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Abstract

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Keyword: Bipolarisation, inequality, two-income distribution

JEL Cassification: D31, D63, O15

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Florent Bresson* Marek Kosny† Gaston Yalonetzky‡

July 25, 2023§

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1 Introduction

Bipolarisation indices and partial orderings have gained traction as methods to measure the growth or disappearance of middle-classes during the last few decades, since the foundational work of Wolfson (1994) and Foster and Wolfson (2010). Essentially, in their original conception, bipolarisation measurement requires partitioning distributions into two groups using the median as dividing percentile, and then distinguishing between transfers across the median and transfers on one side of the median (i.e. within one group). As with inequality measurement, a regressive transfer across the median is deemed to increase the spread of mean attainment between the two groups, thereby increasing bipolarisation. By contrast, unlike inequality measurement, a progressive transfer within any one group is deemed to increase clustering, in the limit leading to perfect bimodality; hence these progressive transfers are deemed to increase bipolarisation.[1]

Perfect bimodality is an interesting situation since the distribution can then be entirely described with the help of just two income levels. Moreover, the extent of bipolarisation can be fully grasped considering the difference between these two incomes or the income share of one of the two groups. For instance, let's consider the two four-person income distributions $A = (4, 4, 6, 6)$ and $B = (2, 2, 8, 8)$. The average income difference between the top half and the bottom half of the income distribution is 2 monetary units for A and 6 for B. The corresponding ordering reflects the larger spread between the two halves when comparing distributions A and B, hence greater bipolarisation in B in comparison to A. Moreover, this income gap is meaningful. The same is true for the income share of the top half that equals 0.6 for distribution A and 0.8 for distribution B.

The opposite effects of between-groups and within-groups transfers that we stressed earlier suggest the possibility of tradeoffs between such transfers. Let's assume that there always exists a sequence of within-group progressive transfers that can perfectly compensate the effect of a between-group progressive transfer. If this hypothesis is true, then, for each observed income distribution, we can consider a perfectly bimodal distribution showing the same level of bipolarisation. Once the two income levels of this equivalent distribution have been identified, then simple figures such as the ones used to describe bipolarisation for distributions A and B can be computed to describe bipolarisation for the observed income distribution. Of course, the estimated values for the absolute income difference or the income share of the top half would not reflect their true value, but they would be representative of the state of bipolarisation in the society in a similar manner as figures based on equally distributed equivalent incomes are representative of the inequality level with the Atkinson-Kolm-Sen approach of inequality measurement.

The present paper elaborates on this simple idea and introduces two classes of (partially) rank-independent and rank-dependent bipolarisation indices spanning the absolute and relative approaches to distributional change. The new classes include some rank-

[1] A famous sibling literature measures a more general notion of polarisation whereby distributions are considered more polarised if their observations are more tightly clustered around multiple modes, or the absolute distances between these modes are wider, or both. Classic papers in this field include Esteban and Ray (1994) and Duclos et al. (2004).

dependent bipolarisation indices from the literature as special cases such as two classes proposed by Wang and Tsui (2000) and modified Foster-Wolfson index, together with numerous novel proposals. Our adopted two-income representation facilitates the interpretation of the proposed bipolarisation indices either as *i)* the representative income gap between individuals from the top part of the distribution and those from the bottom part, or *ii)* the representative relative excess income share of the top part above its population share.

The indices enjoy additional appealing features. First, they can handle any population partition between the p percent poorest individuals and the $1-p$ richest individuals (including the popular p =50% choice). Previous studies have focuses on the differences between two non-overlapping groups of the same size, but other oppositions can be considered, depending on the specific issue that is studied. For instance, the Occupy Wall Street movement focused the attention on the widening gap between the wealthiest 1% and the rest of the US population with its slogan "We are the 99%." Studies investigating the potential threats of the gap between the top incomes and the remaining part of the population on either economic performance or social cohesion, would then be more interested in considering large values for p, e.g. 95% or 99%. Other population quantiles can be of interest for policy makers. For instance, progresses with respect to the 10th objective of the Sustainable Development Goals, namely "Reduce inequality within and among countries," are notably assessed considering the income share of the poorest 40%. Checking wether such progresses are associated with greater income polarisation could justify splitting the population at this population percentage.

Secondly, the suggested indices can be additively decomposed into spread and clustering contributions toward total bipolarisation. Since the spread component reflects the between-group component of inequality, disentangling the changes in spread and clustering may help understand the common or diverging trends in bipolarisation and inequality.

Finally, the proposed indices relate to familiar inequality indices so that bipolarisation levels can easily be connected to the average income and inequality levels relating to each part of the income distribution, using a broad array of both rank-dependent and rank-independent inequality indices (e.g., the Gini class, the generalised entropy and Atkinson-Kolm-Sen classes, the Bosmans-Cowell classes, etc.).

The rest of the paper proceeds as follows. Section 2 provides the required notation and preliminary information, followed by section 3, which states three sets of properties: a group of broadly appealing axioms in the distributional analysis literature; the key defining axioms of bipolarisation measurement; and a batch of special axioms enabling the characterisation of four types of bipolarisation indices: absolute rank-dependent, absolute rank-independent, relative rank-dependent and relative rank-independent. Section 4 starts with the development of the two-income approach to bipolarisation measurement, followed in section 5 by the introduction of the different families of bipolarisation indices and ends with the different available decompositions into clustering and spread component as well as their respective connections to mean and inequality measures. These indices are illustrated in section 6 using consumption series from harmonized household surveys for a sample of nine Sub-Saharan African countries. Finally, the paper concludes with some remarks in section 7.

2 Notation and preliminaries

Let $x_i \in \mathbb{R}_+$ denote the income of individual i. For a population of $n \in \mathbb{N}^* \setminus \{1\}$ persons, $X := (x_1, \dots x_n)$ describes the income distribution of the population. Moreover, let $N := \{1 \dots, n\}$. We define $\mathcal{D}^n$, $n \geq 2$, as the set of income vectors of size n such that there exists at least one strictly positive income, that is $\mathcal{D}^n := \{X \in \mathbb{R}^n_+ | \exists j \in N \text{ s. t. } x_j > 0\}$. Likewise, $\mathcal{D} := \cup_{n=2}^{+\infty} \mathcal{D}^n$ denotes the set of all distributions X with at least two individuals and a strictly positive arithmetic mean μ_X. $\mathcal{D}^* := \cup_{n=2}^{+\infty} \mathcal{R}^n_{++}$ denotes the subset of $\mathcal{D}$ with only strictly positive incomes.

Let $p \in [0, 1]$ be a population percentile and $x(p)$ be the quantile function associated with distribution X. For any possible $p \in]0, 1[$, we also define $b(X; p)$ as the set of individuals that represent the p percent poorest part of the population, and $t(X; p)$ denotes the set of individuals that correspond to the $(1 - p)$ percent richest part of this population. Formally, let $r(i, X)$ be the function that provides the ordinal ranking of the ith element from X, with $r(i, X) = 1$ for the lowest income and $r(i, X) = n$ for the largest income.[2] We then have $b(X; p) := \left\{ i \in N \,\middle|\, \frac{r(i,X)}{n} \leq p \right\}$ and $t(X; p) := \left\{ i \in N \,\middle|\, \frac{r(i,X)}{n} > p \right\}$. We assume $b(X; p) \cup t(X; p) = N$ and $b(X; p) \cap t(X; p) = \emptyset$.[3] As a consequence, we have $x_i \leq x(p)$ $\forall i \in b(X; p)$ and $x_i \geq x(p)$ $\forall i \in t(X; p)$. These two mutually exclusive (and exhaustive) sets enable the definition of $\underline{X}$ as the income vector associated with individuals from $b(X; p)$. Likewise, $\bar{X}$ denotes the income vector of those individuals from $t(X; p)$.

