group i - of the same amount, δ, one from an individual at rank $q+\pi$ to an individual at rank q, and another from rank $p+\pi$ to rank p, with $p \geq q$ and π expressing the equal difference in ranks, and denote the change in social welfare associated with them, respectively, by $\Delta_i W(q, \pi, \delta)$ and $\Delta_i W(p, \pi, \delta)$. W satisfies UPTS if and only if $\Delta_i W(q, \pi, \delta) \leq \Delta_i W(p, \pi, \delta) \; for \; all \; q \leq p$, with $p = q + \varepsilon$. In formal terms, this means that:

$$-\delta v_i(p+\pi) + \delta v_i(p) + \delta v_i(q+\pi) - \delta v_i(q) \geq 0 \qquad for \; any \; p, \pi, \delta, i$$

Hence, we may interpret this condition as the requirement that a combination of a progressive and a regressive transfer of the same amount δ, but taking place respectively from the $(p+\pi)^{th}$ to the p^{th} percentile and from the q^{th} to the $(q+\pi)^{th}$ percentile of the income distribution, with $p \geq q$, does not lead to a welfare loss. We call such combination an 'Elementary Upside Favorable Composite Positional Transfer' (EUFCPT).

We now proceed by simplifying the above expression in order to obtain a condition in terms of the weight function.

Simplifying for δ and for π small enough, we get that:

$$v_i'(q) - v_i'(p) \geq 0 \qquad for\ any\ p, i$$

Having in mind that $p = q + \varepsilon$ and for small ε, this could be written as:

$$v_i^{''}(q) \leq 0 \qquad for\ any\ p, i$$

Therefore, in terms of the weight function, the principle of within-group upside positional transfer sensitivity is equivalent to impose negative second derivatives, namely: $v_i''(p) \leq 0\ for\ any\ i = 1, ..., n,\ for\ any\ p \in [0, 1]$.

Such condition has been introduced by Aaberge et al. (2013) into a unidimensional context. In this paper we extend it to a bidimensional environment. An interesting application of this property is in the context of multidimensional deprivation. If we replace incomes with deprivation counts for each group, the normative interpretation of looking at top of each distribution would mean emphasizing the incidence or severity of deprivation within each group.

One could argue that the condition imposed in Property 6, that is, that progressive transfers in the top part of the distribution would improve welfare more than similar transfers performed among poor people, could be hardly understood by the community and hence hardly endorsed by the policymaker as it might lose intuitive sense. We then introduce a new Property that - through the use of a threshold - makes within group upside and downside positional transfer sensitivity coexist.

Property 7. (Within-Group Threshold Dependent Positional Transfer Sensitivity)

$v_i''(p) \geq 0\, for\ any\ i = 1, ..., n,\ and\ any\ p \in [0, \bar{p}]$ and $v_i''(p) \leq 0\, for\ any\ i = 1, ..., n,\ and\ any\ p \in [\bar{p}, 1]$

This Property applies the idea of upside positional transfer sensitivity with a focus on the rich as it restricts the domain of quantiles where transfers are applicable. It technically says that a social planner is concerned more, up to a given threshold, with inequality at the bottom of the distribution. Beyond that threshold, the social planner becomes more concerned with top inequality. It is clear that Property 5 and 6 become special cases of

Property 7. In particular, Property 7 corresponds to Property 5 when the threshold value $\bar{p} = 1$ and it corresponds to Property 6 when $\bar{p} = 0$.

To better understand the meaning of the TDPT, consider two fixed progressive transfers δ taking place between individuals belonging to group i and showing equal difference in ranks, but located in different positions within their group's distribution. We require - for all individuals in a group ranked below $\bar{p}$ - the transfer taking place at lower ranks to be more equalizing, and thus more welfare improving, than the transfer taking place at higher ranks, for all individuals in a group ranked below $\bar{p}$. We also require - for all individuals in a group ranked above or equal $\bar{p}$ - the transfer taking place at higher ranks to be more equalizing, and thus more welfare improving, than the transfer taking place at lower ranks. In particular, we consider two progressive transfers - within group i for individuals ranked below $\bar{p}$ - of the same amount, δ, one from individuals at rank $q + \pi$ to individuals at rank q, and another from rank $p + \pi$ to rank p, with $q \leq p$ and π expressing the equal difference in ranks. Then we consider other two progressive transfers - within group i for individuals ranked above $\bar{p}$ - of the same amount, δ, one from individuals at rank $q + \pi$ to an individual at rank q, and another from rank $p+\pi$ to rank p, with $p \geq q$ and π expressing the equal difference in ranks, and denote the change in social welfare associated with them, respectively, by $\Delta_i W(q, s, \pi, \delta)$ and $\Delta_i W(p, t, \pi, \delta)$. W satisfies the TPTS if and only if $\Delta_i W(q, s, \pi, \delta) \geq \Delta_i W(p, t\pi, \delta)$ for all $q \leq p$, where $p = q + \varepsilon$. In formal terms, this means that: $\Delta_i W(q, s, \pi, \delta) \geq \Delta_i W(p, t, \pi, \delta)$ for all i and all $q \leq p \leq \bar{p}$ and for all $s \geq t \geq \bar{p}$, which corresponds to saying that:

$$-\delta v_i(q + \pi) + \delta v_i(q) - \delta v_i(s + \pi) + \delta v_i(s) \geq -\delta v_i(p + \pi) + \delta v_i(p) - \delta v_i(t + \pi) + \delta v_i(t)$$

simplifying for δ

$$[v_i(p + \pi) - v_i(p)] - [v_i(q + \pi) - v_i(q)] + [v_i(t + \pi) - v_i(t)] - [v_i(s + \pi) - v_i(s)] \geq 0$$

for small π and for $p = q + \epsilon$ and $s = t + \epsilon$

$$\left[v_i'(q + \epsilon) - v_i'(q)\right] - \left[v_i'(t + \epsilon) - v_i'(t)\right] \geq 0$$

for small ϵ, $v_i''(q) - v_i''(t) \geq 0$ if $v_i''(q) \geq 0$ and $v_i''(t) \leq 0$.

Property 8. (Between Groups Downside Inequality Aversion)

$$\left|v_i''\right| \geq \left|v_{i+1}''\right| \text{ for any } i = 1, ..., n-1 \text{ and any } p \in [0,1]$$

This property implies that the marginal effect of a combination of a progressive and a regressive transfer of the same amount within a group is higher the lower is the rank of the group. A more clear interpretation of this Property can be obtained in combination either with Property 5 or 6 or with Property 7. Starting from Property 5, imposing that a EDFCPT does not lead to a welfare loss, and further adding Property 8 requires that the welfare effect of the same EDFCPT applied over different groups should not be lower the higher is the needs level of the group. To be more precise, we consider two similar EDFCPT that differ only in the group within which they are applied, since one of them takes place within group i, whereas the other takes place within a less needy group, $i+1$, and impose that the former is more welfare improving than the latter, which means that, *for any* p, π, δ, i:

$$|[v_i(p+\pi) - v_i(p)] - [v_i(q+\pi) - v_i(q)]| \geq |[v_{i+1}(p+\pi) - v_{i+1}(p)] - [v_{i+1}(q+\pi) - v_{i+1}(q)]|$$

For small π and having in mind that $p = q + \varepsilon$ with small ε, combining Property 5, this is equivalent to:

$$v_i''(p) \geq v_{i+1}''(p) \geq 0 \text{ for any } i = 1, ..., n-1, \text{ for any } p \in [0,1]$$

