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This paper first shows how to extend the Sen-Shorrocks poverty index to the analysis of multidimensional deprivation, when only dichotomous variables are available to assess deprivation in various domains, the most common case in the literature. More precisely, it introduces the first rank-dependent multidimensional poverty index in the literature, using a counting approach. The resulting multidimensional deprivation index, or MDI in short, has a nice graphical representation ("PUB curve") that turns out to be an extension of the so-called TIP curve of Jenkins and Lambert to the case of multiple deprivations. This graphical representation is similar to the SD curve introduced by Lasso de la Vega (2010), but additionally emphasizes the third "I " of multidimensional deprivation: inequality. The MDI is sensitive to inequality and satisfies quite nice properties, but it cannot be broken down by population subgroups, when a standard decomposition is used, and it does not have the property of dimensional breakdown, as the latter is usually defined in the literature. The paper proves, however, that there exists an alternative decomposition by population subgroups that can be applied to the MDI; it also derives a decomposition by deprivation domain, analogous to the breakdown of the Gini index by factor components. A simple empirical illustration based on deprivation data from four Central American countries (Guatemala, El Salvador, Honduras, and Nicaragua) shows the usefulness of the MDI.

Keyword: Multidimensional poverty analysis, Inequality, Gini index, Dominance.

JEL Cassification: I3, I31, I32, D6, D63, O1, H1.

A rank-dependent multidimensional deprivation index (*MDI*) for binary data*

José Espinoza-Delgado† and Jacques Silber‡

March 2023

(*Revised version*)§

Abstract

This paper first shows how to extend the Sen-Shorrocks poverty index to the analysis of multidimensional deprivation, when only dichotomous variables are available to assess deprivation in various domains, the most common case in the literature. More precisely, it introduces the first rank-dependent multidimensional poverty index in the literature, using a counting approach. The resulting multidimensional deprivation index, or *MDI* in short, has a nice graphical representation ("*PUB curve*") that turns out to be an extension of the so-called *TIP* curve of Jenkins and Lambert to the case of multiple deprivations. This graphical representation is similar to the *SD* curve introduced by Lasso de la Vega (2010), but additionally emphasizes the third "*I*" of multidimensional deprivation: inequality. The *MDI* is sensitive to inequality and satisfies quite nice properties, but it cannot be broken down by population subgroups, when a standard decomposition is used, and it does not have the property of dimensional breakdown, as the latter is usually defined in the literature. The paper proves, however, that there exists an alternative decomposition by population subgroups that can be applied to the *MDI*; it also derives a decomposition by deprivation domain, analogous to the breakdown of the Gini index by factor components. A simple empirical

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illustration based on deprivation data from four Central American countries (Guatemala, El Salvador, Honduras, and Nicaragua) shows the usefulness of the *MDI*.

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1 Introduction

To understand the threat posed by the problem of poverty, it is necessary to know the extent of interdependence between the various dimensions of poverty, its determinants, and the process through which it appears to deepen. In this context, an important question concerns the way poverty, and its changes, should be measured (Chakravarty, 2006). As noted by Thorbecke (2007, p. 4), before poverty can be measured, it has at least to be understood conceptually. In this regard, following the seminal contributions of Amartya Sen and his theoretical framework of "capabilities and functionings" (Sen, 1985, 1992, 1993, 2000), as well as earlier work on the measurement of multidimensional welfare and inequality (see, for example, Kolm, 1977; Atkinson and Bourguignon, 1982; Maasoumi, 1986; Tsui, 1995; Maasoumi, 1999; Bourguignon, 1999), our conceptual understanding of poverty has improved considerably in the last four decades or so. There is now quite a consensus on the multidimensional nature of poverty; as a result, since the pioneering works of Atkinson (2003), Bourguignon and Chakravarty (2003), and Tsui (2002), a number of approaches have been proposed in the literature to analyze and measure multidimensional poverty and deprivation (see, for example, Aaberge, Peluso, and Sigstad, 2019; Alkire and Foster, 2011; Bossert, Chakravarty, and D'Ambrosio, 2013; Datt, 2019; Dhongde, Li, Pattanaik, and Xu, 2016; Duclos, Sahn, and Younger, 2008; Kakwani and Silber, 2008; Lemmi and Betti, 2006, 2013; Pattanaik and Xu, 2018; Permanyer, 2014; Rippin, 2013, 2017).

Of particular interest are the works of Chakravarty, Mukherjee and Renade (1998), Tsui (2002), and Bourguignon and Chakravarty (2003), who defined a poverty line for each dimension and then combined these different poverty thresholds and the domain-specific poverty gaps into a multidimensional poverty measure. Atkinson (2003) also made an important contribution, firstly because his paper focused on the contrast between a social welfare approach and a counting approach to the measurement of multidimensional poverty, secondly because it provided a very thorough discussion on how to integrate the interaction between the various dimensions of poverty into the analysis.

Currently, the most popular measurement method in the literature on multidimensional poverty analysis is the counting approach proposed by Alkire and Foster (2011), largely due to the launch of the Global Multidimensional Poverty Index or *global MPI* (Alkire and Santos, 2010, 2014), the most well-known and influential application of this method (Duclos and Tiberti, 2016; Pogge and Wisor, 2016). This methodology proposes the use of a "dual cutoff method" for the identification of the multidimensional poor (Alkire and Foster, 2011, p. 478), an essential and innovative feature of the method (Datt, 2019), which includes the traditional union and intersection approaches as special cases;[1] it also suggests a "class of multidimensional poverty measures (M_α)" for aggregating the information on the poor (Alkire and Foster, 2011, p. 479), which is an extension of the FGT family of monetary poverty measures (see Foster, Greer, and Thorbecke, 1984). Alkire and Foster's approach has, however, some methodological shortcomings and pays no attention to the distribution of deprivation among the poor, which may challenge, for example, the fulfillment of the overarching concern of the SDGs: "leaving no one behind" (Klasen and Fleurbaey, 2019, p. 1): an inequality insensitive poverty measure "can deflect anti-poverty policy by ignoring the greater misery of the poorer among the poor" (Sen, 1992, p. 105). These deficiencies have been mentioned, for example, by Aaberge and Brandolini (2015) and discussed in depth by Pattanaik and Xu (2018), as well as by Datt (2019), Duclos and Tiberti (2016), Espinoza-Delgado and Silber (2021), and Rippin (2017).

A different view of multidimensional deprivation measurement was adopted by Chakravarty and D'Ambrosio (2006), who followed a counting approach and proposed a measure of social exclusion, while Yalonetzky (2014), as well as Silber and Yalonetzky (2013), proposed a general formulation that includes, as special cases, the approaches of Alkire and Foster (2011), Chakravarty and D'Ambrosio (2006), Rippin (2010) and Bossert et al. (2013). Other interesting contributions are those of Aaberge and Peluso (2012) and Aaberge, Peluso, and Sigstad (2019), who assumed that the social poverty function is directly a function of the proportions of individuals with 1, 2,...D deprivations (see also an extension of this approach by Silber and Yalonetzky, 2013).

In the present paper, we focus on discrete variables, in fact on dichotomous (binary) variables, the most common case in the literature. The key contribution of the paper is that it introduces a rank-dependent and an inequality sensitive multidimensional poverty index for

[1] On the union and intersection approaches, see Atkinson (2003).

multiple binary indicators, using a counting approach. The proposed index is an extension of the famous Sen-Shorrocks unidimensional poverty index (Shorrocks, 1995) to the measurement of multidimensional deprivation; we call this extension "*MDI*": *"Multidimensional Deprivation Index"*.

The Sen-Shorrocks index has many useful properties that turn out to have important policy implications when applied to the multidimensional case. Although, in principle, the *MDI* cannot be broken down by population subgroups when a standard decomposability property is applied, and it does not have the standard property of dimensional breakdown, as the latter is usually defined in the literature, we prove that there exists an alternative decomposition by population subgroups that can be applied to this index, which allows us to conclude that the *MDI* does satisfy a modified subgroup decomposability criterion. We also derive a decomposition of the *MDI* by deprivation domain that is analogous to the breakdown of the Gini index by factor components, thus proving that the *MDI* can also be fully decomposed by dimension to inform and coordinate social policies.

Moreover, since the Sen-Shorrocks index can be interpreted graphically, we can compare the deprivation profiles of various countries or of different age groups and regions. Thus, we also extend the *TIP* curve introduced by Jenkins and Lambert (1997; 1998a; 1998b) to the multidimensional case. Note that the graphical representation we obtain is similar to the *SD* curve introduced by Lasso de la Vega (2010), but presented in another context. We also prove that the *MDI* is related to a specific case of the Aaberge et al. (2019) deprivation measure. Finally, a simple empirical illustration focusing on Central American countries (Guatemala, El Salvador, Honduras, and Nicaragua) shows the usefulness of the *MDI*.

The paper is organized as follows. In Section 2, we summarize some previous attempts to measure multidimensional poverty when only binary variables are available. Section 3 introduces what we call the "Multidimensional Deprivation Index (*MDI*)", which is an extension of the approach of Shorrocks (1995) to the case of multidimensional deprivation with dichotomous variables. Section 4 presents the properties of the *MDI*. Section 5 provides an empirical illustration based on data from Central American countries, while Section 6 offers concluding remarks. An Online Appendix provides simple illustrations of the various properties of the *MDI* and of the similarity between the *MDI* and a specific case of the Aaberge et al. (2019) measure.