Later on, our axiomatic framework will require the use of O^n, namely a vector of n zeroes; and $\mathcal{E}^n$, which is the set of vectors of size n whose elements are all equal and strictly positive. We then can define the set of zero and egalitarian vectors respectively by $\mathcal{O} := \cup_{n=2}^{+\infty} O^n$ and $\mathcal{E} := \cup_{n=2}^{+\infty} \mathcal{E}^n$. Another important set is $\mathcal{B}^n_p := \big(O^{pn}, \mathcal{E}^{(1-p)n}\big)$, which includes all income vectors whose elements from $b\big(X \in \mathcal{B}^n_p, p\big)$ are all equal to zero and whose remaining elements are all equal to some strictly positive real number. Using this definition, we can obtain $\mathcal{B}_p := \cup_{n=2}^{+\infty} \mathcal{B}^n_p$. We finally define the $n-1$ dimensional unit simplex as $\mathcal{S}^n := \{X \in \mathbb{R}^n_+ | \sum_{i=1}^n x_i = 1\}$, and the set of all unit simplexes as $\mathcal{S} := \cup_{n=2}^{+\infty} \mathcal{S}^n$.

Let also $\mathcal{P}^n$ be the set of $n \times n$ permutation matrices. The function $d : \mathcal{D}^n \to \mathcal{D}^{\lambda n}$, with $\lambda \in \mathbb{N}^*$, returns vectors $d_\lambda(X)$ whose elements are λ replications of each element from X. Finally, for two income vectors X and Y of the same size, let $c(X, Y) := \{i \in N | x_i = y_i\}$ be the subset of N for whose elements we observe pairwise equal values in X and Y, and

[2] Unlike other ranking methods, such as dense or fractional, with ordinal ranking there is no pair of element $\{x_i, x_j\}$ from X such that $r(i, X) = r(j, X)$. For instance, with $X = (6, 2, 1, 2)$, we either have $R_X = (4, 2, 1, 3)$ or $R_X = (4, 3, 1, 2)$ where R_X the vector of ranks.

[3] When p is not a multiple of $\frac{1}{n}$, it is obviously difficult to satisfy this assumption. The population axiom, introduced in section 3, renders these two assumptions perfectly plausible $\forall p \in \{p \in \mathbb{Q} | 0 < p < 1\}$. When dealing with household surveys, partitioning the population perfectly into $b(X; p)$ and $t(X; p)$ is very simple using sampling weights. If p is such that an individual whose income equals the chosen quantile $x(p)$ belongs to both the bottom and top parts of the population, we just have to clone this person and share his sampling weight so that $b(X; p)$ and $t(X; p)$ for the resulting $n + 1$ population exactly and respectively represent p and $(1 - p)$ percent of the population.

$\tilde{c}(X,Y)$ be its complement.

3 Axiomatic framework

We consider bipolarisation indices $\Psi : \mathcal{D}\times]0,1[\to \mathbb{R}$ whose value is determined by an income distribution X and a population quantile p used to split the income distribution into a bottom and a top part. We begin with a set of axioms that are not specific to bipolarisation measurement but express widely shared values in the social welfare literature:

Anonymity (ANO): $\forall n \in \mathbb{N}^*$, $X \in \mathcal{D}^n$, $P \in \mathcal{P}^n$, we have $\Psi(PX,p) = \Psi(X,p)$.

Population (POP): $\forall \{n,\lambda\} \subset \mathbb{N}^*$, $X \in \mathcal{D}^n$, we have $\Psi\big(d_\lambda(X),p\big) = \Psi(X,p)$.

Unit consistency (UNC): $\forall Y,X \in \mathcal{D}$, $p \in]0,1[$, $\Psi(Y,p) \geq \Psi(X,p) \Leftrightarrow \Psi(\kappa Y,p) \geq \Psi(\kappa X,p)$ $\forall \kappa \in \mathbb{R}_{++}$.

Weak Independence (IND$_W$): $\forall n \in \mathbb{N}^* \setminus \{1\}$, $\{X,X',Y,Y'\} \subset \mathcal{D}^n$ such that $r(i,X) = r(i,Y)$ $\forall i \in c(X,Y)$, we observe $\Psi(X,p) \geq \Psi(Y,p) \Leftrightarrow \Psi(X',p) \geq \Psi(Y',p)$ if $r(i,X') = r(i,Y')$ $\forall i \in c(X,Y)$, $r(i,X) = r(i,X')$ $\forall i \in \tilde{c}(X,Y)$ and $r(i,Y) = r(i,Y')$ $\forall i \in \tilde{c}(X,Y)$.

Strong Independence (IND$_S$): $\forall n \in \mathbb{N}^* \setminus \{1\}$, $s \in N \setminus \{n\}$, $\{A,B\} \subset \mathcal{D}^{n-s}$, $\{C,D\} \subset \mathcal{D}^s$, $X = (A,C)$, $X' = (A,D)$, $Y = (B,C)$, and $Y' = (B,D)$, we observe $\Psi(X,p) \geq \Psi(Y,p)$ $\Leftrightarrow \Psi(X',p) \geq \Psi(Y',p)$ if: *i)* $b(X,p) \cap c(X,X') = b(X',p) \cap c(X,X')$, *ii)* $t(X,p) \cap c(X,X') = t(X',p) \cap c(X,X')$, *iii)* $b(Y,p) \cap c(Y,Y') = b(Y',p) \cap c(Y,Y')$, and *iv)* $t(Y,p) \cap c(Y,Y') = t(Y',p) \cap c(y,Y')$.

Axiom ANO is commonly used in welfare analyses and is the expression of the necessity for the index Ψ to satisfy horizontal equity. The POP axiom is also traditional and expresses the idea that the index Ψ shall not depend on the size of the population, *i.e.*, that Ψ expresses a degree of bipolarisation and not a quantity of bipolarisation in the economy. A known consequence of satisfying ANO and POP jointly is that the cumulative distribution function of X can be substituted for this vector as a determinant of Ψ without information loss. The UNC axiom sensibly demands that changing the measurement unit for income never changes the bipolarisation ordering of income vectors.

We introduce two versions of the independence axiom. The first one, IND$_W$, inspired by Ebert (1988), argues that rank-preserving changes in the common part of two income distributions do not alter their bipolarisation ordering. The second version, IND$_S$, is stronger as this ordering preservation of the two compared distribution is assumed to be valid for any change in the common part of these distribution that does not alter the belonging of the non-common values to the bottom and top parts of the distributions. While IND$_W$ makes it possible to consider rank-dependent bipolarisation indices as those proposed by Wang and Tsui (2000), the only information regarding ranks that can be taken into account for bipolarisation measurement with IND$_S$ is the inclusion into the bottom or top part of the distribution.

To illustrate the difference between IND_W and IND_S, let's assume $p = \frac{3}{4}$ and consider the four following pairs of income distributions:

$$A = (4, \mathbf{8}, \mathbf{10}, 20), \qquad B = (7, \mathbf{8}, \mathbf{10}, 18), \tag{1}$$
$$A^* = (4, \mathbf{11}, \mathbf{16}, 20), \qquad B^* = (7, \mathbf{11}, \mathbf{16}, 18), \tag{2}$$
$$A' = (4, \mathbf{6}, \mathbf{12}, 20), \qquad B' = (\mathbf{6}, 7, \mathbf{12}, 18), \tag{3}$$
$$A'' = (4, \mathbf{5}, \mathbf{19}, 20), \qquad B'' = (\mathbf{5}, 7, 18, \mathbf{19}). \tag{4}$$

with bold characters here to emphasize the common part for each pair of income vectors. Under the assumption $\Psi\left(A, \frac{3}{4}\right) > \Psi\left(B, \frac{3}{4}\right)$, both IND_W and IND_S entails $\Psi\left(A*, \frac{3}{4}\right) > \Psi\left(B*, \frac{3}{4}\right)$ since changes in the common part of the two initial distributions were rank preserving. Now, consider the change from A to A' and from B to B'; we have $\Psi\left(A', \frac{3}{4}\right) > \Psi\left(B', \frac{3}{4}\right)$ as a consequence of IND_S. On the contrary, under IND_W, it is not possible to make use of the assumed ordering between A and B to get the bipolarisation ordering for A' and B'. Indeed, when moving from B to B', the income level 6, that was initially ranked 2nd in B, is now ranked 1st in B', hence failing to satisfy the condition $r(i, B) = r(i, B')$ $\forall i \in \tilde{c}(A, B)$. Finally, neither IND_W nor IND_S can be used to infer the ordering between A'' and B'' from the initial ordering since, as a result of the change from B to B'', the income level 18 moved from the top part to the bottom part of the income distribution with the chosen threshold p.

