ECINEQ

Society for the Study of Economic Inequality

Working Paper Series

On the measurement of well-being with reference consumption

Domenico Moramarco

François Maniquet

ECINEQ 2022 629

ECINEQ 2022 629
2022 December
www.ecineq.org

On the measurement of well-being with reference consumption

Domenico Moramarco
Université Libre de Bruxelles ,ECARES
François Maniquet
CORE/LIDAM, UCLouvain, and LISER, Luxembourg

Abstract

We axiomatically study how to measure well-being when individuals have heterogeneous preferences and consumption bundles are evaluated by comparison with some reference bundle (a poverty line bundle, an index of needs, the average consumption in a reference group, etc.). If only the reference bundle matters, Deaton (1979)'s distance function turns out to be the only measure satisfying basic axioms. If the measure is allowed to depend on preferences at the reference, we resort to the lattice structure of the set of indifference contours. We require, for instance, that well-being measured at the supremum (resp. infimum) of two indifference contours be equal to the maximum (resp. minimum) of their corresponding well-being levels, for constant indifference contour at the reference. We find these axioms to have different implications depending on the preference domain and we characterize well-being measures that satisfy all possible combinations of these axioms. One prominent measure computes the ratio between equivalent income at consumption and at reference, with prices being chosen to maximize well-being.

Keyword: Well-being measure, heterogeneous preferences, reference, lattice.

JEL Cassification: D63, I30

On the measurement of well-being with reference consumption*

François Maniquet[†] Domenico Moramarco[‡]

December 13, 2022

Abstract

We axiomatically study how to measure well-being when individuals have heterogeneous preferences and consumption bundles are evaluated by comparison with some reference bundle (a poverty line bundle, an index of needs, the average consumption in a reference group, etc.). If only the reference bundle matters, Deaton (1979)'s distance function turns out to be the only measure satisfying basic axioms. If the measure is allowed to depend on preferences at the reference, we resort to the lattice structure of the set of indifference contours. We require, for instance, that well-being measured at the supremum (resp. infimum) of two indifference contours be equal to the maximum (resp. minimum) of their corresponding well-being levels, for constant indifference contour at the reference. We find these axioms to have different implications depending on the preference domain and we characterize well-being measures that satisfy all possible combinations of these axioms. One prominent measure computes the ratio between equivalent income at consumption and at reference, with prices being chosen to maximize well-being.

JEL Classification: D63, I30

Keywords: Well-being measure, heterogeneous preferences, reference, lattice.

*We thank Marc Fleurbaey, Hervé Moulin, Alain Trannoy, John Weymark, Thomas Demuynck, Georg Kirchsteiger, the participants in the Economic Theory seminar, Paris I, the Economic Decisions and Social Preferences conference, Paris Dauphine, the 2022 Francqui Workshop, Brussels, the 2022 Meeting of the Society for Social Choice and Welfare, Mexico City and the 2022 Summer Workshop in Economic Theory, Paris. Domenico Moramarco gratefully acknowledges FNRS for their financial support.

†CORE/LIDAM, UCLouvain, and LISER, Luxembourg - francois.maniquet@uclouvain.be

‡ECARES, ULB - domenico.moramarco@ulb.be

1 Introduction

In this paper, we axiomatically study well-being measures that depend on the consumption of the individual, her preferences and a reference consumption. By doing so, we contribute to solving three problems, depending on whether we interpret the reference consumption as a poverty line bundle, an index of household needs or the average consumption of the group of people to whom one individual is ethically allowed to compare themself.

The first of these problems is that of measuring of poverty in a way that is consistent with preferences. Poverty is measured by aggregating the situation of individuals living below a poverty line. In the typical case when poor individuals face different prices, the income poverty line is computed as the money value at actual prices of a poverty line bundle. It is well known since long, though, that incomes and preference satisfaction don't necessarily go together when prices differ or markets are not perfect. A poor individual may happen to be better-off (that is at a higher level of preference satisfaction) than a non-poor one with the same preferences. Many solutions have been proposed to solve this issue, including solutions based on Samuelson (1974, 1977)'s notion of equivalent income or money-metric utility. The equivalent income of an individual is the minimal amount of money needed to reach a given satisfaction level at some reference prices. If reference prices are fixed, then a larger satisfaction level always goes together with a larger equivalent income. The question of how to choose the reference prices, however, remains largely unanswered (see Fleurbaey & Blanchet, 2013, for a deeper discussion). Another solution is Deaton (1979)'s distance function. It measures well-being as the fraction of the poverty line bundle which an individual is indifferent to. Other solutions include Blackorby & Donaldson (1987)'s welfare ratios and Dimri & Maniquet (2020)'s individualized price equivalent incomes.

We contribute to solving this problem by proposing a complete axiomatic analysis of well-being measures with a reference consumption that can be thought of as the poverty line bundle. One of the measures that turn out to be justified by our axioms is Deaton (1979)'s distance function. It was also axiomatized in Decancq *et al.* (2017), whereas Blackorby & Donaldson (1987)'s welfare ratios and Dimri & Maniquet (2020)'s individualized price equivalent incomes do not satisfy any of the main axioms we study. Most importantly, our study allows us to define new measures, which receive strong justifications and are ready to be used in poverty measurement as well as other social indices.

The second problem we contribute to solving is that of accounting for needs heterogeneity in the evaluation of social policies, in particular when differences in needs stems from differences in household size. How should we compare the consumption bundles of singles and couples, or of couples with and without children? The classical approach consists in applying an equivalence scale to household incomes. Among the many shortcomings of this approach, one is that it does not consider the interactions between preferences and economies of scale: households in which members have a strong preference towards goods that benefit most from scale economies are intuitively better at transforming household resources into well-being, and this is not captured by the equivalence scale approach.

Adding a reference consumption as an argument of well-being solves this problem. Indeed, by calibrating this reference consumption so as to take scale economies into account, one succeeds in comparing well-being across households of different size even when preferences differ across and within groups of households of different size.

The third problem is the definition of normative well-being measures when it is acknowledged that one individual's well-being is affected by what happens to people around them. In positive economics, it has been largely documented since the last decade of the 20th century that other-regarding preferences and distributional concerns have an impact on choices (Kirchsteiger, 1994; Fehr *et al.*, 1998, among others), preferences (see, for example, Fehr *et al.*, 1993; Fehr & Schmidt, 1999; Bolton & Ockenfels, 2000; Sobel, 2005; Dufwenberg *et al.*, 2011) and subjective well-being (Clark *et al.*, 2008; Clark & Senik, 2010; Clark, 2018).

That individuals compare themselves with others, though, does not mean that evaluating social policies *should* recognize that individuals may suffer from an improvement in the situation of others. Sen (1970), for instance, has argued against respecting individual preferences when they are defined over the doings of others. On the other hand, feeling socially included may require to have resources comparable to those with whom one interacts, and this feeling may be considered respectable from a normative point of view.

If one agrees to take the resources obtained by (some) others as a normatively relevant argument of individual well-being, two strategies are possible. The first strategy consists in admitting all possible other-regarding preferences as normatively compelling. Among such a wide class, one should also consider those individuals whose (marginal) utility from consumption depends on the consumption of others. Consequently, two individuals with the same personal consumption, reference consumption and self-centred preferences (defined over own consumption only) may be treated differently by an egalitarian evaluator. For example, if one agent's envy prevents him from enjoying his consumption bundle. The second approach assumes separability between the two components of individual well-being: utility from consumption and (dis)utility from the comparison with other's consumption. Dufwenberg *et al.* (2011) advocate the necessity of such separability to perform consistent comparison of individual consumption. In this paper, we build on this second strategy to focus on (heterogeneous) self-centred preferences, while adding an objective reference consumption in the arguments of the well-being measure. By objective, we mean that it is chosen by the evaluator and not the individuals themselves. It may be the average consumption in the country, the region, or a reference group. We adopt the second strategy, and we axiomatically characterize well-being measures, most of which are new in the literature.

The three sets of literature we contribute to share an important question: should well-being be measured in terms of income, or, more generally, in terms of money value of the individual's (current or equivalent) consumption? If so, which prices should be used to measure the money value of individual consumption? Some studies even posit that well-being is income, without justifying it, such as Blackorby & Donaldson (1987) or Dimri & Maniquet (2020). As a byproduct of our analysis, we answer these questions. Indeed, while starting with basic axioms that do not impose any relationship with prices and incomes, we end up characterizing two well-being measures equal to the ratio between the money value of a bundle equivalent to the individual consumption and the money value of a bundle equivalent to the reference consumption by using prices that are uniquely derived from the preferences. At the same time, we show that similar axioms yield other new measures that cannot be defined in terms of equivalent incomes, providing us with normative grounds to decide whether income is an acceptable measure of well-being. Our results suggest that, besides normative requirements, the answer to this question depends also on the domain of preferences we are interested in.

Our main axioms are divided in three families. The first family contains axioms based on the idea that the special case of homothetic preferences has an obvious solution: given that preferences can be represented by a utility function homogeneous of degree one, it is natural to measure well-being as a function of the relative utility at consumption and at reference. We further build on this idea and we add axioms bearing on how well-being measures should respond to homothetic transforms of preferences either at consumption or at reference bundles. All our measures satisfy these requirements. The most interesting axioms belong to the two other sets and represent the two main normative choices one faces when defining well-being with a reference consumption.

The first normative choice has to do with the interpretation of the reference consumption: is the reference consumption relevant per se or is it relevant as a satisfaction level? In the former case, we prove that Deaton (1979)'s distance function is the only justified way of measuring well-being. According to the Deaton's measure, the well-being of an individual is equal to the fraction of her reference bundle she finds her consumption bundle equivalent to. We then explore the case in which well-being is allowed to depend on preferences at the reference. To do so, we first consider the case in which preferences matter up to some distance from the reference bundle; to capture this idea, we introduce a requirement of continuity of the well-being measure when consumption converges towards reference. Secondly, we investigate the case in which the well-being measure is allowed to depend on the entire indifference contour at reference.

The second normative choice is related to the fact that by making well-being depend on preferences, we typically have to conclude that two individuals consuming the same bundle and having the same reference have unequal well-being levels. There are ways, however, to limit this inequality. We do it by imposing upper and lower bounds on well-being levels by reference to the lattice structure of the set of indifference contours, equipped with the partial order induced by the inclusion operator applied to the corresponding upper contours. We study the requirement that the well-being at the supremum of two indifference contours is equal to their maximum, and the requirement that the well-being at the infimum of two indifference contours is equal to their minimum. Moreover, we apply these two ideas to indifference contours at consumption and at reference.

Requiring the well-being at the supremum to coincide with the maximum well-being at two situations is closely related to comparing bundles using the celebrated no-envy requirement (popularized by Varian (1974)). On the other hand, requiring well-being at the infimum to be equal to the minimum of the well-being at two situations is related to the view that an indifference contour that is the convex combination between two other contours is also intermediary and, therefore, should be associated to an intermediary well-being level as well.

Using lattices in measurement theory has a long history, which includes Kreps (1979); Hougaard & Keiding (1998); Christensen *et al.* (1999); Chambers & Miller (2014b,a). The lattice structure of upper contour sets is used in (Fleurbaey & Maniquet, 2017, 2018a,b) to derive well-being measures. We cannot simply use these results, however, because contour sets at the consumption and at the reference bundles belong to the same preferences, so that the shape of one of them may restrict the domain of admissible shapes of the other. We therefore fail to have a rich enough domain that would allow us to apply these results.

This leads us to studying two domains of preferences, over which results differ. The first one is the domain of preferences that only admit compact lower contours. Over this domain, the incompatibility that is the heart of Fleurbaey & Maniquet (2017) vanishes and more possibilities appear. One prominent well-being measure is the ratio between equivalent incomes at reference and at consumption, when prices are chosen so as to maximize the resulting well-being level.

The second domain we explore is the unrestricted domain. Our axioms have different implications over this domain. Some of the well-being measures we axiomatize work by comparing equivalent incomes at consumption and at reference, but, contrary to the measure mentioned above, with fixed, arbitrary prices. Other measures we axiomatize have no links with equivalent income.

The paper is organized as follows. In Section 2, we introduce the preliminary notation and axioms to define a well-being measure. In Section 3, we define and discuss the well-being measures that will be justified by our axiomatic analysis. In Section 4, we introduce the main axioms, which we use, in Section 5, to characterize our prominent well-being measures. In Section 6, we prove the main results. In Section 7, we give some concluding comments.

2 Notation and definitions

In this section, we introduce the model, define a well-being measure and state basic axioms.

We assume that there are K divisible and cardinally measurable goods, $K \geq 2$. The (closed) consumption set is $\mathcal{X} = \mathbb{R}^K_+$ and its interior is $\mathcal{X}_+ = \mathbb{R}^K_{++}$. Agents have rational, continuous, convex and monotonic preferences over $\mathcal{X}$. We denote the set of such preferences with $\mathcal{R}$. For any two bundles z, $z' \in \mathcal{X}$ we denote weak preference of z over z' as zRz' (z is at least as good as z'), strict preference of z over z' as zPz' (z is strictly preferred to z'), and indifference as zIz' (z is equally good as z'). For any $z \in \mathcal{X}$ and $R \in \mathcal{R}$, we denote lower, upper and indifference (closed) contours as follows: $L(z, R) = \{z' \in \mathcal{X} | zRz'\}$, $U(z, R) = \{z' \in \mathcal{X} | z'Rz\}$, $I(z, R) = U(z, R) \cap L(z, R)$.

We are interested in measuring well-being as a function of what an agent consumes, her preferences, and a reference bundle. Formally, a *well-being measure* is a function $W : \mathcal{X} \times \mathcal{R} \times \mathcal{X}_+ \to \mathbb{R}$. We refer to arguments (x, R, y) of W as situations. Notice that we require the entries of the reference bundle to be all strictly positive;[1] to maintain a lighter notation we will often avoid referring to two different domains for consumption and references, leaving this distinction implicit.

Our first two basic axioms capture the natural intuition that well-being increases with consumption and decreases with reference, the latter being the requirement we introduce in this paper. The first axiom compares well-being when the indifference contour through the reference bundle, and the reference bundles, are the same in two situations. If it is the case that the consumption bundle in one situation is strictly preferred to the other consumption bundle by both preferences, and if, moreover, all consumption bundles indifferent to the former bundle are also strictly preferred to any bundle indifferent to the latter by both preferences, then well-being is larger in the former situation. Observe that this condition on the preferences at the two contemplated consumption bundles is equivalent to requiring that the lower contour at one bundle does not intersect the upper contour at the other.[2]

[1] This last assumption allows us not to discuss the (irrelevant case) in which the reference quantity of a good is zero whereas this good is necessary for the agent.

[2] Consumption Monotonicity is reminiscent of the Nested Contour axiom in Fleurbaey &

Consumption Monotonicity (Mx) - For all $x, x', y \in \mathcal{X}$ and $R, R' \in \mathcal{R}$, if $I(y, R) = I(y, R')$ and $L(x, R) \cap U(x', R') = \emptyset$, then $W(x', R', y) > W(x, R, y)$.

The second axiom requires well-being to be lower in a situation in which the reference bundle dominates the one of another situation, where dominance means that the former reference bundle contains strictly more of all goods than the latter one.

Reference Dominance (Dy) - For all $x, y, y' \in \mathcal{X}$ and $R \in \mathcal{R}$, if $y' \gg y$, then $W(x, R, y) > W(x, R, y')$.

Let us note the asymmetry between these two axioms. With Consumption Monotonicity, we compare two situations in which preferences may be different and the preferred consumption, x', does not necessarily dominate the other one. This is because we want Reference Dominance to be as weak as possible, so that we can study the dilemma about how to treat the reference (see Section 4.2 below).

We then require continuity of the measure with respect to both bundles.

Continuity (Cont) - For all $R \in \mathcal{R}$, $W(\cdot, R, \cdot)$ is continuous in its first and third arguments.

By applying Consumption Monotonicity to $R' = R$, we get that if $x' \, P \, x$, then $W(x', R, y) > W(x, R, y)$. By Continuity, we obtain that $x' \, R \, x$ if and only if $W(x', R, y) \geq W(x, R, y)$. That is, for a fixed y, $W(\cdot, R, y)$ is a utility representation of R.

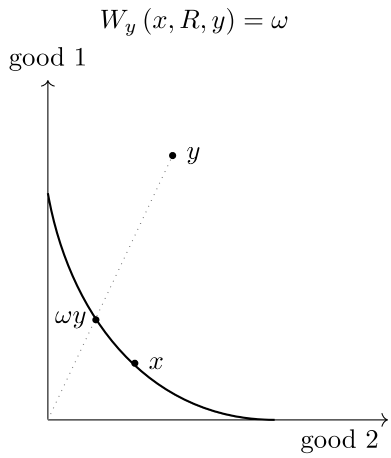
The following lemma proves an important consequence of our basic axioms, which will be used extensively in the following. In words, Consumption Monotonicity and Continuity imply that only the indifference contours at consumption and at reference matter; that is, well-being measures are not allowed to be sensitive to changes in preferences that do not modify these two indifference contours. We formalize this property into the following axiom.

Unchanged Contour Independence (UCIxy) - For all $x, y \in \mathcal{X}$ and $R, R' \in \mathcal{R}$, if $I(x, R) = I(x, R')$ and $I(y, R) = I(y, R')$, then $W(x, R, y) = W(x, R', y)$.

Lemma 1. *If W satisfies Continuity (Cont) and Consumption Monotonicity (Mx), then it satisfies Unchanged Contour Independence (UCIxy).*

Maniquet (2017).

Figure 1: The point-reference well-being measure



Proof. Let $x, y \in \mathcal{X}$ and $R, R' \in \mathcal{R}$ be such that $I(x,R) = I(x,R')$ and $I(y,R) = I(y,R')$. Let $x^n \in \mathcal{X}$, $n \in \mathbb{N}$, be a sequence of bundles so that $x^n P' x$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x^n = x$. By Mx, $W(x^n, R', y) > W(x, R, y)$ for all $n \in \mathbb{N}$. By Cont, $W(x, R', y) \geq W(x, R, y)$. Similarly, with a sequence $x^n \in \mathcal{X}$ such that $x P' x^n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x^n = x$, we get $W(x, R', y) \leq W(x, R, y)$. Gathering the two inequalities, we obtain $W(x, R', y) = W(x, R, y)$, the desired outcome. □

3 Well-Being Measures

In this section, we define the well-being measures that will turn out to be characterized in the paper.

The first measure, W_y, is Deaton's distance function (Deaton, 1979), also used by Samuelson (1977) under the name of ray utility. The well-being of an individual with preferences R consuming bundle x with reference y, is defined as the fraction of y which this individual finds x indifferent to. We call it the point-reference well-being measure. It is illustrated in Fig. 1.

Definition 1. *The point-reference well-being measure is the function* $W_y : \mathcal{X} \times \mathcal{R} \times \mathcal{X}_+ \to \mathbb{R}_+$ *such that, for all* $(x, R, y) \in \mathcal{X} \times \mathcal{R} \times \mathcal{X}_+$

$$W_y\,(x, R, y) = \omega \Leftrightarrow \omega y \in I(x, R).$$

All other measures require the following notation. Let $\mathcal{R}^H \subset \mathcal{R}$ denote the set of homothetic preferences. Let $\mathcal{U}^H$ be the set of all homogeneous of degree one utility functions, which represent homothetic preferences. Moreover, for any indifference contour set $I\,(z, R)$ we call $R_{I(z,R)}$ the homothetic preferences we generate by homothetic transformations of $I(z, R)$ for any positive real number.[3]

There are three equivalent ways to define our second measure. The most intuitive one defines it as the scale by which the indifference contour through consumption has to be homothetically transformed so as to become tangent from below to the indifference contour at reference. This is illustrated in Fig. 2a. The dashed indifference curve is the minimum (homothetic) rescaling of $I(x, R)$ that contains a bundle indifferent to y: bundle z. Consumption x is indifferent to αz and reference y is indifferent to z. Well-being is then equal to α.

Observe that this measure is also the scale by which the indifference contour through reference has to be homothetically transformed so as to become tangent from above to the indifference contour at consumption. In that case, the tangency between the transform of the indifference contour at y and the indifference contour at x would take place at αz.

The measure is also equal to the ratio between the equivalent income at x and that at y, when prices are chosen in such a way as to maximize well-being. This is illustrated in Fig. 2b, in which p is the price vector at which the equivalent income ratio is maximal. Unsurprisingly, tangencies of the black dashed lines to the indifference contours take place at αz and z.

Definition 2. *The maximum well-being measure is the function* $W_{max} : \mathcal{X} \times \mathcal{R}^c \times \mathcal{X} \to \mathbb{R}_+$ *such that, for all* $(x, R, y) \in \mathcal{X} \times \mathcal{R}^c \times \mathcal{X}$,

$$W_{max}(x, R, y) = \frac{u(x)}{min_{y' \in U(y,R)} u(y')}$$

for $u \in \mathcal{U}^H$ *representing* $R_{I(x,R)}$.

The next measure is the dual of the previous one. Well-being is the ratio between the equivalent income at x and that at y, when prices are chosen in such a way as to minimize well-being.