2 Previous attempts to measure multidimensional poverty when only binary variables are available

2.1 The approach of Chakravarty and D'Ambrosio (2006)

These authors derived axiomatically a measure of social exclusion that can also be interpreted as a measure of multidimensional deprivation, as shown by Jayaraj and Subramanian (2010). Let P be the total number of deprivation dimensions, P_i the number of domains in which individual i is deprived and n the size of the population. The set of poor individuals will be defined as $\{i|P_i = P\}$ when adopting an intersection approach, as $\{i|P_i \geq 1\}$ when assuming a union approach, and as $\{i|P_i \geq r\}$ when adopting the Alkire and Foster (2011) intermediate approach, with r referring to the minimum number of domains in which an individual must be deprived to be considered as "overall poor". Following Chakravarty and D'Ambrosio (2006), when choosing a union approach, one can then define an individual deprivation function d_i as $d_i = 0$, if individual i is not deprived in any dimension, and as $d_i = \left(\frac{P_i}{P}\right)^\alpha$ when individual i is deprived in P_i dimensions, with $\alpha > 0$. The level of deprivation in the society as a whole will then be expressed as

$$D = \left(\frac{1}{n}\right)\sum_{i=1}^{n} d_i = \left(\frac{1}{n}\right)\sum_{i=1}^{n}\left(\frac{P_i}{P}\right)^\alpha = \sum_{j=1}^{P} H_j\left(\frac{j}{P}\right)^\alpha \tag{1}$$

where H_j is the proportion of individuals deprived in exactly j dimensions.

In the specific case where $\alpha = 1$, expression (1) will be written as

$$D = \sum_{j=1}^{P} H_j\left(\frac{j}{P}\right) \tag{2}$$

If we adopt the intermediate approach of Alkire and Foster (2011), expression (1) will be written as

$$D = \sum_{j\geq r}^{P} H_j\left(\frac{j}{P}\right) = \sum_{j\geq r}^{P}\left(\frac{n_j}{n}\right)\left(\frac{j}{P}\right) \tag{3}$$

where n_j refers to the number of individuals who have j deprivations.

But (3) may also be written as

$$D = \left(\frac{\sum_{j\geq r}^{P} n_j}{n}\right)\left(\frac{\sum_{j\geq r}^{P} jn_j}{\left(\sum_{j\geq r}^{P} n_j\right)P}\right) = HA \tag{4}$$

where H is the headcount ratio, when adopting the intermediate approach of Alkire Foster (2011) with an overall threshold of r, while A is what Alkire and Foster (2011) call the average deprivation share across those classified as poor (deprived). Expression (4) refers in fact to what Alkire and Foster (2011, p. 479) called "the adjusted headcount ratio" ("M_0").

2.2 Additional approaches to multidimensional deprivation measurement with binary variables

There have been other attempts to measure multidimensional deprivation when only binary variables are available. As stressed by Dhongde et al. (2016), although in the literature on multidimensional poverty there are quite a few studies using discrete data (e.g., Alkire and Foster, 2011; Bossert et al., 2013; Lasso de la Vega, 2010), relatively few stress the specific case of binary data. In this context, Fusco and Dickes (2006) used binary data but did not propose or derive an index; they used a Rasch (1960) model. Rippin (2010) introduced a multidimensional poverty index for the case of discrete data but did not specially focus on binary data. Finally, Dhongde et al. (2016) made an interesting distinction between basic attributes and non-basic attributes, where each basic attribute has priority over the class of non-basic attributes.

2.3 The original approach of Aaberge et al. (2019)

Aaberge et al. (2019) took a dual approach to multidimensional deprivation and poverty measurement and defined deprivation in society via an indicator D where

$$D = P - \sum_{j=0}^{P-1} \Gamma(F_j) \tag{5}$$

In (5), P, as before, refers to the number of possible deprivations suffered by individuals, and F_j is defined as $F_j = \sum_{h=0}^{j} f_h$, with $f_h = \left(\frac{n_h}{n}\right)$ the relative frequency of those who have h deprivations. Finally, Γ is a non-negative and non-decreasing continuous function that represents the preferences of the social planner, with $\Gamma(0) = 0$ and $\Gamma(1) = 1$. Since the mean number of deprivations $\bar{d}$ may be expressed as

$$\bar{d} = P - \sum_{j=0}^{P-1} F_j \tag{6}$$

Combining (5) and (6), we derive that

$$D = \bar{d} + \sum_{j=0}^{P-1} F_j - \sum_{j=0}^{P-1} \Gamma(F_j) \tag{7}$$

However, the mean difference Δ of a distribution $F(t)$ may be expressed as (see, Yitzhaki and Schechtman, 2013, p. 16)

$$\Delta = 2 \int F(t)[1 - F(t)]dt \tag{8}$$

Adapting (8) to the case of discrete data and to the distribution of deprivations, we derive that

$$\Delta_{d_i} = 2\sum_{j=0}^{P} F_j - 2\sum_{j=0}^{P} (F_j)^2 = 2\left[\sum_{j=0}^{P-1} F_j - \sum_{j=0}^{P-1} (F_j)^2\right] \tag{9}$$

where Δ_{d_i} refers to the mean difference of the deprivations, and we recall that $F_P = (F_P)^2 = 1$.

If we assume in (5) that $\Gamma(F_j) = (F_j)^2$, a case indeed discussed by Aaberge et al. (2019), we conclude, using (6) and (9), that in such a case

$$D = \bar{d} + \left(\frac{1}{2}\right) \Delta_{d_i} \tag{10}$$

3 The derivation of a rank-dependent multidimensional deprivation index when only binary variables are available

3.1 On the extension of Sen's poverty index and poverty gap profiles

3.1.1 On Shorrocks' (1995) extension of the Sen (1976) index

Let n denote the population size, x_i the income of individual i, z the poverty line, and q the number of people with income $x_i \leq z$. Sen (1976) derived axiomatically a poverty index that is expressed as

$$P_{Sen} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q} (2q - 2i + 1) \left(\frac{z - x_i}{z}\right) \tag{11}$$

Defining x_i^* as $x_i^* = Min\{x_i, z\}$, Shorrocks (1995) extended Sen's index and proposed to define a poverty index ($P_{Sen-Shorrocks}$) as

$$P_{Sen-Shorrocks} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1) \left(\frac{z - x_i^*}{z}\right) = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q} (2n - 2i + 1) \left(\frac{z - x_i}{z}\right) \tag{12}$$

Shorrocks (1995) stressed that P_{Sen} in (11) is not replication invariant, not a continuous function of individual incomes and fails to satisfy the transfer axiom, whereas the $P_{Sen-Shorrocks}$ index is symmetric, replication invariant, monotonic, homogeneous of degree zero in z (poverty line) and x (income), normalized, continuous and consistent with the transfer axiom.

3.1.2 On poverty gap profiles or the so-called *TIP curve*

There has also been a graphical representation of unidimensional poverty: plot on the horizontal axis the cumulative relative frequencies of the population and on the vertical axis the cumulative values of the expression $\left(\frac{1}{n}\right) Max\{(\frac{z-x_i}{z}), 0\}$, ranking the individual by non-decreasing income; a "poverty gap profile" is then obtained (Shorrocks, 1995), which is also called TIP curve (Jenkins and Lambert, 1997; 1998a; 1998b). Shorrocks (1995) then proved that the Sen-Shorrocks index is equal to twice the area below the poverty gap profile.

3.2 Multi-dimensional deprivation in the case of dichotomous variables

3.2.1 Deriving deprivation profiles in the multi-dimensional case

Assume n individuals, P dimensions of well-being, and a dichotomous variable a_{ij} equal to 1 if individual i has an achievement in domain j (e.g., if j refers to "having a good health", $a_{ij} = 1$ if individual i is in good health, to 0 otherwise). Let a_i be defined as

$$a_i = \sum_{j=1}^{P} w_j a_{ij} \tag{13}$$

where w_j is the weight of dimension j and $\sum_{j=1}^{P} w_j = 1$.

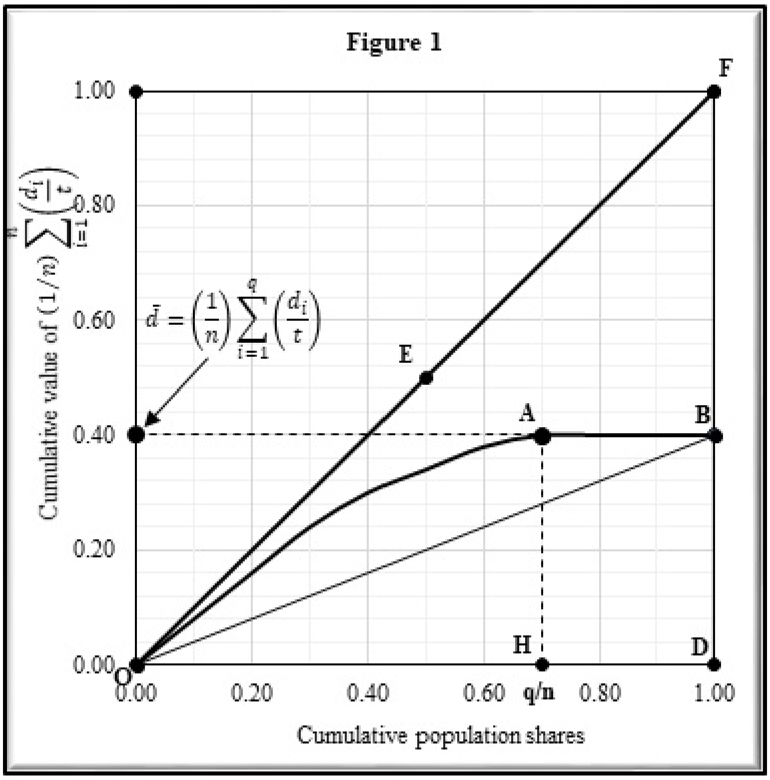
If we define d_{ij} as $d_{ij} = (1 - a_{ij})$, so that $d_{ij} = 1$ if individual i is deprived in domain j, to 0 otherwise, the weighted deprivation score (d_i) for individual i will be expressed as

$$d_i = \sum_{j=1}^{P} w_j d_{ij} \tag{14}$$

The achievement score (a_i) is a "good", so that traditional tools of distributional analysis can be used (e.g., the Lorenz or Generalized Lorenz curves); however, the deprivation score (d_i) is a "bad" (see, Shorrocks, 1998), so that a decrease in an individual's deprivation or in the inequality of the deprivation scores leads to a decrease in "aggregate deprivation".