We now introduce axioms that are specifically relevant to bipolarisation measurement:

Spread increasing transfer (SPT): $\forall n \in \mathbb{N}^* \setminus \{1\}$, $X \in \mathcal{D}^n$, $p \in]0, 1[$, $i \in b(X; p)$, $j \in t(X; p)$, and $\delta \in]0, x_i]$, we have $\Psi(Y, p) \geq \Psi(X, p)$ if $y_i = x_i - \delta$, $y_j = x_j + \delta$, and $y_k = x_k$ $\forall k \in N \setminus \{i, j\}$.

Spread increasing changes (SPC): $\forall n \in \mathbb{N}^* \setminus \{1\}$, $X \in \mathcal{D}^n$, $p \in]0, 1[$, we have $\Psi(Y, p) \geq \Psi(X, p)$ if $y_i = x_i + \delta$, $y_k = x_k$ $\forall k \in N \setminus \{i\}$ and either:

i) $i \in b(X; p)$ and $\delta \in [-x_i, 0]$, or

ii) $i \in t(X; p)$ and $\delta \in \mathbb{R}_+$.

Clustering increasing transfer (CLU): $\forall n \in \mathbb{N}^* \setminus \{1, 2\}$, $X \in \mathcal{D}^n$, $p \in]0, 1[$, $x_i < x_j$ and $\delta \in \left]0, \frac{x_j - x_i}{2}\right]$, we have $\Psi(Y, p) \geq \Psi(X, p)$ if either:

i) $pn \geq 2$, $\{i, j\} \subseteq b(X; p)$, $y_i = x_i + \delta$, $y_j = x_j - \delta$, and $y_k = x_k$ $\forall k \in N \setminus \{i, j\}$, or

ii) $(1 - p)n \geq 2$, $\{i, j\} \subseteq t(X; p)$, $y_i = x_i + \delta$, $y_j = x_j - \delta$, and $y_k = x_k$ $\forall k \in N \setminus \{i, j\}$.

Axiom SPT states that regressive transfers from a person from the bottom part of the distribution to a person from the top part increase bipolarisation. This axiom captures an essential shared feature of inequality and bipolarisation indices since it is generally assumed that regressive transfers increase inequality. Meanwhile, the SPC axiom only considers simple income increments and decrements. Since, arguably, an income change within one part of the distribution away from the other part of the distribution widens the gap between the bottom and the upper part of the distribution, the axiom demands a larger value of the bipolarisation index from such changes. As the regressive transfers considered for SPT are combinations of income decrements within the bottom part and increments within the top part, it is easy to show that the satisfaction of SPC entails the satisfaction of SPT while the converse is not necessarily true.

Axiom CLU considers regressive transfers that either take place within the bottom or within the top part of the distribution. Such transfer reduce clustering and consequently are associated with a decrease in bipolarisation. This behaviour clearly distinguish bipolarisation indices from inequality indices as the latter do not decrease with any regressive transfers.

A last set of axioms is introduced either for practical purposes or to refine some of the core axioms:

Continuity (CON): $\forall n \in \mathbb{N}^*$, Ψ has continuous first-order partial derivatives $\frac{\partial \Psi}{\partial x_i}$ over $\mathcal{D}^n$.

Scale invariance (SCI): $\forall X \in \mathcal{D}$, $p \in]0,1[$, and $\kappa \in \mathbb{R}_{++}$, we have $\Psi(\kappa X, p) = \Psi(X, p)$.

Translation invariance (TRI): $\forall n \in \mathbb{N}^*$, $X \in \mathcal{D}^n$, $E \in \mathcal{E}^n$, and $p \in]0,1[$, we have $\Psi(X + E, p) = \Psi(X, p)$.

Equality normalisation (ENO): $\forall p \in]0,1[$, $\Psi(X,p) = 0$ if $X \in \mathcal{E}$.

Maximum bipolarity normalisation (MNO): $\forall p \in]0,1[$, $\Psi(X,p) = 1$ if $X \in \mathcal{B}_p$ and $n \in \mathbb{N}^* \setminus \{1\}$.

Linearity (LIN): $\forall n \in \mathbb{N}^* \setminus \{1\}$, $p \in]0,1[$, $\alpha \in [0,1]$, $X \in \mathcal{E}^n$, $Y \in \mathcal{B}_p^n$, such that $\mu_X = \mu_Y$, we have $\Psi(Z,p) = \alpha \Psi(X,p) + (1-\alpha)\Psi(Y,p)$ if $Z = \alpha X + (1-\alpha)Y$.

The usual justification for CON is not ethical but practical as it is often used to prevent marginal errors to result in non-marginal variations of the index. Axioms SCI and TRI are stronger versions of UNC. The former says that a proportionate change in all incomes does not change the value of the bipolarisation index, while the latter assumes that an equal addition to all incomes preserves the value of the index. As discussed by Kolm (1976), these axioms do have an ethical content as they indicate how an extra income should be shared among the population in order to preserve the considered distributional feature.

Ubiquitous in the inequality and poverty measurement literatures, the ENO axiom is innocuous as it simply sets a value, namely zero, for bipolarisation indices when the income distribution shows no bipolarity; that is, when all income are equal. In combination with CLU and SPC, it can be checked that any departure from equality cannot result in a negative value for Ψ, so that 0 stands as a lower bound for Ψ. The MNO also indicates a specific value, namely one, for bipolarisation indices, but now in the case of extremely bipolarised income distribution.[4] In combination with CLU and SPC, it can then be seen

[4]A related axiom was proposed by Subramanian (2010) for inequality measurement with maximum inequality corresponding to income distributions where everybody but one person has zero income. Subramanian showed that this axiom was not compatible with POP because the replication of a maximum inequality distribution is not a maximum inequality distribution. Since replicating a maximum bipolarity distribution yields a maximum bipolarity distribution, we do not observe any conflict between MNO and POP for bipolarisation measurement.

that Ψ is necessarily lower than or equal to 1 when MNO is assumed. Moreover, for a given population size and a given partition of the income distribution into a bottom and a top part, it is worth stressing that all extremely bipolarised distributions are related by a scale factor (*i.e.* $\lambda X \in \mathcal{B}_p^n \; \forall X \in \mathcal{B}_p^n$ and $\lambda \in \mathbb{R}_{++}$). Consequently, MNO cannot be assumed if SCI is rejected.

Finally, the LIN axiom means that, considering distributions that only show inequality between the bottom and top parts, moving halfway from perfect equality to maximum bipolarity is valued the same as doing the rest of the way. In other words, when considering variations of Ψ, we assume the social evaluator does not show a differential sensitivity to a given change in the spread between the two parts of the income distribution depending on the initial level of bipolarisation.

4 The two-income approach

Since the seminal works of Kolm (1969), Atkinson (1970) and Sen (1973), a traditional approach for distributional evaluation is to link indices to individualistic social evaluation functions à la Bergson-Samuelson (Bergson, 1938, Samuelson, 1947). These functions may be used to generate an *equally distributed equivalent income* that, if earned by everyone in the society, would generate, from the point of view of the social evaluator, the same level of welfare as the observed income distribution. Kolm, Atkinson and Sen then showed how inequality indices could be proposed by comparing this representative income with the average income. To the best of our knowledge, there is no successful attempt to ground bipolarisation indices on this approach since it is considered that the concept "does not fit into the framework of the traditional Bergson-type Social Welfare Function" (Yitzhaki, 2010, page 7).[5]

We attribute part of this failure to the inappropriateness of using a *single* representative income (along with mean income) to assess bipolarisation. Indeed, at the heart of the concept of bipolarisation is the idea of two more or less distant poles around which incomes are distributed. As within-group and between-groups regressive transfers are assumed to show effects of opposite signs, it may be possible to combine them so as to preserve the estimated level of bipolarisation. Moreover, a sequence of such combined transfers is likely to result in a perfectly bimodal distribution. So we can imagine that, for any income distribution $X \in \mathcal{D}$, there always exists a unique two-income distribution $\tilde{X} := (\underline{x}, \bar{x})$ with $\underline{x} \leq \bar{x}$, such that $\tilde{X}$ has the same average income as X and can be used to describe the latter's level of bipolarisation. More specifically, we assume that the perfectly bimodal distribution $\tilde{X}$ with pn persons all having $\underline{x}$ and $(1-p)n$ persons receiving $\bar{x}$ would result in the same level of bipolarisation as $(\underline{X}, \bar{X})$. Within this framework, $\underline{x}$ would be regarded as the representative income of the bottom part (poorest $p\%$ of the population) and $\bar{x}$ would play the same role for the top part (richest $(1-p)\%$).