Definition 3. *The minimum well-being measure is the function* $W_{min} : \mathcal{X} \times \mathcal{R}^c \times \mathcal{X} \to \mathbb{R}_+$ *such that, for all* $(x, R, y) \in \mathcal{X} \times \mathcal{R}^c \times \mathcal{X}$,

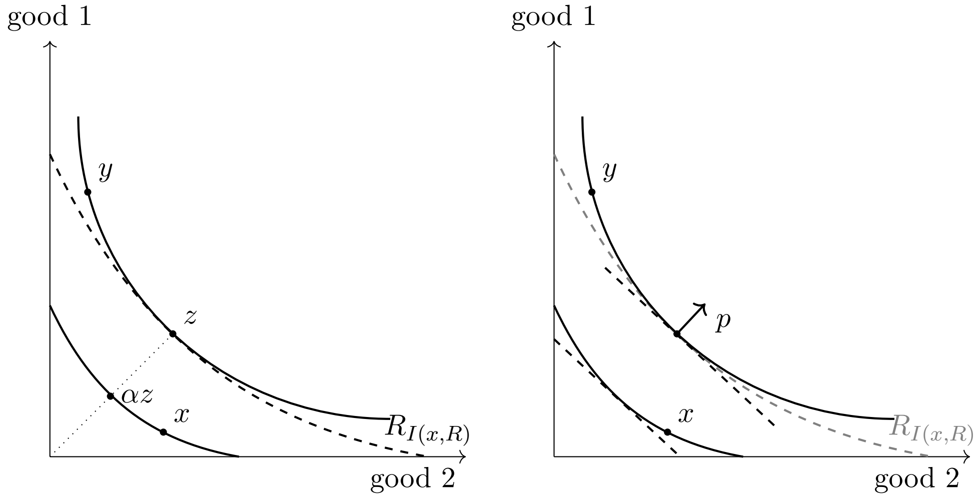
$$W_{min}(x, R, y) = \frac{u(x)}{max_{y' \in L(y,R)} u(y')}$$

for $u \in \mathcal{U}^H$ *representing* $R_{I(x,R)}$.

[3]If $I(z', R)$ is an homothetic transformation of $I(z, R)$ for a scalar $\lambda \in \mathbb{R}_+$, then $x \in I(z, R)$ if and only if $\lambda x \in I(z', R)$. Notice that $R \in \mathcal{R}$ implies $R_{I(z,R)} \in \mathcal{R}$ and $R = R_{I(z,R)}$ if and only if $R \in \mathcal{R}^H$.

Figure 2: The maximum well-being measure.

(a) $W_{max}(x, R, y) = \alpha$ (b) The endogenous prices

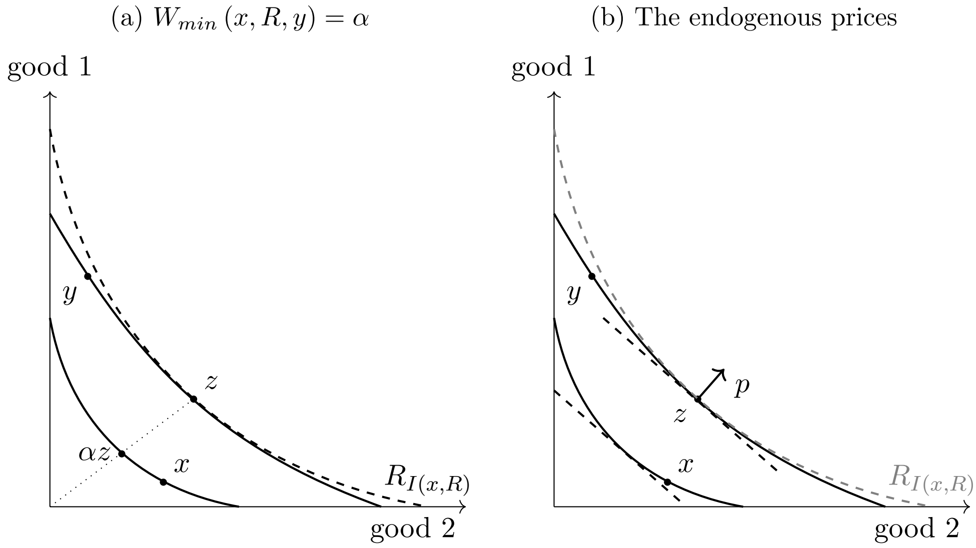


It is straightforward to check that $W_{min}(x, R, y) = (W_{max}(y, R, x))^{-1}$. This implies that both measures are well-defined on the same domain. They are not defined, however, on the full domain $\mathcal{R}$. The need for a domain restriction for W_{min} is evident if we consider the example of an individual with linear indifference curve through x and Leontief indifference curve at y: in this case there is no rescaling of the indifference contour through consumption that is tangent from above to the indifference contour through reference. In order to have a well-defined and finite well-being level for all x and all y, preferences R need to be such that all lower contour sets are compact. We call this domain $\mathcal{R}^c$. Observe that what we have called tangencies in our informal definitions may take place at corner bundles. It is easy to check that the absence of a unique supporting price does not prevent well-being from being uniquely defined.

Our next two well-being measures are actually families of well-being measures. Each measure in a family is parameterized by some *evaluating preferences.* These evaluating preferences are homothetic. A well-being measure in the first (resp. second) family works like this: consider the upper (resp. lower) contour sets at consumption and reference, and identify the lowest (resp. highest) utility level at bundles in these sets, for a utility function representing the evaluating preferences. The well-being level is the ratio between utilities associated to the consumption and the reference.

Let us focus on the first family, illustrated in Fig. 4a. The evaluating preferences, named R^i, are represented in dashed. We see two indifference contours: one

Figure 3: The mininum well-being measure.



that is tangent from below to the indifference contour at x and one at y. Given that R^i are homothetic, the ratio between the utility at the two contours is equal to the ratio between αz and z, that is α. This is, therefore, the well-being level of both individuals R and R^i in Fig. 4a. In the special case in which evaluating preferences are linear, represented in Fig. 4b, well-being is equal to the ratio of equivalent incomes at x and y, with a price vector proportional to the (unique) gradient of the utility function.

Not all homothetic preferences can be used as evaluating preferences. Evaluating preferences in the first family of measures need to have a least preferred bundle (*an infimum*) in each upper contour set generated by preferences in $\mathcal{R}$. We denote with $\mathcal{R}^i \subset \mathcal{R}$ the set of such preferences that we call *infimum (or inf) preferences.*[4] In a dual fashion, evaluating preferences for the second family of measures need to have a most preferred bundle (*a supremum*) in any lower contour set generated by preferences in $\mathcal{R}$. We denote with $\mathcal{R}^s \subset \mathcal{R}$ the set of such preferences that we call *supremum (or sup) preferences.*[5] Observe that Leontief preferences are members of $\mathcal{R}^s$. Fig. 5a illustrates such a measure for general evaluating preferences and Fig. 5b for the case of Leontief evaluating preferences. We are now equipped to formally define our next two families of well-being measures.

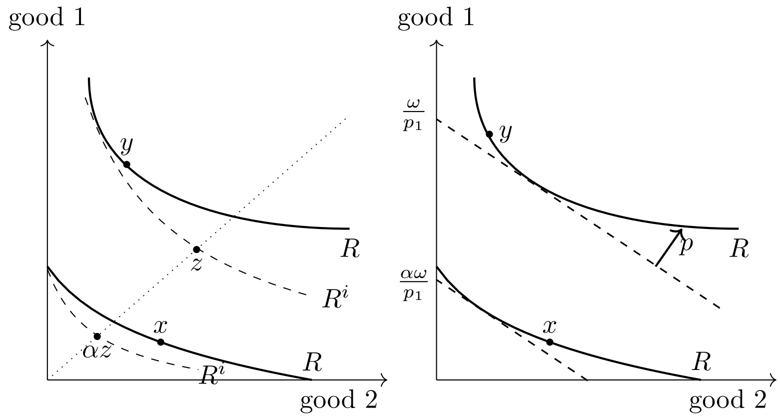
Definition 4. *The infimum preference well-being measure (parameterized by $R^i \in \mathcal{R}^i \cap \mathcal{R}^H$) is the function $W_{inf}^{R^i} : \mathcal{X} \times \mathcal{R} \times \mathcal{X}_+ \to \mathbb{R}_+$ such that, for all $(x, R, y) \in$*

[4]Fleurbaey & Maniquet (2017) call them best preferences.

[5]Fleurbaey & Maniquet (2017) call them worst preferences.

Figure 4: Infinum preference well-being measure

(a) $W_{inf}^{R^i}(x,R,y)=\alpha$ (b) $W_{inf}^{p}(x,R,y)=\alpha$



$\mathcal{X}\times\mathcal{R}\times\mathcal{X}_+$,

$$W_{inf}^{R^i}(x,R,y)=\frac{\min_{x'\in U(x,R)}u(x')}{\min_{y'\in U(y,R)}u(y')}$$

for $u\in\mathcal{U}^H$ representing R^i.

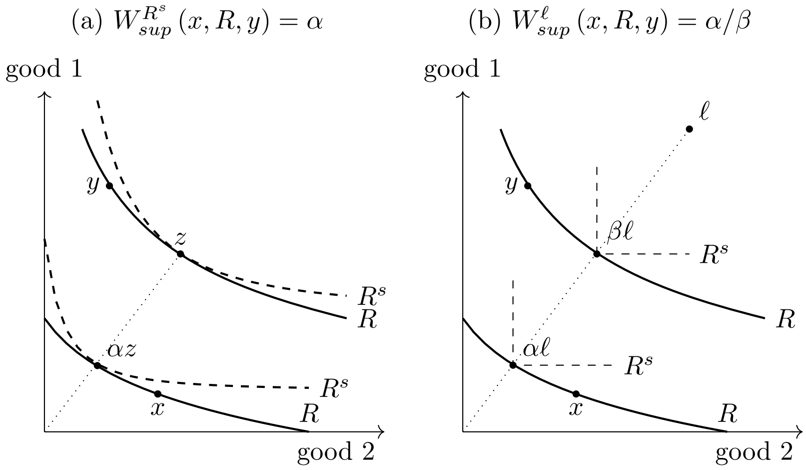
Definition 5. *The supremum preference well-being measure (parameterized by $R^s\in\mathcal{R}^s\cap\mathcal{R}^H$) is the function $W_{sup}^{R^s}:\mathcal{X}\times\mathcal{R}\times\mathcal{X}_+\to\mathbb{R}_+$ such that, for all $(x,R,y)\in\mathcal{X}\times\mathcal{R}\times\mathcal{X}_+$,*

$$W_{sup}^{R^s}(x,R,y)=\frac{\max_{x'\in L(x,R)}u(x')}{\max_{y'\in L(y,R)}u(y')}$$

for $u\in\mathcal{U}^H$ representing R^s.

The definitions above describe families of well-being measures. Intuitively, individuals with inf preferences are more efficient in extracting utility from a given set of bundles (i. e. they are easier to please) while those with sup preferences are characterized by lower ability of substituting between goods, which makes it harder to extract utility from the same set of available options. The families of measure in Definitions 4 and 5 compare individuals by looking at how a particular (inf or sup) evaluation preference would evaluate their relative situations. Observe that this is different from what the measures in Definitions 2 and 3 do: roughly speaking, these measures compare two individuals considering their own preferences, without relying on exogenous evaluation preference.

Figure 5: Supremum preference well-being measure



The special instance W^{ℓ}_{sup} in Figure 5b evaluates contour sets looking at their intersection with ray through ℓ. If we impose such a ray to coincide with one of the axis, we can define a new family of measures. For $k \in \{1,\dots,K\}$, let $A^k \subset \mathcal{X}$ denote the k-th axis, that is, $x \in A^k \Leftrightarrow x_\ell = 0 \, \forall \, \ell \in \{1,\dots,K\} \setminus \{k\}$. For $k \in \{1,\dots,K\}$ and $(x,R) \in \mathcal{X} \times \mathcal{R}^c$, let $A^k(I(x,R))$ denote the intersection of $I(x,R)$ with the k-th axis.

Definition 6. *The k-th axis well-being measure is the function $W_k : \mathcal{X} \times \mathcal{R}^c \times \mathcal{X}_+ \to \mathbb{R}_+$ such that, for all $(x,R,y) \in \mathcal{X} \times \mathcal{R}^c \times \mathcal{X}_+$,*

$$W_k(x,R,y) = \frac{A^k(I(x,R))}{A^k(I(y,R))}$$

Observe that the k-th axis measure is well-defined only if the intersection between an indifference contour and the axis is a singleton. This motivates the restriction of W_k to the set of preferences with compact indifference contour sets. Similarly to Definitions 2 and 3, a k-th axis measure coincides with the ratio between equivalent income at consumption and reference; in this case, however, the multiplicity of supporting prices is the norm, more then a special case.

4 Axioms

In this section, we introduce our main axioms. For the sake of exposition, we divide the axioms in three groups. The first group focuses on homothetic indifference contours to define desirable basic properties of relative well-being measures. The second group deals with the choice of considering references as a bundle or as a satisfaction level, that is an indifference contour set. The third group strengthens well-being monotonicity to deal with intersecting indifference contours.

4.1 Basic axioms

First, we introduce axioms inspired by the idea that well-being has a clear interpretation when preferences are homothetic and consumption is a fraction of the reference bundle. In this case, there is no ground on which to distinguish among homothetic agents, so that they should all have the same well-being. We also require that changing consumption and reference so that the agent finds the new situation equivalent to the previous one and consumption is still the same fraction of the reference does not affect well-being.

Homotheticity (H) - For all $x, x' \in \mathcal{X}$ and $\lambda > 0$, if $R, R' \in \mathcal{R}^H$ and xIx', then (1) $W(x, R, \lambda x) = W(x, R', \lambda x)$ and (2) $W(x, R, \lambda x) = W(x', R, \lambda x')$.

It would have been stronger to require $W(x, R, \lambda x) = W(x', R, \lambda x')$ without the proviso that xIx'. When the well-being satisfies Consumption Monotonicity and Continuity, however, Homotheticity implies the even stronger requirement that $W(x, R, \lambda x) = W(x', R', \lambda x')$ for all x and x' and all homothetic preferences R and R', as proven in the following lemma.

Lemma 2. *If a well-being measure W satisfies Consumption Monotonicity (Mx), Continuity (Cont) and Homotheticity (H), then for all $x, x' \in \mathcal{X}$ and $R, R' \in \mathcal{R}^H$, $\lambda > 0$:*

$$W(x, R, \lambda x) = W(x', R', \lambda x').$$

Proof. Let $x, x' \in \mathcal{X}$, $R, R' \in \mathcal{R}^H$, $\lambda > 0$. Let $z \in \mathcal{X}$ and $R^1, R^2 \in \mathcal{R}$ be such that: *(i)* $z = (x_1, x'_2, 0, \ldots, 0)$; *(ii)* R^1 can be represented by $u^1(x) = x_1$; *(iii)* R^2 can be represented by $u^2(x) = x_2$.

By H1, $W(x, R, \lambda x) = W(x, R^1, \lambda x)$. By H2, $W(x, R^1, \lambda x) = W(z, R^1, \lambda z)$. By H1, $W(z, R^1, \lambda z) = W(z, R^2, \lambda z)$. By H2, $W(z, R^2, \lambda z) = W(x', R^2, \lambda x')$. By H1, $W(x', R^2, \lambda x') = W(x', R', \lambda x')$. Gathering the equalities, $W(x, R, \lambda x) = W(x', R', \lambda x')$, the desired outcome.

$\square$

Let $R \in \mathcal{R}^H$ and $u \in \mathcal{U}^H$. Lemma 2, applied to $R' = R$, implies that

$$W(x, R, y) = f\left(\frac{u(x)}{u(y)}\right)$$

for some strictly increasing real-valued function f. In the aggregation of individuals' well-being levels into a measure of social welfare, f would be crucial. For instance, its curvature would measure the degree of inequality aversion of the social welfare function. As we do not build social welfare functions in this paper, for the sake of simplicity of the exposition, we assume that $f(u) = u$. Nothing in what we say below depends on this assumption. Any f function can be used to aggregate the well-being measures we obtain here. This will allow us to skip mentioning after each result that the relevant well-being measure is ordinally equivalent to this or that well-being measure satisfying the $f(u) = u$ property. We come back to the possible shapes of f in the conclusion.

We are now equipped to state and prove our first main result. The maximum and minimum well-being measures presented in the previous section are actually bounds to all well-being measures satisfying the axioms we have defined.

Theorem 1. *If a well-being measure W satisfies Consumption Monotonicity (Mx), Continuity (Cont) and Homotheticity (H), then for all $x, y \in \mathcal{X}$, all $R \in \mathcal{R}$,*

$$W_{\min}(x, R, y) \leq W(x, R, y) \leq W_{\max}(x, R, y). \tag{1}$$

Proof. Let us start by considering preferences $R \in \mathcal{R}^c \subset \mathcal{R}$, so that W_{min} and W_{max} are well-defined. We prove the first inequality.

By contradiction, let W be such that $W(x, R, y) < W_{min}(x, R, y)$ for some situation $(x, R, y) \in \mathcal{X} \times \mathcal{R}^c \times \mathcal{X}$. Let $R' \in \mathcal{R}^c \cap \mathcal{R}^H$ and $\omega \in \mathbb{R}$ be such that: *(i)* $I(y, R') = I(y, R)$ and *(ii)* $I(\omega y, R')$ is tangent from below to $I(x, R)$. By H, $W(\omega y, R', y) = \omega$ which, by Definition 3, coincides with $W_{min}(x, R, y)$. Gathering all this, we have

$$W(x, R, y) < \omega = W(\omega y, R', y) = W_{min}(x, R, y) \tag{2}$$

By Cont, the first inequality in eq.(2) implies that there exists $x' \in \mathcal{X}$ such that $x'Px$ and $W(x', R, y) = \omega$. Observe that, by *(ii)* we must have $U(x', R) \cap L(\omega y, R') = \emptyset$, so that *(i)* and Mx imply

$$\omega = W(x', R, y) > W(\omega y, R', y) \tag{3}$$

A contradiction. The proof of the second inequality follows the same logic, with $I(\omega y, R')$ tangent from above to $I(x, R)$ and xPx'.

Let us now consider the case $R \in \mathcal{R} \setminus \mathcal{R}^c$. Observe that if W_{min} (resp. W_{max}) is not defined, then the first (resp. second) inequality in eq.(1) is vacuously true. If W_{min} is well-defined, then R' defined above exists and the last equality in eq.(2) is maintained. It is then sufficient to notice that the previous proof continues to hold despite $R \notin \mathcal{R}^c$. Similarly, if $R \notin \mathcal{R}^c$ but W_{max} is well-defined, then we can adapt the previous proof to show the last inequality in eq.(1) $\square$

In empirical applications, Theorem 1 may be all we need to compare distributions of well-being. If the estimation of preferences is parametric, then $W_{\max}$ and $W_{\min}$ may moreover take very simple shapes. For instance, if estimated preferences are of the Stone-Geary type, that is they are represented by utility functions[6]

$$u^{a,b}(x) = \prod_{i=1}^{K}(x_i - b_i)^{a_i},$$

with $-b \in \mathbb{R}^K_{++}$, $a \geq 0$ and $\sum_{i=1}^K a_i = 1$, it is intuitive that the price vector leading to $W_{\max}$ is the supporting vector along the ray of bundles proportional to $-b$ and the one that leads to $W_{\min}$ supports bundles on one of the axis. Simple computations yield (assuming $W_{\min}$ is computed by reference to the price that supports consumption on the good-1 axis)

$$\begin{aligned} W_{\max}(x, R^{a,b}, y) &= \frac{\prod_{i=1}^K \left(\frac{x_i - b_i}{-b_i}\right)^{a_i} - 1}{\prod_{i=1}^K \left(\frac{y_i - b_i}{-b_i}\right)^{a_i} - 1} \\ W_{\min}(x, R^{a,b}, y) &= \frac{(x_1 - b_1)\prod_{i=2}^K \left(\frac{x_i - b_i}{-b_i}\right)^{\frac{a_i}{a_1}} + b_1}{(y_1 - b_1)\prod_{i=2}^K \left(\frac{y_i - b_i}{-b_i}\right)^{\frac{a_i}{a_1}} + b_1}. \end{aligned}$$

When the difference between $W_{\min}(x, R, y)$ and $W_{\max}(x, R, y)$ is large, evaluating well-being requires to go beyond Theorem 1. To this end, we begin by pushing the role of Homotheticity further, to restrict the well-being measures deserving to be studied, while still keeping the axioms sufficiently weak so that all our measures of interest do satisfy them. Assume we have equal well-being in two situations in which the reference bundle is the same and the indifference contours through it are also the same. We can interpret this well-being equality by saying that the distance between the two consumption indifference contours and the reference consumption is the same. As a result, if we apply an homothetic transformation to this reference bundle indifference contour, then its distance to the two consumption indifference contours should remain the same.

Reference Rescaling (Ry) - Let $x \in \mathcal{X}$, $R, R' \in \mathcal{R}$, $\alpha, \lambda \in \mathbb{R}_{++}$ such that either $1 < \alpha < \lambda$ or $\lambda < \alpha < 1$, if $I(\alpha x, R) = I(\alpha x, R')$, $I(\lambda x, R) = I(\lambda x, R')$ and $I(\lambda x, R)$ is homothetic to $I(\alpha x, R)$, then $W(x, R, \alpha x) = W(x, R', \alpha x) \Leftrightarrow$

[6] Parameters b are sometimes presented as minimal consumption, beyond which preferences are behaving like Cobb-Douglas preferences. The case $b \geq 0$, though, is problematic. Indeed, for bundles x such that $0 \leq x_k < b_k$ for some $k \in \{1, \ldots, K\}$, preferences are not monotonic. Observe that the condition $-b \in \mathbb{R}^K_+$ is also the one that guarantees that $R^{a,b} \in \mathcal{R}^c$.