The concept of poverty gap profile or TIP curve previously mentioned may be also applied in the context of multidimensional deprivation. Define an achievement threshold t, compute the normalized achievement gaps $\left(\frac{d_i^*}{t}\right) = Max\{(\frac{t-a_i}{t}), 0\} = Max\{\frac{d_i}{t}, 0\}$, and then plot on the horizontal axis the cumulative population shares and on the vertical axis the cumulative sum of the expressions $m_i = (\frac{1}{n}) \sum_{i=n}^{1} (\frac{d_i^*}{t}) = (\frac{1}{n}) \sum_{i=1}^{q} (\frac{d_i}{t})$, the d_i^*'s being ranked by non-increasing values; we obtain a rising curve whose slope is non-decreasing and equal to 0 when we reach the $(n - q)$ individuals with no deprivation (there are q individuals with at least one deprivation). The curve is similar to the TIP curve previously mentioned (see Figure 1).

Note that if $t = 1$, $\left(\frac{d_i^*}{t}\right) = Max\{\left(\frac{1-a_i}{1}\right), 0\} \leftrightarrow d_i^* = Max\{(1 - a_i), 0\} = Max\{d_i, 0\}$



- In Figure 1, OH refers to the *"prevalence"* (P) or *incidence* of deprivation [proportion (q/n) of individuals having some deprivation].

- The slope BOD equals $(BD/OD) = \left[\frac{(1/n)\sum_{i=1}^{n} d_i^*}{1}\right] = \frac{(1/n)\left[\sum_{i=1}^{q} d_i\right]}{1} = \left(\frac{q}{n}\right)\left(\frac{\sum_{i=1}^{q} d_i}{q}\right) = \left(\frac{q}{n}\right)\bar{d_q}$

where $\bar{d_q}$ represents the average percentage of deprivations among those who have at least one deprivation; $\bar{d_q}$ could be labeled the *"breadth" (B)* or $''intensity''$ of deprivation.

- The curvature of the OA curve indicates the extent of inequality among those deprived in at least one dimension or the *"unevenness" (U)* or inequality of deprivation.

This "deprivation curve" (OAB) is actually an adaptation of the TIP curve to multidimensional deprivation with dichotomous variables; given that this curve takes into account the "prevalence", the "unevenness" and the "breadth" of deprivation, we propose to call it the "*PUB curve*".[2]

[2] It is worthy to note that Lasso de la Vega (2010) also introduced deprivation curves, derived from deprivation counts, and called them the FD and the SD curves. The focus of the FD curve is on the multidimensional headcount ratio, while the SD curve shows on the same graph the "headcount ratio, the adjusted headcount ratio, and the average deprivation share according to Alkire and Foster (2007)" (p. 156). The PUB curve is similar to the SD curve, but, in addition, it does emphasize the third "I" of multidimensional deprivation: inequality ("unevenness"). Furthermore, it should be observed that Alkire and Foster's methodology (2007, 2011), from which the FD and SD curves of Lasso de la Vega (2010) are derived, pays no attention to the

3.2.2 Deriving a rank-dependent multidimensional deprivation index (MDI) when only binary variables are available

As Shorrocks (1995) showed for the uni-dimensional case, it is possible to prove that twice the OABDHO area in Figure 1 is equal to a "Multidimensional Deprivation Index" (MDI).

More precisely, we may write that

$$MDI = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1) \left(\frac{d_i^*}{t}\right) = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q} (2n - 2i + 1) \left(\frac{d_i}{t}\right) \tag{15}$$

With a union approach (an individual is deprived even if in only one domain), $t = 1$ and then

$$MDI_{union} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1)\, d_i^* = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{q} (2n - 2i + 1)\, d_i \tag{16}$$

Using (15), the contribution ($Cont_i$) of individual i to the overall deprivation is expressed as

$$Cont_i = 2\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{d_i}{t}\right)\left[\left(\frac{2n+1}{2}\right) - i\right] \tag{17}$$

Following Shorrocks' (1995) work, it is easy to show that

$$MDI = \bar{d}\,(1 + G_{d_i}) = \bar{d}\left[1 + \left(\frac{d_{EQ} - \bar{d}}{\bar{d}}\right)\right] = d_{EQ} = \bar{d} + (\frac{1}{2})\Delta_{d_i} \tag{18}$$

where $\bar{d}$ and G_{d_i} are respectively the average level of deprivation and the Gini index of the deprivation scores in the whole population (including those who have no deprivation), $\Delta_{d_i} = 2\bar{d}G_{d_i}$ is the mean difference of the deprivations, and d_{EQ} is the "equally distributed equivalent deprivation score".[3]

We may observe that expressions (10) and (18) are identical, so that the MDI is a specific case of the deprivation measure of Aaberge et al. (2019), the one where $\Gamma(F_k) = (F_k)^2$.

deprivation distribution, and it is hence insensitive to the extent of inequality among the multidimensionally poor people.

[3]It is well known that the Gini index of incomes I_G, like several other income inequality indices that can be related to a welfare function, may be expressed as $I_G = (\frac{\bar{y}-y_E}{\bar{y}})$, where $\bar{y}$ refers to the average income and y_E to Atkinson's (1970) concept of "equally distributed equivalent level of income" applied to the Gini welfare function. While income is a "good", deprivation is a "bad" so that the Gini index of the deprivation scores is defined as $G_{d_i} = \frac{(d_{EQ}-\bar{d})}{\bar{d}}$.

Rather than using the traditional Gini index G_{d_i}, we can also use the generalized Gini index introduced by Donaldson and Weymark (1980) and apply it to the deprivation scores. The "equally distributed equivalent deprivation score" $d_{EQ,GEN}$ in such a case uses the concept of "ill-fare ranking" (Donaldson and Weymark, 1980) so that

$$d_{EQ,GEN} = \sum_{i=1}^{n} \left(\frac{i^{\beta} - (i-1)^{\beta}}{n^{\beta}} \right) d_i \tag{19}$$

with $0 \leq \beta \leq 1$ and, evidently, $d_1 \geq ... \geq d_q \geq ...0$.

In the case of tied ranks, we can apply the procedure described in Deutsch and Silber (2005) in the case of occupational segregation.[4]

3.3 Estimating the contribution of different population subgroups to the *MDI*

Assume K population subgroups, each subgroup k with n_k individuals. Using (15), we write

$$MDI = \left(\frac{1}{n} \right)^2 2 \sum_{k=1}^{K} \sum_{i \in k} \left(\frac{d_i}{t} \right) \left[\left(\frac{2n+1}{2} \right) - i \right] \tag{20}$$

i being the ranking of the individual in the whole population and not in his/her subgroup.

The contribution C_k of population subgroup k to multidimensional deprivation is hence

$$C_k = \left(\frac{1}{n} \right) \left(\frac{1}{n} \right) 2 \sum_{i \in k} \left(\frac{d_i}{t} \right) \left[\left(\frac{2n+1}{2} \right) - i \right] \tag{21}$$

3.4 Making assumptions concerning the weight of the different deprivation domains

Let j refer to a given deprivation domain with $j = 1$ to J. Combining (14) and (15), we derive

[4] Let us rank the deprivation scores d_i by decreasing values. Call f_i the population frequency of deprivation score d_i and s_i the share of deprivation score d_i in the total amount of deprivation in the society. Define a variable a_i as $a_i = \left(\sum_{j=1}^{i} f_j \right)^{\beta} - \left(\sum_{j=1}^{i-1} f_j \right)^{\beta}$. Deutsch and Silber (2005) have then shown that the generalized Gini index (I_{GG}) may be expressed as $I_{GG} = 1 - \left[\sum_i a_i \left(\frac{s_i}{f_i} \right) \right]$. A similar procedure may be applied to the MDI.

$$MDI = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n}\sum_{j=1}^{J} \frac{w_j d_{ij}}{t}\left(2n - 2i + 1\right) \tag{22}$$

so that the contribution $CONTR_j$ of deprivation in domain j to the overall deprivation becomes

$$CONTR_j = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} w_j \left(\frac{d_{ij}}{t}\right)\left(2n - 2i + 1\right) \tag{23}$$

There are quite a few possibilities as far as the choice of the weights w_j of the various dimensions are concerned. In a recent paper, Dutta et al. (2021) have however shown that endogenous (data driven) weights violate key properties of poverty indices, namely "monotonicity" and "subgroup consisteny". They hence recommend using exogenous weights, the simplest case being that where all the deprivation domains have the same weight. We will make this assumption so that we rewrite (23) as

$$CONTR_j = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \left(\frac{1}{J}\right)\left(\frac{d_{ij}}{t}\right)\left(2n - 2i + 1\right) \tag{24}$$

3.5 Comparing the approach of Chakravarty and D'Ambrosio with that of the MDI

There is a clear parallelism between expressions (1) and (16). In (1), deprivation in society is defined as the arithmetic average of the individual deprivations, each individual deprivation d_i being a function of the percentage of possible deprivations suffered by individual i. When the parameter α is equal to 2, for example, this individual deprivation is not only higher the higher the number of domains in which the individual is deprived, but this individual deprivation also increases at an increasing rate with the number of deprivations suffered.

In expression (16), deprivation in society is a weighted average of the individual deprivations; here the individual deprivation d_i is simply a weighted or unweighted average of the number of deprivation domains in which the individual is deprived. But the weight of each individual deprivation d_i is higher the higher the number of deprivations, hence the term "rank-dependent multidimensional deprivation index" that appears in the title of our paper. In expression (16), these weights increase in a linear way, but in expression (19), the parameter β may be chosen in such a way that the weights increase at an increasing rate.

4 Properties of the *MDI*

In this section, we discuss the properties of the MDI in a simple and intuitive way, based on Alkire and Foster (2016). As stressed previously, the MDI is an extension of the Sen-Shorrocks poverty index applied to the weighted deprivation scores d_i; therefore, all the properties of the Sen-Shorrocks index stated by Shorrocks (1995) and mentioned above hold for the MDI as well.

Alkire and Foster (2016) have stated that the properties of multidimensional poverty methodologies can be classified into three categories: $invariance$, $subgroup$, and $dominanceproperties$.