[5]Rodríguez (2015) proposed a social welfare index that is consistent with polarisation measurement but it is based on the notion of identification with and alienation from multiple local modes (Duclos et al., 2004), which significantly differs from the proposal by Wolfson (1994), Foster and Wolfson (2010) that serves as the basis for the present paper.

The idea of representing distributional features with a two-income distribution is not new as it was proposed for inequality measurement by Subramanian (2002). His so-called *dichotomously allocated equivalent distribution* is a two-person distribution meant to show the same inequality level as the current observed income distribution.[6] The concept is particularly helpful for interpretation purposes as it facilitates relating our observed distributional feature to the appealing image of a cake-sharing problem. In the context of inequality measurement, Subramanian (2013) argued that ratios $\frac{\underline{x}}{\underline{x}+\bar{x}}$ and $\frac{\bar{x}}{\underline{x}+\bar{x}}$ can interchangeably be used as a relative inequality indices in the spirit of those suggested by Kolm, Atkinson and Sen while being easily interpreted as the share of total income respectively obtained by the poorest and richest half of the population in a two-person economy that would exhibit the same level of inequality as the one currently observed in the population. Although not considered by Subramanian (2010), an absolute index based on the difference $\bar{x} - \underline{x}$ could also be proposed to measure inequality. Again, the interpretation is appealing since the measure indicates a representative income gap between a person from the top half and a person from the bottom half of the distribution.

Both types of indices, with specific definitions for $\underline{x}$ and $\bar{x}$ tailored to fit our axiomatic framework, can easily be repurposed for bipolarisation measures. The conversion is straightforward for absolute indices of the form:

$$\Psi_p^A := \bar{x} - \underline{x}. \tag{5}$$

In the case of relative indices, as we may divide the population into two groups of possibly different size, our indices should be based on the share of one of the two groups in total income, e.g. $\bar{s}_p := \frac{(1-p)\bar{x}}{p\underline{x}+(1-p)\bar{x}}$, the share of total income obtained by the richest $(1-p)$ percents of the population that, in a two-income economy, would result in the same level of bipolarisation as the one estimated for the observed income distribution.

An appealing requirement for $\underline{x}$ and $\bar{x}$ is joint idempotence, that is $\underline{x}(X) = \mu_{\underline{X}}$ and $\bar{x}(\bar{X}) = \mu_{\bar{X}}$ whenever X is such that $\{\underline{X}, \bar{X}\} \subset \mathcal{E}$. The property is desirable both for the satisfaction of POP and LIN, and to ease the interpretation of $\underline{x}$ and $\bar{x}$. Assuming idempotence, it can easily be checked that $\bar{s}_p$ satisfies MNO but violates ENO. In order to satisfy both axioms and obtain a relative index Ψ^R whose values are within the unit interval, we consider the transformation $\Psi_p^R := \frac{\bar{s}_p-(1-p)}{p} = \frac{(1-p)(\bar{x}-\underline{x})}{p\underline{x}+(1-p)\bar{x}}$. Since $\underline{x}$ and $\bar{x}$ are chosen so that $p\underline{x} + (1-p)\bar{x} = \mu$, we derive an ethical index of the form:

$$\Psi_p^R := \frac{(1-p)(\bar{x} - \underline{x})}{\mu}. \tag{6}$$

The index Ψ_p^R can be interpreted as the relative excess share in total income of the $(1-p)$ top percent of the population in comparison with a perfectly equally distributed, hence non-bipolarized, income distribution. For instance, if the population is split be-

[6] Since the maximum level of inequality for a population of two individuals is likely to be lower than the estimated level of inequality with a larger population size, Subramanian (2010) proposed a generalization of his concept to three and more persons. Our proposal of a two-income vector association with a population share p is an elegant way of generalizing Subramanian's concept without taking population size into account for the design of the equivalent income vector.

tween the bottom quintile and the four higher quintiles, the excess share of the richest 80% in the two-income equivalent distribution can be between 0 and 20% of total income. If $\Psi^R_{0.2} = 0.6$, it means that the excess share is 60% of its maximum value, that is the one that would result in the two-income equivalent distribution being included in $\mathcal{B}_{0.2}$. Alternatively, the excess share of the richest group can easily be expressed in percentage points by multiplying Ψ^R by p, that is $0.6 \times 0.2 = 0.12$ in our example. Please note that in the case of both $\underline{X}$ and $\bar{X}$ showing no inequality, Ψ^R_p is precisely the relative excess share of the top part of the population in comparison with the perfect equality state.

(a) A translation-invariant index. (b) A scale-invariant index.

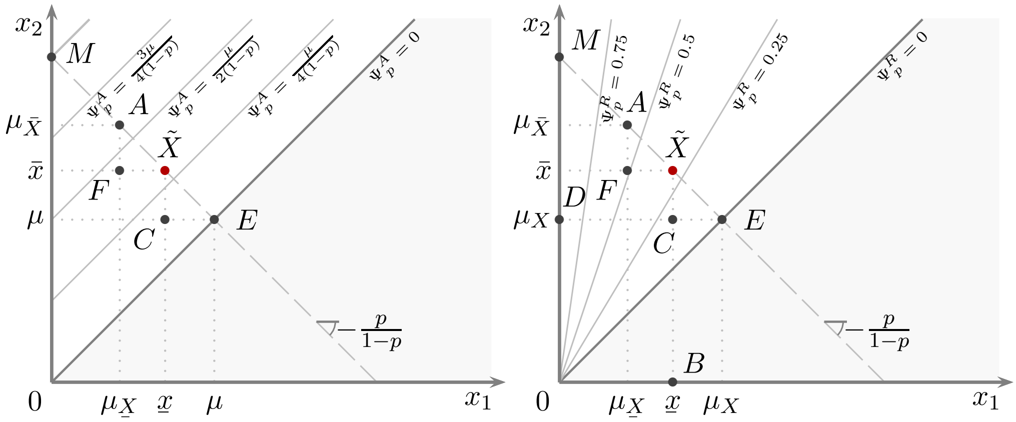


Figure 1: Graphical representations of bipolarisation with the two-income approach.

For the rest of the present paper, we follow the literature and focus on absolute indices that comply with TRI and relative indices satisfying SCI. Figure 1 shows simple representations of $\tilde{X}$, μ, Ψ^A, and Ψ^R. Since $\tilde{X}$ is a two-element vector, we can depict it with a single point with $\underline{x}$ on the horizontal axis and $\bar{x}$ on the vertical axis. The point E indicates the absence of bipolarisation (i.e., $\underline{x} = \bar{x} = \mu$), M corresponds to the case of maximum bipolarity (i.e., $\underline{x} = 0$ and $\bar{x} = \frac{\mu}{1-p}$), and A is the situation of perfect clustering (i.e., $\underline{x} = \mu_{\underline{X}}$ and $\bar{x} = \mu_{\bar{X}}$). M, A, $\tilde{X}$, and E are all aligned on the same straight dashed line that brings together all the two-income vectors whose arithmetic mean is μ for the chosen partition p. As we assume $\underline{x} \leq \bar{x}\ \forall X \in \mathcal{D}$, $\tilde{X}$ is necessarily on the segment ME. Moreover, if we assume that bipolarisation increases with clustering, then for given values $\mu_{\underline{X}}$ and $\mu_{\bar{X}}$, $\tilde{X}$ necessarily coincides with A or is closer than A to E.

The first diagonal brings together all two-income vectors associated with income distributions from $\mathcal{E}$. The distance of $\tilde{X}$ from this diagonal of perfect equality indicates the value of Ψ^A_p. Parallel lines above this diagonal on Figure 1a enable the identification of all the two-income vectors showing the same level of absolute bipolarisation. In the case of relative bipolarisation, the value of Ψ^R_p is given by the ratio of the distance $\tilde{X}E$ over the distance ME. Since we assume scale invariance for Ψ^R_p, all two-income vectors featuring the same level of relative bipolarisation are aligned on the same ray through the origin and above the no-bipolarity diagonal.