$W(x, R, \lambda x) = W(x, R', \lambda x)$.

We apply the same idea to an homothetic transformation of the consumption indifference contour.

Consumption Rescaling (Rx) - Let $x \in \mathcal{X}$, $R, R' \in \mathcal{R}$, $\alpha, \lambda \in \mathbb{R}_{++}$ such that either $1 < \alpha < \lambda$ or $\lambda < \alpha < 1$, if $I(\alpha x, R) = I(\alpha x, R')$, $I(\lambda x, R) = I(\lambda y, R')$ and $I(\lambda x, R)$ is homothetic to $I(\alpha x, R)$, then $W(\alpha x, R, x) = W(\alpha x, R', x) \Leftrightarrow W(\lambda x, R, x) = W(\lambda x, R', x)$.

Reference Rescaling and Consumption Rescaling are defined in a way that preserves the relative position of consumption and reference bundles. For example, when consumption is preferred to the reference, the rescaling does not invert this relation. An interesting strengthening of this requirement consists in allowing the reference to become preferred to consumption after the rescaling. Formally, this amounts to considering $\alpha, \lambda \in \mathbb{R}_{++}$ without imposing restrictions on their relative size. We call the resulting axioms Rescaling Across Consumption and Rescaling Across Reference, and refer to Section 6 for the formal definition.

All well-being measures introduced in the previous section satisfy all the axioms defined in this section and the previous ones. Therefore, abusing terminology and to save on notation, throughout the paper we rely on the following definition.

Definition 7. *A* well-being measure *is a function W satisfying Consumption Monotonicity (Mx), Reference Dominance (Dy), Continuity (Cont), Homotheticity (H), Consumption Rescaling (Rx) and Reference Rescaling (Ry).*

4.2 First normative dilemma: reference bundle or reference satisfaction level?

When taking individual reference into account, we face the dilemma of considering it as a bundle or as a level of preference satisfaction.

In the first case, we would be comparing the consumed bundle, and its indifference contour set, with a specific reference bundle. Following this logic, if two individuals have the same consumption indifference contour and the same reference bundle, then they should have the same well-being. The following axiom formalizes this requirement.

Consumption Focus (Fx) - For all $x, y \in \mathcal{X}$ and $R, R' \in \mathcal{R}$, if $I(x, R) = I(x, R')$, then $W(x, R, y) = W(x, R', y)$

Only one of our well-being measures satisfies this axiom: the point-reference measure W_y. Moreover, we have the following characterization result.

Theorem 2. *W_y is the only well-being measure that satisfies Consumption Focus (Fx).*

Proof. The proof that W_y satisfies Fx is simple and omitted here.

Let W be a well-being measure satisfying Fx. We need to show that for all $x, y \in \mathcal{X}$, $R \in \mathcal{R}$, $W(x, R, y) = W_y(x, R, y)$.

Let $x' \in \mathcal{X}$ and $R' \in \mathcal{R}^H$ be defined by: $x' \, I \, x$, $x' = \alpha y$ for some $\alpha \geq 0$, and $I(x, R') = I(x, R)$. By H, $W(x', R', y) = \alpha^{-1}$. By Mx and Cont, $x \, I' \, x'$ implies that $W(x, R', y) = W(x', R', y)$. By Fx, $W(x, R, y) = W(x, R', y)$. Gathering these equalities, $W(x, R, y) = \alpha^{-1}$, the desired outcome. $\square$

If we consider references as indifference contours, it is natural to impose that replacing the reference bundle with another bundle that the agent finds equivalent does not change well-being.

Reference Indifference (Iy) - For all $x, y, y' \in \mathcal{X}$ and $R \in \mathcal{R}$, if yIy' then $W(x, R, y) = W(x, R, y')$.

It is easy to see that W_y does not satisfy Reference Indifference, whereas all other measures presented in the previous section do. Notice that, while Consumption Focus completely disregards preferences at the reference bundle, Reference Indifference considers any other bundle equivalent to the reference, when measuring well-being. Between these two extremes, we find interesting to also explore axioms in which preferences at the reference matter up to the case in which the consumption bundle converges to the reference bundle. In the context of poverty measurement, for instance, if the consumption of an individual converges towards the poverty line, it is natural that the contribution of this individual to poverty converges to zero, independently of the preferences over other bundles. The following axiom formalizes this idea, when the reference bundle is preferred to the current one.

Convergence from Below (Cb) - For all $y \in \mathcal{X}$ and $R \in \mathcal{R}$, if there exists a sequence $(x^n, R^n)_{n \in \mathbb{N}}$ such that, for all $n \in \mathbb{N}$, (i) $I(y, R^n) = I(y, R)$, (ii) $L(x^n, R^n) \cap U(x^{n+1}, R^{n+1}) = \emptyset$, (iii) $y \, P \, x^n$, and (iv) $x^n \to y$, then $W(x^n, R^n, y) \to 1$.

We also consider the dual case in which the current consumption bundle is preferred to the reference one.

Convergence from Above (Ca) - For all $y \in \mathcal{X}$ and $R \in \mathcal{R}$, if there exists a sequence $(x^n, R^n)_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, (i) $I(y, R^n) = I(y, R)$, (ii) $L(x^{n+1}, R^{n+1}) \cap U(x^n, R^n) = \emptyset$, (iii) $x^n \, P \, y$, and (iv) $x^n \to y$, then $W(x^n, R^n, y) \to 1$.

It is worth underlining here that Consumption Focus is stronger than both Convergence from Below and Convergence from Above, in the sense that if a well-being measure satisfies Consumption Focus, then it satisfies both convergence axioms. On the other hand, if W satisfies Reference Indifference, then it cannot satisfy both Convergence from Below and Convergence from Above. Nevertheless, combining Reference Indifference with Convergence from Below (resp. Convergence from Above) allows us to extend the convergence requirement to a sequence of reference bundles that converges to the consumption one from above (resp. below). For the interested reader, we provide all these results in Appendix A.

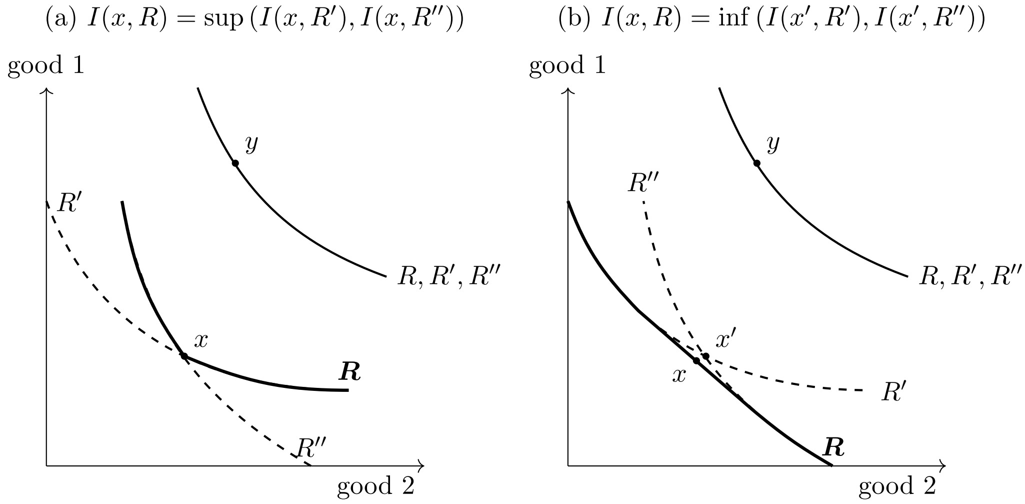
4.3 Second normative dilemma: bounds on well-being inequality

Out of the basic axioms, only Consumption Monotonicity forces us to take actual individuals' preferences into account. This has a consequence that we did not underline before: two individuals consuming identical bundles and having the same reference may have different well-being levels. This may be viewed as a drawback of the approach (a drawback that is shared by all normative analysis that impose Pareto efficiency or any other way of respecting individual preferences). The best way to fight this drawback consists in minimizing well-being inequality *at identical or similar bundles*. The axioms that we study in this section impose bounds on well-being inequality while still respecting individual preferences.

These axioms build on the lattice structure of the set of indifference contours, or, to present it differently, on the lattice structure of the set of upper contour sets equipped with the inclusion partial order.[7] This is represented in Figures 6a and 6b. In Fig. 6a, the indifference curve at x for preferences R is the supremum of indifference curves at x of preferences R' and R''. Observe that by Consumption Monotonicity, any indifference curve above $I(x, R)$ would be associated to a strictly higher well-being than the one at x of preferences R' and R'' (assuming they have the same indifference curves at y, as it is the case in the figure). Therefore, well-being at x with preferences R is either equal or strictly higher than with preferences R' or R''. Our first axiom, Supremum Contour at Consumption, imposes an upper

[7] A set equipped with a partial order is a lattice if for any two elements of this set there exists a smallest element that is larger than both, their supremum, and a largest element that is smaller than both, their infimum.

Figure 6: Supremum and Infimum Contour at Consumption.



bound on the well-being at x for preferences R: it should be *equal* to the maximum between well-being at x with preferences R' and R''.

This upper bound can be further justified by the following reasoning. Because $I(x,R)$ is the supremum of $I(x,R')$ and $I(x,R'')$, any bundle that an individual with preferences R deems equally good as x is also deemed equally good as x by an individual with preferences R' or R'', which gives us a reason not to claim that well-being at x with preferences R is higher than well-being at x with both preferences R' and R''.

We can apply the same idea by looking at indifference contours at reference across situations in which the indifference contours at consumption are the same. We obtain a lower bound on well-being. We call this second axiom Supremum Contour at Reference.

Supremum Contour at Consumption (Sup(x|y)) - For all $x, y \in \mathcal{X}$ and $R, R', R'' \in \mathcal{R}$, if $L(x,R) = [L(x,R') \cup L(x,R'')]$ and $I(y,R) = I(y,R') = I(y,R'')$, then $W(x,R,y) = \max\{W(x,R',y), W(x,R'',y)\}$.

Supremum Contour at Reference (Sup(y|x)) - For all $x, y \in \mathcal{X}$ and $R, R', R'' \in \mathcal{R}$, if $L(y,R) = [L(y,R') \cup L(y,R'')]$ and $I(x,R) = I(x,R') = I(x,R'')$, then $W(x,R,y) = \min\{W(x,R',y), W(x,R'',y)\}$.

Let us now build the dual axioms by looking at infimum of two indifference contours. In Fig. 6b, the indifference curves at x for preferences R is the infimum of indifference curves at x' of preferences R' and R''. We see that convexity of R'

and R'' prevents x' from belonging to the infimum of $I(x', R')$ and $I(x', R'')$. If we consider that x is *similar* to x', we can again create a lower bound on well-being inequality by imposing that the well-being at x for an individual with preferences R be exactly equal to the minimum between the well-being of both an individual with preferences R' and an individual with preferences R'' at x'. We call this axiom Infimum Contour at Consumption.

Again, there is an additional motivation for imposing this requirement. Any bundle that an individual with preferences R considers at least as good as x, indeed, is the convex combination between a bundle that an individual with preferences R' considers at least as good as x' and a bundle that an individual with preferences R'' considers at least as good as x'. As a result, the indifference curve of R through x can be considered as intermediary between the indifference curves of R' and R'' through x', so that the well-being at x with preferences R cannot be claimed to be strictly lower than the two others.

A similar axiom can be defined if we fix indifference contours at consumption and we look at infimum contours at reference. We call this axiom Infimum Contour at Reference.

Let the *ch* operator refer to the convex hull of its argument.

Infimum Contour at Consumption (Inf(x|y)) - For all $x, x', y \in \mathcal{X}$ and $R, R', R'' \in \mathcal{R}$, if $U(x, R) = ch\ [U(x', R') \cup U(x', R'')]$ and $I(y, R) = I(y, R') = I(y, R'')$, then $W(x, R, y) = \min\{W(x', R', y), W(x', R'', y)\}$.

Infimum Contour at Reference (Inf(y|x)) - For all $x, y, y' \in \mathcal{X}$ and $R, R', R'' \in \mathcal{R}$, if $U(y, R) = ch\ [U(y', R') \cup U(y', R'')]$ and $I(x, R) = I(x, R') = I(x, R'')$, then $W(x, R, y) = \max\{W(x, R', y'), W(x, R'', y')\}$.

The consequences of each one of the above four axioms depend on the preference domain. In a way, these axioms imply the existence of evaluating preferences, in $\mathcal{R}^s$ or $\mathcal{R}^i$, that we should use when measuring well-being. In a model without reference bundles one could refer to Fleurbaey & Maniquet (2017) which uses the richness of the preference domain do construct the equivalent of our inf and sup preferences in Definitions 4 and 5. In the current setting, preference rationality imposes the indifference contours at consumption and reference not to cross. This restricts preferences in a way that does not allow us to borrow these results, neither to immediately construct evaluating preferences. The rescaling axioms, however, help us circumvent the domain restriction and characterize well-being measures which find strong normative support in our Supremum and Infimum Contour axioms. The following section illustrates our main results.

Table 1: Summary of the results

Domain	$\mathcal{R}^c$			$\mathcal{R}$		
	W_k	W_{max}	W_{min}	W_y	W_{inf}	W_{sup}
Fx				+		
Cb		+		+		
Ca			+	+		
Iy	+	+	+		+	+
Sup(x|y)	+	+		+		
Sup(y|x)	+		+	+		
Inf(x|y)	+		+		+	+
Inf(y|x)	+	+			+	+

Note: the positive (resp. absence of positive) sign indicates that the measure satisfies (resp. violates) the axiom.

5 Combining the axioms

We can deduce from Fleurbaey & Maniquet (2017) that Supremum and Infimun Contour at Consumption (resp. Reference) are incompatible with each other in the general domain $\mathcal{R}$: we cannot simultaneously impose a lower and an upper bound at both supremum and infimum of two indifference contours at consumption (resp. reference) for a fixed indifference contour at reference (resp. consumption). Moreover, Supremum Contour at Consumption (resp. Reference) and Infimum Contour ar Reference (resp. Consumption) are incompatible in $\mathcal{R}$.[8] However, when the preferences domain is restricted to $\mathcal{R}^c$, these impossibilities are resolved. The measures we introduced in Section 3 all satisfy one possible combination of axioms, in the two domains we are considering. This is summarized in Table 1 in which (+) means that the column measure satisfies the row axiom. For the sake of completeness, we have added Deaton's distance function, which does not satisfy Reference Indifference but does satisfy the two Supremum Contour axioms. The proofs for each of these results follow almost directly from the axioms. To maintain a brief exposition, we leave them to the reader.

The following theorems prove that these well-being measures are also the *only* ones that satisfy each possible combination of axioms. All proofs can be found in the following section.

Theorem 3. *Let W be a well-being measure on $\mathcal{R}^c$ satisfying Reference Independence (Iy). W satisfies any three of Supremum Contour at Consumption (Sup($x|y$)), Supremum Contour at Reference (Sup($y|x$)), Infimum Contour at Con-*

[8]To maintain a short exposition, we omit the proof of this claim, which is available upon request to the authors.

sumption (Inf(x|y)) and Infimum Contour at Reference (Inf(x|y)), only if there exists $k \in \{1, \ldots, K\}$ *such that* $W = W_k$.

The next theorems characterize the two well-being measures in Definition 2 and 3.

Theorem 4. *Let* W *be a well-being measure on* $\mathcal{R}^c$ *satisfying Reference Independence (Iy) and Convergence from below (Cb).*

If W *satisfies either Supremum Contour at Consumption (Sup(x|y)) or Infimum Contour at Reference (Inf(y|x)), then for all* $x, y \in \mathcal{X}$ *and* $R \in \mathcal{R}^c$ *such that* yRx, $W(x, R, y) = W_{max}(x, R, y)$.

If, in addition, W *satisfies either Rescaling Across Consumption (RAx) or Rescaling Across Reference (RAy), then* $W = W_{max}$.

The previous theorem, besides characterizing W_{max}, provides a powerful result for contexts, like poverty measurement, in which the reference consumption is above the current one. More precisely, because of the typical focus axiom, in the context of poverty measurement, the well-being of any situation in which the current consumption is preferred to the poverty line is normalized to zero. Therefore, the only situations for which one needs to compare individual well-beings are those in which the reference consumption is preferred to the current one. In such a framework one of the two contour axioms is sufficient to characterize this measure which, following Theorem 1, coincides with the smallest poverty gap.

Theorem 5. *Let* W *be a well-being measure on* $\mathcal{R}^c$ *satisfying Reference Independence (Iy) and Convergence from above (Ca).*

If W *satisfies either Supremum Contour at Reference (Sup(y|x)) or Infimum Contour at Consumption (Inf(x|y)), then for all* $x, y \in \mathcal{X}$ *and* $R \in \mathcal{R}^c$ *such that* xRy, $W(x, R, y) = W_{min}(x, R, y)$.

If, in addition, W *satisfies either Rescaling Across Consumption (RAx) or Rescaling Across Reference (RAy), then* $W = W_{min}$.

This results is the dual of Theorem 4 and is particularly useful in frameworks, like the critical level utilitarianism (see Blackorby *et al.*, 1997, for example), in which comparisons are restricted to situations in which current consumption is preferred to the reference one.

The next two theorems complete our characterizations.

Theorem 6. *Let* W *be a well-being measure on* $\mathcal{R}$ *satisfying Reference Independence (Iy) and either Rescaling Across Consumption (RAx) or Rescaling Across Reference (RAy).* W *satisfies Infimum Contour at Consumption (Inf(x|y)) and Infimum Contour at Reference (Inf(x|y)), only if there exists* $R^i \in \mathcal{R}^i \cap \mathcal{R}^H$ *such that* $W = W_{inf}^{R^i}$.

Theorem 7. *Let W be a well-being measure on $\mathcal{R}$ satisfying Reference Independence (Iy) and either Rescaling Across Consumption (RAx) or Rescaling Across Reference (RAy). W satisfies Supremum Contour at Consumption (Sup(x|y)) and Supremum Contour at Reference (Sup(y|x)) only if there exists $R^s \in \mathcal{R}^s \cap \mathcal{R}^H$ such that $W = W_{sup}^{R^s}$.*

We have thus explored all the possible combinations of the four contour axioms, and identified sufficient restrictions on preferences to escape their incompatibilities in the general domain.

Next section provides the proof of the theorems above.

6 Proofs

When a well-being measure satisfies Reference Indifference, using Lemma 1, the relevant information about individual situations are a pair of non intersecting indifference contour sets. In this sense, $W(x, R, y)$ can be written as $W(I(x, R), I(y, R))$. Therefore, abusing notation, we will sometimes let W be a function of two indifference curves belonging to the set $\mathcal{I}$ of all admissible indifference contours. For $J \in \mathcal{I}$, $R \in \mathcal{R}$, we write $J \in R$ to denote that for all $x, x' \in J$, x is indifferent to x' according to R. For $I, J \in \mathcal{I}$, $\lambda > 0$, we write $I = \lambda J$ if I is the homothetic transformation of J by a factor λ. We also write $I \asymp_B J$ (resp., $I \asymp^A J$) if I is tangent to J from below (resp., above). Finally, we let $U(I)$ (resp. $L(I)$) denote the upper (resp. lower) contour of all points on I.

The proofs below use the following Reference Monotonicity axiom, which is the proper adaptation of Consumption Monotonicity: if the indifference contours at consumption are identical in two situations and the indifference contour at one reference is everywhere higher than in the other situation (in the sense that the lower contour of the latter does not intersect with the upper contour of the former), then well-being is lower.

Reference Monotonicity (My) - For all $x, y, y' \in \mathcal{X}$ and $R, R' \in \mathcal{R}$, if $I(x, R) = I(x, R')$ and $L(y, R) \cap U(y', R') = \emptyset$, then $W(x, R, y) > W(x, R', y')$.

This axiom turns out to be a consequence of Continuity, Consumption Monotonicity, Reference Dominance and Reference Indifference, as proven in the following lemma.

Lemma 3. *If $W : \mathcal{X} \times \mathcal{R} \times \mathcal{X}_+ \to \mathbb{R}$ satisfies Continuity (Cont), Consumption Monotonicity (Mx), Reference Dominance (Dy) and Reference Indifference (Iy), then it satisfies Reference Monotonicity (My).*

Proof. By Lemma 1, W satisfies UCIxy. Let $x, y, y' \in \mathcal{X}$ and $R, R' \in \mathcal{R}$ be such that $I(x, R) = I(x, R')$ and $L(y, R) \cap U(y', R') = \emptyset$. Let $y'' \in \mathcal{X}$ be such that $y'' \gg y$ and $y'' I' y'$. Let $R'' \in \mathcal{R}$ be such that $I(x, R'') = I(x, R)$, $I(y, R'') = I(y, R)$ and $I(y', R'') = I(y', R')$. By UCIxy, $W(x, R'', y) = W(x, R, y)$. By Dy, $W(x, R'', y) > W(x, R'', y'')$. By Iy, $W(x, R'', y') = W(x, R'', y'')$. By UCIxy, $W(x, R'', y') = W(x, R', y')$. Gathering these (in)equalities, we get $W(x, R, y) > W(x, R', y')$, the desired outcome. □

For future reference, let us formalize the stronger versions of our rescaling axioms.