4.1 Invariance properties

Invariance properties include those of $symmetry$, $replicationinvariance$, $deprivationfocus$ and $povertyfocus$.

Symmetry

The reference here is to "permutations of achievement vectors across individuals". As stressed by Shorrocks (1995), the Sen-Shorrocks poverty index has this property.

Population replication

Assume a "cloning" of the whole population, so that the total population and the number of deprived individuals are now respectively equal to (λn) and (λq), with λ an integer greater than 1. We assume no change in the number of dimensions. In addition, any deprived individual (i) with a deprivation score d_i will be replaced by λ individuals with this deprivation score d_i. Here again, Shorrocks (1995) stated that such a property holds for the Sen-Shorrocks poverty index.

Poverty focus

This assumption says that an increment in the achievement of a non-deprived person, that is, of an individual who is not deprived in any dimension, will not affect the value of the multidimensional deprivation index (MDI). This should be clear from equations (15) and (16), since the MDI is only a function of the deprivation of the deprived individuals.

Deprivation focus

This property assumes that the multidimensional deprivation index (MDI) will be invariant to an increment in a non-deprived achievement. It is easy to check this property too, since if an individual i improves his/her achievement in a dimension j in which he/she was not deprived, the value of the dichotomous variable d_{ij} will not vary and remain equal to 0.

4.2 Subgroup properties

Alkire and Foster (2016) have also mentioned the properties of *subgroup consistency* and *subgroup decomposability*.

Subgroup decomposability

The expression for the contribution of subgroup k to the overall deprivation (MDI) is given in (21). Combining (20) and (21), we conclude that

$$MDI = \sum_{k=1}^{K} C_k \tag{25}$$

We can therefore compute the contribution of each subgroup to the overall level of deprivation. Note however that C_k in (22) is not identical to what would be the definition of an MDI limited to group k. This is so because the coefficient $\left[\left(\frac{2n+1}{2}\right) - i\right]$ associated to the deprivation component $\left(\frac{d_i}{t}\right)$ of individual i depends on the rank of individual i in the whole population, and not in subgroup k. A subgroup decomposable deprivation index would be expressed as the sum of a between and a within groups deprivations. But this is not what (21) is expressing; consequently, we cannot conclude that the "Multidimensional Deprivation Index (MDI)" is subgroup decomposable in the traditional interpretation of such a breakdown. This is also the case of the Gini index, since it is well known that, as soon as there is some overlap between the population subgroups, the decomposition of the Gini index will include three components: a between and a within groups inequality, and also a residual, which has been shown to be a measure of the overlap between the different distributions (see, for example, Silber, 1989).

It is however possible to take an alternative view of the breakdown of the MDI by population subgroups. To derive such an alternative decomposition, we borrow ideas from the literature on alternative decompositions of the Gini index. Deutsch and Silber (1999) have indicated that there is no unique way of decomposing inequality by population subgroups; these scholars have mentioned a decomposition of the Gini index, originally proposed by Lerman and Yizthaki (1991) and Sastry and Kelkar (1994), where the Gini index turns out to be the sum of a between and within groups components, but these two components are not defined in the traditional way. The idea is to keep the original ranking of the individuals, when computing these between and within group components; this idea may be also applied to the breakdown of the MDI into a between and a within groups components. Therefore, the alternative between groups MDI is then defined as

$$MDI^{Alternative}_{BETWEEN} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1) \left(\frac{\bar{d}_i}{t}\right) \tag{26}$$

where i refers to the original rank of an individual,while $\bar{d}_i$ refers to the average deprivation level in the population subgroup to which individual i belongs.

The alternative within groups component is then expressed as

$$MDI^{Alternative}_{WITHIN} = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} (2n - 2i + 1) \left[\frac{(d_i - \bar{d}_i)}{t}\right] \tag{27}$$

In Appendix A, we give a simple empirical illustration of what we called the traditional and the alternative decompositions of the MDI. Figure A.1 also shows a graphical representation of the alternative decomposition.

In consequence, when using the alternative approach, it is possible to affirm that the MDI is decomposable by population subgroups.[5]

Subgroup consistency

Shorrocks (1995, p. 1226) stressed that, like the Sen poverty index P_{Sen}, the Sen-Shorrocks poverty index ($P_{Sen-Shorrocks}$) is not subgroup consistent, although "it is an ideal measure of poverty in all other respects". Since the MDI is equivalent to the $P_{Sen-Shorrocks}$ index, but applied to multidimensional deprivation, we can conclude that the MDI is not subgroup consistent. However, although the MDI violates this "standard" decomposability, it does satisfy, as mentioned previously, a "modified" decomposability criterion, in which subgroup ranks of individuals are replaced by the ranks of these same individuals in the overall population, as was already stressed by Podder (1993).[6] As stressed by Foster and Sen (Sen, 1997), two routes can be taken to measure poverty. "The first is to go for subgroup consistency in an emphatic way,

[5] One may however wonder how convenient this alternative approach is for policy purposes. It has been pointed to us that India has recently released its computations of multidimensional poverty for more than 600 districts. Since our approach suggests that the population should be ranked at the country level and then those ranks should be used at all subgroup level, for a very large country, like India, this may not be easy to implement, particularly if census data are used.

[6] We thank Subbu Subramanian for drawing our attention to this paper of Podder and for suggesting that we could label this decomposability property "modified subgroup consistency".

which certainly makes it easy to relate the poverty of each group to the poverty of its constituent subgroups. The other route is to try to capture the interdependence in people's perception of poverty and perhaps even actual well-being, and build these interdependences into the measure of poverty itself" (Sen, 1997, pp. 183-184) ... Then "each person's deprivation is judged by taking into account not only the gap from the externally given poverty line, but also the relative positioning of any poor *vis-à-vis* others" (Sen, 1997, p. 186).[7]

Dimensional breakdown

The dimensional breakdown or factor decomposability property technically requires that "after identification has taken place" and the poverty status of each person has been fixed, multidimensional poverty can be expressed as a "weighted sum of dimensional components" (Alkire and Foster, 2019, p. 13). This implies, following Chakravarty et al. (1998), that the overall poverty index is a weighted sum of the poverty measures of the various dimensions, these measures being only function of the distribution of the individual achievements in the corresponding dimension and of the threshold selected for this dimension.

However, this is not the case for the MDI, since the individual level weight $(2n - 2i + 1)$ we use to compute each dimensional component depends on the overall ranks of the poor, so any change in the joint distribution is likely to change the rank of the individuals and thus the value of each component. But, in the traditional decomposition of the Gini index by income sources, it is generally stated that each source's contribution is "the product of its own inequality, its share of total income, and its correlation with the rank of total income" (Lerman and Yitzhaki, 1985, p. 153).[8] Now clearly the poverty dimensions in a multidimensional framework play the role of the income sources in a unidimensional analysis of inequality; therefore, while it cannot be said that our MDI has the property of dimensional breakdown in the way Chakravarty et al. (1998) and Alkire and Foster (2019) have defined this feature, we can assert that when using the MDI to measure (multidimensional) poverty, the contribution of each dimension to the overall value of the MDI can easily be computed, so we can conclude that the MDI does satisfy an alternative dimensional breakdown property, which is analogous to the breakdown of the Gini index by factor components.

[7] Here again we thank Subbu Subramanian for having reminded us of what Foster and Sen wrote on the normative significance of sub-group consistency in the expanded version of Sen's *"On Economic Inequality"* (Sen, 1997).

[8] See also, Fei, Ranis and Kuo (1978) for a previous presentation of the decomposition of the Gini index by factor components, that is, by income sources.

4.3 Dominance

Alkire and Foster (2016) have included here two properties. There is first the concept of *Weak Monotonicity* according to which an increase in the achievement of an individual cannot increase deprivation. Then, there is the notion of *Weak Rearrangement* that requires that a progressive transfer among the deprived individuals, which is the consequence of an "association-decreasing rearrangement", cannot increase deprivation.

Monotonicity

Shorrocks (1995) stated that the index $P_{Sen-Shorrocks}$ is monotonic. We can therefore conclude that the (MDI) has the property of monotonicity.

Transfers

Let us first state that in the context of uni-dimensional poverty measurement, Shorrocks (1995) stressed that the $P_{Sen-Shorrocks}$ index is consistent with the transfer axiom. When applying this property to multidimensional deprivation analysis, we can therefore conclude that if, within a given deprivation domain j, a transfer takes place from a more to a less deprived individual, assuming no change in the ranking of the individuals, the MDI will decrease. More precisely, assume that originally individual i, as a whole, was more deprived than individual m and was deprived in domain j while individual m was not. After the "transfer" individual i remains more deprived than individual m, but he/she has one deprivation less, while individual m has one more deprivation than originally. In such a case, the MDI will decrease.

The same kind of reasoning applies when a transfer takes place between individuals and across domains. Assume, for example, that individual h has n_h deprivations and that individual i has n_i deprivations with $n_h > n_i$, that individual h is deprived in domain j but not in domain k and individual i in domain k but not in domain j. If, for some reason, a change occurs such that individual h is not deprived any more in domain j while individual i, who was deprived in domain k, becomes also deprived in domain j. Assume, however, that, after such a "transfer" of deprivations, individual h has still more deprivations than individual i. If we assume that all the domains have the same weight, it is easy to observe, using (23), that the MDI will decrease.

Given that in the formulation of the MDI in (23), which refers to the case of equal weights, only the number of deprivations of each individual is taken into account, no matter in which domains these deprivations take place, the notion of *"Weak Dimensional Rearrangement among* *the deprived individuals"*, which was discussed by Alkire and Foster (2016), is not relevant.

Rather than analyzing the impact of a transfer of deprivations between two individuals h and i, let us assume that these two individuals switch their deprivations. In other words, using the example given previously, we would observe that in the new situation individual h is deprived in domain k but not in domain j and individual i in domain j but not in domain k. Clearly such a switch will not affect the number of deprivations of each individual and hence there will be no change in the value of the MDI.