Point A suggests a simple decomposition of Ψ^A_p and Ψ^R_p into three components related to the spread (the difference between average incomes of the top and bottom part) and clustering (the differences between these averages and their corresponding representative incomes). More specifically, we have:

$$\Psi_p^A = (\mu_{\bar{X}} - \mu_{\underline{X}}) + (\bar{x} - \mu_{\bar{X}}) + (\mu_{\underline{X}} - \underline{x}), \tag{7}$$

$$\Psi_p^R = \frac{(1-p)(\mu_{\bar{X}} - \mu_{\underline{X}})}{\mu} + \frac{(1-p)(\bar{x} - \mu_{\bar{X}})}{\mu} + \frac{(1-p)(\mu_{\underline{X}} - \underline{x})}{\mu}. \tag{8}$$

By definition, the first right-hand side element in (7) and (8) is non-negative. Because CLU assumes that mean-preserving spreads within the bottom or the top part reduce clustering and consequently decrease bipolarisation, the second and third elements are necessarily non-positive. All those components can be read on Figure 1. Indeed the distance AE corresponds to the absolute spread component, while $\tilde{X}F$ and FA can be used to assess jointly the clustering component.[7]

Figure 3, in section 6 illustrates the usefulness of the representations in the spirit of Figure 1 using income data for nine Sub-Saharan African countries in 2018–2019.

5 Estimation with quasi-linear means

It is straightforward to see how indices Ψ_p^A and Ψ_p^R depend on the spread between incomes from the bottom and the top parts. One can also intuitively note that $\underline{x}$ and $\bar{x}$ must be sensitive to inequality within the corresponding part of the income distribution in order to render the indices sensitive to clustering. More specifically, since clustering is akin to a decrease in inequality over a given income interval and CLU assumes that Pigou-Dalton transfers within the bottom or the top part increase bipolarity, one can deduce from (5) and (6) that $\underline{x}$ has to increase as the result of a mean-preserving spread within the bottom part of the income distribution while $\bar{x}$ must decrease after a mean-preserving spread within the top part.

However, the definitions of $\underline{x}$ and $\bar{x}$ must meet the constraints $p\underline{x} + (1-p)\bar{x} = \mu$. Consequently this equality is not likely to hold if, for instance, $\underline{x}$ increases after a mean-preserving spread within $\underline{X}$ but $\bar{x}$ is left unaffected. To keep the average income unchanged, we need $\bar{x}$ to decrease at the same time so as to perfectly compensate for the increase in $\underline{x}$. The same reasoning shows that $\underline{x}$ would also have to adjust after a Pigou-Dalton transfer within the top part of the distribution and the resulting change in $\bar{x}$. In other words, $\underline{x}$ and $\bar{x}$ are both determined by $\underline{X}$ and $\bar{X}$, and they should never be independently considered.

Since, for a given value of p, IND_W and IND_S both assume some form of separability between the bottom and top parts of the income distribution, it would make sense to build

[7] As explained in section 5, $\underline{x}$ and $\bar{x}$ are not independent. As a consequence, a mean preserving-spread within the bottom part of the distribution will not only result in a change in $\underline{x}$ but also in $\bar{x}$. This would create a deviation of $\bar{x}$ from $\mu_{\bar{X}}$ even if there is no inequality within the top part of the income distribution! This is why it is sound to consider jointly the second and third elements in (7) and (8) as capturing the overall clustering component and not try to disentangle it into parts than can be directly related either to cluster in the bottom or in the top part of the income distribution.

bipolarisation indices using, possibly along with mean income, one function that is only determined by $\underline{X}$ and another one that is only determined by $\bar{X}$. As noted earlier, this would generate two-income distributions whose arithmetic mean differ from the one of the observed distribution X but that nevertheless could show the same bipolarisation level and consequently may be used to estimate directly either Ψ_p^A or Ψ_p^R.

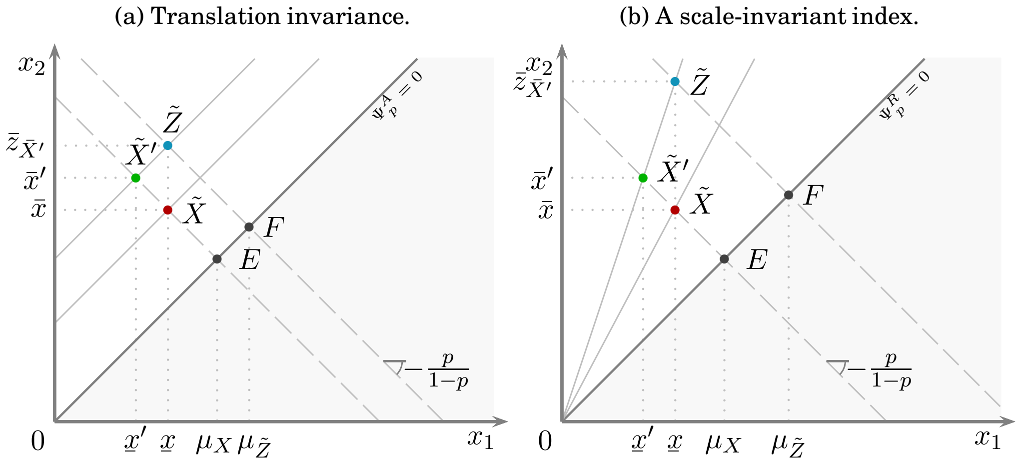


Figure 2: Invariance properties and the estimation of Ψ_p^A and Ψ_p^R.

For this purpose, invariance properties such as SCI and TRI are of crucial importance. To understand why, consider a sequence of progressive transfers within the upper part of an income distribution that transforms a two-income equivalent distribution $\tilde{X}$ into $\tilde{X}'$. Now, let TRI be satisfied, so that all two-income distributions along the straight line that passes through $\tilde{X}'$ and that is parallel to the no-bipolarisation line, will show the same level of bipolarisation as $\tilde{X}'$ (cf Figure 2a). Among these distributions, we can see that there exists a two-income distribution $\boldsymbol{Z}$ whose representative income for the bottom part of the income distribution equals $\underline{x}$. The representative income for the upper part is $\bar{z}_{\bar{X}'}$ and the difference $\bar{z}_{\bar{X}'} - \bar{x}$ captures the clustering effect of the considered sequence of progressive transfers within the top part of the income distribution. Of course, $\tilde{Z}$ is not the two-income equivalent distribution associated with X', but it can directly be used to assess its level of bipolarisation while $\tilde{X}'$ can easily be computed from $\tilde{Z}$ by substracting the difference $\mu_{\tilde{Z}} - \mu_X$ from $\underline{x}$ and $\bar{z}_{\bar{X}'}$. Figure 2b illustrates how, thanks to SCI, such an intermediate distribution like $\tilde{Z}$ can be used to estimate both Ψ_p^R and $\tilde{X}'$. In this case, the ratio $\frac{\mu_X}{\mu_{\tilde{Z}}}$ will be used to scale down the income from $\tilde{Z}$ so as to get $\tilde{X}'$.

Here, we show how these intermediate distributions and values of Ψ_p^A and Ψ_p^R can be obtained using the following estimates:

$$\hat{\Psi}^A(X;p) := \bar{\tau}(\bar{X}) - \underline{\tau}(\underline{X}), \tag{9}$$

$$\hat{\Psi}^R(X;p) := \frac{(1-p)\big(\bar{\tau}(\bar{X}) - \underline{\tau}(\underline{X})\big)}{p\underline{\tau}(\underline{X}) + (1-p)\bar{\tau}(\bar{X})}, \tag{10}$$

where $\underline{\tau} : \mathbb{R}_+^{pn} \to \mathbb{R}_+$ and $\bar{\tau} : \mathcal{D}^{(1-p)n} \to \mathbb{R}_{++}$ are quasi-linear means. Quasi-linear means are a generalization of the concept of mean introduced by Bonferroni in the 1920s which can be expressed as follows:[8]