Alternatively, we may be interested in upside inequality aversion with respect to the income component, thereby assuming the validity of Property 6; we might still endorse downside inequality aversion with respect to the non-income component, thus requiring that a EUFCPT be more welfare enhancing if applied to needier groups. To say it in other words, this perspective reflects the idea that we give priority to a progressive transfer at higher ranks in the income distribution rather than at lower ranks, hence imposing that a EUFCPT does not lead to a welfare loss, but, at the same time, we want such EUFCPT to be more effective the higher is the needs level of the group within which it takes place. This implies that a EUFCPT has a greater effect if applied to group i rather than group $i+1$, that is, for any p, π, i:

$$|-v_i(p+\pi)+v_i(p)+v_i(q+\pi)-v_i(q)| \geq |-v_{i+1}(p+\pi)+v_{i+1}(p)+v_{i+1}(q+\pi)-v_{i+1}(q)|$$

For small π and having in mind that $p = q + \varepsilon$ with small ε, combining with property 6:

$$v_i''(p) \leq v_{i+1}^{''}(p) \leq 0 \ for \ any \ i = 1, ..., n-1, \ for \ any \ p \in [0,1]$$

It follows that Property 8 is compatible with Property 7 as it is shown in the following derivation. From Property 7 we have:

$$|[v_i(p+\pi)-v_i(p)]-[v_i(q+\pi)-v_i(q)]+[v_i(s+\pi)]-[v_i(t+\pi)-v_i(t)]| \geq$$

$$|[v_{i+1}(p+\pi)-v_{i+1}(p)]-[v_{i+1}(q+\pi)-v_{i+1}(q)]+[v_{i+1}(s+\pi)]-[v_{i+1}(t+\pi)-v_{i+1}(t)]|$$

for small π

$$\left|v_i^{'}(p)-v_i^{'}(q)+v_i^{'}(s)-v_i^{'}(t)\right| \geq \left|v_{i+1}^{'}(p)-v_{i+1}^{'}(q)+v_{i+1}^{'}(s)-v_{i+1}^{'}(t)\right|$$

if $p = q + \epsilon$ and $s = t + \epsilon$, for small ϵ and combining Property 8 and Property 7 we have that the above relation is true if

$$v_i^{''}(q) \geq v_{i+1}^{''}(q) \geq 0 \ and \ v_i^{''}(t) \leq v_{i+1}^{''}(t) \leq 0 \ for \ all \ i \ and \ q \leq \bar{p} \ and \ t \geq \bar{p}.$$

Property 9. (Between Groups Upside Inequality Aversion)

$$\left|v_i''\right| \leq \left|v_{i+1}''\right| \ for \ any \ i = 1, ..., n-1 \ and \ p \in [0,1].$$

Property 9 is dual to Property 8 as it implies that the marginal effect of a combination of a progressive and a regressive transfer of the same amount within a group is higher the higher is the rank of the group. Though being more concerned about inequalities occurring at lower ranks of the income distribution, thereby allowing for the validity of Property 5, we might still be more sensitive to this distinction the better ranked is the needs-based group considered, as expressed in Property 9. Imposing Property 5, a EDFCPT does not lead to a welfare loss, but, at the same time, we want such EDFCPT to be more effective the lower is the needs level of the group within which it takes place. In more formal terms, this amounts to imposing that, between two similar EDFCPT taking place respectively within group i and within group $i+1$, the former is less welfare improving than the latter, that is, for any p, π, i:

$$\left|[v_i(p+\pi) - v_i(p)] - [v_i(q+\pi) - v_i(q)]\right| \leq \left|[v_{i+1}(p+\pi) - v_{i+1}(p)] - [v_{i+1}(q+\pi) - v_{i+1}(q)]\right|.$$

For small π and having in mind that $p = q + \varepsilon$ for small ε and combining Property 5 with Property 9:

$$v''_{i+1}(p) \geq v''_i(p) \geq 0 \; for \; any \; i = 1, ..., n-1, \; for \; any \; p \in [0,1].$$

Now, we endorse the alternative view, encompassing upside inequality aversion both with respect to income and needs, thereby allowing for Property 6 and 9 to coexist. In particular, the interest in the inequalities affecting the upper part of the distribution would require to devote a greater attention not only to individuals at higher ranks of the income distribution, thus endorsing the view contained in Property 6, but also to groups characterized by a lower level of needs. In this case, the welfare effect of the same EUFCPT applied over different groups, should not be lower the lower is the needs level of the group, that is *for any* p, π, i:

$$\left|-v_i(p+\pi) + v_i(p) + v_i(q+\pi) - v_i(q)\right| \leq \left|-v_{i+1}(p+\pi) + v_{i+1}(p) + v_{i+1}(q+\pi) - v_{i+1}(q)\right|.$$

For small π and having in mind that $p = q + \varepsilon$ for small ε, combining Property 6 with 9, this is equivalent to:

$$v''_{i+1}(p) \leq v''_i(p) \leq 0 \; for \; any \; i = 1, ..., n-1, \; for \; any \; p \in [0,1]$$

Similarly, Property 9 is also compatible with property 7,

$$|[v_i(p+\pi)-v_i(p)]-[v_i(q+\pi)-v_i(q)]+[v_i(s+\pi)]-[v_i(t+\pi)-v_i(t)]| \leq$$

$$|[v_{i+1}(p+\pi)-v_{i+1}(p)]-[v_{i+1}(q+\pi)-v_{i+1}(q)]+[v_{i+1}(s+\pi)]-[v_{i+1}(t+\pi)-v_{i+1}(t)]|$$

for small π

$$v_i^{'}(p)-v_i^{'}(q)+v_i^{'}(s)-v_i^{'}(t) \leq v_{i+1}^{'}(p)-v_{i+1}^{'}(q)+v_{i+1}^{'}(s)-v_{i+1}^{'}(t)$$

if $p=q+\epsilon$ and $s=t+\epsilon$, for small ϵ and combining with Property 7, the above relation is true if:

$$v_{i+1}^{''}(q) \geq v_i^{''}(q) \geq 0 \textit{ and } v_{i+1}^{''}(t) \leq v_i^{''}(t) \leq 0 \textit{ for all } i \textit{ and } q \leq \bar{p} \textit{ and } t \geq \bar{p}.$$

The following families of social welfare functions can be identified on the basis of the properties introduced above:

$\mathbf{W}_1$ is the class of SWFs satisfying Properties 1 and 2;
$\mathbf{W}_2$ is the class of SWFs satisfying Properties 1, 2, 3, 4;
$\mathbf{W}_3$ is the class of SWFs satisfying Properties 1, 2, 3, 4, 5, 8;
$\mathbf{W}_4$ is the class of SWFs satisfying Properties 1, 2, 3, 7;
$\mathbf{W}_5$ is the class of SWFs satisfying Properties 1, 2, 3, 4, 7, 8;
$\mathbf{W}_{5^*}$ is the class of SWFs satisfying Properties 1, 2, 3, 4, 6, 8;
$\mathbf{W}_6$ is the class of SWFs satisfying Properties 1, 2, 3, 4, 7, 9;
$\mathbf{W}_{6^*}$ is the class of SWFs satisfying Properties 1, 2, 3, 4, 5, 9;
$\mathbf{W}_{6^{**}}$ is the class of SWFs satisfying Properties 1, 2, 3, 4, 6, 9;

3 Results

In this section we present the sequential inverse stochastic dominance conditions for the classes of social welfare functions previously defined. We start by reviewing the dominance conditions introduced in the literature for the bidimensional case (section 3.1). We then move to the main results of our paper (section 3.2) that define different versions of the third order sequential inverse stochastic dominance and emerge along with the existing sequential dominance tests proposed in the literature.