The conclusions are different when examining the case of unequal weights. It should be clear that even in the case where the various dimensions have different weights, a transfer of deprivations between two individuals of the kind described above, whether it takes place within a given domain or across domains, will lead to a decrease in the MDI, as long as the ranking of the individuals by the number of deprivations suffered by them is not affected. However, when the deprivation domains have not the same weight, the switch of deprivations between two individuals and two domains with unequal weights will lead either to an increase or a decrease in the value of the MDI, depending on the assumption made concerning the weights of domains j and k.

4.4 Comparing deprivation profiles and comparing *MDI* indices

Lasso de la Vega (2010) defined what she called a FD curve, a curve that represents the multidimensional headcount ratio for all possible dimension cutoffs. She stressed the similarity between this curve and the deprivation distribution profile introduced by Jayaraj and Subramanian (2010). She then defined also what she called a SD curve, which represents in the same picture the headcount ratio, the adjusted headcount ratio, and the average deprivation share defined by Alkire and Foster in a 2009 working paper that was published in 2011 (Alkire and Foster, 2011). Lasso de la Vega then proved the equivalence between dominance of one of the SD curves over another, and the values of the corresponding multidimensional poverty measures MP that she also defined and that were assumed to obey the following five axioms:

- Poverty focus: the multidimensional poverty measure MP remains unchanged if the poverty score of an individual defined as "overall non-poor" decreases. - Dimensional monotonicity: the multidimensional poverty measure MP will decrease if the poverty score of any individual defined as "overall poor" decreases.

- Symmetry: No other characteristic, except the number of weighted dimensions in which an individual is deprived, will affect the multidimensional poverty measure MP.

- Replication invariance: A "cloning" of the deprivation vector of all the individuals will not affect the multidimensional poverty measure MP.

- Distribution sensitivity: A decrease in poverty, due to a decrease in the poverty score of a poor individual, should be greater, the higher the poverty score of this individual.

In other words, when the SD curve of a deprivation vector d' lies above the SD curve of a deprivation vector d with the same or different population sizes, any poverty measure having the five properties listed above will rank in the same way these two deprivation vectors.

Note that the dimension adjusted headcount ratio [the ratio of the number of weighted deprivations suffered by those defined as "overall deprived" and the total (maximum) number of weighted deprivations] introduced by Alkire and Foster (2011) violates the distribution sensitivity axiom. Lasso de la Vega proved however that if two deprivation vectors (corresponding to two different societies) can be unanimously ranked by the dimension adjusted headcount ratio, whatever the value of the dimension cutoff, then all poverty counting measures satisfying the property of distribution sensitivity will rank societies in the same way. Lasso de la Vega (2010) also examined the case of intersecting SD curves and showed that it is possible to obtain robust conclusions provided one restricts the set of identification cutoffs.

The question is whether we can find a similar correspondence between the ranking of SD curves and the MDI. The ordinal approach to uni-dimensional poverty analysis seems to have been originally introduced by Spencer and Fisher (1992). Jenkins and Lambert (1997, p. 317) then introduced the concept of TIP ("Three I's of Poverty") curves. Subsequently, Jenkins and Lambert (1998b, p. 47) stated in their Theorem 3 that "given any two income distributions x and y and poverty lines z_x and z_y, TIP dominance of the normalized poverty gap distribution Γ_y over the normalized poverty gap distribution Γ_x is necessary and sufficient to ensure $Q(x \mid k.z_x) \leq Q(y \mid k.z_y)$ for all $k \in (0, 1]$ and for all povety measures $Q \in \mathbf{Q}$", the latter being replication invariant and increasing Schur-convex functions of the normalized gaps. These deprivation profiles or TIP curves may naturally be used when adopting the $P_{Sen-Shorrocks}$ rather than the P_{Sen} index, as shown in Shorrocks (1995).

The MDI introduced in the present paper is an extension of the $P_{Sen-Shorrocks}$ index to the case of multidimensional deprivation. Moreover, we have mentioned previously that Lasso de la Vega's SD curve is a simple adaptation of the notion of TIP curve to the multidimensional case, when one assumes that deprivation in a given domain is only measured via dichotomous variables. It seems therefore that the theorem of Jenkins and Lambert (1998b) stated above could be applied, provided that the deprivation profiles of the distributions we compare do not intersect.[9]

5 A simple empirical illustration

In this section, we present a simple empirical illustration of the PUB curve, the MDI and its decomposition by deprivation indicator, using data from four Central American countries, namely, Guatemala, El Salvador, Honduras, and Nicaragua (for previous work on multidimensional poverty in these countries, using other approaches, see Espinoza-Delgado and Silber, 2018, 2021).

To estimate multidimensional poverty in these Central American countries, we used data from the Guatemala National Survey of Living Conditions (2014) (GUA-ENCOVI2014), the El Salvador Multipurpose Household Survey (2016) (ELS-EHPM2016), the Honduras Multipurpose Household Survey (2013) (HON-EPHPM2013), and the Nicaragua National Household Survey on Living Standards Measurement (2014) (NIC-EMNV2014), which are nationally representative. In our exercise, we focus on individuals who are between 18 and 59 years old, are identified as household members and completed a full interview; in other words, we use the individual, rather than the household, as the unit of analysis and focus on the adult members of the households, approximately 50% of the population in the countries studied (from a low of 47.7% in Honduras up to a maximum of 59.3% in El Salvador).

Regarding the empirical design of the MDI, we considered five deprivation dimensions (education, employment, water and sanitation, energy and electricity, and the quality of the dwelling) with ten indicators, which are certainly among the most significant aspects of individual well-being (Stiglitz et al., 2009a, 2009b). The specific indicators chosen for each of the five dimensions and the corresponding deprivation definitions are presented in Table 1; this table also shows the weighting structure that we used: equal-nested weights.

[9] In a recent paper, Azpitarte et al. (2020) introduced fundamental conditions whose fulfilment is both necessary and sufficient to ensure that poverty comparisons are robust to changes in individual poverty functions, dimensional weights and poverty cut-off. As stated by the authors, these conditions may be cumbersome when the number of variables is large. This is the reason why they also derive conditions whose fulfilment is necessary, but insufficient for robust first- and second-order poverty comparisons. The extension of the Sen-Shorrocks index to multidimensional poverty proposed in the present paper might be a simpler way of analyzing dominance.

The PUB curve: prevalence (P), unevenness (U) and deprivation breadth (B) curve

We assumed that the threshold t was equal to 1. Figure 2 displays the PUB curve for Guatemala, El Salvador, Honduras, Nicaragua, and Central America as a whole; in this figure, the cumulative population frequencies are plotted on the X-axis, while the cumulative values of $\left(\frac{1}{n}\right)\sum_{i=1}^{n}\left(\frac{d_i}{1}\right)$ are plotted on the Y-axis. Overall, the left side of Figure 2 suggests that in the Central American region, the highest and lowest levels of multidimensional poverty are found in Guatemala and El Salvador, respectively. The PUB curve of Honduras dominates that of El Salvador, so that multidimensional poverty in the former country is always higher than in the latter, regardless of the population decile we choose. The cases of the Guatemalan and Nicaraguan curves are interesting. Figure 2 shows that the Nicaraguan curve crosses the Guatemalan curve once from above around the 25% point on the horizontal axis (see the right side of the figure), suggesting that overall multidimensional poverty is higher in Guatemala than in Nicaragua only from this point on, i.e., the poorest of the poor are in Nicaragua.

Table 2 illustrates the contribution of the different domains to the overall deprivation for the case of Guatemala, El Salvador, Honduras, and Nicaragua, as well as for Central American as a whole. Table 2 presents the absolute and relative contributions to the overall estimate of multidimensional poverty of each of the ten indicators used to measure multidimensional poverty in Central America; the overall estimates are shown in the last column of the table. The table indicates that in Central America, education is the largest contributor to multidimensional poverty; deprivations in this dimension accounts for one-third of the estimated MDI in each of the countries.

Table 1: Dimensions [in parenthesis the related Sustainable Development Goal (SDG)], indicators, weigh

Dimensions	Indicators	Weights (%)	Deprivation indicators: He / Sh
1. Education (Goal 4 of the SDGs)	1.1. Schooling achievement	20	has not completed lower seconda imately).
2. Employment (Goal 8 of the SDGs)	2.1. Employment status	20	is unemployed, employed withou tic worker or an unpaid care w have a job" but was available to
3. Water and sanitation (Goal 6 of the SDGs)	3.1. Improved water source	10	does not have access to an impr out of the house and yard/plot.
	3.2. Improved sanitation	10	only has access to an unimpro without treatment or a toilet f ravine) or to a shared toilet faci
4. Energy and electricity (Goal 7 of the SDGs)	4.1. Type of cooking fuel	10	is living in a household which us cooking fuel.
	4.2. Access to electricity	10	does not have access to electrici
5. Quality of dwelling (Goal 11 of the SDGs)	5.1. Housing materials	5	is living in a house with dirt floor and similar, other precarious m (waste, cardboard, tin, cane, pal
	5.2. People-per-bedroom	5	has to share a bedroom with tw
	5.3. Housing tenure	5	is living in an illegally occupied
	5.4. Assets	5	does not have access to more th Radio, TV, Refrigerator, Motor