$$\tau(X;W) := \rho^{-1}\left(\sum_{i=1}^{n} w_i \rho(x_i)\right), \tag{11}$$

where $W := (w_1, \dots w_n) \in \mathcal{S}^n$ is a vector of strictly positive weights and $\rho : \mathbb{R} \to \mathbb{R}$ is bijective. Weights are important from a practical point of view when dealing with survey data with unequal sampling probabilities, but here they essentially reflect ethical considerations as we assume X refers to the whole population. As long as it does not result in misleading interpretation, we use τ_X as shorthand for $\tau(X;W)$. Quasi-linear means notably include power means ($w_i = \frac{1}{n}\ \forall i \in N$ and $\rho(x_i) = x_i^\alpha$) and Gini means ($w_i = \frac{2\big(n-r(i,X)\big)+1}{n^2}$ and $\rho(x_i) = x_i$) that are widely used in welfare and inequality analyses. The two following functional forms feature prominently below:

$$\tau_X^{\gamma,W} := \begin{cases} \frac{1}{\gamma}\log\left(\sum_{i=1}^n w_i e^{\gamma x_i}\right) & \text{if } \gamma \neq 0 \\ \sum_{i=1}^n w_i x_i & \text{if } \gamma = 0 \end{cases}, \tag{12}$$

$$\tau_X^{\alpha,W} := \begin{cases} \left(\sum_{i=1}^n w_i x_i^\alpha\right)^{\frac{1}{\alpha}} & \text{if } \alpha \neq 0 \\ \prod_{i=1}^n x_i^{w_i} & \text{if } \alpha = 0 \end{cases}, \tag{13}$$

with $\sum_{i=1}^n w_i = 1$.

For $n = 2$, Aczél (1948, Theorem 2, p.399) demonstrated that quasi-linear means (11) are the unique family of functions $\chi : \mathbb{R}^n \to \mathbb{R}$ complying with the following axiomatic framework:[9]

Idempotence (IN): $\chi(X;W) = \mu_X\ \forall X \in \mathcal{E}^n$, $W \in \mathcal{D}^n$.

Monotonicity (MO): $\chi(Y;W) \geq \chi(X;W)\ \forall \{X, Y, W\} \subset \mathcal{D}^n$ such that $Y \geq X$ but $Y \neq X$.

Continuity (CO): χ is continuous over its domain.

Bisymmetry (BS): $\forall (x_1, x_2, x_3) \in \mathbb{R}^3$, $(w_1, w_2, w_3) \in \mathcal{D}^3$, we have $\chi\big((y_{12}, x_3); (w_1+w_2, w_3)\big) = \chi\big((y_{13}, x_2); (w_1+w_3, w_2)\big) = \chi\big((y_{23}, x_1); (w_2+w_3, w_1)\big)$ where $y_{ij} := \chi\big((x_i, x_j); (w_i, w_j)\big)$.

As shown below, the estimation of Ψ_p^A and Ψ_p^R using quasi-linear means requires using the TRI and SCI axioms, respectively, and focusing on specific families of quasi-linear means.

5.1 Absolute indices

We consider the case of absolute bipolarisation indices first. Let $\underline{\tau}^A$ and $\bar{\tau}^A$ be quasi-linear means used to estimate Ψ_p^A. The estimator $\hat{\Psi}^A$ of Ψ_p^A, as given by (9), is valid if and only if $\hat{\Psi}^A(X;p) = \Psi_p^A(X)\ \forall X \in \mathcal{D}$, $p \in]0,1[$. This validity notably requires the satisfaction of TRI that can be observed if and only if, $\forall X \in \mathcal{D}^n$, $D \in \mathcal{E}^n$, $\underline{\tau}^A_{X+D} = \underline{\tau}^A_X + f(\mu_D)$ and

[8]For a review, see notably Muliere and Parmigiani (1993).

[9]Because of the bisymmetry property, extension to $n > 2$ is straightforward.

$\bar{\tau}^A_{X+D} = \bar{\tau}^A_X + f(\mu_D)$. Because of CO and MO, the derivative $\frac{\partial f}{\partial x}$ necessarily exists and is non-negative. Combined with IN, we necessarily have $\underline{\tau}^A$ and $\bar{\tau}^A$ being unit-translatable, that is $f(x) = x$. Indeed, if $\underline{\tau}^A_X = \mu_X$ (respectively $\bar{\tau}^A_X = \mu_X$) whenever $X \in \mathcal{E}$, we necessarily have $\underline{\tau}^A_{X+D} = \mu_{X+D} = \mu_X + \mu_D$ (respectively $\bar{\tau}^A_{X+D} = \mu_{X+D} = \mu_X + \mu_D$).

Let $\eta := p\underline{\tau}_{\underline{X}} + (1-p)\bar{\tau}_{\bar{X}}$. With both $\underline{\tau}^A$ and $\bar{\tau}^A$ being unit translatable, we obtain $\underline{x} = \underline{\tau}^A_{\underline{X}} - (\eta - \mu)$ and $\bar{x} = \bar{\tau}^A_{\bar{X}} - (\eta - \mu)$ as $(\underline{\tau}^A_{\underline{X}}, \bar{\tau}^A_{\bar{X}})$ is supposed to show the same level of bipolarisation as $(\underline{x}, \bar{x})$. Then proposition 1 characterises our proposed class of rank-dependent absolute 'two-income' bipolarisation indices:

Proposition 1. $\hat{\Psi}^A$ *is a valid estimator of* Ψ^A_p *and complies with ANO, POP, IND*$_W$*, SPC, CLU, CON, TRI, ENO, and LIN if* $\underline{\tau}^A$ *and* $\bar{\tau}^A$ *are of the form* (12) *with* $w_i = \pi\big(r(i,X)\big)$, $\pi : \mathbb{N}^* \to [0,1]$*, and:*

i) for $\underline{\tau}$*, we have* $\gamma \leq 0$ *and* π *is a non-decreasing function,*

ii) for $\bar{\tau}$*, we have* $\gamma \geq 0$ *and* π *is a non-increasing function.*

Proof. See appendix A.1. □

It is worth pointing out that $\hat{\Psi}^A$ boils down to the first class of rank-dependent bipolarisation indices proposed by Wang and Tsui (2000, p. 356) when $p = \frac{1}{2}$ and $\gamma = 0$ for both $\underline{\tau}$ and $\bar{\tau}$. Meanwhile, proposition 2 characterises a novel class of rank-independent absolute 'two-income' bipolarisation indices:

Proposition 2. $\hat{\Psi}^A$ *is a valid estimator of* Ψ^A_p *and complies with ANO, POP, IND*$_S$*, SPC, CLU, CON, TRI, ENO, and LIN if* $\underline{\tau}^A$ *and* $\bar{\tau}^A$ *are of the form* (12) *with* $w_i = \frac{1}{n}$*, and:*

i) for $\underline{\tau}$*, we have* $\gamma \leq 0$*,*

ii) for $\bar{\tau}$*, we have* $\gamma \geq 0$*.*

Proof. See appendix A.1. □

5.2 Relative indices

We now turn to the case of relative bipolarisation indices. Let $\underline{\tau}^R$ and $\bar{\tau}^R$ be the two quasi-linear means used to compute our estimate $\hat{\Psi}^R(X;p)$ of Ψ^R_p. $\hat{\Psi}^R(X;p)$ is valid if and only if $\hat{\Psi}^R(X;p) = \Psi^R_p \; \forall X \in \mathcal{D}$, $p \in]0,1[$.

It can easily be checked that $\hat{\Psi}^R(X;p)$ complies with SCI if, $\forall \kappa > 0$ and $X \in \mathcal{D}^n$, we have $\underline{\tau}_{\kappa X} = f(\kappa)\underline{\tau}_X$ and $\bar{\tau}_{\kappa X} = f(\kappa)\bar{\tau}_X$. As shown by Eichhorn (1978, Corollary 1.9.20) and assuming $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$, this property is satisfied if and only if $f(\kappa) = \kappa^r$ with $r \in \mathbb{R}_{++}$. Because of CO and MO, the function f is necessarily continuous, strictly positive and increasing, so that Eichhorn's result holds. We can focus on the case $r = 1$, that is linearly homogeneous generalized means, thanks to IN. Indeed, if $\underline{\tau}_Y = \mu_Y$ (respectively $\bar{\tau}_Y = \mu_Y$) whenever $Y \in \mathcal{E}$, we necessarily have $\underline{\tau}_{\kappa Y} = \mu_{\kappa Y} = \kappa\mu_Y$ (respectively $\bar{\tau}_Y = \mu_{\kappa Y} = \kappa\mu_Y$).