Rescaling Across Consumption (RAx) - Let $x \in \mathcal{X}$, $R, R' \in \mathcal{R}$, $\alpha, \lambda \in \mathbb{R}_{++}$, if $I(\alpha x, R) = I(\alpha x, R')$ and $I(\lambda x, R)$ and $I(\lambda x, R')$ are homothetic to $I(\alpha x, R)$, then

$$W(x, R, \alpha x) = W(x, R', \alpha x) \Leftrightarrow W(x, R, \lambda x) = W(x, R', \lambda x).$$

Rescaling Across Reference (RAy) - Let $y \in \mathcal{X}$, $R, R' \in \mathcal{R}$, $\alpha, \lambda \in \mathbb{R}_{++}$, if $I(\alpha y, R) = I(\alpha y, R')$ and $I(\lambda y, R)$ and $I(\lambda y, R')$ are homothetic to $I(\alpha y, R)$, then

$$W(\alpha y, R, y) = W(\alpha y, R', y) \Leftrightarrow W(\lambda y, R, y) = W(\lambda y, R', y).$$

It is useful to recall the following result, stated without proof.

Lemma 4. *W satisfies Rescaling Across Consumption (RAx) and Consumption Rescaling (Rx) if and only if it satisfies Reference Rescaling (Ry) and Rescaling Across Reference (RAy)*

The following lemma, stated without proof, is a simple yet important consequence of these rescaling axioms. Considering, for example, Rescaling Across Consumption, this axiom imposes consistency when rescaling the same reference indifference contour of two agents with (possibly) different consumption indifference contours. The following lemma shows that this consistency holds also if we rescale (possibly) different consumption indifference curves of individuals with the same reference one.

Lemma 5. *Let W be a well-being measure satisfying either Rescaling Across Consumption (RAx) or Rescaling Across Reference (RAy). Then, for all $X, X', Y, Y' \in \mathcal{I}$, all $\lambda > 0$,*

1. $W(X, Y) = W(X, Y') \Rightarrow W(X, \lambda Y) = W(X, \lambda Y')$,
2. $W(X, Y) = W(X', Y) \Rightarrow W(\lambda X, Y) = W(\lambda X', Y)$.

If W does not satisfy Rescaling Across Consumption nor Rescaling Across Reference, then we have a weaker version of Lemma 5, in which λ must be such that the relative order between current and reference indifference contours is preserved. This is Lemma 10 in Appendix A.

To maintain a compact exposition, section 6.1 contains the proofs of the results for the domain of preferences with compact indifference contour sets. Section 6.2 focuses on the general domain.

6.1 Compact indifference contours

The following two lemmas formalize the consequence of the Supremum and Infimum Contour axioms for the measurement of well-being in $\mathcal{R}^c$. In words, for a fixed consumption (resp. reference) indifference contour, our axioms allow us to define nested subsets of $\mathcal{X}$ which evaluate the reference (resp. consumption) indifference contours they contain. Those subsets have different shapes, depending on whether they are in the upper or the lower contour of the fixed consumption (resp. reference) and may not cover the whole consumption set $\mathcal{X}$.

Lemma 6. *Let $W : \mathcal{X} \times \mathcal{R}^c \times \mathcal{X}_+ \to \mathbb{R}_+$ be a well-being measure satisfying Reference Indifference (Iy).*

(a) If W satisfies Supremum Contour at Consumption (Sup(x|y)), then for all $Y \in \mathcal{I}^c$, $w \in \mathbb{R}_+$, $w \neq 1$, there exists $L_w^Y \subset \mathcal{X}$ such that for all $X \in \mathcal{I}^c$ such that $X \cap Y = \emptyset$, $W(X,Y) \leq w \Leftrightarrow L(X) \subseteq L_w^Y$.

(b) If W satisfies Supremum Contour at Reference (Sup(y|x)), then for all $X \in \mathcal{I}^c$, $w \in \mathbb{R}_+$, $w \neq 1$, there exists $L_w^X \subset \mathcal{X}$ such that for all $Y \in \mathcal{I}^c$ such that $Y \cap X = \emptyset$,$W(X,Y) \geq w \Leftrightarrow L(Y) \subseteq L_w^X$.

(c) If W satisfies Infimum Contour at Consumption (Inf(x|y)), then for all $Y \in \mathcal{I}^c$, $w \in \mathbb{R}_+$, $w \neq 1$, there exists a closed and convex $U_w^Y \subset \mathcal{X}$ such that for all $X \in \mathcal{I}^c$ such that $X \cap Y = \emptyset$,$W(X,Y) \geq w \Leftrightarrow U(X) \subseteq U_w^Y$.

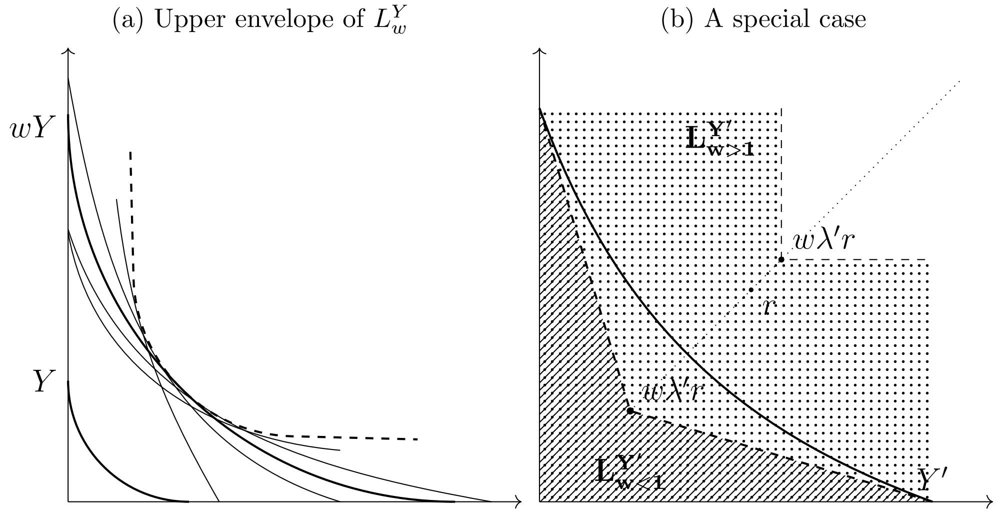
(d) If W satisfies Infimum Contour ar Reference (Inf(y|x)), then for all $X \in \mathcal{I}^c$, $w \in \mathbb{R}_+$, $w \neq 1$, there exists a closed and convex $U_w^X \subset \mathcal{X}$ such that for all $Y \in \mathcal{I}^c$ such that $Y \cap X = \emptyset$, $W(X,Y) \leq w \Leftrightarrow U(Y) \subseteq U_w^X$.

Proof of (a). Let $Y \in \mathcal{I}^c$ and $w \in \mathbb{R}_+$, $w \neq 1$. Let $L_w^Y \subset \mathcal{X}$ be defined by

$$L_w^Y = \bigcup_{X \in \mathcal{I}^c, W(X,Y) \leq w} L(X).$$

See Figure 7a for an illustration, in which the dashed line is an example of the upper envelope of L_w^Y. Observe that the constraint $X \in \mathcal{I}^c$ implies that we have

Figure 7: Union of lower contours.



a restricted set of X's, with the consequence that L_w^Y is not necessarily compact.[9] For instance, let $r \in \mathcal{X}$, and let W be defined by

$$W(X,Y) = \frac{\max\{\lambda|\lambda r \in L(X)\}}{\max\{\lambda|\lambda r \in L(Y)\}}.$$

Let $Y' \in \mathcal{I}^c$ be such that $\max\{\lambda|\lambda r \in L(Y')\} = \lambda'$ and $(y'_k)_{k\in\{1,\ldots,K\}}$ are the intersections between Y' and the K axes. Then, for $w < 1$, $L_w^{Y'} = \mathcal{X} \setminus ch\{(y'_k), w\lambda' r\} \cup \{w\lambda' r\}$, which is bounded but not closed. For $w > 1$, $L_w^{Y'} = \mathcal{X} \setminus \{y|y \gg w\lambda' r\}$, which is closed but not bounded (see Figure 7b for an example). This prevents us from following the proof of Fleurbaey & Maniquet (2017).

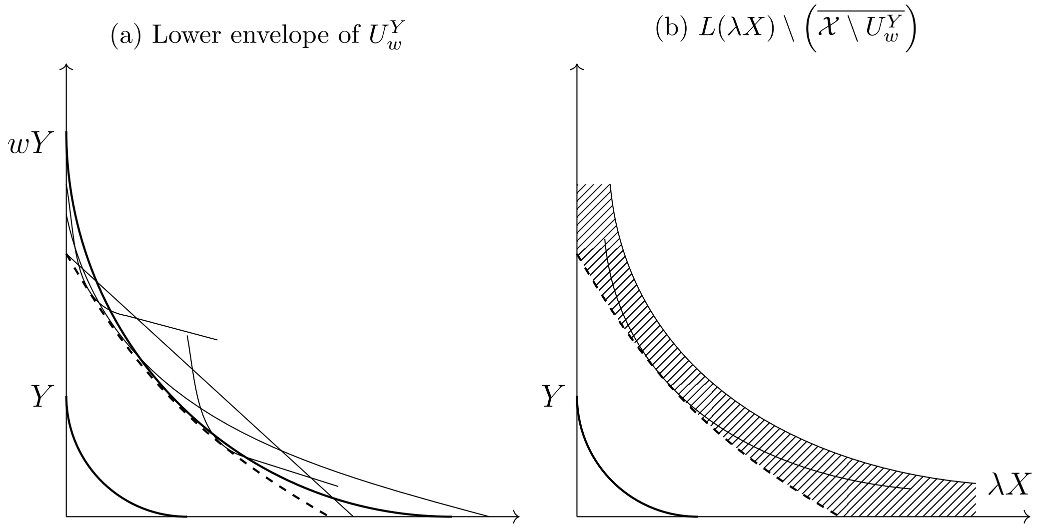
By construction, for all $X' \in \mathcal{I}^c$, $W(X',Y) \leq w \Rightarrow L(X') \subseteq L_w^Y$. Let us assume $L(X') \subseteq L_w^Y$ for some $X' \in \mathcal{I}$. Assume, by way of contradiction, that $W(X',Y) > w$. Then, by continuity of W with respect to consumption, there exists $\lambda < 1$ such that $W(\lambda X',Y) > w$. Observe that, since $L(\lambda X')$ is compact and $L(\lambda X') \subset L(X')$, its frontier is bounded away from the frontier of L_w^Y. As a result,

$$L(\lambda X') \subset \bigcup_{X\in\mathcal{I}^c, W(X,Y)\leq w} Int(L(X)),$$

that is $Int(L(X))$, for all $X \in \mathcal{I}^c$ such that $W(X,Y) \leq w$ are an open cover of $L(\lambda X')$, which is compact. Therefore, there exists a finite sequence $X^1, X^2, \ldots, X^N$,

[9] In words, since preferences must be rational, X cannot intersect Y.

Figure 8: Union of lower contours.



such that $W(X^n, Y) \leq w$ for all $n \in \{1, \dots, N\}$, and

$$L(\lambda X') \subset \bigcup_{n\in\{1,\dots,N\}} Int(L(X^n)), \tag{4}$$

that is $Int(L(X^n))$ are an open subcover of $L(\lambda X')$. Let $X^* = \vee_{n\in\{1,\dots,N\}} X^n$.[10] By Sup(x|y), $W(X^*, Y) \leq w$. By (4), $L(\lambda X') \subset L(X^*)$, which, together with $W(\lambda X', Y) > w$, contradicts Mx. □

Proof of (c). Let $Y \in \mathcal{I}^c$ and $w \in \mathbb{R}$, $w \neq 1$. Let $U_w^Y \subset \mathcal{X}$ be defined by

$$U_w^Y = \bigcup_{X\in\mathcal{I}^c, W(X,Y)\geq w} U(X).$$

Note that, by H, $W(wY, Y) = w$, so that $U(wY) \subseteq U_w^Y$. Therefore, $\mathcal{X} \setminus U_w^Y$ is bounded. Also note that U_w^Y is the closure of $\bigcup_{X\in\mathcal{I}^c, W(X,Y)>w} U(X)$. Therefore, U_w^Y is closed.

Assume it is not convex. Then there exist $x, x' \in \mathcal{X}$ and $X^x, X^{x'} \in \mathcal{I}^c$ such that $W(X^x, Y) \geq w$, $W(X^{x'}, Y) \geq w$, $x \in X^x$, $x' \in X^{x'}$ and $\lambda x + (1-\lambda)x' \notin U_w^Y$. Let $X^{xx'} \in \mathcal{I}^c$ be defined by $U(X^{xx'}) = ch\left(U(X^x) \cup U(X^{x'})\right)$. By construction, $\lambda x + (1-\lambda)x' \in U(X^{xx'})$. By Inf(x|y), $W(X^{xx'}, Y) = \min\{W(X^x, Y), W(X^{x'}, Y)\} \geq w$, so that $U(X^{xx'}) \subseteq U_w^Y$, a contradiction proving that U_w^Y is convex.

[10]We refer here to the standard convention of denoting $\vee$ the join or supremum.

We need to prove that $W(X,Y) \geq w \Leftrightarrow U(X) \subseteq U_w^Y$. By construction, for all $X' \in \mathcal{I}$, $W(X',Y) \geq w \Rightarrow U(X') \subseteq U_w^Y$. Let us prove the converse relationship. Let us assume $U(X') \subseteq U_w^Y$ for some $X' \in \mathcal{I}$. Assume, by way of contradiction, that

$$W(X',Y) < w. \tag{5}$$

Then, by continuity of W with respect to consumption, there exists $\lambda > 1$ such that $W(\lambda X', Y) < w$. Observe that, as also shown by the dashed line in Figure 8b, the lower frontier of $U(\lambda X')$ is bounded away from the lower frontier of U_w^Y (that is $U(\lambda X') \cap \overline{\mathcal{X} \setminus U_w^Y} = \emptyset$, where $\overline{A}$ stands for the closure of A). As a result, there exists a finite sequence $X^1, X^2, \ldots, X^N$, such that $W(X^n, Y) \geq w$ for all $n \in \{1, \ldots, N\}$, and

$$U(\lambda X') \subset int\left(ch\left(\bigcup_{n \in \{1,\ldots,N\}} U(X^n)\right)\right). \tag{6}$$

Let $X^* = \wedge_{n\in\{1,\ldots,N\}} X^n$. By Inf(x|y), $W(X^*, Y) \geq w$. By (6), $U(\lambda X') \subset U(X^*)$, which, together with $W(\lambda X', Y) < w$, contradicts Mx. □

Proof of (b) and (d). The results are obtained permuting X and Y in the previous proofs. □

We are now endowed with the necessary notions to prove Theorems 3 to 5.

Theorem 3

Proof. Let $k \in \{1, \ldots, K\}$. In the terminology of lemma 6, W_k satisfies Sup(x|y) with

$$L_w^Y = \{x \in \mathcal{X} | x_k \leq wA^k(Y)\}$$

such that $W(X,Y) \leq w$ if and only if $\{L(X) \cap A^k\} \subseteq L_w^Y$. Similarly, we can define

$$\begin{aligned} L_w^X &= \{x \in \mathcal{X} | x_k \leq w^{-1}A^k(X)\} \\ U_w^Y &= \mathcal{X} \setminus \{x \in A^k | x_k < wA^k(Y)\} \\ U_w^X &= \mathcal{X} \setminus \{x \in A^k | x_k < w^{-1}A^k(X)\} \end{aligned}$$

to show that W_k satisfies the other three axioms.

To prove the converse, let us assume that W satisfies Sup(x|y) and Inf(x|y). A similar proof holds if we start from Sup(y|x) and Inf(y|x). First, observe that if, for some $Y \in \mathcal{I}^c$ and some $w \in \mathbb{R}_+$, $L_w^Y = \{x \in \mathcal{X} | x_k \leq wA^k(Y)\}$, then $U_w^Y = \{x \in A^k | x_k \geq wA^k(Y)\}$, otherwise we have two competing ways of measuring $W(X,Y)$.

Claim 1: Let $Y \in \mathcal{I}$. There exists $k \in \{1, \ldots, K\}$ such that for all $X, X' \in \mathcal{I}$, if $W(X, Y) = W(X', Y)$, then $A^k(X) = A^k(X')$.

Let $\mathcal{K}^0 \subseteq \{1, \ldots, K\}$ be defined by: $k \in \mathcal{K}^0$ if and only if there exist $X, X' \in \mathcal{I}$ such that

$$W(X, Y) = W(X', Y) \text{ and } (1)\ A^k(X) = A^k(X'), (2)\ A^{k'}(X) \neq A^{k'}(X') \forall k' \neq k.$$

We need to prove that $|\mathcal{K}^0| = 1$. Assume $\mathcal{K}^0 = \emptyset$. Then there exist $X, X' \in \mathcal{I}$ such that $W(X, Y) = W(X', Y)$ whereas $A^k(X) \neq A^k(X')$ for all $k \in \{1, \ldots, K\}$. By Sup(x|y), $W(X \vee X', Y) = W(X, Y) = W(X', Y)$. By Inf(x|y), $W(X \wedge X', Y) = W(X, Y) = W(X', Y)$. We therefore have $W(X \vee X', Y) = W(X \wedge X', Y)$, whereas $L(X \wedge X') \cap U(X \vee X') = \emptyset$, violating Mx (see the dashed lines in Figure 9a for an example). Assume $|\mathcal{K}^0| > 1$. Let us assume, wlog, that $\mathcal{K}^0 = \{1, \ldots, K^0\}$. Then, by Rx, we can find $X^1, X'^1, \ldots, X^{K^0}, X'^{K^0} \in \mathcal{I}$ such that $W(X^1, Y) = W(X'^1, Y) = \ldots = W(X^{K^0}, Y) = W(X'^{K^0}, Y) = w$, with, for all $k \in \mathcal{K}^0$: $A^k(X^k) = A^k(X'^k)$ and $A^{k'}(X^k) \neq A^{k'}(X'^k) \forall k' \neq k$ (see Figure 9b for an example). In a similar way as above, by Sup(x|y), $W(X^1 \vee X'^1 \vee \ldots \vee X^{K^0} \vee X'^{K^0}, Y) = w$, and by Inf(x|y), $W(X^1 \wedge X'^1 \wedge \ldots \wedge X^{K^0} \wedge X'^{K^0}, Y) = w$ whereas $L(X^1 \wedge X'^1 \wedge \ldots \wedge X^{K^0} \wedge X'^{K^0}) \cap U(X^1 \vee X'^1 \vee \ldots \vee X^{K^0} \vee X'^{K^0}) = \emptyset$, violating Mx.

Claim 2: There exists $k \in \{1, \ldots, K\}$ such that for all $Y \in \mathcal{I}$, all $X, X' \in \mathcal{I}$, if $W(X, Y) = W(X', Y)$, then $A^k(X) = A^k(X')$.

Assume not. That is, assume there exists $Y, Y' \in \mathcal{I}$ and $k, k' \in \{1, \ldots, K\}$ such that claim 1 holds with k for Y and with k' for Y'. By Rx and Ry, we know that Y and Y' cannot be homothetic to each other, because if they were homothetic to each other, then L_w^Y (resp. U_w^Y) should be homothetic to $L_w^{Y'}$ (resp. $U_w^{Y'}$) so that $k = k'$. Also, by Ry again, we may assume that $Y \cap Y' \neq \emptyset$, because otherwise we can rescale Y so that it crosses Y'. Then we can find $X, X' \in \mathcal{I}$ such that $W(X, Y) = W(X', Y') = w$ whereas $A^k(X) < A^k(X')$ and $A^{k'}(X') < A^{k'}(X')$, so that $W(X', Y) > w$ and $W(X, Y') > w$. By Sup(y|x), $W(X, Y \vee Y') = w = W(X', Y \vee Y')$, violating claim 1. If we want to use Inf(y|x) rather than Sup(y|x), then we can find $X, X' \in \mathcal{I}$ such that $W(X, Y) = W(X', Y') = w$ whereas $A^k(X) > A^k(X')$ and $A^{k'}(X') > A^{k'}(X')$, so that $W(X', Y) < w$ and $W(X, Y') < w$. By Inf(y|x),$W(X, Y \wedge Y') = w = W(X', Y \wedge Y')$, violating claim 1. □

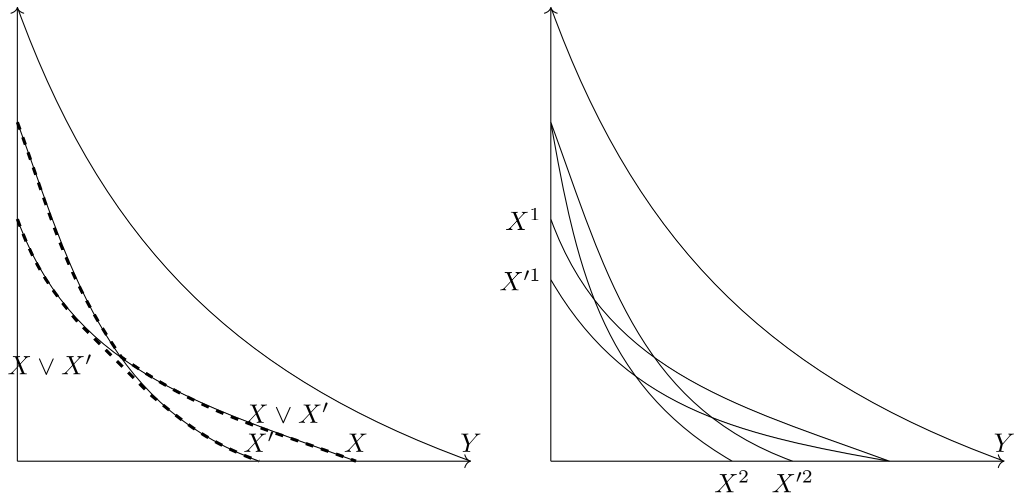
Theorem 4

Proof. It is easy to check that $W_{\max}$ satisfies the axioms. In the terminology of Lemma 6, $W_{\max}$ satisfies the axioms with $L_w^Y = L(wY)$ and $U_w^X = U(w^{-1}X)$.