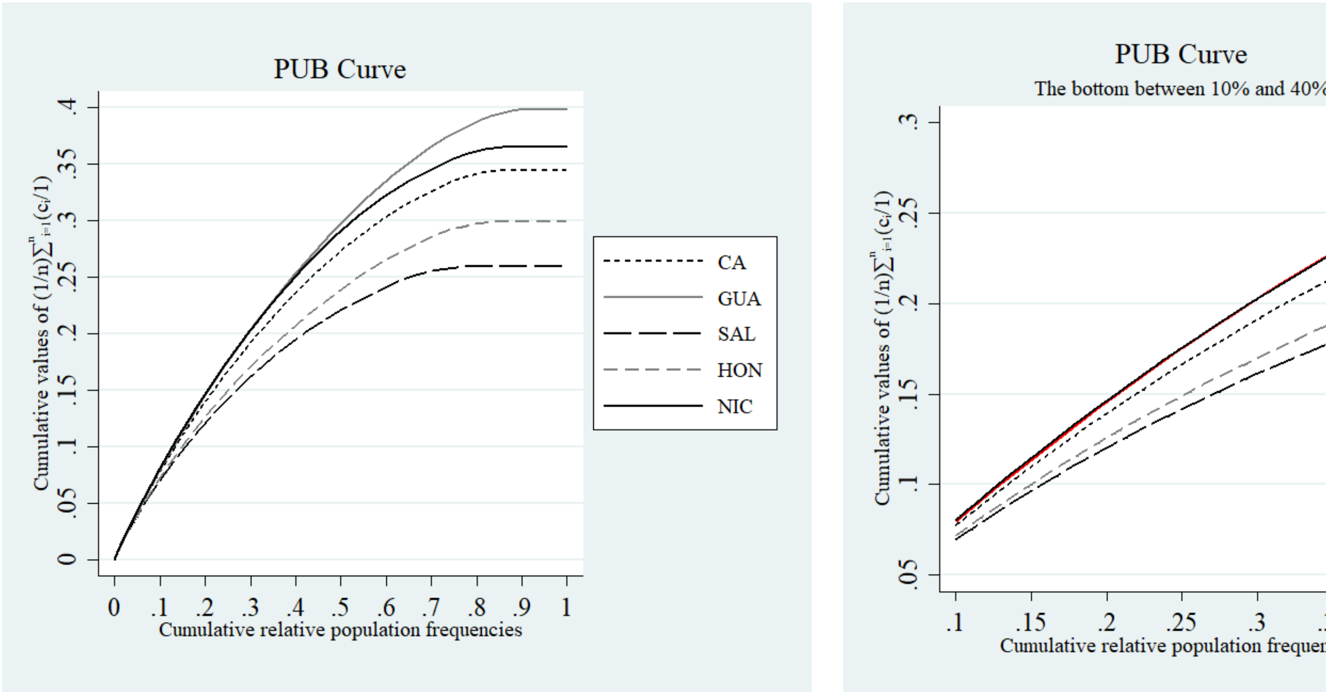


Figure 2: "*PUB curve*" for Central American as a whole (CA), Guatemala (GUA), El Salvador (SAL), Ho
(NIC). *Source*: Authors' estimates based on GUA-ENCOVI2014, ELS-EHPM2016, HON-EPHPM2013, and N

Table 2: Absolute and relative contributions of each indicator to the overall MDI. *Sources*: Authors' estimates
ELS-EHPM2016, HON-EPHPM2013, and NIC-EMNV2014.

Guatemala

Contrib.	**Education**	**Employment**	**Water**	**Sanitation**	**Energy**	**Electricity**	**Housing**	**Overcrowding**	**Ho**
Absolute	0.2645	0.0664	0.0471	0.1107	0.1444	0.0360	0.0309	0.0495	0.0
Relative	33.2%	8.3%	5.9%	13.9%	18.2%	4.5%	3.9%	6.2%	1.3

El Salvador

Contrib.	**Education**	**Employment**	**Water**	**Sanitation**	**Energy**	**Electricity**	**Housing**	**Overcrowding**	**Ho**
Absolute	0.1736	0.0726	0.0417	0.0849	0.0204	0.0256	0.0184	0.0450	0.0
Relative	33.6%	14.0%	8.1%	16.4%	3.9%	4.9%	3.6%	8.7%	3.5

Honduras

Contrib.	**Education**	**Employment**	**Water**	**Sanitation**	**Energy**	**Electricity**	**Housing**	**Overcrowding**	**Ho**
Absolute	0.2365	0.0644	0.0247	0.0466	0.1100	0.0242	0.0177	0.0439	0.0
Relative	39.6%	10.8%	4.1%	7.8%	18.4%	4.1%	3.0%	7.3%	0.9

Nicaragua

Contrib.	**Education**	**Employment**	**Water**	**Sanitation**	**Energy**	**Electricity**	**Housing**	**Overcrowding**	**Ho**
Absolute	0.2247	0.0786	0.0676	0.0860	0.1051	0.0265	0.0394	0.0535	0.0
Relative	30.8%	10.8%	9.3%	11.8%	14.4%	3.6%	5.4%	7.3%	2.2

Central America as a whole

Contrib.	**Education**	**Employment**	**Water**	**Sanitation**	**Energy**	**Electricity**	**Housing**	**Overcrowding**	**Ho**
Absolute	0.2342	0.0693	0.0448	0.0876	0.1066	0.0298	0.0272	0.0481	0.0
Relative	34.0%	10.1%	6.5%	12.7%	15.5%	4.3%	3.9%	7.0%	1.7

Note: surveys weights used.

6 Concluding comments

In this paper, we have introduced a new Multidimensional Deprivation Index (MDI) that is an extension of the Sen-Shorrocks index of unidimensional poverty to the multidimensional case. Interestingly, it turns out that the MDI is a particular case of a measure of multidimensional deprivation recently introduced by Aaberge et al. (2019). In addition, by linking the MDI to the Sen-Shorrocks index, we have been able to derive a simple graphical representation that takes into account the prevalence (incidence), unevenness (inequality) and breadth (intensity) of deprivation. This curve is an extension of the TIP curve of Jenkins and Lambert (1997) to the multidimensional case and turns out to be similar to the SD curve introduced by Lasso de la Vega (2010), although based on a different approach. It is therefore possible to compare the deprivation profiles of two or more countries, or of a country during various periods, and to derive dominance relationships. The MDI can be broken down by population subgroup, although it is not a subgroup consistent index, but "it is an ideal measure of poverty in all other respects" (Shorrocks, 1995, p. 1226). We also showed that while the MDI does not have the property of dimensional breakdown, we can compute the contribution of each domain to the overall deprivation, in the same way as in the literature on the Gini index one can compute the contribution of each income source to the Gini index or income inequality. These two decompositions, which may be considered as not standard, should allow policy makers to detect the population subgroups and the deprivation domains that require special attention. The empirical illustration of the paper, which looked at four Central American countries (Guatemala, El Salvador, Honduras, and Nicaragua), allowed us to conclude that education is the largest contributor to multidimensional deprivation, since it accounts for one-third of the MDI in each of the countries.

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Appendix A: The decomposition of the *MDI* by population subgroups

Assume a population of four individuals. Three of them have a certain number of deprivations and one is without any deprivation so that $n = 4$ and $q = 3$. Suppose that there are 5 domains of deprivation ($j = 1$ to 5). Individual 1 is deprived in domains 1, 2, 4, 5 so that ($d_1 = (4/5)$), individual 2 in domains 3 and 4 ($d_2 = (2/5)$) and individual 3 in domain 5 ($d_3 = (1/5)$). Individual 4 has no deprivation. Suppose that individuals 1 and 3 belong to group A and individuals 2 and 4 to group B. Let us also assume that the threshold t is equal to 1. Finally, define p_i as that $p_i = (1/n)d_i = (1/4)d_i$. Figure A-1 illustrates this case.

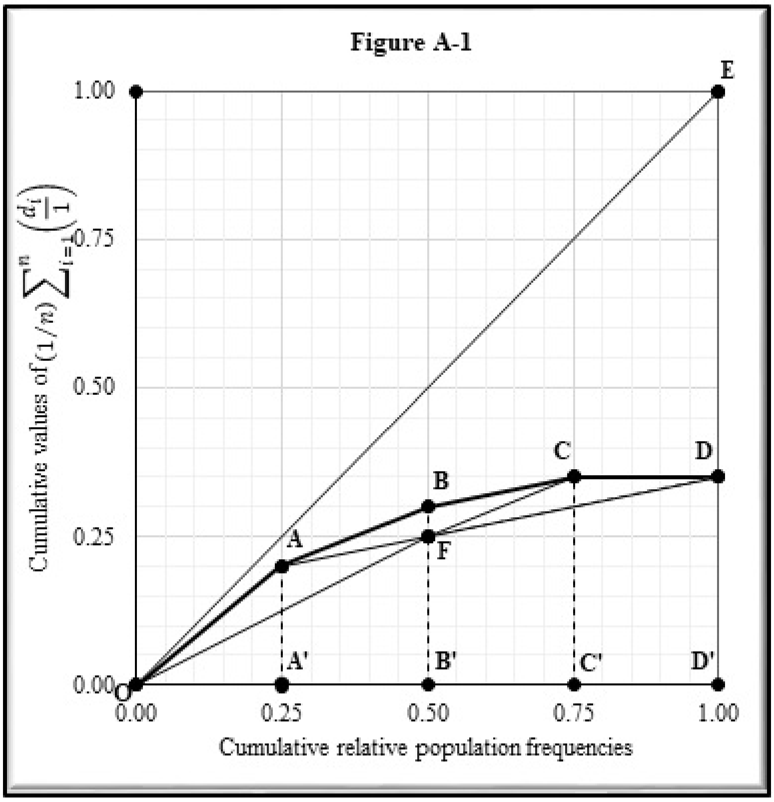

Figure A-1: Illustration of the decompostion of the MDI by population subgroups

Using (20) the MDI is expressed as

$MDI = \left(\frac{1}{16}\right)\{[(7)(0.8)] + [(5)(0.4)] + [(3)(0.2)]\} = \frac{(5.6+2+0.6)}{16} = \frac{8.2}{16}$

Using (27) we then derive that the contributions C_A and C_B of groups A and B are expressed as

$C_A = (1/16)\{[(7)(0.8)] + [(3)(0.2)]\} = \frac{6.2}{16}$

$C_B = (1/16)\{[(5)(0.4)]\} = \frac{2}{16}$

It is easy to observe that, as expected, the sum of these two contributions is equal to $\frac{6.2+2}{16} = \frac{8.2}{16}$, which is the value of the MDI for the whole population.