With both $\underline{\tau}^R$ and $\bar{\tau}^R$ being linearly homogeneous, we obtain $\underline{x} = \underline{\tau}^R_{\underline{X}}\frac{\mu}{\eta}$ and $\bar{x} = \bar{\tau}^R_{\bar{X}}\frac{\mu}{\eta}$ as $(\underline{\tau}^R_{\underline{X}}, \bar{\tau}^R_{\bar{X}})$ is assumed to show the same level of bipolarisation as $(\underline{x}, \bar{x})$. Then proposition 3 characterises a class of rank-dependent relative 'two-income' bipolarisation indices:

Proposition 3. *$\hat{\Psi}^R$ is a valid estimator of Ψ_p^R and complies with ANO, POP, IND$_W$, SPC, CLU, CON, SCI, ENO, MNO, and LIN if $\underline{\tau}^R$ and $\bar{\tau}^R$ are of the form* (13) *with $w_i = \pi\big(r(i,X)\big)$, $\pi : \mathbb{N}^* \to [0,1]$, and:*

i) for $\underline{\tau}$, we have $\alpha \geq 1$ and π is a non-decreasing function,

ii) for $\bar{\tau}$, we have $\alpha \leq 1$ and π is a non-increasing function.

Proof. See appendix A.1. □

Note that when $p = \frac{1}{2}$, $\alpha = 1$ for both $\underline{\tau}$ and $\bar{\tau}$, $\hat{\Psi}^R$ becomes a mean-normalized version of the second family of bipolarisation indices proposed by Wang and Tsui (2000, p. 356). Meanwhile, proposition 4 characterises a novel class of rank-independent relative 'two-income' bipolarisation indices:

Proposition 4. *$\hat{\Psi}^R$ is a valid estimator of Ψ_p^R and complies with ANO, POP, IND$_S$, SPC, CLU, CON, SCI, ENO, MNO, and LIN if $\underline{\tau}^R$ and $\bar{\tau}^R$ are of the form* (13) *with $w_i = \frac{1}{n}$, and:*

i) for $\underline{\tau}$, we have $\alpha \geq 1$,

ii) for $\bar{\tau}$, we have $\alpha \leq 1$.

Proof. See appendix A.1. □

Whenever $\exists i \in N$ such that $x_i = 0$, it is necessary to chose $\alpha \in]0,1]$ for $\underline{\tau}$ in both Proposition 3 and 4.

5.3 Mean-inequality expressions of quasi-linear means

An appealing feature of quasi-linear means is that they can easily be expressed as functions of the arithmetic mean and an inequality index (Chakravarty, 2009). Indeed, in the case where τ is Schur-concave, τ_X can be interpreted as an equally distributed equivalent income x^E in the spirit of the Kolm, Atkinson and Sen's approach. Defining an absolute inequality index as $\Theta^A := \mu - x^E$ and a relative index as $\Theta^R = 1 - \frac{x^E}{\mu}$, we naturally have, by definition, $\tau_X = \mu - \Theta^A$ or $\tau_X = (1-\Theta^R)\mu$. Here, we observe that the same approach can be used for Schur-convex quasi-linear means with appropriate definitions of the inequality indices. Indeed, in this case, we necessarily have $\tau_X \geq \mu \; \forall X$ and the difference between τ_X and μ can be used to assess the extent of inequality. Let us redefine Θ^A and Θ^R to include the possibility of having these indices being also defined with respect to a Schur-convex estimation of the EDE income:

$$\Theta^A := \begin{cases} \mu - x^E & \text{if } x^E = \tau_X \text{ with } \tau_X \text{ being S-concave} \\ x^E - \mu & \text{if } x^E = \tau_X \text{ with } \tau_X \text{ being S-convex} \end{cases}, \tag{14}$$

$$\Theta^R := \begin{cases} 1 - \frac{x^E}{\mu} & \text{if } x^E = \tau_X \text{ with } \tau_X \text{ being S-concave} \\ \frac{x^E}{\mu} - 1 & \text{if } x^E = \tau_X \text{ with } \tau_X \text{ being S-convex} \end{cases}. \tag{15}$$

Using results from the previous section, definitions (14) and (15) then provide a generalization of families of inequality indices suggested by Ebert (1988), that is:

$$\Theta_X^{\gamma,W} := \begin{cases} \mu_X - \frac{1}{\gamma}\log\left(\sum_{i=1}^n w_i e^{\gamma x_i}\right) & \text{if } \gamma < 0 \\ \mu_X - \sum_{i=1}^n w_i x_i & \text{if } \gamma = 0 \\ \frac{1}{\gamma}\log\left(\sum_{i=1}^n w_i e^{\gamma x_i}\right) - \mu_X & \text{if } \gamma > 0 \end{cases}, \tag{16}$$

$$\Theta_X^{\alpha,W} := \begin{cases} 1 - \frac{\left(\sum_{i=1}^n w_i x_i^\alpha\right)^{\frac{1}{\alpha}}}{\mu_X} & \text{if } \alpha \in]-\infty, 0[\cup]0,1] \\ 1 - \prod_{i=1}^n \left(\frac{x_i}{\mu_X}\right)^{w_i} & \text{if } \alpha = 0 \\ \frac{\left(\sum_{i=1}^n w_i x_i^\alpha\right)^{\frac{1}{\alpha}}}{\mu_X} - 1 & \text{if } \alpha > 1 \end{cases}, \tag{17}$$

with $w_i = \pi\big(r(i,X)\big)$, $\pi : \mathbb{N}^* \to [0,1]$ being an non-increasing function of rank if $\gamma \leq 0$ or $\alpha \leq 1$ and a non-decreasing function otherwise. Famous inequality measures can be obtained as special cases of (16) or (17). For instance, Kolm and Pollak absolute indices are obtained for $w_i = \frac{1}{n}\ \forall i \in N$ and $\gamma < 0$. Atkinson (1970)'s relative indices correspond to the case $w_i = \frac{1}{n}\ \forall i \in N$ and $\alpha < 1$. Generalized Gini indices (Weymark, 1981) can be obtained either with (16) or (17). Imposing $w_i \geq w_j\ \forall\{i,j\} \in N$ such that $r(i,X) < r(j,X)$, absolute generalized Gini indices are obtained in the case $\gamma = 0$ and relative generalized Gini indices with $\alpha = 1$. In both cases, the family of S-Gini relative indices (Donaldson and Weymark, 1980) is obtained for $w_i = \frac{1}{n^\delta}\Big(\big(n+1-r(i,X)\big)^\delta - \big(n-r(i,X)\big)^\delta\Big)$, $\delta > 1$. Another example is the class of 'metallic' indices recently proposed by Subramanian (2021) and that includes in addition to the Gini index, a Fibonacci index that corresponds to the case $\alpha = 1$ and $w_i = \frac{\sum_{i=1}^n (\mathcal{F}[n+3-r(i,X)]-1)}{\mathcal{F}[n+4]-n-3}$ with $\mathcal{F}[j]$ being the jth element of the sequence of Fibonacci numbers.

Among new indices defined by (17), one can find the complement D of Atkinson's indices, that is:

$$D(X;\alpha) := \left(\frac{1}{n}\sum_{i=1}^n \left(\frac{x_i}{\mu_X}\right)^\alpha\right)^{\frac{1}{\alpha}} - 1, \quad \text{with } \alpha > 1. \tag{18}$$

It can easily be seen that $D(X;\tau) = 0\ \forall X \in \mathcal{E}$ and $D(X;\tau) > 0$ otherwise. Moreover, the index complies with the transfer axiom (since it is S-concave) as well a with the usual anonymity, population and scale invariance axioms. Contrary to Atkinson's family, it can always be used with zero income, while this is not possible with Atkinson's indices for $\alpha \leq 0$. On the other hand, a possibly undesirable feature is that the upper bound of D does not tend to 1 as population size tends to infinity.[10]

With the appropriate choices regarding the different parameters of $\hat{\Psi}^A$ and $\hat{\Psi}^R$, it is possible to consider bipolarisation indices that use the same inequality index to assess inequalities within the bottom and the top parts of the income distribution. For instance,

[10] Indeed, when $x_i = z > 0$ and $x_j = 0\ \forall j \neq i$, we have $\mu = \frac{z}{n}$ and so the maximum value of D is $n^{1-\frac{1}{\alpha}} - 1$.