3.1 Existing dominance conditions

Proposition 1 *Given two distributions* F *and* $G \in \mathcal{F}$, $W(F) \geq W(G)$ $\forall W \in \boldsymbol{W}_1$ *if and only if*

$$\sum_{i=1}^{k} \Phi_i(p) \geq 0 \ \forall k = 1, ..., n, \forall p \in [0, 1]$$

where $\Phi_i(p) = q_i^F F_i^{-1}(p) - q_i^G G_i^{-1}(p)$.

That is, the necessary and sufficient condition for welfare dominance for social welfare functions belonging to the class $\mathbf{W}_1$ is a sequential inverse stochastic dominance condition of the first order. This requires to carry out a comparison between the two distributions F and G of an average of incomes of every subgroup, weighted according to the relative population share q_i, and sequentially aggregated starting from the neediest group, then adding the following, and so on, until all the subgroups have been aggregated. Notice that the comparison has to be conducted at every percentile p. Hence the dominance of F on G must hold at every p and every stage k.

Proposition 2 *Given two distributions* F *and* $G \in \mathcal{F}$, $W(F) \geq W(G)$ $\forall W \in \boldsymbol{W}_2$ *if and only if*

$$\sum_{i=1}^{k} \Psi_i(p) \geq 0 \ \forall k = 1, ..., n, \forall p \in [0, 1]$$

where $\Psi_i(p) = \int_0^p \Phi_i(q)dq$.

That is, the necessary and sufficient condition for welfare dominance for social welfare functions belonging to the class $\mathbf{W}_2$ is a sequential inverse stochastic dominance condition of the second order. To interpret this condition, we define the Generalized Lorenz curve for F following the formulation by Gastwirth (1971): $GL_F(p) = \int_0^p F^{-1}(t)dt$. We rewrite $\Psi_i(p)$ explicitly in terms of inverse distributions to have that:

$$\Psi_i(p) = \int_0^p q_i^F F_i^{-1}(t)dt - \int_0^p q_i^G G_i^{-1}(t)dt = q_i^F \int_0^p F_i^{-1}(t)dt - q_i^G \int_0^p G_i^{-1}(t)dt = q_i^F GL_{F_i}(p) - q_i^G GL_{G_i}(p)$$

where $GL_{F_i}(p)$ and $GL_{G_i}(p)$ corresponds to the Generalized Lorenz curves associated to distributions F and G. Hence, the dominance of F on G for all SWF in $\mathbf{W}_2$, requiring that:

$$\sum_{i=1}^{k} q_i^F GL_{F_i}(p) \geq \sum_{i=1}^{k} q_i^G GL_{G_i}(p)\ \forall k = 1, ..., n,\ \forall p \in [0,1]$$

means that we have to compare, at every percentile p and every stage k of the sequential procedure, the Generalized Lorenz curves of every subgroup, weighted according to the relative population share q_i.

Proposition 3 *Given two distributions F and $G \in \digamma$, $W(F) \geq W(G)\ \forall W \in \boldsymbol{W}_3$ if and only if*

$$\sum_{i=1}^{k} \Psi_i(1) \geq 0\ \forall k = 1, ..., n \textit{ and} \sum_{i=1}^{k} \Gamma_i(p) \geq 0\ \forall k = 1, ..., n, \forall p \in [0,1]$$

where $\Gamma_i(p) = \int_0^p \Psi_i(q)dq$.

That is, the necessary and sufficient condition for welfare dominance for social welfare functions belonging to the class $\mathbf{W}_3$ is a sequential upward inverse stochastic dominance condition of the third order.[4] In order to check whether such dominance criterion holds, we have to perform two different tests. As far as the first test is concerned, recall that, generally speaking, $GL(1) = \mu$. Therefore $\sum_{i=1}^{k} \Psi_i(1) \geq 0\ \forall k = 1, ..., n$ means that we have to compare, between the two distributions F and G, the weighted averages of mean incomes of every subgroup, sequentially aggregated starting from the neediest group, then adding the second, and so on, that is $\sum_{i=1}^{k} q_i^F \mu(F_i) \geq \sum_{i=1}^{k} q_i^G \mu(G_i)\ \forall k = 1, ..., n$. As regards the second test, checking whether $\sum_{i=1}^{k} \Gamma_i(p) \geq 0\ \forall k = 1, ..., n, \forall p \in [0,1]$, amounts to comparing, at every percentile p and every stage k of the sequential procedure, the

[4]The proofs of Proposition 1, 2, and 3 have been provided in a number of existing works and therefore are omitted from this paper. For references, see Aaberge et al. (2019), Palmisano and Peragine (2015), Zoli (2000), Zoli and Lambert (2012).

integrated Generalized Lorenz curves of every subgroup, integrated from below, weighted according to the relative population share q_i. This could be expressed also as:

$$\sum_{i=1}^{k} q_i^F \int_0^p GL_{F_i}(p) \geq \sum_{i=1}^{k} q_i^G \int_0^p GL_{G_i}(p) \ \forall k = 1, ..., n, \ \forall p \in [0, 1].$$

Notice that, endorsing downside inequality aversion both with respect to income and needs, leads us to:

i) integrate Generalized Lorenz curves starting from the poorest income percentile, that is $p = 0$;

ii) aggregate needs groups in the sequential procedure starting from the neediest one, that is $i = 1$.

3.2 New dominance conditions

Proposition 4 *Given two distributions F and $G \in \digamma$, $W(F) \geq W(G)$ $\forall W \in \boldsymbol{W}_4$ such that $v_i'(1) = 0$ $\forall i = 1, ..., n$ if and only if*

$$\Gamma_i(p) \geq 0 \ \forall p \in [0, \bar{p}], \forall i = 1, ..., n \text{ and} \sum_{i=1}^{k} \Psi_i(\bar{p}) \geq 0 \ \forall k = 1, ..., n \tag{2}$$

and

$$\Omega_i(p) \geq 0 \ \forall p \in [\bar{p}, 1], \forall i = 1, ..., n \text{ and} \sum_{i=1}^{k} X_i(1) \geq 0 \ \forall k = 1, ..., n \tag{3}$$

where $\Gamma_i(p) = \int_0^p \Psi_i(q)dq$, $\Omega_i(p) = \int_p^1 \Psi_i(q)dq$, $\Psi_i(\bar{p}) = \int_0^{\bar{p}} \phi_i(p)dp$ and $X_i(1) = \int_{\bar{p}}^1 \phi_i(p)dp$.

Proof. We want to find a necessary and sufficient condition for

$$\Delta W = \sum_{i=1}^{n} \int_0^1 v_i\,(p)\, \phi_i(p)dp \geq 0, \ \forall W \in \mathbf{W}_4 \tag{4}$$

In order to prove sufficiency, we will make use of Lemma 1, known as Abel's Lemma, that we now state.

Lemma 1 *If $v_1 \geq ... \geq v_i \geq ... \geq v_n \geq 0$, a sufficient condition for $\sum_{i=1}^n v_i w_i \geq 0$ is $\sum_{i=1}^k w_i \geq 0$ $\forall k = 1, ..., n$. If $v_1 \leq ... \leq v_i \leq ... \leq v_n \leq 0$, the same condition is sufficient for $\sum_{i=1}^n v_i w_i \leq 0$.*

We now turn to the sufficiency proof.