The graphical representation of a traditional decomposition

In Figure A-1, the curve $OABCD$ represents what we previously called the PUB curve. The line OE is the deprivation curve that would be obtained if everyone had the same and maximal level of deprivation, namely (5/5) so that the height ED' is, as expected, equal to $4(1/4)(5/5) = 1$. It is easy to check that the heights AA', BB', CC' and DD are respectively equal to 0.2, 0.3, 0.35 and 0.35 and that the areas OAA', $AA'B'B$, $BB'C'C$ and $CC'D'D$ are respectively equal to 0.025, 0.0625, 0.08125 and 0.0875. The sum of these 4 areas, which corresponds to the area $OABCDD'C'B'A'O$, is then equal to 0.25625. Twice this sum gives us $0.5125 = (8.2/16)$, which is, as expected and shown previously, the value of the MDI when all the domains have the same weight.

Given that individuals 1 and 3 belong to group A and individual 2 and 4 to group B, it is easy to check that the average number of deprivations in group A is $(4+2)/2 = 3$ and in group B it is $((2+0)/2) = 1$. We can therefore draw in Figure A-1 a broken curve OFD. On the section OF, the height of point F corresponds to the total deprivation in group A, which includes individuals 1 and 3 and hence it is equal to $[(1/4)(4/5)]+[(1/4)(1/5)] = (5/20) = 0.25$. Similarly, the difference between the height of point D and that of point F corresponds to the deprivation in group B and is hence expressed as $[(1/4)(2/5)] + [(1/4)(0/5)] = (2/20) = 0.1$. The height of point D is therefore $0.25 + 0.1 = 0.35$. The area below the curve $OFDD'O$ is, therefore, computed as $[(1/2)(0.5)(0.25)] + \{(1/2)(0.5)[0.25 + 0.35]\} = 0.0625 + 0.150 = 0.2125$. Twice this area, that is, 0.425, is hence the between groups A and B components of multidimensional deprivation.

We can also compute the within groups A and B components of multidimensional deprivation. The within group A deprivation is evidently the area OAF while that within group B is the area FCD. Now $OAF = [(OAA') + (AA'B'F)] - (OFB')$ with $OAA' = [(1/2)(0.25)(0.2)] = 0.025$; $AA'B'F = [(1/2)(0.25)(0.2{+}0.25)] = 0.05625$; $OFB' = [(0.5)(0.5)(0.25)] = 0.0625$. We therefore derive that the area OAF is equal to $(0.025 + 0.05625) - 0.0625 = 0.01875$. Twice this number gives us the within group A multidimensional deprivation and it is equal to 0.0375.

The within group B deprivation is given by the triangle FCD whose area is equal to $[(FB'C'C + CC'D'D) - FB'D'D]$. But $FB'C'C = (1/2)(0.25)(0.25 + 035) = 0.075$; $CC'D'D = (0.250.35) = 0.0875$; and $FB'D'D = (1/2)(0.5)(0.25 + 0.35) = 0.15$. The area FCD is hence equal to $(0.075 + 0.0875) - 0.15 = 0.0125$. Twice this area is therefore equal to the within group B multidimensional deprivation, that is, to 0.025.

Let us now compute the area $ABCF$ that corresponds to the overlap between group A and group B. We may write that $ABCF = (AA'B'B + BB'C'C)$ - $(AA'B'F + FB'C'C)$. $AA'B'B = (0.5)(0.25)(0.2{+}0.3) = 0.0625$; $BB'C'C = (0.5)(0.25)(0.3 + 0.35) = 0.08125$; $AA'B'F = (0.5)(0.25)(0.2 + 0.25) = 0.05625$; $FB'C'C = (0.5)(0.25)(0.25 + 0.35) = 0.075$. Therefore, $ABCF = (0.0625 + 0.08125)$ - $(0.05625 + 0.075) = 0.0125$. Twice this area will be the overlap component of the MDI, and it is equal to 0.025.

The sum of the three components (between groups, within groups and overlap deprivation) is then equal to $(0.425 + 0.0375 + 0.025 + 0.025) = 0.5125 = (8.2/16) = MDI$.

The graphical representation of an alternative decomposition of the MDI

Figure A-2 gives a graphical representation of this alternative decomposition.

As in Figure A-1, the curve $ABCD$ represents the actual PUB curve, and it is drawn by ranking the individuals by decreasing level of deprivation This ranking will be kept when drawing the deprivation curve that would be observed if each individual's deprivation was the average deprivation of the group to which he/she belongs. We saw previously that the average deprivation in group A, which includes individuals 1 and 3, is $(4+1)/2 = 2.5$, while the average deprivation in group B is $(2+0)/2 = 1$. Keeping the original ranking of the individual, we conclude that the height of point A'', which corresponds to this deprivation of individual 1, will be $(1/4)(2.5/5) = (2.5/20) = 0.125$. To reach the second point (B'') on this "alternative average deprivation curve", we add to the height of point A' the average deprivation in group B (equal to 1) since individual 2 belongs to group B so that the height of point B'' is $0.125 + [(1/4)(1/5)] = 0.125 + 0.050 = 0.175$. The same idea is applied to compute the height of point C''. Starting from B'', we have to add a height which corresponds to the average deprivation in group A, since individual 3 belongs to group A and so the height of point C'' is $0.175 + [(1/4)(2.5/5)] = 0.175 + 0.125 = 0.3$. Finally, by adding to the height of point C'' a height corresponding to the average deprivation in group B (individual 4 belongs to group B), we end up with $0.3 + [(1/4)(1/5)] = 0.3 + 0.05 = 0.35$, which is indeed the height of point D. Clearly, the area $OA''B''C''DD'O$ corresponds to half the value of the alternative between groups deprivations while the area $OABCDC''B''A''O$ represents half the value of the within groups deprivation.

It is easy to find out that the area $OA''B''C''DD'O$ is equal to $(0.5 * 0.25 * 0.125) + [0.5 * 0.25 * (0.125 + 0.175)] + [0.5 * 0.25 * (0.175 + 0.3)] + [0.5 * 0.25 * (0.3 + 0.35)] = 0.015625 + 0.0375 + 0.059375 + 0.08125 = 0.19375$. Twice this value (0.3875) is hence the value of the alternative between groups deprivation.

This result can also be obtained by applying (22) to the average deprivation of the group to which each individual belongs, giving each individual his/her original rank. We then obtain: $(1/16)\{[(7)(2.5/5)] + [(5)(1/5)] + [(3)(2.5/5)] + [(1)(1/5)]\} = (1/80)(17.5 + 5 + 7.5 + 1) = (31/80) = 0.3875$.

The within groups deprivation (the area $OABCDC''B''A''O$) is computed as $[0.5 * 0.25 * (0.2 - 0.125)] + \{0.5 * 0.25 * [(0.2 - 0.125) + (0.3 - 0.175)]\} + \{0.5 * 0.25 * [(0.3 - 0.175) + (0.35 - 0.3)]\} + \{0.5 * 0.25 * [(0.35 - 0.3)]\} = 0.009375 + 0.025 + 0.021875 + 0.00625 = 0.0625$. Twice this area is hence equal to 0.125.

This result may be obtained by applying (20) to the difference for each individual between his/her actual deprivation and the average deprivation of the group to which he/she belongs, each individual being assigned again his/her original rank. We then get $(1/16)\{[(7)((4-2.5)/5)] + [(5)((2-1)/5)] + [(3)((1-2.5)/5)] + [(1)((0-1)/5)]\} = (1/80)((10.5+5)-(4.5+1)) = 10/80 = 0.125$.

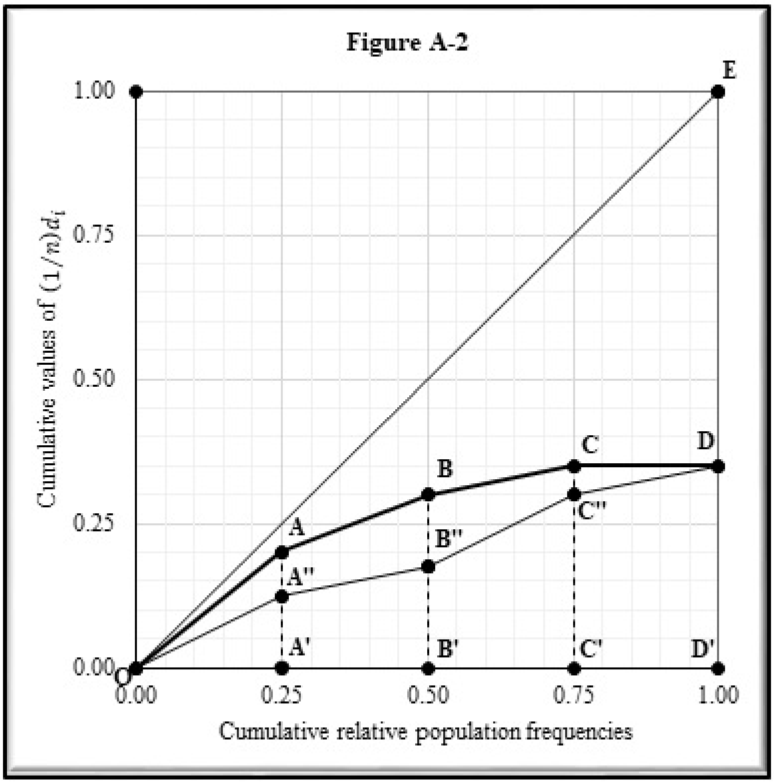

Figure A-2: Graphical representation of an alternative decomposition of the MDI

The sum of these alternative between and within group's deprivation is hence equal to $0.3875 + 0.125 = 0.5125 = 8.2/16 = MDI$.