Let us prove that if W satisfies Cb and Inf(y|x) then $W = W_{max}$ whenever reference is above consumption.

Figure 9: Illustration of the proof of Theorem 3

(a) Case 1: $\mathcal{K}^0 = \emptyset$ (b) Case 2: $|\mathcal{K}^0| > 1$



By Lemma 6 (d), for all $X \in \mathcal{I}^c$, $w \in \mathbb{R}_+$, $w \neq 1$, there exists closed and convex $U_w^X \subset \mathcal{X}$ such that for all $Y \in \mathcal{I}^c$ such that $X \cap Y = \emptyset$, $W(X,Y) \leq w \Leftrightarrow U(Y) \subseteq U_w^X$. Let us fix $X \in \mathcal{I}^c$.

We need to show that for all $w < 1$, $U_w^X = U(w^{-1}X)$. Assume not. Then there exists $k < 1$ such that

$$U_k^X \neq U(k^{-1}X). \tag{7}$$

Note that, by H, $W(X, k^{-1}X) = k$, so that by Lemma 6 (d), $U(k^{-1}X) \subseteq U_k^X$ and U_k^X is closed. Together with Eq. 7, this implies, using Cont, that there exists $k' > k$ such that

$$k'^{-1}X \asymp_B U_k^X. \tag{8}$$

By Rx,

$$\forall\, \alpha \in]1, k'[\colon U_{\frac{k}{\alpha}}^{\alpha^{-1}X} = U_k^X. \tag{9}$$

See Figure 10a for an example.

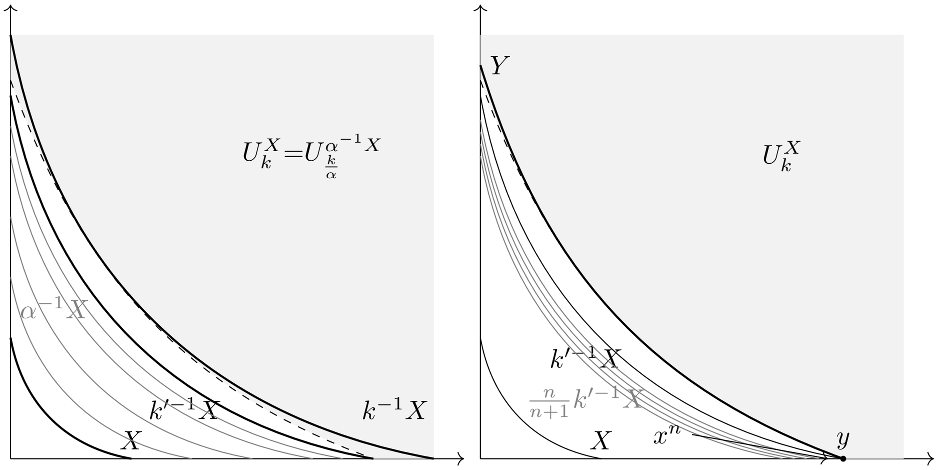
Let $y \in \mathcal{X}$ be such that $y \in L(k^{-1}X) \cap U_k^X$. There exists $Y \in \mathcal{I}^c$ such that: $U(Y) \subset U_k^X$; $U(\lambda Y) \not\subset U_k^X$, for all $\lambda < 1$; $k'^{-1}X \asymp_B Y$ and $k'^{-1}X \cap Y = \{y\}$. See Figure 10b for an example. Because $U(Y) \subset U_k^X$ and $y \in Y$, y being at the boundary of U_k^X, $W(X,Y) = k$. Therefore, by H and Rx,

$$W(\alpha^{-1}X, Y) = \frac{k}{\alpha}, \forall\, 1 < \alpha < k'. \tag{10}$$

Let sequence $(x^n, R^n)_{n \in \mathbb{N}} \in \mathcal{X} \times \mathcal{R}^c$ be defined by, for all $n \in \mathbb{N}$: $I(y, R^n) = Y$, $L(x^n, R^n) = L(\frac{n}{n+1} k'^{-1} X)$, $x^n = \frac{n}{n+1} y$.

Figure 10: Illustration of the proof of Theorem 4

(a) Eq. (9) (b) An example of Y



Note that $W(x^n, R^n, y) = W(\frac{n}{n+1}k'^{-1}X, Y)$. By Eq. 10, $W(\frac{n}{n+1}w'^{-1}X, Y) = \frac{n}{n+1}\frac{k}{k'}$. Therefore, $W(x^n, R^n, y) \to \frac{k}{k'} < 1$, violating Cb. That proves that for all $w < 1$, $U_w^X = U(w^{-1}X)$.

The proof that if W satisfies Cb and Sup(x|y) then $W = W_{max}$ whenever the reference is above current consumption is similar to the proof above, provided we replace Cb with CRa, which is possible, by Lemma 11.

Finally, to prove the second statement of the theorem, we need to we prove that the previous property holds for all $w > 1$. This amounts to prove that for all $Y, Y' \in \mathcal{I}^c$ such that $Y \asymp^A w^{-1}X$ and $Y' \asymp^A w^{-1}X$ for some $w > 1$, $W(X, Y) = W(X, Y')$. This follows immediately from the fact that it holds for $w < 1$ and W satisfies RAx. □

Theorem 5

Proof. The proof is similar to the one of Theorem 4. For the sake of completeness, we prove here here only one part of the first statement. Let W satisfy Sup$(y|x)$ and Ca. By Lemma 6 (b), for all $X \in \mathcal{I}^c$, $w \in \mathbb{R}$, $w \neq 1$, there exist $L_w^X \subset X$ such that, for all $X \in \mathcal{I}^c$ such that $X \cap Y = \emptyset$, $W(X, Y) \geq w \iff L(Y) \subseteq L_w^X$. Remember that L_w^X is not necessarily compact.

Fix $X \in \mathcal{I}^c$. Observe that the statement is proven if the following statement is true. For all $w > 1$, if $L(Y) \subseteq L_w^X$ then $L(Y) \subseteq L(w^{-1}X)$. Notice that it must be $L(w^{-1}X) \subseteq L_w^X$. Assume that our claim is wrong. Then there exist $Y \in \mathcal{I}^c$ such that $L(Y) \subseteq L_w^X$ and $int\,(U(w^{-1}X)) \cap L(Y) \neq \emptyset$.

Since $L(Y)$ is compact, there exist $w' < w$ such that $w'^{-1}X \asymp^A Y$. Let

$y \in \mathcal{X}$ be such that $y \in Y \cap w'^{-1}X$ and $(x_n, R_n)_{n\in\mathbb{N}}$ be a sequence defined by, for all $n \in \mathbb{N}$: $I(y, R_n) = Y$, $L(x_n, R_n) = L\left(\frac{n+1}{n} w'^{-1}X\right)$, $x_n = \frac{n+1}{n} y$. Notice that $W(x_n, R_n, y) = W\left(\frac{n+1}{n} w'^{-1}X, Y\right)$. By Ca, $W\left(\frac{n+1}{n} w'^{-1}X, Y\right) \to 1 = W(w'^{-1}X, w'^{-1}X)$. Therefore, by Rx, $W\left(\frac{n+1}{n} X, Y\right) \to W(X, w'^{-1}X) = w'$: a contradiction to $L(Y) \subseteq L_w^X$. $\square$

6.2 The unrestricted domain

The following Lemma builds on Lemma 6 to derive similar results as in Fleurbaey & Maniquet (2017). Recalling that Lemma 6 implies the existence of evaluating contours (subset of $\mathcal{X}$), the following lemma refines those sets in such a way that they constitute a map of nested convex upper contours, that covers $\mathcal{X}$. Such a map, induces a preference relation in $\mathcal{R}^H$; more precisely, particular instances of preferences which we call sup or inf preferences. Lemma 7 concerns homothetic preferences, but its consequence for the general domain $\mathcal{R}$ is formalized in Corollary 1. The latter states that, for any indifference curve, $I \in \mathcal{I}$ there exist two sup and two inf evaluating preferences which measure the well-being of situations in which I is either the consumption or reference indifference contour. These four evaluating preferences are in principle different from each other and depend on the contour axiom they stem from, as well as on the role played by I (consumption or reference). The proofs of Theorems 6 and 7 consist then in showing that the two sup (resp. inf) preferences must coincide for the same I but also across all the other admissible indifference contours.

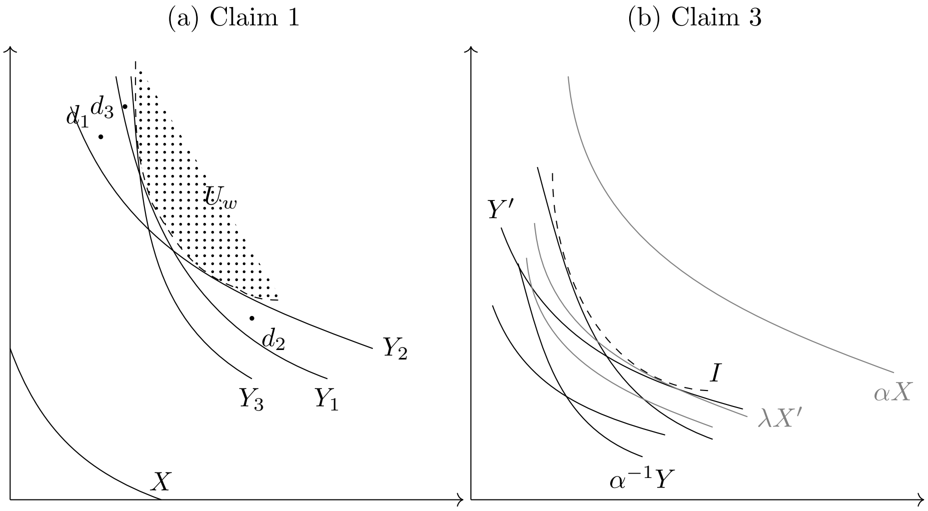
Lemma 7. *Let $W : \mathcal{X} \times \mathcal{R} \times \mathcal{X}_+ \to \mathbb{R}_+$ be a well-being measure satisfying Reference indifference (Iy) and either Rescaling Across Consumption (RAx) or Rescaling Across Reference (RAy).*

(a) If W satisfies Supremum Contour at Consumption (Sup(x|y)), then then for all $R \in \mathcal{R}^H$, there exists $R_r^s(R) \in \mathcal{R}^H$ such that for all $Y \in R$, $X \in \mathcal{I}$, $w \in \mathbb{R}_+$, $W(X, Y) \leq w \Leftrightarrow wY \asymp_B I$ and $L(X) \subseteq L(I)$ for some $I \in R_r^s(R)$.

(b) If W satisfies Supremum Contour at Reference (Sup(y|x)), then then for all $R \in \mathcal{R}^H$, there exists $R_c^s(R) \in \mathcal{R}^H$ such that for all $X \in R$, $Y \in \mathcal{I}$, $w \in \mathbb{R}_+$, $W(X, Y) \geq w \Leftrightarrow w^{-1}X \asymp_B I$ and $L(Y) \subseteq L(I)$ for some $I \in R_c^s(R)$.

(c) If W satisfies Infimum Contour at Consumption (Inf(x|y)), then for all $R \in \mathcal{R}^H$, there exists $R_r^i(R) \in \mathcal{R}^H$ such that for all $Y \in R$, $X \in \mathcal{I}$, $w \in \mathbb{R}_+$, $W(X, Y) \leq w \Leftrightarrow w^{-1}X \asymp^A I$ and $U(Y) \subseteq U(I)$

Figure 11: Illustration of the proof of Lemma 7(b)



(d) *If W satisfies Infimum Contour at Reference (Inf(y|x)), then for all $R \in \mathcal{R}^H$, there exists $R^i_c(R) \in \mathcal{R}^H$ such that for all $X \in R$, $Y \in \mathcal{I}$, $w \in \mathbb{R}_+$, $W(X,Y) \geq w \Leftrightarrow wY \succsim^A I$ and $U(X) \subseteq U(I)$*

Proof of (b). Let W be a well-being measure satisfying Iy. If it satisfies RAx then, by Lemma 4 , it also satisfies RAy and vice versa. By Lemma 1, W satisfies UCIxy, so that, by Iy, we can reduce any $W(x,R,y)$ into $W(X,Y)$ for $X = I(x,R)$ and $Y = I(y,R)$.

Let $R \in \mathcal{R}^H$ and $X \in R$. Let $0 < w < 1$. By H, $W(X, w^{-1}X) = w$. Let $\mathcal{Y}_w = \{Y \in \mathcal{I} | W(X,Y) = w\}$, and

$$U_w = \bigcap_{Y \in \mathcal{Y}_w} U(Y).$$

Because U_w is the intersection of closed and convex sets, it is closed and convex. Let Y_w be the lower envelop of U_w. Let us start by making the following claim.

Claim 1: $W(X, Y_w) = w$.

Notice that by construction, $U_w \subseteq U(w^{-1}X)$, so that $L(X) \cap U_w = \emptyset$, which means that there exists $R' \in \mathcal{R}$ such that both $X \in R'$ and $Y_w \in R'$. Therefore, $W(X, Y_w)$ is well-defined.

Let D be a countable and dense subset of $intL(Y_w)$. Let $D = \{d_n, n \in \mathbb{N}\}$. Let the sequence $Y_n \in \mathcal{Y}_w$, $n \in \mathbb{N}$, be such that $U(Y_n)$ is closed and does not contain d_n. Such a sequence exists because if some d_n belongs to all $L(Y)$, $Y \in \mathcal{Y}_w$, then $d_n \in U_w$, a contradiction. See Figure 11a for a graphical intuition.

Let the sequence $Z_n \in \mathcal{I}, n \in \mathbb{N}$, be defined by $Z_1 = Y_1$ and $U(Z_n) = U(Y_n) \cap U(Z_{n-1})$ for all $n \geq 2$. Observe that $\lim_{n\to\infty} U(Z_n) = U_w$. Indeed, if it is not the case, then $U_w \setminus \lim_{n\to\infty} U(Z_n)$ contains an open set containing some d_n, a contradiction. Because $Y_n \in \mathcal{Y}_w$, $W(X, Y_n) = w$, which implies that $W(X, Z_n)$ is well-defined. By Sup(y|x), for all $n \in \mathbb{N}$, $W(X, Z_n) = w$.

Let $T_n \in \mathcal{I}, n \in \mathbb{N}$, be defined by $L(T_n) = \frac{n}{n+1} L(Z_n)$. Again, $\lim_{n\to\infty} U(T_n) = U_w$. Therefore, there exists $\tilde{n} \geq 1$ such that for all $n \geq \tilde{n}$, $L(X) \cap U(T_n) = \emptyset$, so that $W(X, T_n)$ is well-defined. By Ry and because $W(X, Z_n) = w$, $W(X, T_n) = \frac{nw}{n+1}$ for all $n \geq \tilde{n}$.

Let $R^* \in \mathcal{R}$ be such that $X \in R^*$, $T_n \in R^*$ for all $n \geq \tilde{n}$ and $Y_w \in R^*$. Such R^* exists because $L(T_n) \cap U(T_{n+1}) = \emptyset$ and $L(T_n) \cap U_w = \emptyset$ for all $n \geq \tilde{n}$.[11] By Cont, $W(X, Y_w) = \lim_{n\to\infty} W(X, T_n) = w$, which proves the claim.

Observe that, by construction, $W(X, Y) = w$ implies $L(Y) \subseteq L(Y_w)$, and, because $W(X, Y_w) = w$, we cannot have, by Dy, $L(Y) \subsetneq intL(Y_w)$, which means that $Y \asymp_B Y_w$. Moreover, $Y \asymp_B Y_w$ implies $W(X, Y) = w$. Assume not: if $W(X, Y) > w$, then, by continuity, there exist Y' such that $L(Y) \cap U(Y') = \emptyset$ and $W(X, Y') = w$, whereas $L(Y) \cap U(Y') = \emptyset$ and $Y \asymp_B Y_w$ imply $U(Y') \not\subseteq U_w$, a contradiction. If $W(X, Y) < w$, then there exist $\alpha < 1$ such that $W(X, \alpha Y) \leq w$. However, $L(Y) \subseteq L(Y_w)$ implies $L(\alpha Y) \subset L(Y_w)$ which, by Lemma 3, implies $w = W(X, Y_w) < W(X, \alpha Y)$: contradiction.

Let $R_c^s(X) \in \mathcal{R}^H$ be defined by $Y_w \in R_c^s(X)$. Our second claim extends the previous reasoning to all $w \in \mathbb{R}_+$

Claim 2: For all $Y \in \mathcal{I}$, $w' \in \mathbb{R}_+$,

$$W(X, Y) \geq w' \Leftrightarrow \exists\ I \in R_c^s(X) \text{ s. t. } w'^{-1} X \asymp_B I \text{ and } L(Y) \subseteq L(I).$$

Part 1 of the proof of the claim, $\Leftarrow$: Assume that, contrary to the claim, there exist $Y \in \mathcal{I}$, $w' \in \mathbb{R}_+$, and $I \in R_c^s(X)$ such that $w'^{-1} X \asymp_B I$ and $L(Y) \subseteq L(I)$ whereas $W(X, Y) < w'$. By H, $W(X, w'^{-1} X) = w'$, so that $W(X, Y) < W(X, w'^{-1} X)$. Let $\alpha = \frac{w'}{w}$, where w is the number used in Claim 1 (and for which, therefore, we know the claim is true). By RAx, $W(X, \alpha^{-1} Y) < W(X, w^{-1} X) = w$. Observe that $L(Y) \subseteq L(I)$ implies $L(\alpha^{-1} Y) \subseteq L(Y_w)$ (because both I and $Y_w \in R_c^s(X)$ and $R_c^s(X) \in \mathcal{R}^H$), so that $W(X, \alpha^{-1} Y) = w$, a contradiction.

Part 2 of the proof of the claim, $\Rightarrow$: Let $Y \in \mathcal{I}$ and $w' \in \mathbb{R}_+$ be such that $W(X, Y) = w'$. By H, $W(X, w'^{-1} X) = w'$. Let $I \in R_c^s(X)$ be such that $w'^{-1} X \asymp_B I$. Assume, contrary to what needs to be proven, that $L(Y) \not\subseteq L(I)$. Let λ be defined by

$$\lambda = \max\{\lambda' > 0 | L(\lambda' Y) \subseteq L(I)\}.$$

By Part 1 of the proof, $W(X, \lambda Y) = w'$, violating Dy.

We now claim that the $R_c^s(X) \in \mathcal{R}^H$ constructed above for $X \in R$ must be the same for all other consumption indifference curves of R.

[11] Observe that, by construction, T_n is bounded away by Z_n. Therefore, while $U(Z_n)$ converges to U_w, we have that $U_w \subset intU(T_n)$.

Claim 3: For all $X' \in R$, $R^s_c(X') = R^s_c(X)$.

Let $R^s_c(X)$ be defined as in claims 1 and 2. Let $Y, Y' \in \mathcal{I}$ be such that there exists $I \in R^s_c(X)$ such that $Y \asymp_B I$ and $Y' \asymp_B I$ and $W(X', Y)$ and $W(X', Y')$ are well-defined. Claim 3 amounts to claim that $W(X', Y) = W(X', Y')$. The need to prove this claim comes from the fact that $W(X, Y)$ and $W(X, Y')$ may not be well-defined.

Let $\alpha, \lambda \in \mathbb{R}_{++}$ be defined by $X' = \alpha X$ and $\lambda X' \asymp_B I$. If $\alpha > 1$ and $\lambda > 1$ or if $\alpha < 1$ and $\lambda < 1$, then $W(X, Y)$ and $W(X, Y')$ are well-defined and the claim follows from RAy. If $\alpha > 1$ and $\lambda < 1$ as in Figure 11b, then $W(X, \alpha^{-1}Y)$ and $W(X, \alpha^{-1}Y')$ are well-defined and $\alpha^{-1}Y, \alpha^{-1}Y' \asymp_B \alpha^{-1}I$. By claim 2, $W(X, \alpha^{-1}Y) = W(X, \alpha^{-1}Y')$. By RAy, $W(X', \alpha^{-1}Y) = W(X', \alpha^{-1}Y')$. By RAx, $W(X', Y) = W(X', Y')$. If $\alpha < 1$ and $\lambda > 1$, then $W(X, \lambda Y)$ and $W(X, \lambda Y')$ are well-defined and $\lambda Y, \lambda Y' \asymp_B \lambda I$. By claim 2, $W(X, \lambda Y) = W(X, \lambda Y')$. By RAy, $W(X', \lambda Y) = W(X', \lambda Y')$. By RAx, $W(X', Y) = W(X', Y')$.