Rewrite equation (4) as:

$$\Delta W = \sum_{i=1}^{n} \int_{0}^{\bar{p}} v_i(p)\,\phi_i(p)dp + \sum_{i=1}^{n} \int_{\bar{p}}^{1} v_i(p)\,\phi_i(p)dp \geq 0. \tag{5}$$

Sufficiency can be shown as follows. First, integrate by parts both components:

$$\Delta W = \sum_{i=1}^{n} \left[v_i(\bar{p}) \int_{0}^{\bar{p}} \phi_i(p)\,dp \right] - \sum_{i=1}^{n} \int_{0}^{\bar{p}} v_i'(p) \int_{0}^{p} \phi_i(q)\,dqdp+ \tag{6}$$

$$+\sum_{i=1}^{n} \left[v_i(1) \int_{\bar{p}}^{1} \phi_i(p)\,dp \right] - \sum_{i=1}^{n} \int_{\bar{p}}^{1} v_i'(p) \int_{\bar{p}}^{p} \phi_i(q)\,dqdp.$$

Rewrite $\int_0^{\bar{p}} \phi(p)dp = \Psi(\bar{p})$ and $\int_{\bar{p}}^{1} \phi(p)dp = X(1)$, $\int_0^{p} \phi(p)dp = \Psi(p)$ and $\int_{\bar{p}}^{p} \phi(p)dp = X(p)$, so to rewrite Equation (6) as follows:

$$\Delta W = \sum_{i=1}^{n} v_i(\bar{p})\,\Psi(\bar{p}) - \sum_{i=1}^{n} \int_{0}^{\bar{p}} v_i'(p)\,\Psi(p)dp+ \tag{7}$$

$$+\sum_{i=1}^{n} v_i(1)\,X(1) - \sum_{i=1}^{n} \int_{\bar{p}}^{1} v_i'(p)\,X(p)ds.$$

Integrate again by parts the second and fourth components:

$$\Delta W = \sum_{i=1}^{n} v_i(\bar{p})\Psi_i(\bar{p}) - \sum_{i=1}^{n} v_i^{'}(\bar{p}) \int_{0}^{\bar{p}} \Psi_i(p)dp + \sum_{i=1}^{n} \int_{0}^{\bar{p}} v_i^{''}(p) \int_{0}^{p} \Psi_i(q)dqdp+ \tag{8}$$

$$\sum_{i=1}^{n} v_i(1)X_i(1) - \sum_{i=1}^{n} v_i^{'}(1) \int_{\bar{p}}^{1} X_i(p)dp + \sum_{i=1}^{n} \int_{\bar{p}}^{1} v_i^{''}(p) \int_{\bar{p}}^{p} X_i(q)dqdp.$$

The last component of equation (8) can be rewritten as follows:

$$\sum_{i=1}^{n} \int_{\bar{p}}^{1} v_i^{''}(p) \left[\int_{\bar{p}}^{1} X_i(q) - \int_{p}^{1} X_i(q) \right] dqdp = \sum_{i=1}^{n} \int_{\bar{p}}^{1} v_i^{''}(p) \int_{\bar{p}}^{1} X_i(p) - \int_{\bar{p}}^{1} v_i^{''}(p) \int_{p}^{1} X_i(s)dqds.$$

Noting that $\int_{\bar{p}}^{1} v'' = v'(1) - v'(\bar{p})$ and that $v'(1) = 0$ we have

$$-\sum_{i=1}^{n} v_i^{'}(\bar{p}) \int_{\bar{p}}^{1} X_i(p) - \int_{\bar{p}}^{1} v_i^{''}(p) \int_{p}^{1} X_i(p) dq dp.$$

Given $v'(1) = 0$, and rewriting $\int_0^p \Psi_i(q)dq = \Gamma_i(p)$ and $\int_p^1 X_i(q)dq = \Omega_i(p)$, ΔW can now be rewritten as follows:

$$\Delta W = \sum_{i=1}^{n} v_i(\bar{p})\Psi_i(\bar{p}) - \sum_{i=1}^{n} v_i^{'}(\bar{p})\Gamma_i(\bar{p})dp + \sum_{i=1}^{n} \int_0^{\bar{p}} v_i^{''}(p)\Gamma_i(p)dp+ \tag{9}$$

$$\sum_{i=1}^{n} v_i(1)X_i(1) - \sum_{i=1}^{n} v_i^{'}(\bar{p})\Omega_i(\bar{p}) - \sum_{i=1}^{n} \int_{\bar{p}}^{1} v_i^{''}(p)\Omega_i(p)dp.$$

- by Property 7 and Lemma 2 in Chambaz and Maurin (1998): $\sum_{i=1}^{n} \int_0^{\bar{p}} v_i^{''}(p)\Gamma_i(p)dp \geq 0$ if $\Gamma_i(p)\,dp \geq 0, \forall i = 1, ..., n$ and $\forall p \in [0, \bar{p}]$. By Property 3 this also implies that $-\sum_{i=1}^{n} v_i^{'}(\bar{p})\Gamma_i(\bar{p})dp \geq 0$. Similarly, $-\sum_{i=1}^{n} \int_{\bar{p}}^{1} v_i^{''}(p)\Omega(p)dp \geq 0$ if $\Omega_i(p)\,dp \geq 0, \forall i = 1, ..., n$ and $\forall p \in [\bar{p}, 1]$. By Property 3 this also implies that $-\sum_{i=1}^{n} v_i^{'}(\bar{p})\Omega_i(\bar{p})dp \geq 0$.

- by Property 1 and 2 and application of the Abel's lemma: $\sum_{i=1}^{n} v_i(\bar{p})\Psi_i(\bar{p}) \geq 0$ if $\sum_{i=1}^{k} \Psi_i(\bar{p}) \geq 0$ for all $k = 1, ..., n$. Similarly, $\sum_{i=1}^{n} v_i(1)X_i(1) \geq 0$ if $\sum_1^k X_i(1) \geq 0$ for all $k = 1, ..., n$.

For the necessity, suppose by contradiction that $\Delta W \geq 0$ but $\exists h \in \{1, ..., n\}$ and $\exists I \equiv [a, b] \subseteq [0, \bar{p}]$ and $\exists Z \equiv [c, d] \subseteq [\bar{p}, 1]$ such that $\Gamma_h(p)\,dp \leq 0, \forall p \in I$ and $\Omega_h(p)\,dp \leq 0, \forall p \in Z$. Then, by choosing a combination of $v_i''(p)\Gamma_i(p)$ and $v_i''(p)\Omega_i(p)$ such that $v_i''(p)\Gamma_i(p) - v_i''(p)\Omega_i(p) \searrow 0$ for all $i \neq h$ and $v_h''(p)\Gamma_h(p) - v_h''(p)\Omega_h(p) \searrow 0$ for all $p \in [0, \bar{p}]\backslash I$ and $p \in [\bar{p}, 1]\backslash Z$ and choosing a combination of $v_i'(\bar{p})\Gamma_i(\bar{p})$ and $v_i'(\bar{p})\Omega_i(\bar{p})$ such that $-v_i'(\bar{p})\Gamma_i(\bar{p}) - v_i'(\bar{p})\Omega_i(\bar{p}) \searrow 0$ for all $i \neq h$ and $-v_h'(\bar{p})\Gamma_h(\bar{p}) - v_h'(\bar{p})\Omega_h(\bar{p}) \searrow 0 p \in [0, 1]\backslash I, Z$, ΔW becomes

$$\Delta W = \sum_{i=1}^{n} v_i(\bar{p})\Psi_i(\bar{p}) + \int_a^b v_h^{''}(p)\Gamma_h(p)dp + \sum_{i=1}^{n} v_i(1)X_i(1) - \int_c^d v_h^{''}(p)\Omega_h(p)dp$$