Appendix B: The *MDI* as a specific case of the deprivation index of Aaberge et al. (2019): a simple illustration

Let us assume that there are 5 individuals and 10 deprivation domains. Each deprivation has the same weight. Table B-1 indicates how many deprivations each individual has. The Gini index G_{d_i} of the distribution of the deprivations may then be computed [see expression (4) in Berrebi and Silber, 1983] as $G_{d_i} = [(4/5)(\frac{10-0}{25}) + (2/5)(\frac{7-2}{25})] = (\frac{40+10}{125}) = 0.4$, where 25 in the denominator refers to the total number of deprivations in the population and 5 is the number of individuals. Using (15), we conclude that $MDI = \bar{d}(1 + G_{d_i}) = 5(1 + 0.4) = 7$. Note that it is also possible to compute the G_{d_i} index using the following formulation of the Gini index (see, Yitzhaki and Schechtman, 2013, p. 15):

$$G_{d_i} = 2\int [1 - F(k)]dk - 2\int [1 - F(k)]^2 dk \tag{B-1}$$

Using the data of Table B-1, we conclude that $\int_0^9 [1 - F(k)]dk = 5$ and that $\int_0^9 [1 - F(k)]^2 dk = 3$. We also conclude that $G_{d_i} = 2(5 - 3)(\frac{1}{10}) = 0.4$.

Since the mean difference Δ_{d_i} of the deprivations is expressed (see Kendall and Stuart, 1969) as

$$\Delta_{d_i} = 2\bar{d}G_{d_i} \tag{B-2}$$

where $\bar{d}$ is the mean number of deprivations, which is here equal to $(2 + 6 + 7 + 10)/5 = 5$, we conclude that $\Delta_{d_i} = 2 * 5 * 0.4 = 4$.

Aaberge et al. (2019) have suggested using as measure of deprivation in a society, an index $D_\Gamma(F)$ defined [see their expression (2.4)] as

$$D_\Gamma(F) = r - \sum_{k=0}^{r-1} \Gamma(F_k) \tag{B-3}$$

where r refers to the maximum number of deprivation (in our simple illustration $r = 10$). If we take a "union approach", the function Γ has to be convex. A simple convex function would be $\Gamma(F_k) = (F_k)^2$, so that we end up with:

$$D_\Gamma(F) = r - \sum_{k=0}^{r-1} (F_k)^2 \tag{B-4}$$

Using the data of Table B-1, we easily find that $\sum_{k=0}^{r-1}(F_k)^2 = \sum_{k=0}^{9}(F_k)^2 = 3$. Since $r = 10$, we conclude that $D_\Gamma(F) = 10 - 3 = 7$.

Table B-1: A simple numerical illustration

Number k of deprivations	**Number of individuals deprived**	**Relative frequency f_k of deprivations**	**Cumulative relative frequency F_k of deprivations**	$(F_k)^2$	$(1-F$
0	1	0.20	0.20	0.04	0.80
1	0	0.00	0.20	0.04	0.80
2	1	0.20	0.40	0.16	0.60
3	0	0.00	0.40	0.16	0.60
4	0	0.00	0.40	0.16	0.60
5	0	0.00	0.40	0.16	0.60
6	1	0.20	0.60	0.36	0.40
7	1	0.20	0.80	0.64	0.20
8	0	0.00	0.80	0.64	0.20
9	0	0.00	0.80	0.64	0.20
10	1	0.20	1.00	1.00	0.00

Appendix C: Decomposition of MDI by domains

Recall that in (22) MDI is expressed as

$$MDI = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \sum_{j=1}^{J} \frac{w_j d_{ij}}{t} (2n - 2i + 1) \tag{C-1}$$

Assume to simplify equal weights, so that $w_j = (1/J) \forall j$. In addition, let us take a "union approach", so that $t = 1$. We can then express (C-1) as

$$MDI = \left(\frac{\sum_{j=1}^{J} \sum_{i=1}^{n} d_{ij}}{nJ}\right) \sum_{j=1}^{J} \left(\frac{\sum_{i=1}^{n} d_{ij}}{\sum_{j=1}^{J} \sum_{i=1}^{n} d_{ij}}\right) \left[\sum_{i=1}^{n} \frac{d_{ij}}{\sum_{i=1}^{n} d_{ij}} \frac{(2n - 2i + 1)}{n}\right] \tag{C-2}$$

Let us define now b_{ij} as $b_{ij} = \frac{d_{ij}}{\sum_{i=1}^{n} d_{ij}}$, so that b_{ij} refers to the share of individual i in the total amount of deprivation in the population in domain j. Let us also define s_j as $s_j = \left(\frac{\sum_{i=1}^{n} d_{ij}}{\sum_{j=1}^{J} \sum_{i=1}^{n} d_{ij}}\right)$. In other words, s_j represents the share of domain j in the total amount of deprivation in the population (all domains included). Finally, let us call $\bar{d}$ the ratio $\left(\frac{\sum_{j=1}^{J} \sum_{i=1}^{n} d_{ij}}{nJ}\right)$, so that refers to the average level of deprivation per individual and per domain in the population.

We can now rewrite (C-2) as

$$MDI = \bar{d} \sum_{j=1}^{J} s_j \left[\sum_{i=1}^{n} b_{ij} \left(1 + \left(\frac{n - 2i + 1}{n}\right)\right)\right] \tag{C-3}$$

$$\leftrightarrow MDI = \left\{(\bar{d}) \sum_{j=1}^{J} s_j \left[\sum_{i=1}^{n} b_{ij}\right]\right\} + \bar{d} \sum_{j=1}^{J} s_j \left[\sum_{i=1}^{n} b_{ij} \left(\frac{n - 2i + 1}{n}\right)\right] \tag{C-4}$$

It is then easy to check that

$$\left\{(\bar{d}) \sum_{j=1}^{J} s_j \left[\sum_{i=1}^{n} b_{ij}\right]\right\} = \bar{d} \tag{C-5}$$

so that

$$MDI = \bar{d} \left\{1 + \sum_{j=1}^{J} s_j \left[\sum_{i=1}^{n} b_{ij} \left(\frac{n - 2i + 1}{n}\right)\right]\right\} \tag{C-6}$$

While i is the rank of individual i in the distribution of total deprivation (all domains included) in the population, let us call i^j the rank of individual i in the distribution of deprivations in domain j. It is then easy to check (see, Berrebi and Silber, 1987) that $\left[\sum_1^n b_{ij}\left(\frac{n-2i^j+1}{n}\right)\right]$ represents the Gini index G_j of the deprivations in domain j while $\left[\sum_1^n b_{ij}\left(\frac{n-2i+1}{n}\right)\right]$ is called the Pseudo-Gini PG_j of the deprivations in domain j (see, Fei et al., 1979).

We may therefore rewrite (C-6) as

$$MDI = \bar{d}\left\{1+\sum_{j=1}^{J} s_j\left[PG_j\right]\right\} = \bar{d}\left\{1+\sum_{j=1}^{J} s_j\left[G_j\left(\frac{PG_j}{G_j}\right)\right]\right\} = \bar{d}\left\{1+\sum_{j=1}^{J} s_j\left[G_j\left(GC_j\right)\right]\right\} \tag{C-7}$$

where $GC_j = \left(\frac{PG_j}{G_j}\right)$ is called the Gini correlation coefficient (see, Yitzhaki and Schechtman, 2013).

In other words, a domain j of deprivation contributes more to the MDI the higher the share s_j of domain j in the total amount of deprivation in the population (all domains included); the higher the Gini index G_j of the deprivations in domain j; and the higher the Gini correlation coefficient GC_j for domain j (the higher the correlation between the distribution of the deprivations in domain j and the distributioon of the total deprivations, all domains included, in the population, this correlation being measured not via the Pearson correlation coefficient but via the Gini correlation coefficient).

Working with mean differences

Let us first recall that $\left[\sum_1^n b_{ij}\left(\frac{n-2i^j+1}{n}\right)\right]$ represents the Gini index G_j of the deprivations in domain j while $\left[\sum_1^n b_{ij}\left(\frac{n-2i+1}{n}\right)\right]$ is called the Pseudo-Gini PG_j of the deprivations in domain j. Moreover, as already mentioned by Kendall and Stuart (1969), we know that the Gini index is equal to half the ratio of the mean difference over the corresponding mean. We may therefore write that

$$G_j = \left(\frac{1}{2}\right)\left(\frac{\Delta_j}{\frac{\left(\sum_{i=1}^n d_{ij}\right)}{n}}\right) \tag{C-8}$$

where Δ_j is the mean difference of the deprivations d_{ij} (within the deprivation domain j). We can similarly define a "Pseudo Mean Difference" and write that

$$PG_j = \left(\frac{1}{2}\right)\left(\frac{P\Delta_j}{\frac{\left(\sum_{i=1}^n d_{ij}\right)}{n}}\right) \tag{C-9}$$

where Δ_j is the mean difference of the deprivations d_{ij} (within the deprivation domain j).

Combining (C-7), (C-8) and (C-9), we derive that

$$MDI = \left\{ \left(\frac{1}{nJ}\right) \sum_{j=1}^{J} \left[\sum_{i=1}^{n} d_{ij} \right] \right\} + \left\{ \bar{d} \left[\sum_{j=1}^{J} s_j P G_j \right] \right\} = \left\{ \left(\frac{1}{J}\right) \sum_{j=1}^{J} \left[\frac{\sum_{i=1}^{n} d_{ij}}{n} \right] \right\} +$$

$$\left\{ \left(\frac{1}{J}\right) \left(\frac{\sum_{j=1}^{J} \sum_{i=1}^{n} d_{ij}}{n} \right) \left[\sum_{j=1}^{J} \left(\frac{\frac{\left(\sum_{i=1}^{n} d_{ij}\right)}{n}}{\frac{\left(\sum_{j=1}^{J} \sum_{i=1}^{n} d_{ij}\right)}{n}} \right) \left(\frac{1}{2}\right) \left(\frac{P\Delta_j}{\frac{\left(\sum_{i=1}^{n} d_{ij}\right)}{n}} \right) \right] \right\} = \bar{d} + \left(\frac{1}{2}\right) \left(\frac{1}{J}\right) \sum_{j=1}^{J} P\Delta_j =$$

$$\bar{d} + \left(\frac{1}{2}\right) \left(\frac{1}{J}\right) \sum_{j=1}^{J} \Delta_j \left(\frac{P\Delta_j}{\Delta_j} \right) = \bar{d} + \left(\frac{1}{2}\right) \left(\frac{1}{J}\right) \sum_{j=1}^{J} \Delta_j GC_j \tag{C-10}$$