□

The proof of 7(a) is obtained in a symmetric way. The other two results follow a similar logic; for the sake of completeness we include the proof of 7(c) in Appendix B. To prove Theorems 6 and 7 we will make extensive use of the following corollary of Lemma 7.

Corollary 1. *Let $W : \mathcal{X} \times \mathcal{R} \times \mathcal{X}_+ \to \mathbb{R}_+$ be a well-being measure satisfying Reference indifference (Iy) and either Rescaling Across Consumption (RAx) or Rescaling Across Reference (RAy).*

(a) If W satisfies Supremum Contour at Consumption (Sup(x|y)), then then for all $Y \in \mathcal{I}$, there exists $R^s_r(Y) \in \mathcal{R}^H$ such that for all $\alpha > 0$, $X \in \mathcal{I}$,

$$W(X, \alpha Y) = \max_{I \in R^s_r(Y),\, U(I) \cap L(X) \neq \emptyset} W(I, \alpha Y).$$

(b) If W satisfies Supremum Contour at Reference (Sup(y|x)), then then for all $X \in \mathcal{I}$, there exists $R^s_c(X) \in \mathcal{R}^H$ such that for all $\alpha > 0$, all $Y \in \mathcal{I}$,

$$W(\alpha X, Y) = \min_{I \in R^s_c(X),\, U(I) \cap L(Y) \neq \emptyset} W(\alpha X, I).$$

(c) If W satisfies Infimum Contour at Consumption (Inf(x|y)), then for all $Y \in \mathcal{I}$, there exists $R^i_r(Y) \in \mathcal{R}^H$ such that for all $\alpha > 0$, $X \in \mathcal{I}$,

$$W(X, \alpha Y) = \min_{I \in R^r_i(Y),\, L(I) \cap U(X) \neq \emptyset} W(I, \alpha Y).$$

(d) *If W satisfies Infimum Contour at Reference (Inf(y|x)), then for all $X \in \mathcal{R}^H$, there exists $R_c^i(X) \in \mathcal{R}^H$ such that for all $\alpha > 0$, $Y \in \mathcal{I}$,*

$$W(\alpha X, Y) = \max_{I \in R_i^c(X),\, L(I) \cap U(Y) \neq \emptyset} W(\alpha X, I).$$

Proof of (b). By Lemma 7(b), we know that for all $R \in \mathcal{R}^H$, there exists $R_c^s(R) \in \mathcal{R}^H$ such that for all $X \in R$, $Y \in \mathcal{I}$, $w \in \mathbb{R}_+$,

$$W(X, Y) \geq w \Leftrightarrow w^{-1} X \asymp_B I \text{ and } L(Y) \subseteq L(I)$$

for some $I \in R_c^s(R)$. Let $X \in \mathcal{I}$. We know that there is only one $R^X \in \mathcal{R}^H$ such that $X \in R^X$. Moreover, for all $\alpha > 0$, $\alpha X \in R^X$. Therefore, we can define $R_c^s(X) = R_c^s(R^X)$. Let $Y \in \mathcal{I}$, and let $I \in R_c^s(X)$ be such that $Y \asymp_B I$. By Cont and Dy,

$$\min_{\lambda I \in R_c^s(X), U(\lambda I) \cap L(Y) \neq \emptyset} W(\alpha X, \lambda I) = W(\alpha X, I).$$

Now, let $\beta > 0$ be defined by $\beta X \asymp_B I$. By H, $W(\alpha X, \beta X) = \frac{\alpha}{\beta}$. We need to prove that $W(\alpha X, Y) = \frac{\alpha}{\beta}$. Because $Y \asymp_B I$, $L(Y) \subseteq L(I)$, by Lemma 7(b), $W(\alpha X, Y) \geq \frac{\alpha}{\beta}$. If $W(\alpha X, Y) > \frac{\alpha}{\beta}$, then, by Cont, $W(\alpha X, \lambda Y) > \frac{\alpha}{\beta}$ as well for some $\lambda > 1$, whereas $L(\lambda Y) \not\subseteq L(I)$, contradicting Lemma 7(b).

The rest of the proof - statements (a), (c) and (d) - follows the same logic. □

We are now endowed with the necessary notation and results to prove Theorems 6 and 7.

Theorem 6

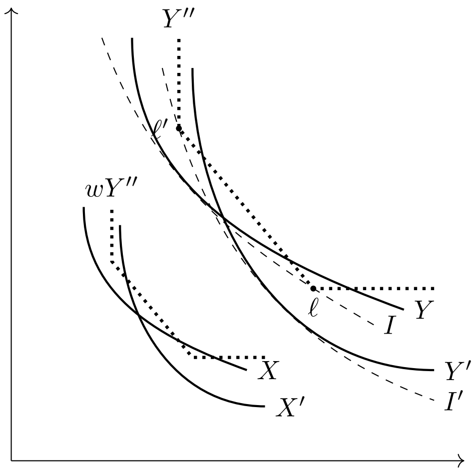
Let W satisfy Inf(x|y) and Inf(y|x). Recalling corollaries 1(c) and (d), we need to prove that for all $X, Y \in \mathcal{I}$, $R_c^i(X) = R_r^i(X) = R_c^i(Y) = R_r^i(Y)$.

We begin by proving an important consequence of Inf(x|y), Inf(y|x) which will be used several times in the proof of the Theorem.

*Property P**: Let W satisfy Inf(x|y) and Inf(y|x). Let $R, R' \in \mathcal{R}^H$. Let $X, Y \in R$, $X', Y' \in R'$ be such that $W(X, Y) = W(X', Y')$. Let $I, I' \in \mathcal{I}$ be defined by: $I \asymp_B Y$, $I \in R_c^i(X)$, and $I' \asymp_B Y'$, $I' \in R_c^i(X')$. Then, either $Y \subseteq U(I')$ or $Y' \subseteq U(I)$.

Proof of property P.* Assume not, as illustrated in Fig. 12. Then, we can find $\ell, \ell' \in \mathcal{X}$, with $\ell \not\geqslant \ell'$ and $\ell' \not\geqslant \ell$, such that $\ell \in I \cap int(U(Y'))$ and $\ell' \in I' \cap int(U(Y))$, where the int operator refers to the interior of its argument. Let $Y'' \in \mathcal{I}$ be defined by $U(Y'') = ch(\{x \in X | x \geqslant \ell\}, \{x \in X | x \geqslant \ell'\})$. As a consequence, Y''

Figure 12: Proof of property P*



is such that: *(i)* $I \asymp_B Y''$,*(ii)* $I' \asymp_B Y''$, and*(iii)* $Y'' \subset int\,[ch\,(U(Y) \cup U(Y'))]$. Let w be defined by $W(X,Y) = W(X',Y') = w$. By *(i)* and *(ii)*, $W(X,Y'') = W(X',Y'') = w$. By H, $W(wY'',Y'') = w$. Observe that *(iii)* is equivalent to: *(iv)* $wY'' \subset int\,[ch\,(U(X) \cup U(X'))]$.

Since $W(X,Y'') = W(X',Y'') = W(wY'',Y'')$, by Inf(x|y), there exist $J \in R_r^i(Y'')$ such that $J \asymp_B X, X', wY''$, which is impossible, given *(iv)* and convexity of preferences.

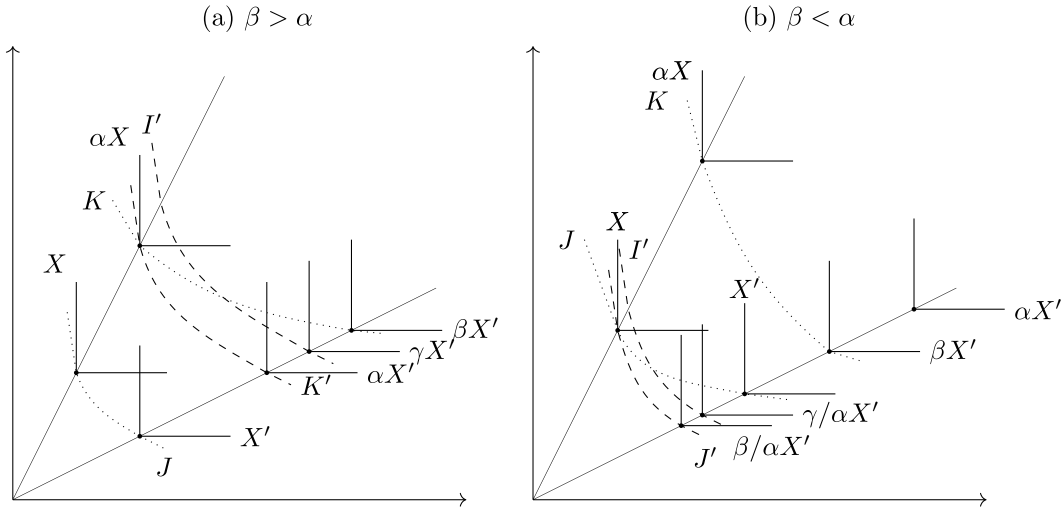
□

In the same way, we can prove the dual of the previous property.

*Property P***: Let W satisfy Inf(x|y) and Inf(y|x). Let $R, R' \in \mathcal{R}^H$. Let $X, Y \in R$, $X', Y' \in R'$ be such that $W(X,Y) = W(X',Y')$. Let $I, I' \in \mathcal{I}$ be defined by: $I \asymp_B X$, $I \in R_r^i(Y)$, and $I' \asymp_B X'$, $I' \in R_r^i(Y')$. Then, either $X \subseteq U(I')$ or $X' \subseteq U(I)$.

The proof is divided in several steps. (1) We start by showing that, independently whether it is a consumption or reference one, to a Leontief indifference contour we should assign the same inf preference. (2) We then extend this to all homothetic preferences. (3) We show that inf preference associated to different homothetic preferences must be *similar* to each other. (4) There exist a non-empty set of bundles in $\mathcal{X}$ such that, in their neighbourhoods, the inf preferences indifference contours associated to different Leontief preferences cannot cross. (5) The set in the previous step is $\mathcal{X}$ so that, to all Leontief preferences, we must assign the same inf preference. (6) We conclude that one should assign the same inf preference to all preferences in $\mathcal{R}$.

Figure 13: Step 1 of the proof of Theorem 6



Step 1

Let us denote $\mathcal{R}^L$ the set of all Leontief preferences.[12]

Claim. For all $R \in \mathcal{R}^L$ and all $X \in R$, $R^i_c(X) = R^i_r(X)$.

Proof. Assume not. The proof is illustrated in Figures 13a and 13b. Let $R \in \mathcal{R}^L$ and $X \in R$ be such that $R^i_c(X) \neq R^i_r(X)$. Let $\alpha > 1$. Using Lemma 7(c), we get $R^i_r(X) = R^i_r(\alpha X)$. Therefore, $R^i_c(X) \neq R^i_r(\alpha X)$. Let $J \in R^i_r(\alpha X)$ be such that $J \asymp_B X$. Let $K \in R^i_c(X)$ be such that $K \asymp_B \alpha X$. Because $R^i_c(X) \neq R^i_r(\alpha X)$, J is not homothetic to K. Therefore, there exists $x' \in \mathcal{X}$ and $\beta \neq \alpha$ such that $x' \in J$, $\beta x' \in K$. Let $R' \in \mathcal{R}^L$ be such that there exists $X' \in R'$ that has its kink at x'. Clearly we also have $\beta X' \in R'$ with $\beta X'$ having its kink at $\beta x'$. Observe that $J \asymp_B X'$ and $K \asymp_B \beta X'$, and:

$$\begin{aligned} J \asymp_B X' \quad &\Rightarrow \quad W(X, \alpha X) = W(X', \alpha X) && (11) \\ K \asymp_B \beta X' \quad &\Rightarrow \quad W(X, \alpha X) = W(X, \beta X') = \alpha^{-1}, && (12) \end{aligned}$$

the last equality being obtained by H.

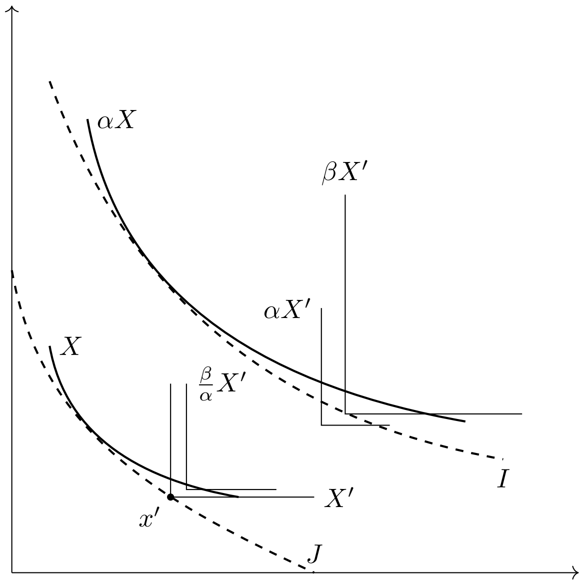
By H again, $W(X, \alpha X) = W(X', \alpha X') = \alpha^{-1}$. Therefore, $W(X', \alpha X) = W(X', \alpha X')$, so that there exists $K' \in R^i_c(X')$ such that $K' \asymp_B \alpha X, \alpha X'$. Moreover, by H, $W(\frac{\beta}{\alpha} X', \beta X') = \alpha^{-1} = W(X, \beta X')$, so that here exists $J' \in R^i_r(\beta X')$ such that $J' \asymp_B X, \frac{\beta}{\alpha} X'$.

[12]Preferences R^ℓ are Leontief if there exist $\ell \in \Delta^{K-1}$ (where Δ^{K-1} is the $K-1$ dimensional simplex) such that $xR^\ell x'$ if and only if $\min_k \frac{x_k}{\ell_k} \geq \min_k \frac{x'_k}{\ell_k}$.

Let us assume $\beta > \alpha$. This is illustrated in Fig. 13a. Take γ such that $\alpha < \gamma < \beta$. As before, $R_c^i(X') = R_c^i(\frac{\gamma}{\alpha}X')$. Moreover, by H, $W(X, \alpha X) = W(\frac{\gamma}{\alpha}X', \gamma X')$. Now, let $I \in R_c^i(X)$ and $I' \in R_c^i(\frac{\gamma}{\alpha}X')$ be defined by $I \asymp_B \alpha X$ and $I' \asymp_B \gamma X'$. Observe that we must have $I = K$, while I' should be homothetic to K' and nowhere below it. Since $\gamma < \beta$, $\gamma X' \nsubseteq U(K)$, while $\alpha < \gamma$ implies $\alpha X \nsubseteq U(I')$. This is a contradiction to P*. For the case of $\beta < \alpha$ (Fig. 13b), taking γ such that $\beta < \gamma < \alpha$, we can use a similar reasoning to obtain $\gamma/\alpha X' \nsubseteq U(J)$ and $X \nsubseteq U(I')$: a contradiction to P**. □

<u>Step 2</u>
Claim: For all $R \in \mathcal{R}^H$ and all $X \in R$, $R_c^i(X) = R_r^i(X)$.

Figure 14: Step 2 of the Proof of Theorem 6.



Proof. Assume not. Then, there exist $R \in \mathcal{R}^H$, $X \in R$, $\alpha > 1$, $I \in R_c^i(X)$, $J \in R_r^i(\alpha X)$ such that $I \asymp_B \alpha X$ and $J \asymp_B X$ whereas I is not homothetic to J. Therefore, there exist $x' \in \mathcal{X}$ and $\beta \neq \alpha$ such that $x' \in J$, $\beta x' \in I$. Let $R' \in \mathcal{R}^L$ be such that there exists $X' \in R'$ that has its kink at x'. Clearly, we also have $\beta X' \in R'$ and $\beta X'$ has its kink at $\beta x'$. Observe that $J \asymp_B X'$ and $I \asymp_B \beta X'$. By H,

$$W(X, \alpha X) = W(X', \alpha X') = \alpha^{-1}, \tag{13}$$

and

$$W(X', \alpha X') = W(\frac{\beta}{\alpha}X', \beta X'). \tag{14}$$

Fig. 14 shows an example for the case of $\beta > \alpha$. By construction, we have

$$J \asymp_B X' \;\Rightarrow\; W(X, \alpha X) = W(X', \alpha X) \tag{15}$$

$$I \asymp_B \beta X' \;\Rightarrow\; W(X, \alpha X) = W(X, \beta X'). \tag{16}$$

Combining eq.(13) and eq.(15), we get $W(X', \alpha X') = W(X', \alpha X)$, so that there exists $I' \in R^i_c(X')$ such that $I' \asymp_B \alpha X, \alpha X'$. Similarly, from eq.(13), eq.(14) and eq.(16) we get $W(\frac{\beta}{\alpha}X', \beta X') = W(X, \beta X')$, so that there exists $J' \in R^i_r(\beta X')$ such that $J' \asymp_B X, \frac{\beta}{\alpha}X'$. Since $\alpha \neq \beta$, I' is not homothetic to J', so that $R^i_c(X') \neq R^i_r(\beta X')$. A contradiction to Step 1.

□

Gathering the results, we know that, for homothetic preferences, the inf preferences associated to each indifference curve, whenever it contains the consumption or the reference bundle, are all the same. Therefore, we can mention the dependence of best preferences on homothetic preferences as follows: for all $R \in \mathcal{R}^H$, there exists $R^i(R) \in \mathcal{R}^i \cap \mathcal{R}^H$ such that for all $X, Y \in R$ and $I, J \in R^i(R)$, for all $Z \in \mathcal{I}$, if $I \asymp_B Z$ (resp. $J \asymp_B Z$), then $W(X, Z) = W(X, I)$ (resp. $W(Z, Y) = W(J, Y)$).

We can also merge P * and P** into the following.

*Property P ****: Let W satisfy Inf(x|y) and Inf(y|x). Let $R, R' \in \mathcal{R}^H$. Let $I \in R$, $I' \in R'$. Let $J \in R^i(R)$ and $J' \in R^i(R')$ be such that $J \asymp_B I$ and $J' \asymp_B I'$. Then, either $I \subseteq U(J')$ or $I' \subseteq U(J)$.

The next step of the proof shows that inf preferences of two different homothetic preferences cannot be too different from each other in the following sense. If two indifference curves of these homothetic preferences are tangent to the same inf preference indifference curve of one preference then they are tangent to the same inf indifference curve of the other preference as well.

Step 3

Claim: For all $R, R' \in \mathcal{R}^H$, all $X \in R$ and $X' \in R'$, if there exists $I \in R^i(R)$ such that $I \asymp_B X, X'$, then there exists $I' \in R^i(R')$ such that $I' \asymp_B X, X'$.

Proof. Let R, R', X, X', I satisfy the properties of the claim and take $\alpha > 0$. Observe that $I \asymp_B X, X'$ implies $W(\alpha X, X) = W(\alpha X, X')$. By H, $W(\alpha X, X) = W(\alpha X', X')$, so that $W(\alpha X, X') = W(\alpha X', X')$. Consequently, by Inf (x|y), there exists $I' \in R^i(R')$ such that $I' \asymp_B \alpha X, \alpha X'$. This, given that $R^i(R') \in \mathcal{R}^H$, proves the claim. □

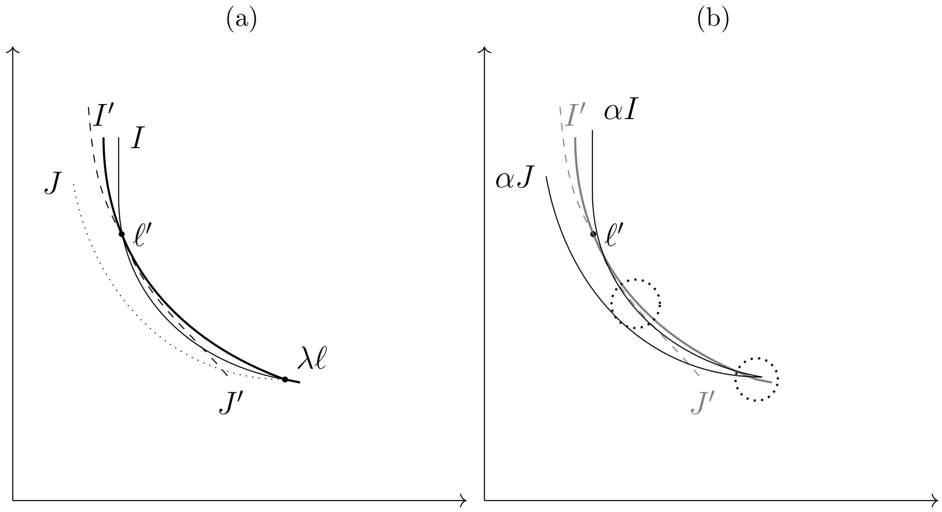
In the following two steps, we prove that all Leontief preferences have the same inf preferences. The proof technique is somehow different. We heavily rely on continuity and convexity of the preferences so that inf preference indifference curves are continuous and differentiable almost everywhere.