Writing $\int_a^b v_h^{''}(p)\Gamma_h(p)dp - \int_c^d v''(p)\Omega_h(p)dp = T_h(p)$; because of the previous arguments $T_h(p) \leq 0$, ΔW becomes

$$\Delta W = \sum_{i=1}^{n} v_i(\bar{p})\Psi_i(\bar{p}) + \sum_{i=1}^{n} v_i(1)X_i(1) + T_h(p)$$

Furthermore suppose that $\sum_1^k \Psi_i(\bar{p}) \leq 0$ for some $r \in \{1,...,n\}$ and $\sum_1^k X_i(1) \leq 0$ for some $r \in \{1,...,n\}$, then it possible to find combination of $v_i(\bar{p})\Psi_i(\bar{p})$ and $v_i(1)X_i(1)$ such that $\sum_{i=1}^j v_i(\bar{p})\Psi_i(\bar{p}) + v_i(1)X_i(1) \searrow 0$ for all $j \neq r$. Rewriting $\sum_1^r \Psi_i(\bar{p}) + \sum_1^r X_i(1) = R_r \leq 0$. Then ΔW reduces to $\Delta W = R_r + T_h(p) \leq 0$ which is a contradiction. ■

Two conditions of the third order and two sequential conditions of the second order are characterized by this proposition. They are combined such that one third order test and one sequential test are applied on the bottom part of the distribution withing each group and the other two are applied on the top part of the distribution withing each group. The distinction between the two parts of the distribution depends on the value of the threshold $\bar{p}$ that we assume it is defined exogenously.[5] In particular, given a value for the threshold $\bar{p}$, within each group we have to check that $\Gamma_i(p) \geq 0$ $\forall i = i, ...n, \forall p \in [0, \bar{p}]$, that is, we have to compare the integrated GL, integrated from below, and weighted according to q_i, at every group, $i = 1, ..., n$ and every p up to the threshold $\bar{p}$. This could be expressed as:

$$q_i^F \int_0^p GL_{F_i}(p) \geq q_i^G \int_0^p GL_{G_i}(p) \ \forall i = 1, ..., n, \ \forall p \in [0, \bar{p}].$$

Then, for the same portion of the distribution within each group, we have to compare, between the two distributions F and G, the weighted averages of mean incomes, sequentially aggregated starting from the lowest ranked group, then adding the second, and so on, that is $\sum_{i=1}^k q_i^F \mu_i(\bar{p}, F_i) \geq \sum_{i=1}^k q_i \mu_i(\bar{p}, G_i)$ $\forall k = 1, ..., n$, where $\mu_i(\bar{p}, F_i) = \int_0^{\bar{p}} F_i^{-1}(p)dp$.

The second part of the proposition introduces two tests to be applied on the upper part of the distribution within each group. The first imposes to check whether $\Omega_i(p) \geq 0$ $\forall i = i, ...n, \forall p \in [\bar{p}, 1]$, which is a third order downward inverse stochastic dominance test to be applied to each group of the population. It amounts to compare at every p the integrated GL, integrated from above, and weighted according to q_i. This could be expressed also as:

$$q_i^F \int_p^1 GL_{F_i}(p) \geq q_i^G \int_p^1 GL_{G_i}(p) \ \forall i = 1, ..., n \ \forall p \in [\bar{p}, 1].$$

Last step requires to compare, between the two distributions F and G, the weighted av-

[5]It could be for instance the poverty line.

erages of mean incomes of the top part of the distribution, sequentially aggregated starting from the lowest ranked group, then adding the second, and so on, that is $\sum_{i=1}^{k} q_i^F \mu_i(1, F_i) \geq \sum_{i=1}^{k} q_i \mu_i(1, G_i)\ \forall k = 1, ..., n$, where $\mu_i(1, F_i) = \int_{\bar{p}}^{1} F_i^{-1}(p)dp$.

Special cases of Proposition 4 are gathered in the following two corollaries and are obtained by selecting two particular values for the threshold $\bar{p}$.

Corollary 1. Given two distributions F and $G \in \digamma$, $W(F) \geq W(G)\ \forall W \in \mathbf{W}_4$ such that $v_i'(1) = 0\ \forall i = 1, ..., n$ and $\bar{p} = 1$ if and only if

$$\Gamma_i(p) \geq 0\ \forall p \in [0,1], \forall i = 1, ..., n \text{ and} \sum_{i=1}^{k} \Psi_i(1) \geq 0\ \forall k = 1, ..., n \tag{10}$$

where $\Psi_i(1) = \int_0^1 \phi_i(p)dp$.

Corollary 2. Given two distributions F and $G \in \digamma$, $W(F) \geq W(G)\ \forall W \in \mathbf{W}_4$ such that $v_i'(1) = 0\ \forall i = 1, ..., n$ and $\bar{p} = 0$ if and only if

$$\Omega_i(p) \geq 0\ \forall p \in [0,1], \forall i = 1, ..., n \text{ and} \sum_{i=1}^{k} X_i(1) \geq 0\ \forall k = 1, ..., n \tag{11}$$

where $X_i(1) = \int_0^1 \phi_i(p)dp$.

Corollary 1 boils down to a standard upward inverse stochastic dominance of the third order to be applied to each group of the population. Whereas, Corollary 2 boils down to a downward inverse stochastic dominance of the third order to be applied to each group of the population and represents a generalization of Theorem 2.2 of Aaberge et al. (2013), in fact the two results are equivalent if we assume a population only partitioned in one group.

Proposition 5 *Given two distributions F and $G \in \digamma$, $W(F) \geq W(G)\ \forall W \in \mathbf{W}_5$ such that $v_i'(1) = 0\ \forall i$ if and only if*

$$\sum_{i=1}^{k} \Gamma_i(p) \geq 0\ \forall p \in [0, \bar{p}], \forall k = 1, ..., n \text{ and} \sum_{i=1}^{k} \Psi_i(\bar{p})\ \forall k = 1, ..., n \tag{12}$$

and

$$\sum_{i=1}^{n} \Omega_i(p) \geq 0 \ \forall p \in [\bar{p}, 1], \forall k = 1, ..., n \text{ and} \sum_{i=1}^{k} X_i(1) \ \forall k = 1, ..., n. \tag{13}$$

Proof. We want to find a necessary and sufficient condition for

$$\Delta W = \sum_{i=1}^{n} \int_0^1 v_i\,(p)\,\phi_i(p)dp \geq 0,\ \ \forall W \in \mathbf{W}_5 \tag{14}$$

In order to obtain the necessary and sufficient conditions for equation (14), we start from equation (9) of the Proof of Proposition 4, reversing the order of integration and summation.

- By Properties 1 and 2, a sufficient condition for $\sum_{i=1}^{n} v_i(\bar{p})\Gamma_i(\bar{p}) \geq 0$ is that $\sum_{i=1}^{k} \Psi_i(\bar{p}) \geq 0\ \forall k = 1, ..., n$.

- By Properties 7 and 8 and application of the Abel's Lemma a sufficient condition for $\int_{\bar{p}}^{1} \sum_{i=1}^{n} v_i''(p)\Gamma_i(p)dp\ \ \geq 0$ is that $\sum_{i=1}^{k} \Gamma_i(p) \geq 0\ \forall k = 1, ..., n,\ \forall p \in [0, \bar{p}]$. Now, since $\mathbf{W}_5$ satisfies also properties 3 and 4, $\sum_{i=1}^{k} \Gamma_i(p) \geq 0\ \forall k = 1, ..., n,\ \forall p \in [0, \bar{p}]$ is sufficient also for $\sum_{i=1}^{n} v_i'(\bar{p})\Gamma_i(\bar{p}) \geq 0$.

- Using similar arguments, by Properties 1 and 2, a sufficient condition for $\sum_{i=1}^{n} v_i(1)X_i(1) \geq 0$ is that $\sum_{i=1}^{k} X_i(1) \geq 0\ \forall k = 1, ..., n$.

- By Property 7 and 8 and application of the Abel's Lemma, a sufficient condition for $-\int_0^1 \sum_{i=1}^{n} v_i''(p)\Omega_i(p)dp\ \ \geq 0$ is that $\sum_{i=1}^{n} \Omega_i(p) \geq 0\ \forall k = i, ..., n\ \forall p \in [\bar{p}, 1]$. Now, since $\mathbf{W}_5$ satisfies also properties 3 and 4, $\sum_{i=i}^{k} \Omega_i(p) \geq 0\ \forall k = i, ...n,\ \forall p \in [\bar{p}, 1]$ is sufficient also for $\sum_{i=1}^{n} v_i'(\bar{p})\Omega_i(\bar{p}) \geq 0$.

To show necessity, we resort to reduction to absurd arguments and make use of Lemma 1 in Chambaz and Maurin (1998), Lemma 1 in Atkinson and Bourguignon (1987) and Abel's decomposition, according to which $\sum_{i=1}^{n} v_i w_i = v_n \sum_{i=1}^{n} w_i + \sum_{i=1}^{n-1}(v_i - v_{i+1}) \sum_{k=1}^{i} w_k$.

Let $\varepsilon_n(p) = v_n''(p)\ \forall p \in [0, \bar{p}]$ and $\omega_i(p) = (v_i''(p) - v_{i+1}''(p))\ \forall i,\ \forall p \in [0, \bar{p}]$. Then let $\tau_n(p) = -v_n''(p)\ \forall p \in [\bar{p}, 1]$ and $\varsigma_i(p) = -(v_i''(p) - v_{i+1}''(p))\ \forall i, \forall p \in [\bar{p}, 1]$. We can apply the following decompositions to equation (9), reversing the order of integration and summation, and rewrite it as follows

$$\Delta W = \sum_{i=1}^{n} v_i(\bar{p})\Psi_i(\bar{p}) - \sum_{i=1}^{n} v_i'(\bar{p})\Gamma_i(\bar{p}) + \int_0^{\bar{p}} \varepsilon_n(p) \sum_{i=1}^{n} \Gamma_i(p)dp + \int_0^{\bar{p}} \sum_{i=1}^{n-1} \omega_i(p) \sum_{k=1}^{i} \Gamma_k(p)dp +$$

$$\sum_{i=1}^{n} v_i(1)X_i(1) - \sum_{i=1}^{n} v_i'(\bar{p})\Omega_i(\bar{p}) + \int_{\bar{p}}^{1} \tau_n(p) \sum_{i=1}^{n} \Omega_i(p)dp + \int_{\bar{p}}^{1} \sum_{i=1}^{n-1} \varsigma_i(p) \sum_{k=n}^{j} \Omega_k(p)dp$$

Suppose for a contradiction that $\Delta W \geq 0$, but $\exists\ h \in \{1, ..., n-1\}$ and $\exists\ h = n$ and an interval $I \equiv [a, b] \subseteq [0, \bar{p}]$ and an interval $Z \equiv [c, d] \subseteq [\bar{p}, 1]$ such that $\sum_{i=1}^{h} \Gamma_i(p) < 0\ \forall p \in I$ and $\sum_{i=1}^{h} \Omega_i(p) < 0\ \forall p \in Z$.

Given that $\{\omega_i(p) \geq 0\}_{i\in\{1,...,n-1\}}$ and $\{\varsigma_i(p) \geq 0\}_{i\in\{1,...,n-1\}}$ by Lemma 1 in Chambaz and Maurin (1998) we have that $\sum_{i=1}^{n-1} \omega_i(p) \left(\sum_{k=1}^{i} \Gamma_k(p)dp\right) < 0 \forall p \in I$ and $\sum_{i=1}^{n-1} \varsigma_i(p) \left(\sum_{k=1}^{i} \Omega_k(p)dp\right) < 0\ \forall p \in Z$ and given that $\varepsilon_n(p) \geq 0$ and $\tau_n(p) \geq 0$, by Lemma 1 in Atkinson and Bourguignon (1987) we have that $\int_0^{\bar{p}} \varepsilon_n(p) \sum_{k=1}^{n} \Gamma_k(p)dp < 0$ and $\int_{\bar{p}}^{1} \tau_n(p) \sum_{k=1}^{n} \Omega_k(p)dp < 0$. Denoting $T(p) = \sum_{i=1}^{n-1} \omega_i(p) \left(\sum_{k=1}^{i} \Gamma_k(p)dp\right)$, $S(p) = \varepsilon_n(p) \sum_{i=1}^{n} \Gamma_i(p)dp$, $R(p) = \sum_{i=1}^{n-1} \varsigma_i(p) \left(\sum_{k=1}^{i} \Omega_k(p)dp\right)$ and $Q(p) = \tau_n(p) \sum_{i=1}^{n} \Omega_i(p)dp$, we have

$$\Delta W = \sum_{i=1}^{n} v_i(\bar{p})\Psi_i(\bar{p}) + \sum_{i=1}^{n} v_i(1)X_i(1) - \sum_{i=1}^{n} v_i'(\bar{p})\Gamma_i(\bar{p}) - \sum_{i=1}^{n} v_i'(\bar{p})\Omega_i(\bar{p}) + \int_0^{\bar{p}} T(p)dp +$$

$$\int_0^{\bar{(}p)} S(p)dp + \int_{\bar{p}}^{1} Q(p)dp + \int_{\bar{p}}^{1} R(p)dp$$

If we choose $T(p)$ and $R(p)$ such that $T(p) \to 0$ for all $p \in [0, \bar{p}] \setminus I$ $R(p) \to 0$ and all $p \in [\bar{p}, 1] \setminus Z$, we have that:

$$\Delta W = \sum_{i=1}^{n} v_i(\bar{p})\Psi_i(\bar{p}) + \sum_{i=1}^{n} v_i(\bar{p})X_i(\bar{p}) - \sum_{i=1}^{n} v_i'(\bar{p})\Gamma_i(\bar{p}) -$$

$$\sum_{i=1}^{n} v_i'(\bar{p})\Omega(\bar{p})_i(\bar{p}) + \int_0^{\bar{p}} S(p)dp + \int_{\bar{p}}^{1} Q(p)dp + \int_a^b T(p)dp + \int_c^d R(p)dp$$

Furthermore, suppose for a contradiction that there exists some $j \in \{1, ..., n\}$ for which $\sum_{i=1}^{j} \Psi_i(\bar{p}) < 0$ and $\sum_{i=1}^{j} X_i(1) < 0$.