Step 4

Claim: Let $R, R' \in \mathcal{R}^L$ have kinks occurring at bundles proportional to ℓ and ℓ' respectively. Let $I \in R^i(R)$ and $I' \in R^i(R')$ be defined by $\ell' \in I \cap I'$. If I is differentiable at ℓ', then $U(I') \subseteq U(I)$ in a neighbourhood of ℓ'.

Proof. Let $R, R', \ell, \ell', I, I'$ satisfy the above conditions. Let $\lambda > 0$ be defined by $\lambda\ell \in I$. By Step 3, $\lambda\ell \in I'$. Let $X \in R$ be defined by $\lambda\ell \in X$. By definition, $I \asymp_B X$. Applying Step 3 to X and I, there exists $J \in R^i(I)$ such that $J \asymp_B X$ and $\lambda\ell \in J$.[13] Let $X' \in R'$ be defined by $\ell' \in X'$. By definition, $I' \asymp_B X'$. Applying Step 3 to X' and I', there exists $J' \in R^i(I')$ such that $J' \asymp_B X'$ and $\ell' \in J'$.

Figure 15: Step 4 of the Proof of Theorem 6

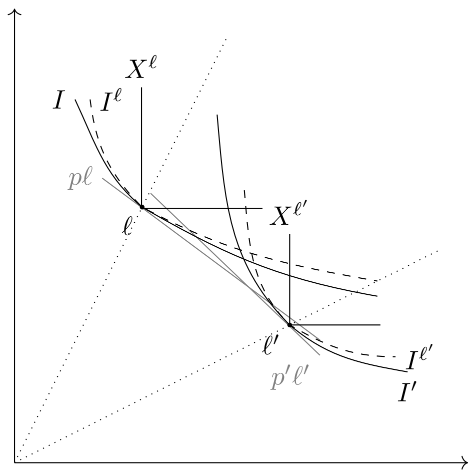


Now, assume that the claim is not true, like in Fig. 15a. This implies that I' crosses I at ℓ'. Hence, there exists $\alpha > 1$ such that I' crosses αI in a neighborhood of ℓ'. Recall that $\lambda\ell \in I' \cap I \cap J$ and $\ell' \in I \cap I' \cap J'$. Let us consider αI and I', for which we have $J' \asymp_B I'$ and $\alpha J \asymp_B \alpha I$. By P*** it must be either $I' \subseteq U(\alpha J)$ or $\alpha I \subseteq U(J')$. Observe that $\alpha\lambda\ell \in \alpha J$ and $\alpha\lambda\ell > \lambda\ell \in I'$, so that $I' \not\subseteq U(\alpha J)$. Moreover, αI crossing I' implies $\alpha I \not\subseteq U(J')$. A contradiction. A graphical example is given in Fig. 15b, where the intersections between I' and αJ, and between αI and J' are highlighted with a dotted circle around them

□

[13]Observe that, $I \in R^i(R)$, $X \in R$ and $I \asymp_B I$. By step 3, $I \asymp_B I, X$ implies that the inf preference associated to I must be tangent from below to both I and X.

Figure 16: Step 5 of the proof of Theorem 6



Observe that $U(I') \subseteq U(I)$ in a neighbourhood of ℓ' and I being differentiable at ℓ' imply that either I' is differentiable at ℓ', so that I and I' have the same marginal rates of substitution at ℓ', or I' is not differentiable at ℓ' and yet the same marginal rates of substitution along I at ℓ' are a supporting price vector for I'.

Step 5

Claim: Let $R, R' \in \mathcal{R}^L$. Then $R^i(R) = R^i(R')$.

Proof. Let $R, R' \in \mathcal{R}^L$. Let $I \in R^i(R)$ and $I' \in R^i(R')$. Since preferences are continuous and convex, and preferences in $\mathcal{R}^i$ have bounded lower contours, I and I' can be represented by real-valued functions that are differentiable almost everywhere. Intuitively, if these two functions cross at one point but have the same derivatives whenever they both are differentiable, then they should coincide.

Formally, let us assume, by contradiction, that $R^i(R) \neq R^i(R')$. Then, it is possible to find $I \in R^i(R)$ and $I' \in R^i(R')$ and two points ℓ and ℓ' such that: *(i)* $\ell \in I$ and $\ell' \in I'$, *(ii)* I is differentiable at ℓ and I' is differentiable at ℓ', *(iii)* $\alpha'\ell \in I'$ and $\alpha\ell' \in I$ for some $\alpha, \alpha' > 1$, and *(iv)* $p\ell > p\ell'$ and $p'\ell' > p'\ell$, with p (resp. p') the supporting price at ℓ along I (resp. ℓ' along I'). This is illustrated on Fig. 16.

Let $R^\ell, R^{\ell'} \in \mathcal{R}^L$ be the Leontief preferences having their kinks proportional to ℓ and ℓ' respectively. Let $X^\ell \in R^\ell$, $X^{\ell'} \in R^{\ell'}$ be such that $\ell \in X^\ell$ and $\ell' \in X^{\ell'}$. Let $I^\ell \in R^i(R^\ell)$, $I^{\ell'} \in R^i(R^{\ell'})$ be such that $I^\ell \asymp_B X^\ell$ and $I^{\ell'} \asymp_B X^{\ell'}$. By Step 4, $I^\ell \subseteq U(I)$ in a neighborhood of ℓ, so that, given $p\ell > p\ell'$, $\ell' \notin U(I^\ell)$, and $I^{\ell'} \subseteq U(I')$ in a neighborhood of ℓ', so that, given $p'\ell' > p'\ell$, $\ell \notin U(I^{\ell'})$. This implies that P *** fails when we apply it to X^ℓ and $X^{\ell'}$, because $\ell' \notin U(I^\ell)$ implies $X^{\ell'} \nsubseteq U(I^\ell)$ and $\ell \notin U(I^{\ell'})$ implies $X^\ell \nsubseteq U(I^{\ell'})$.

□

Step 6
Claim: For all $R, R' \in \mathcal{R}$, $R^i(R) = R^i(R')$.

Proof. Let I be a best preference indifference curve associated to a Leontief preference. If the claim is false, then there exist $R \in \mathcal{R}$, $X \in R$ and $J \in R^i(X)$ such that $I \asymp_B X$, $J \asymp_B X$ and $J \neq I$. As a consequence, we can find $z \in J \setminus I$. Let $R' \in \mathcal{R}^L$ and $X' \in R'$ be such that X' has its kink at z. By Step 4 $J \asymp_B X'$ implies that there exist $I' \in R^i(R')$ such that $I' \asymp_B X, X'$. As a consequence, I' must intersect I. However, by Step 5, $I \in R^i(R')$. A contradiction. □

Theorem 7

Let W satisfy Sup(x|y) and Sup(y|x). We need to prove that, for all $X, Y \in \mathcal{I}$, $R^s_c(X) = R^s_r(X) = R^s_c(Y) = R^s_r(Y)$.

We begin the proof by stating the dual of P*.

*Property P′**: Let W satisfy Sup(x|y) and Sup(y|x). Let $R, R' \in \mathcal{R}^H$. Let $X, Y \in R$, $X', Y' \in R'$ be such that $W(X, Y) = W(X', Y')$. Let $I, I' \in \mathcal{I}$ be defined by: $I \asymp^A Y$, $I \in R^s_c(X)$, and $I' \asymp^A Y'$, $I' \in R^s_c(X')$. Then, either $I \subseteq U(Y')$ or $I' \subseteq U(Y)$.

Moreover, by replacing X, X' with Y, Y', and $R^s_c(X), R^s_c(X')$ with $R^s_r(Y), R^s_r(Y')$ in P′*, we obtain a symmetric version of P**: *property P′***.

The first part of the proof follows a similar logic of the one for Theorem 6. Steps 1 to 3, and Step 6 parallels those of Theorem 6, once we substitute Leontief with linear preferences. To maintain a brief exposition, we omit the proof of these steps and of the two properties P′*, P′**. Here, we prove only steps 4 and 5.

Step 1 - Let us denote $\mathcal{R}^{lin}$ the set of all linear preferences. *Claim*: For all $X \in R \in \mathcal{R}^{lin}$, $R^s_c(X) = R^s_r(X)$.

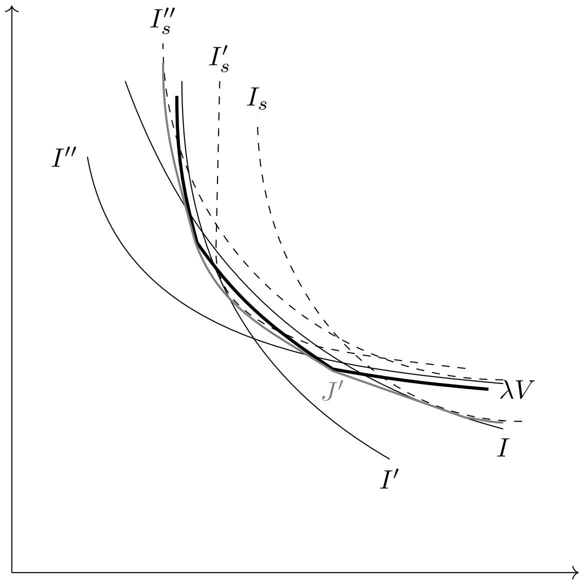
Step 2 - *Claim*: For all $X \in R \in \mathcal{R}^H$, $R^s_c(X) = R^s_r(X)$.

We can then use the same simplified notation to refer to sup preferences associated to homothetic preferences.

Step 3 - *Claim*: For all $R, R' \in \mathcal{R}^H$, all $X \in R$ and $X' \in R'$, if there exists $I \in R^s(R)$ such that $I \asymp^A X, X'$, then there exists $I' \in R^s(R')$ such that $I' \asymp^A X, X'$.

We now claim that the sup preferences associated to three (different) indifference contours must have an indifference contour that is always above all of them.

Figure 17: Step 4 of the proof of Theorem 7



Step 4

Claim: Let $R, R', R'' \in \mathcal{R}^H$. Let $I \in R$, $I' \in R'$, $I'' \in R''$, and $I_s \in R^s(R)$, $I'_s \in R^s(R')$, $I''_s \in R^s(R'')$ be such that $I_s \asymp^A I$, $I'_s \asymp^A I'$ and $I''_s \asymp^A I''$. Then, there exists $J \in \{I_s, I'_s, I''_s\}$ such that $J \subseteq (U(I) \cap U(I') \cap U(I''))$.

Proof. Let $R, R', R'', I, I', I'', I_s, I'_s, I''_s$ satisfy the conditions of the claim. If P$'$* fails for any pair among I_s, I'_s, I''_s, then the claim is proven. Hence, let us assume that P$'$* is satisfied for all pairs but the claim does not hold true. We can assume, w.l.o.g., that $I_s \subseteq (U(I) \cap U(I'))$, $I'_s \subseteq (U(I') \cap U(I''))$ and $I''_s \subseteq (U(I) \cap U(I''))$. This is illustrated in Fig. 17.

Let $V = U(I) \cap U(I') \cap U(I'')$ and $\lambda < 1$ be sufficiently close to 1 so that for all $J \in \{I_s, I'_s, I''_s\}$, $J \not\subset \lambda V$. Let $V' = ch(\lambda V, U(I_s), U(I'_s), U(I''_s))$, and let J' be the lower envelop of V'. Observe that $J' \cap V = \emptyset$. By construction: (a) $I_s \asymp^A J'$; (b) $I'_s \asymp^A J'$; (c) $I''_s \asymp^A J'$. Let $\alpha > 0$. By (a), (b) and (c), we have $W(\alpha I, J') = W(\alpha I', J') = W(\alpha I'', J') = \alpha$. By H, $W(\alpha J', J') = \alpha$. By Sup(x|y), there exists $J_s \in R^s(J')$ such that $J_s \asymp^A \alpha I, \alpha I', \alpha I''$, as well as $J_s \asymp^A \alpha J'$. Observe that this is impossible because the first three conditions imply $J_s \subseteq \alpha V$ whereas $\alpha J' \cap \alpha V = \emptyset$: a contradiction.

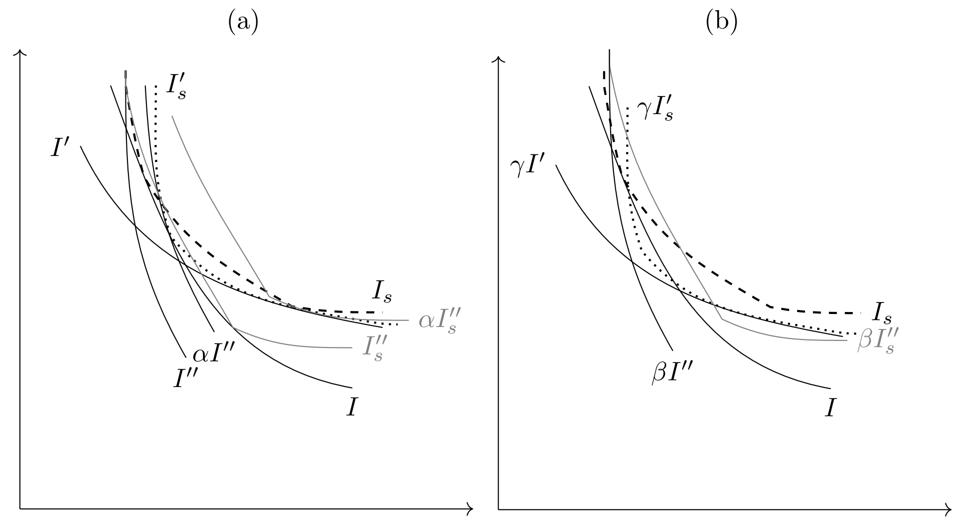
□

We now use the previous result to show that all homothetic preferences must have the same sup preference.

Step 5

Claim: Let $R, R' \in \mathcal{R}^H$, $I \in R$, $I' \in R'$, $I_s \in R^s(R)$ and $I'_s \in R^s(R')$ be such that $I_s \asymp^A I$ and $I'_s \asymp^A I'$. If $I_s \asymp^A I'$ (so that $I'_s \asymp^A I$ by step 3) then $I_s = I'_s$, which is equivalent to $R^s(R) = R^s(R')$.

Figure 18: Step 5 of the proof of Theorem 7



Proof. Assume not. Then we can find $R'' \in \mathcal{R}^H$, $I'' \in R''$, $\alpha \neq 1$ such that $I_s \asymp^A I''$ and $I'_s \asymp^A \alpha I''$. As a consequence, there must exist $I''_s \in R^s(R'')$ such that $I''_s \asymp^A I, I''$ and $\alpha I''_s \asymp^A I'$. See Fig. 18a for a graphical example.

Let us assume, w.l.o.g., that $\alpha > 1$. Let $\beta > 1$ be such that $\beta < \alpha$, and $\gamma < 1$ be close to 1. This is illustrated in Fig. 18b

Let us consider three indifference curves: I, $\gamma I'$ and $\beta I''$. By step 4, there must exist $J \in \{I_s, \gamma I'_s, \beta I''_s\}$ such that $J \subseteq (U(I) \cap U(\gamma I') \cap U(\beta I''))$.

Since $I''_s \asymp^A I, I''$; $\beta > 1$ implies $\beta I''_s \subseteq U(I) \cap U(\beta I'')$. However, since $\alpha I''^w \asymp^A I'$ and $\beta < \alpha$, we have $\beta I''_s \not\subseteq U(I')$. Therefore, since $\gamma < 1$ is close to 1, $\beta I''_s \not\subseteq U(\gamma I')$.

Since $I'_s \asymp^A \alpha I''$ and $\beta < \alpha$, we have $I'_s \subseteq U(I') \cap U(\beta I'')$, which, given $\gamma < 1$ close to 1, implies $\gamma I'_s \subseteq U(\gamma I') \cap U(\beta I'')$. Yet, $\gamma I'_s \not\subseteq U(I)$, because $I'_s \asymp^A I$.

Finally, $I_s \not\subset U(\beta I'')$, because $\beta > 1$ and $I_s \asymp^A I''$.

To sum up, none of I_s, $\gamma I'_s$ or $\beta I''_s$ lies in the intersection of $U(I)$, $U(\gamma I')$ and $U(\beta I'')$, contradicting Step 4. $\square$

Step 7 - *Claim*: For all $R, R' \in \mathcal{R}$, $R^s(R) = R^s(R')$.

7 Conclusion

We have explored how to measure well-being when individuals differ in terms of their consumption, their preferences and their reference consumption. We have studied three sets of axioms. The first set of axioms, which we call the basic axioms, are satisfied by all our measures (as well as by known measures that end up excluded from the analysis, such as Blackorby & Donaldson (1987)'s welfare ratios and Dimri & Maniquet (2020)'s individualized price equivalent incomes). Most of them are grounded on the idea that when preferences are homothetic, then well-being should be measured by reference to the homogenous of degree one utility function that represents these preferences. The second set of axioms captures the choice that has to be done between taking account of the reference bundle only (reference as a bundle) versus taking the entire indifference contour through the reference into account (reference as a satisfaction level). The third set of axioms captures the idea that well-being inequality at one bundle should be minimized, and this can be done by imposing bounds on well-being either at the supremum or at the infimum of indifference contours.

This axiomatic study allowed us to identify six prominent ways of measuring well-being. The first three ways are three well-being measures, Deaton's point-reference measure, the maximum and the minimum well-being measures, the last two being new. The last three ways of measuring well-being are captured in families of measures, each individual measure being parameterized by some reference preferences - or some good in the case of W_k - that are used to evaluate indifference contours at consumption and at reference.

In this conclusion, we comment on six features of our well-being measures. First, we need to comment on the special role played by homothetic preferences in our analysis. The postulate is that when preferences are homothetic, we *know* how to measure well-being: if the reference bundle is a multiple of the consumption bundle, then well-being should only depend on this multiple. This is strongly related to the situations to which this theory is supposed to be applied: all goods should be considered equally important, no distinction can be made across goods. This strikes us as a legitimate assumption when goods are sufficiently aggregated, so that we deal with food, clothes and housing, for instance.

The theory, however, easily generalizes to other situations and models. The most obvious alternative is the one in which one good plays a particular role so that the most natural preferences are quasi-linear preferences. Let us assume that good 1 is this special good, the consumption of which can be unboundedly negative or positive (and we redefine $\mathcal{X}$ accordingly). The domain of quasi-linear preferences is denoted $\mathcal{R}^{ql}$, and for $R \in \mathcal{R}^{ql}$, we use utility function u to represent R provided there exists some $v : \mathbb{R}_+^{K\setminus\{1\}} \to \mathbb{R}$ such that $u(x) = v(x_{-1}) + x_1$, where x_{-1} is obtained from x after removing its first component. In this new model, we need to replace our axiom of Homotheticity with the following axiom of Quasi Linearity: when we compare two situations in which the reference bundle is equal to the consumption bundle except for the quantity of good 1, then well-being should be the same as soon as preferences are quasi linear:

Quasi Linearity (QL) - For all $x = (x_1, x_{-1}), x' = (x'_1, x'_{-1}) \in \mathcal{X}$ and $\lambda \in \mathbb{R}$, if $R, R' \in \mathcal{R}^{ql}$ and xIx', then

1. $W(x, R, (x_1 + \lambda, x_{-1})) = W(x, R', (x_1 + \lambda, x_{-1}))$, and
2. $W(x, R, (x_1 + \lambda, x_{-1})) = W\left(x', R, (x'_1 + \lambda, x'_{-1})\right)$.

With this new axiom, all the reasonings that are developed in the paper remains valid provided homothetic transforms are replaced with translation along the good 1-axis. For instance, the analogue of Deaton's measure is defined as follows: the well-being of an agent is measured by the distance by which reference bundle y has to be translated to become indifferent to x:

$$W_y(x, R, y) = \omega \Leftrightarrow (y_1 - \omega, y_{-1}) \in I(x, R).$$

The other measures require to define the quasi-linear preferences obtained by translation of a given indifference contour. By analogy with what we write in the previous sections, let $R_{I(x,R)} \in \mathcal{R}^{ql}$ denote the quasi-linear preferences obtained by translation of the indifference contour at x for preferences R. Then, we can rewrite the variant of $W_{\max}$, for instance, as

$$W_{\max}(x, R, y) = u(x) - \min_{y' \in U(y,R)} u(y'),$$

with u representing quasi-linear preferences $R_{I(x,R)}$.

All our results can be adjusted to this modified model provided definitions and proofs are adjusted accordingly.

Second, we comment on the role of incomes in our measures. Prices that agents actually face in their consumption do not play any role in any of our measures. This is not surprising, as actual incomes do not lead to well-being comparisons that are consistent with individual preferences as soon as individuals face different prices. Equivalent incomes, on the contrary, play a crucial role here. The maximum (resp. minimum) well-being measures that we introduce in this paper defines well-being as the ratio between equivalent income at consumption and at reference, when the price vector that is used to compute equivalent income is the one that maximizes (resp. minimizes) well-being. If linear preferences are chosen as parametric preferences for a best preference well-being measure, then well-being is also a ratio between equivalent incomes but prices are, as in Samuelson (1974, 1977), arbitrary.