Given that $\int_a^b T(p)dp < 0$ and $\int_a^b R(p)dp < 0$ and that $\int_0^{\bar{p}} S(p)dp < 0$ and $\int_{\bar{p}}^{1} Q(p)dp < 0$, we can always choose a combination of $v_i(\bar{p})\Psi_i(\bar{p})$ and $v_i(1)X_i(1) \searrow 0$ and a combination of $v_i'(\bar{p}\Gamma_i(\bar{p}))$ and $v_i'(\bar{p})\Omega_i(\bar{p})$ such that $v_i(\bar{p})\Psi_i(\bar{p}) + v_i(1)X_i(1) - v_i'(\bar{p}\Gamma_i(\bar{p})) - v_i'(\bar{p})\Omega_i(\bar{p}) \searrow 0$ for all $i \neq j$, then $\Delta W < 0$, a contradiction. ■

Also Proposition 5 characterizes two tests to be applied to the bottom part group-specific distribution and two tests to be applied to the top part group-specific distribution. The second and fourth conditions are identical to the second and fourth conditions of Proposition 4, while the first and third conditions are second order sequential versions of the first and third tests presented in Proposition 4. In particular, the first test of Proposition 5 is identical to Proposition 3, except for the fact that the check of the dominance is restricted to part of group-specific distribution with upper-bound $\bar{p}$. Hence, first we have to compare, at every p up to $\bar{p}$, integrated GL, integrated from below, and weighted according to q_i starting from $i = 1$, the lowest ranked group, than adding the second lowest ranked group and so on up to the highest ranked group and perform the same check at every step. This could be expressed as follows:

$$\sum_{i=n}^{k} q_i^F \int_0^p GL_{F_i}(p) \geq \sum_{i=n}^{k} q_i^G \int_0^p GL_{G_i}(p) \; \forall k = i, ...n, \; \forall p \in [0, \bar{p}].$$

The third test of Proposition 5 requires an upward sequential aggregation of the third order downward dominance conditions presented in the third test of Proposition 4. We start by comparing, at every p beginning form $\bar{p}$, integrated GL, integrated from above, and weighted according to q_i starting from $i = 1$, the lowest ranked group, than adding the second lowest ranked group and so on up to the highest ranked group and perform the same check at every step. This could be expressed as follows:

$$\sum_{i=1}^{k} q_i^F \int_p^1 GL_{F_i}(p) \geq \sum_{i=1}^{k} q_i^G \int_p^1 GL_{G_i}(p) \; \forall k = 1, ...n, \; \forall p \in [\bar{p}, 1].$$

As stated in Section 1 and 2, the framework developed in this paper is flexible enough to be applied for performing different kinds of comparisons. For instance, a social planner could implement it to evaluate the welfare effects of two alternative policies when the relevant characteristic for identifying individuals, in addition to income, is the area of residence, for instance urban or rural. In this case, the group of individuals living in rural areas can be considered to be more 'needy' than those living in urban areas. If the social planner is more concerned with fighting extreme poverty and inequality in the more disadvantaged areas (i.e. rural) of the country, he could base its decision on the result of testing Proposition 5.

Proposition 3 in Zoli (2000) is a special case of Proposition 5 that is obtained when $\bar{p} = 1$. A second special case is obtained when $\bar{p} = 0$ and is formalized in the next corollary.

Corollary 3 Given two distributions F and $G \in \digamma$, $W(F) \geq W(G)$ $\forall W \in \mathbf{W}_5$ such that $\bar{p} = 1$ and $v_i'(1) = 0$ $\forall i = 1, ..., n$ and $\forall W \in \mathbf{W}_{5^*}$ such that $v_i'(1) = 0$ $\forall i = 1, ..., n$ if and only if

$$\sum_{i=1}^{k} \Omega_i(p) \geq 0 \ \forall p \in [0,1], \forall k = 1, ...n \text{ and} \sum_{i=1}^{k} \Psi(1) \geq 0 \forall k = 1, ..., n. \qquad (15)$$

Corollary 3 boils down to an upward sequential second order condition (across groups) of the third order downward inverse stochastic dominance (within groups) to be applied to the whole distribution.

In this case, we are assuming upside inequality aversion with respect to income and downside inequality aversion with respect to needs. This attitude is suitable for a framework in which, in order to check for third order inverse sequential stochastic dominance, we have to:

i) integrate Generalized Lorenz curves starting from the richest income percentile, i.e. $p = 1$;

ii) aggregate needs groups in the sequential procedure starting from the neediest one, i.e. $i = 1$.

Proposition 6 *Given two distributions F and $G \in \digamma$, $W(F) \geq W(G)$ $\forall W \in \boldsymbol{W}_6$ such that $v_i'(1) = 0$ $\forall i = 1, ..., n$ if and only if*

$$\sum_{i=n}^{k} \Gamma_i(p) \geq 0 \ \forall p \in [0, \bar{p}], \forall k = n, ..., 1 \text{ and} \sum_{i=1}^{k} \Psi(\bar{p}) \forall k = 1, ..., n \qquad (16)$$

and

$$\sum_{i=n}^{k} \Omega_i(p) \geq 0 \ \forall p \in [\bar{p}, 1], \forall k = n, ..., 1 \text{ and} \sum_{i=1}^{k} X(1) \forall k = 1, ..., n \qquad (17)$$

Proof.

We want to find a necessary and sufficient condition for

$$\Delta W = \sum_{i=1}^{n} \int_{0}^{1} v_i\,(p)\,\phi_i(p)dp \geq 0,\ \forall W \in \mathbf{W}_6 \tag{18}$$

Before finding these conditions, we propose Lemma 2, representing an alternative formulation of Abel's Lemma:

Lemma 2 *If $v_n \geq ... \geq v_i \geq ... \geq v_1 \geq 0$, a sufficient condition for $\sum_{i=1}^{n} v_i w_i \geq 0$ is $\sum_{i=n}^{k} w_i \geq 0\ \forall k = n, n-1, ..., 1$. If $v_n \leq ... \leq v_i \leq ... \leq v_1 \leq 0$, the same condition is sufficient for $\sum_{i=1}^{n} v_i w_i \leq 0$.*

In order to obtain these conditions, we start from equation (9) of the Proof of Proposition 4. For the sufficiency part, notice that:

- By Properties 1 and 2, a sufficient condition for $\sum_{i=1}^{n} v_i(\bar{p})\Gamma_i(\bar{p}) \geq 0$ is that $\sum_{i=1}^{k} \Psi_i(\bar{p}) \geq 0\ \forall k = 1, ..., n$ and a sufficient condition for $\sum_{i=1}^{n} v_i(1)X_i(1) \geq 0$ is that $\sum_{i=1}^{k} X_i(1) \geq 0\ \forall k = 1, ..., n$.

- By Property 7 and 9, we can apply Lemma 2 to get that a sufficient condition for $-\sum_{i=1}^{n} \int_0^1 v_i''(p)\Gamma_i(p)dp\ \geq 0$ is that $\sum_{i=n}^{k} \Gamma_i(p) \geq 0\ \forall k = n, n-1, ..., 1,\ \forall p \in [0, \bar{p}]$. Now, since $\mathbf{W}_6$ satisfies also properties 3 and 4, $\sum_{i=n}^{k} \Gamma_i(p) \geq 0\ \forall k = n, ...1,\ \forall p \in [0, \bar{p}]$ is sufficient also for $\sum_{i=1}^{n} v_i'(\bar{p})\Gamma_i(\bar{p}) \geq 0$.

- By Property 7 and 9 and application of Lemma 2 a sufficient condition for $\sum_{i=1}^{n} \int_{\bar{p}}^{1} v_i''(p)\Omega_i(p)dp \geq 0$ is that $\sum_{i=n}^{k} \Omega_i(p) \geq 0\ \forall k = n, ..., 1,\ \forall p \in [\bar{p}, 1]$. Now, since $\mathbf{W}_5$ satisfies also properties 3 and 4, $\sum_{i=n}^{k} \Omega_i(p) \geq 0\ \forall k = n, ..., 1,\ \forall p \in [\bar{p}, 1]$ is sufficient also for $\sum_{i=1}^{n} v_i'(\bar{p})\Omega_i(\bar{p}) \geq 0$.