All in all, this means that measuring well-being using relative incomes receives some normative justification, especially when prices are endogenous to preferences, such as with the maximum and minimum well-being measures. The reader may consider such endogeneity somehow *artificial*, as it follows from the choice of the reference bundle. It is nevertheless the case that justifying a specific reference bundle can be easier than choosing a price vector. Coming back to the issue of measuring poverty, it is easier to quantify the minimum amount of, say bread, an individual needs rather then the market price he has to pay for it. In this sense, our framework can help researchers in providing stronger normative bases for defining the price vector at which to compute equivalent incomes.

Our third comment refers to the information that is needed to compute well-being. Among the measures that are axiomatized in this paper, Deaton's point-reference measure only requires to know one indifference contour, whereas all other measures require to know two indifference contours. We did not introduce information parsimony as a normative requirement, but Deaton's measure is clearly the most parsimonious measure among the ones that turn out to be normatively justified. In practice, however, once preferences are estimated, there is no difference between making well-being depend on one or two indifference contours.

Our fourth comment has to do with the cardinalization of our measures. As explained after Lemma 2 above, we must have $W(x, R, y) = f\left(\frac{u(x)}{u(y)}\right)$ for all homothetic preferences R, with u being the homogenous of degree 1 utility function representing R. That leaves f unspecified. Of course, f does not matter when we make *comparisons* of levels of well-being, but f will matter if we want to compare *differences* in well-being. This refers to the well-being inequality aversion that the social welfare function should embody. The final choice of f may itself be the topic of an axiomatic study. This is what Fleurbaey & Maniquet (2018a) studies when there is no reference bundle. By reference to the solution they give to this problem, we can suggest the natural solution consisting of using a linear f and an inequality averse aggregator (that is one that satisfies the Pigou-Dalton principle in well-being).

Our fifth comment is related to the idea, which we defended in the introduction, that the reference bundle is defined by the social evaluator. As a consequence, only self-centered preferences are relevant for welfare analysis. As it has been largely documented in the behavioral economics literature, though, individuals' actual preferences are other-regarding. This raises the question of how to go from actual to self-centered preferences. Other-regarding preferences, however, are often assumed to be separable in own and others' consumption, as in Dufwenberg *et al.* (2011), so that the relevant self-centered preferences are obtained by simply disregarding their social arguments.

Finally, our analysis has been developed in a classic and standard framework with non-satiation and convex preferences. Future research may focus on extending the current analysis to context in which these restrictions are not binding, in line with Fleurbaey & Maniquet (2018b).

Appendix A

Lemma 8. *If a well-being measure W satisfies Consumption Focus (Fx) and Continuity (Cont), then it satisfies Convergence from Below (Cb) and Convergence from Above (Ca).*

Proof. Let W satisfy Fx and Cont. Let $y \in \mathcal{X}$ and $R \in \mathcal{R}$. Let $(x^n, R^n)_{n\in\mathbb{N}}$ satisfy the conditions of Cb. A similar proof holds for Ca. Because $L(x^n, R^n) \cap U(x^{n+1}, R^{n+1}) = \emptyset$, there exists $R' \in \mathcal{R}$ such that $I(x^n, R^n) \in R'$ for all $n \in \mathbb{N}$. By Fx, $W(x^n, R^n, y) = W(x^n, R', y)$ for all $n \in \mathbb{N}$. By Cont, $W(x^n, R', y) \to W(y, R', y) = 1$. $\square$

Lemma 9. *If a well-being measure W satisfies Continuity (Cont), Consumption Monotonicity (Mx), Reference Indifference (Iy) and Rescaling Across Consumption (RAx), then it does not satisfy both Convergence from Below (Cb) and Convergence from Above (Ca).*

Proof. Assume W satisfies Cont, Mx, Iy, RAx, Cb and Ca. We need to derive a contradiction. Because W satisfies Cont and Mx, by Lemma 1, it satisfies UCIxy. By UCIxy, we can restrict the arguments of W to two indifference contours. Let $y \in \mathcal{X}$ and $R \in \mathcal{R}$ be such that $I(y, R)$ is strictly convex and $L(y, R)$ is compact. Let $p \in \mathbb{R}^K_{++}$ be a supporting price at y. Let $y' \in I(y, R)$ be such that $y' \neq y$. Let $(x^n, R^n)_{n\in\mathbb{N}}$ satisfy the conditions of Cb, and, moreover, $I(x^n, R^n) = \{x \in \mathcal{X} | px = px^n\}$. By Cb, $W(I(x^n, R^n), I(y, R)) \to 1$. Let $(x'^n, R'^n)_{n\in\mathbb{N}}$ satisfy the conditions of Ca with $x'^n \to y'$, and, moreover, $x'^n = \lambda^n y'$, $\lambda^n > 1$, for all $n \in \mathbb{N}$, and there exist $p', p'' \in \mathbb{R}^K_{++}$ such that $I(x'^n, R'^n) = \{x \in \mathcal{X} | (p'x = p'x'^n \text{ and } p'x \geq p''x) \text{ or } (p''x = p''x'^n \text{ and } p''x \geq p'x)\}$. In words, in words $I(x'^n, R'^n)$ are piecewise linear with kinks at x'^n. By Ca, $W(I(x^n, R^n), I(y, R)) \to 1$. See Figure 19 for a graphic illustration.

Let $\lambda > 0$ be such that $L(\lambda I(y, R)) \cap U(I(x^1, R^1)) = \emptyset$.[14] Let w^n (resp. w'^n), $n \in \mathbb{N}$, be defined by

$$W(w^n I(y, R), I(y, R)) = W(I(x^n, R^n), I(y, R)) \tag{17}$$

(resp. $W(w'^n I(y', R), I(y', R)) = W(I(x'^n, R'^n), I(y', R)))$. Observe that $w^n \to 1$ and $w'^n \to 1$. By RAx and Eq. 17,

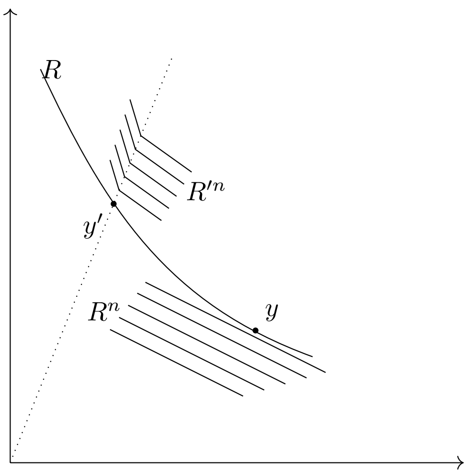
$$W(w^n I(y, R), \lambda I(y, R)) = W(I(x^n, R^n), \lambda I(y, R))$$

Let $R' \in \mathcal{R}^c$ be such that: (i) $\lambda I(y, R) \in R'$, (ii) $I(x^n, R^n) \in R'$, for all $n \in \mathbb{N}$, (iii) $I(x'^n, R'^n) \in R'$, for all $n \in \mathbb{N}$. Observe that $I(y, R') \neq I(y, R) \neq I(y', R')$, and, necessarily,

$$L(I(y, R')) \cap U(I(y', R')) = \emptyset. \tag{18}$$

[14] The existence of λ is ensured by linearity of R^n and compactness of $L(y, R)$

Figure 19: Proof of Lemma 9



We get

$$W(I(x^n, R'), \lambda I(y, R)) = W(I(x^n, R^n), \lambda I(y, R)),$$

and

$$W(I(x'^n, R'), \lambda I(y, R)) = W(I(x'^n, R'^n), \lambda I(y, R)),$$

which implies

$$W(I(x^n, R'), \lambda I(y, R)) \to W(I(x'^n, R'), \lambda I(y, R)),$$

which, together with Eq. 18, violates Mx. □

Lemma 10. *Let W be a well-being measure satisfying Reference Indifference (Iy). Then, for all $R \in \mathcal{R}$, all $x, y \in \mathcal{X}$, all $\lambda > 0$,*

$$W(x, R, y) = W(\lambda x, R^\lambda, \lambda y)$$

for $R^\lambda \in \mathcal{R}$ such that $I(\lambda x, R^\lambda) = \lambda I(x, R)$ and $I(\lambda y, R^\lambda) = \lambda I(y, R)$.

Proof. Let W be a well-being measure. By Lemma 1, W satisfies UCIxy, so that, by Iy, we can reduce any $W(x, R, y)$ into $W(X, Y)$ for $X = I(x, R)$ and $Y = I(y, R)$.

We need to prove that for all $X, Y \in \mathcal{I}$, all $\lambda > 0$, $W(\lambda X, \lambda Y) = W(X, Y)$. If X is homothetic to Y, then the claim follows from H and Lemma 2. Let us assume that X is not homothetic to Y. Let $w = W(X, Y)$. We have four cases to consider.

Case 1: $\lambda < 1$ and $w < 1$. By H, $W(X, w^{-1}X) = w$. Therefore, $W(X, Y) = W(X, w^{-1}X)$. By Rx, $W(\lambda X, Y) = W(\lambda X, w^{-1}X)$. By H, $W(\lambda X, w^{-1}X) =$

λw, so that $W(\lambda X, Y) = \lambda w$. By H, $W(\lambda w Y, Y) = \lambda w$, so that $W(\lambda X, Y) = W(\lambda w Y, Y)$. By Ry, $W(\lambda X, \lambda Y) = W(\lambda w Y, \lambda Y)$. By H, $W(\lambda w Y, \lambda Y) = w$, so that $W(\lambda X, \lambda Y) = w$, the desired outcome.

The three other cases - (2) $\lambda > 1$ and $w < 1$, (3) $\lambda < 1$ and $w > 1$, (4) $\lambda > 1$ and $w > 1$ - are treated in a similar way, where the order of the change from X into λX and Y into λY must guarantee that $\lambda X \cap Y = \emptyset$ when, like in case 1, X is transformed first, and $\lambda Y \cap X = \emptyset$ when Y is transformed first (cases 2 or 3). $\square$

Convergence of the Reference from Above (CRa) - For all $x \in \mathcal{X}$ and $R \in \mathcal{R}$, if there exists a sequence $(y^n, R^n)_{n \in \mathbb{N}}$ such that, for all $n \in \mathbb{N}$, (i) $I(x, R^n) = I(x, R)$, (ii) $L(y^{n+1}, R^{n+1}) \cap U(y^n, R^n) = \emptyset$, (iii) $y^n \, P \, x$, and (iv) $y^n \to x$, then $W(x, R^n, y^n) \to 1$.

Convergence of the Reference from Below (CRb) - For all $x \in \mathcal{X}$ and $R \in \mathcal{R}$, if there exists a sequence $(y^n, R^n)_{n \in \mathbb{N}}$ such that, for all $n \in \mathbb{N}$, (i) $I(x, R^n) = I(x, R)$, (ii) $L(y^n, R^n) \cap U(y^{n+1}, R^{n+1}) = \emptyset$, (iii) $x \, P \, y^n$, (iv) and $y^n \to x$, then $W(x, R^n, y^n) \to 1$.

Lemma 11. *If W is a well-being measure that satisfies Reference Indifference (Iy) and Convergence from Below (Cb), then it satisfies Convergence of the Reference from Above (CRa) . Similarly, if W is a well-being measure that satisfies Reference Indifference (Iy) and Convergence from Above (Ca), then it satisfies Convergence of the Reference from Below (CRb).*

Proof. We prove the first part of the lemma. The second part can be proved in a similar way. Observe that W satisfies the property of Lemma 10. Let sequence $(y^n, R^n)_{n \in \mathbb{N}}$ be such that, for all $n \in \mathbb{N}$, (i) $I(x, R^n) = I(x, R)$, (ii) $L(y^{n+1}, R^{n+1}) \cap U(y^n, R^n) = \emptyset$, (iii) $y^n \, P \, x$, and (iv) $y^n \to x$. By Iy, we can assume, w.l.o.g., that for all $n \in \mathbb{N}$, there exists $\lambda_n > 0$ such that $y^n = \lambda^n x$. Indeed, if it is not true, then we can replace y^n in the sequence with $\lambda^n x$ such that $\lambda^n x \in I(y^n, R^n)$ because, by Iy, $W(x, R^n, y^n) = W(x, R^n, \lambda^n x)$. Observe that $y^n \to x$ is equivalent to $\lambda_n \to 1$.

By Lemma 10, $W(x, R^n, \lambda^n x) = W((\lambda^n)^{-1} x, (\lambda^n)^{-1} R^n, x)$.

Therefore, $\lim_{n \to \infty} W(x, R^n, y^n) = \lim_{n \to \infty} W((\lambda^n)^{-1} x, (\lambda^n)^{-1} R^n, x)$.

By Cb, $W((\lambda^n)^{-1} x, (\lambda^n)^{-1} R^n, x) \to 1$, so that $W(x, R^n, y^n) \to 1$, the desired outcome. $\square$

Appendix B

Proof of Lemma 7(c) . Let W be a well-being measure satisfying Iy. If it satisfies RAx then, by Lemma 4 , it also satisfies RAy and vice versa. By Lemma 1, W satisfies UCIxy, so that, by Iy, we can reduce any $W(x, R, y)$ into $W(X, Y)$ for $X = I(x, R)$ and $Y = I(y, R)$.

Let $R \in \mathcal{R}^H$, $Y \in R$, and $0 < w < 1$. By H, $W(wY, Y) = w$. Let $\mathcal{I}_w(Y) = \{X \in \mathcal{I} | W(X, Y) = w\}$ and

$$L_w(Y) = \bigcap_{X \in \mathcal{I}_w(Y)} L(X)$$

or, alternatively, taking account of the fact that $L(wY) \subseteq L_w(Y)$,

$$L_w(Y) = \bigcap_{X \in \mathcal{I}_w(Y)} (L(X) \cap L(wY)),$$

showing that $L_w(Y)$ is closed (and, therefore, compact). Let $X_w(Y)$ be the upper envelop of $L_w(Y)$.

Claim 1: By Inf(x|y), $U(X_w(Y))$ is convex.

Assume not. Then there exist, $\lambda \in (0, 1)$, $x, x' \in \mathcal{X}$ and $X^x, X^{x'} \in \mathcal{I}_w(Y)$ such that: (i) $x \in X^x$, (ii) $x' \in X^{x'}$, and (iii) $\lambda x + (1 - \lambda)x' \notin U(X_w(Y))$.

Let $X^{xx'} \in \mathcal{I}$ be defined by $U(X^{xx'}) = ch\left(U(X^x) \cup U(X^{x'})\right)$. By construction, $\lambda x+(1-\lambda)x' \in U(X^{xx'})$. By Inf(x|y), $W(X^{xx'}, Y) = \min\{W(X^x, Y), W(X^{x'}, Y)\} = w$, so that $U(X^{xx'}) \subseteq U(X_w(Y))$, a contradiction.

Because $U(X_w(Y))$ is convex, $X_w(Y) \in \mathcal{I}$. Our second claim is that following.

Claim 2: $W(X_w(Y), Y) = w$.

Assume not. First, let $W(X_w(Y), Y) < w$. Then, there exists $\lambda > 1$ such that $W(\lambda X_w(Y), Y) < w$. Let us write $\overline{X} = \lambda X_w(Y)$. Note that $\overline{X}$ is compact. Because $\lambda > 1$, the set of $Int\,(U(X))$, for all $X \in \mathcal{I}_w(Y)$, is an open cover of $\overline{X}$. Consequently, there is a finite subcover, $U(X^1), \ldots, U(X^n)$. Let $\overline{\overline{X}} \in \mathcal{I}$ be defined by

$$U\left(\overline{\overline{X}}\right) = ch\left(\bigcup_{t \in \{1,\ldots,n\}} U(X^t)\right).$$

By Inf(x|y), $W\left(\overline{\overline{X}}, Y\right) = w > W(X_w(Y), Y)$ whereas $X_w(Y) \subset Int\left(U(\overline{\overline{X}})\right)$, violating Mx.

Second, let $W(X_w(Y), Y) > w$. Then, by Cont, there exists $\lambda < 1$ such that $W(\lambda X_w(Y), Y) = w$, whereas $\lambda X_w(Y) \notin U(X_w(Y))$, a contradiction.

Let $R_r^i(Y) \in \mathcal{R}^H$ be defined by $X_w(Y) \in R_r^i(Y)$.

The rest of the proof mimics the one of Lemma 6(b). We state here two claims which can be proved using RAx and RAy similarly to Claims 2 and 3 in the proof of Lemma 6(b). We leave the formal proof to the reader.

Claim 3 - For all $X \in \mathcal{I}$, $w' \in \mathbb{R}_+$, $W(X,Y) \geq w'$ if and only if there exists $I \in R^i_r(Y)$ such that $w'Y \asymp^A I$ and $U(X) \subseteq U(I)$.

Claim 4 - For all $Y' \in R$, $R^i_r(Y') = R^i_r(Y)$. $\square$

References

Blackorby, Charles, & Donaldson, David. 1987. Welfare ratios and distributionally sensitive cost-benefit analysis. *Journal of Public Economics*, **34**(3), 265–290.

Blackorby, Charles, Bossert, Walter, & Donaldson, David. 1997. Critical-level utilitarianism and the population-ethics dilemma. *Economics & Philosophy*, **13**(2), 197–230.

Bolton, Gary E, & Ockenfels, Axel. 2000. ERC: A theory of equity, reciprocity, and competition. *American economic review*, **90**(1), 166–193.

Chambers, Christopher P, & Miller, Alan D. 2014a. Inefficiency measurement. *American Economic Journal: Microeconomics*, **6**(2), 79–92.

Chambers, Christopher P, & Miller, Alan D. 2014b. Scholarly influence. *Journal of Economic Theory*, **151**, 571–583.

Christensen, Flemming, Hougaard, Jens Leth, & Keiding, Hans. 1999. An axiomatic characterization of efficiency indices. *Economics Letters*, **63**(1), 33–37.

Clark, Andrew E. 2018. Four decades of the economics of happiness: Where next? *Review of Income and Wealth*, **64**(2), 245–269.

Clark, Andrew E, & Senik, Claudia. 2010. Who compares to whom? The anatomy of income comparisons in Europe. *The Economic Journal*, **120**(544), 573–594.

Clark, Andrew E, Frijters, Paul, & Shields, Michael A. 2008. Relative income, happiness, and utility: An explanation for the Easterlin paradox and other puzzles. *Journal of Economic literature*, **46**(1), 95–144.

Deaton, Angus. 1979. The distance function in consumer behaviour with applications to index numbers and optimal taxation. *The Review of Economic Studies*, **46**(3), 391–405.

Decancq, Koen, Fleurbaey, Marc, & Schokkaert, Erik. 2017. Wellbeing inequality and preference heterogeneity. *Economica*, **84**(334), 210–238.

Dimri, Aditi, & Maniquet, François. 2020. Income poverty measurement in India: defining group-specific poverty lines or taking preferences into account? *The Journal of Economic Inequality*, **18**(2), 137–156.

Dufwenberg, Martin, Heidhues, Paul, Kirchsteiger, Georg, Riedel, Frank, & Sobel, Joel. 2011. Other-regarding preferences in general equilibrium. *The Review of Economic Studies*, **78**(2), 613–639.

Fehr, Ernst, & Schmidt, Klaus M. 1999. A theory of fairness, competition, and cooperation. *The quarterly journal of economics*, **114**(3), 817–868.

Fehr, Ernst, Kirchsteiger, Georg, & Riedl, Arno. 1993. Does fairness prevent market clearing? An experimental investigation. *The quarterly journal of economics*, **108**(2), 437–459.

Fehr, Ernst, Kirchsteiger, Georg, & Riedl, Arno. 1998. Gift exchange and reciprocity in competitive experimental markets. *European Economic Review*, **42**(1), 1–34.

Fleurbaey, Marc, & Blanchet, Didier. 2013. *Beyond GDP: Measuring welfare and assessing sustainability*. Oxford University Press.

Fleurbaey, Marc, & Maniquet, François. 2017. Fairness and well-being measurement. *Mathematical Social Sciences*, **90**, 119–126.

Fleurbaey, Marc, & Maniquet, François. 2018a. Inequality-averse well-being measurement. *International Journal of Economic Theory*, **14**(1), 35–50.

Fleurbaey, Marc, & Maniquet, François. 2018b. Well-being measurement with non-classical goods. *Economic Theory*, 1–22.

Hougaard, Jens Leth, & Keiding, Hans. 1998. On the functional form of an efficiency index. *Journal of Productivity Analysis*, **9**(2), 103–111.

Kirchsteiger, Georg. 1994. The role of envy in ultimatum games. *Journal of economic behavior & organization*, **25**(3), 373–389.

Kreps, David M. 1979. A representation theorem for" preference for flexibility". *Econometrica: Journal of the Econometric Society*, 565–577.

Samuelson, Paul A. 1974. Complementarity: An essay on the 40th anniversary of the Hicks-Allen revolution in demand theory. *Journal of Economic literature*, **12**(4), 1255–1289.

Samuelson, Paul A. 1977. Reaffirming the existence of" reasonable" bergson-samuelson social welfare functions. *Economica*, 81–88.

Sen, Amartya. 1970. The impossibility of a Paretian liberal. *Journal of political economy*, **78**(1), 152–157.

Sobel, Joel. 2005. Interdependent preferences and reciprocity. *Journal of economic literature*, **43**(2), 392–436.

Varian, Hal. 1974. Equity, Envy, and Efficiency. *Journal of Economic Theory*, **9**(1), 63–91.