For the necessity part, using the same reasoning as in the Proof of Proposition 5. Let $\varepsilon_1(p) = v_1''(p)\ \forall p$ and $\pi_i(p) = (v_{i+1}''(p) - v_i''(p))\ \forall i,\ \forall p \in [0, \bar{p}]$ and let $\tau_1(p) = -v_1''(p)\ \forall p$ and $\varsigma_i(p) = -(v_{i+1}''(p) - v_i''(p))\ \forall i,\ \forall p \in [\bar{p}, 1]$, we rewrite equation (9) as follows

$$\Delta W = \sum_{i=1}^{n} v_i(\bar{p})\Psi_i(\bar{p}) - \sum_{i=1}^{n} v_i'(\bar{p})\Gamma_i(\bar{p}) + \int_0^{\bar{p}} \varepsilon_1(p) \sum_{i=1}^{n} \Gamma_i(p)dp + \int_0^{\bar{p}} \sum_{i=1}^{n-1} \pi_i(p) \sum_{k=1}^{i} \Gamma_i(p)dp+ \tag{19}$$

$$\sum_{i=1}^{n} v_i(1)X_i(1) - \sum_{i=1}^{n} v_i'(\bar{p})\Omega_i(\bar{p}) + \int_{\bar{p}}^{1} \varepsilon_1(p) \sum_{i=1}^{n} \Omega_i(p)dp + \int_{\bar{p}}^{1} \sum_{i=1}^{n-1} \pi_i(p) \sum_{k=n}^{j} \Omega_i(p)dp$$

Thus, by arguments like those in the Proof of Proposition 5 the results of Proposition 6 are obtained. ■

The second and fourth conditions characterized by Proposition 6 are identical to second and fourth components of Proposition 4 and 5. The first and third conditions are dual to the first and third conditions of Proposition 5 insofar as they endorse the opposite view about the integration procedure. In particular, the first test of Proposition 6 is identical to Proposition 4, except for the fact that it requires to carry out the sequential procedure starting from the last group rather than from the first. Hence, first we have to compare, at every p and up to $\bar{p}$ the integrated GL, integrated from below, and weighted according to q_i, for the least needy group, $i = n$. Then, we have to add group $i = n - 1$, and so on, up to the neediest group, $i = 1$, and perform the same check at every step. This could be expressed also as:

$$\sum_{i=n}^{k} q_i^F \int_0^p GL_{F_i}(p) \geq \sum_{i=n}^{k} q_i^G \int_0^p GL_{G_i}(p) \ \forall k = n, n-1, ..., 1, \ \forall p \in [0, \bar{p}].$$

The third test of Proposition 6 requires a downward sequential aggregation of the third order downward dominance conditions presented in the third test of Proposition 4. We start by comparing, at every p beginning form $\bar{p}$, integrated GL, integrated from above, and weighted according to q_i starting from $i = n$ and so on up to the highest ranked group and perform the same check at every step. This could be expressed as follows:

$$\sum_{i=n}^{k} q_i^F \int_p^1 GL_{F_i}(p) \geq \sum_{i=n}^{k} q_i^G \int_p^1 GL_{G_i}(p) \ \forall k = n, ...1, \ \forall p \in [\bar{p}, 1].$$

A social planner could, for instance, implement the test proposed in Proposition 6 to evaluate the welfare effects of two alternative policies when the relevant characteristic for identifying individuals, in addition to income, is the level of education, so that it is possible to distinguish the group of individuals with middle or low level of education and the group of individuals with high level of education. In this case, the group of individuals having a high level of education can be considered to be less 'needy' than those having a middle or low level of education. Nevertheless, the social planner could be more concerned with fighting poverty and inequality especially among the highly educated individuals, for instance, to make higher education more attractive, he could base his decision on the result of testing Proposition 6.

Two special cases of Proposition 6 are formalized in the following two corollaries.

Corollary 4. Given two distributions F and $G \in \digamma$, $W(F) \geq W(G)$ $\forall W \in \mathbf{W}_6$ such that $\bar{p} = 1$ and $v_i'(1) = 0$ $\forall i = 1, ..., n$ and $\forall W \in \mathbf{W}_{6^*}$ such that $v_i'(1) = 0$ $\forall i = 1, ..., n$ if and only if

$$\sum_{i=n}^{k} \Gamma_i(p) \geq 0 \;\forall p \in [0,1], \forall k = n, ..., 1 \text{ and} \sum_{i=1}^{k} \Psi(1) \;\forall k = 1, ..., n. \tag{20}$$

If we endorse downside inequality aversion with respect to income and upside inequality aversion with respect to needs, then we come up with a third order procedure in which we:

i) integrate Generalized Lorenz curves starting from the poorest income percentile, that is $p = 0$;

ii) aggregate needs groups in the sequential procedure starting from the least needy one, that is $i = n$.

Corollary 5. Given two distributions F and $G \in \digamma$, $W(F) \geq W(G)$ $\forall W \in \mathbf{W}_6$ such that $\bar{p} = 0$ and $v_i'(1) = 0$ $\forall i = 1, ..., n$ and $\forall W \in \mathbf{W}_{6^{**}}$ such that $v_i'(1) = 0$ $\forall i = 1, ..., n$ if and only if

$$\sum_{i=1}^{k} \Omega_i(p) \geq 0 \;\forall p \in [0,1] \text{ and} \sum_{i=1}^{k} \Psi(1) \forall k = 1, ..., n. \tag{21}$$

This is the last possible attitude towards inequality aversion, supporting upside inequality aversion both with respect to income and needs. In this case, to check whether third order inverse sequential stochastic dominance holds, we have to:

i) integrate Generalized Lorenz curves starting from the richest income percentile $p = 1$;

ii) aggregate needs groups in the sequential procedure starting from the least needy one $i = n$.

4 Concluding remarks

Several contributions in the economic literature show the need to modify the standard framework to rank income distributions, in order to take into account non-income aspects of well-being. At the same time, an increasing interest has been shown for inequalities affecting the upper part of the income distribution. In this context, the choice between upside and downside inequality aversion specifies whether, in the evaluation of social welfare, one should give priority to equalizing transfers between poorer *vis-a-vis* richer individuals.

This work represents an attempt to bring together these issues, by introducing upside inequality aversion considerations within a bidimensional evaluation of social welfare. In particular, we adopt a rank-dependent and needs-based social welfare function and develop third order dominance conditions to rank bidimensional distributions when a concern for upside inequality aversion is embedded.

To this end, we propose some properties that our SWF should satisfy. These properties have the common feature of presenting different formulations of a principle, introduced by Aaberge (2009) and called upside positional transfer sensitivity, that has been discussed only in terms of the unidimensional distribution of income. We enrich the literature by adapting such principle to a bidimensional context, in order to provide an alternative framework for those who might be more concerned about differences in the upper part of the distribution rather than in the lower part of it. In particular, we propose different combinations of upside and downside inequality aversion referred alternatively to the monetary and non-monetary component, to come up with a complete framework allowing for various possible normative choices. We introduce three different third order stochastic dominance conditions, holding for classes of social welfare functions that are built upon such choices. We show that our results represent a generalization of existing dominance conditions in the literature and discuss a number of special situations.

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