the Gini index can be written either as:

$$G_X = 1 - \frac{\sum_{i=1}^{n} \frac{2\left(n-r(i,X)\right)+1}{n^2} x_i}{\mu_X}. \tag{19}$$

or equivalently:

$$G_X = \frac{\sum_{i=1}^{n} \frac{2r(i,X)-1}{n^2} x_i}{\mu_X} - 1. \tag{20}$$

Consequently, with the appropriate weighing schemes in $\underline{\tau}$ and $\bar{\tau}$, we have $\underline{\tau}_{\underline{X}} = (1 + G_{\underline{X}})\mu_{\underline{X}}$ and $\bar{\tau}_{\bar{X}} = (1 - G_{\bar{X}})\mu_{\bar{X}}$. We thus obtain the rank-dependent relative bipolarisation index:

$$\hat{\Psi}^{RG}(X;p) = \frac{(1-p)\left((1-G_{\bar{X}})\mu_{\bar{X}} - (1+G_{\underline{X}})\mu_{\underline{X}}\right)}{\eta}. \tag{21}$$

Since the Gini index is a compromise index, *i.e.* its weighing system can used to design both absolute and relative inequality indices, a Gini-based estimate $\hat{\Psi}^{AG}$ of Ψ_p^A can be written as:

$$\hat{\Psi}^{AG}(X;p) = (1-G_{\bar{X}})\mu_{\bar{X}} - (1+G_{\underline{X}})\mu_{\underline{X}}. \tag{22}$$

Indices $\hat{\Psi}^{RG}$ and $\hat{\Psi}^{AG}$ are reminiscent of the weighing scheme used for the Foster-Wolfson index as shown by Wang and Tsui (2000). Indeed $\hat{\Psi}^{AG}$ is an absolute version of the Foster-Wolfson index when $p = \frac{1}{2}$.[11] At first sight, $\hat{\Psi}^{RG}$ looks like the mean-normalized version of the Foster-Wolfson index, up to a multiplicative term, proposed by Rodríguez and Salas (2003) in the case $p = \frac{1}{2}$. However, both indices are ordinally equivalent only in the specific case of X being a linear combination of $\mathcal{O}^n$ and an element from $\mathcal{B}^n_{\frac{1}{2}}$.[12]

It is worth pointing that such indices can be appealing because it notably helps reducing the choice of parameters for $\underline{\tau}$ and $\bar{\tau}$ when choosing a bipolarisation index among the classes suggested in Propositions 1 and 3. Moreover, it satisfies our natural inclination for symmetry. However, from an ethical point of view, this should not be deemed a compelling argument for preferring $\hat{\Psi}^A$ and $\hat{\Psi}^R$ based on the Gini index in comparison with those based on Kolm-Pollack or Atkinson indices and their counterparts computed with the help of S-convex generalized means. It is notably worth reminding that if one claims a bipolarisation index should comply with IND$_S$ instead of IND$_W$, then the use of bipolarisation indices built upon rank-dependent generalized means shall be ruled out.

[11] Let $G^B = \frac{\mu_{\bar{X}} - \mu_{\underline{X}}}{4\mu}$ and $G^W = \frac{\mu_{\underline{X}}}{4\mu} G_{\underline{X}} + \frac{\mu_{\bar{X}}}{4\mu} G_{\bar{X}}$ respectively be the between and within component of the Gini index when the income distribution is split into two non overlapping groups of the same size. Using (22), we have:

$$\begin{aligned}\hat{\Psi}^{AG}\left(X;\frac{1}{2}\right) &= \mu_{\bar{X}} - \mu_{\underline{X}} - \mu_{\bar{X}} G_{\bar{X}} - \mu_{\underline{X}} G_{\underline{X}} = 4\mu \frac{\mu_{\bar{X}} - \mu_{\underline{X}}}{4\mu} - 4\mu\left(\frac{\mu_{\underline{X}}}{4\mu} G_{\underline{X}} + \frac{\mu_{\bar{X}}}{4\mu} G_{\bar{X}}\right), \\ &= 4\mu\left(G^B - G^W\right) = 2\mu FW_\mu, \end{aligned} \tag{23}$$

where FW_μ is the mean-normalized version of the Foster-Wolfson index (as defined in Wolfson, 1994).

[12] Using (23) and defining $\underline{G}^W = \frac{\mu_{\underline{X}}}{4\mu} G_{\underline{X}}$ and $\bar{G}^W = \frac{\mu_{\bar{X}}}{4\mu} G_{\bar{X}}$ respectively as the bottom and top components of G^W, simple algebra makes it possible to see that $\hat{\Psi}^{RG}\left(X;\frac{1}{2}\right) = \frac{FW_\mu}{1+2\underline{G}^W - 2\bar{G}^W}$. In the absence of inequality within each part of the income distribution and for $p = \frac{1}{2}$, $\hat{\Psi}^{RG}$ boils down to the mean-normalized version of the Foster-Wolfson index.

6 Bipolarisation in a sample of Sub-Saharan African countries

Indices and decompositions introduced in the previous pages are now illustrated using data from the 2018–2019 *Enquête Harmonisée sur les Conditions de Vie des Ménages* (EHCVM) for Benin, Burkina Faso, Chad, Cote d'Ivoire, Guinea Bissau, Mali, Niger, Senegal, and Togo.[13] These nationally representative household surveys are the result of a joint program by the World Bank and the West Africa Economic Monitary Union (WAEMU). The aim of this project is notably to produce harmonized, hence comparable, figures on household well-being. Per-capita consumption aggregates from these surveys are quite comprehensive as they include imputed rents for housing in addition to usual items such as food, garment, utilities, health, education, durable goods... Spatial deflators are used to make consumption levels perfectly comparable across, countries, regions and areas. Finally, since data is collected in two waves with each wave covering half of the sample–the first wave was fielded in Fall 2018, while the second wave occurred during Spring 2019—, consumption aggregates are deflated so as to take inflation into account. Usual caveats apply due to the possible under-representation of the richest households in household surveys. As in the case of inequality estimates, the absence of top-income earners is likely to result in a downward bias of our bipolarisation estimates. However, since we focus on cross-country comparisons and since sampling procedures are fully harmonized, we can reasonably assume that missing information on the right-hand tail of the consumption distributions has limited effect on the estimated bipolarisation orderings.

Kernel estimates of the consumption distributions (Figure 6), show that the nine distributions are essentially single peaked. Although distributions broadly show a typical lognormal shape, Kolmogorov-Smirnov tests result in the rejection of the lognormality assumption. Observed distributions are not only more rightly skewed than under lognormality, but they also show fatter tails.[14] Positive excess kurtosis is indicative of a relatively low clustering component when p is set at intermediate values. Nevertheless, we can also see that many distributions show a small bump within the interval $[7 \times 10^6, 12 \times 10^6]$ CFA Francs, that is approximately between €1,050 and €1,800 per year and per person.

Table 1 reports estimated values for mean income as well as for the Gini and Atkinson indices of relative inequality. Consumption inequalities are moderate for our sample of countries, with Guinea Bissau being the less unequal country and Burkina Faso being the most unequal. Lorenz dominance tests (Figure 7 and Table 3) show that 20 out the 36 possible comparisons result in a robust inequality ordering. Failures to observe a Lorenz dominance relationship mostly happen when Chad or Niger are considered for the pairwise

[13] Institut National de la Statistique et de l'Analyse Économique (INSAE) (2018–19), Institut National de la Statistique et de la Démographie (INSD) (2018–19), Institut National de la Statistique (INS) (2018–19a), Instituto Nacional de Estatística (INE) (2018–19), Institut National de la Statistique (INSTAT) (2018–19), Institut National de la Statistique (INS) (2018–19b), Agence National de la Statistique et de la Démographie (ANSD) (2018–19), Institut National de la Statistique, des Études Économiques et Démographiques (INSEED (2018–19) and Institut National de la Statistique et des Études Économiques et Démographiques (INSEED) (2018–19).

[14] Skewness and kurtosis coefficient computed for the logarithm of per capita consumption respectively range $[0.31, 0.69]$ and $[3.11, 4.19]$.

Table 1: Mean consumption and relative inequality levels in selected Sub-Saharan African countries, 2018–2019.

	Mean		Gini		Atk. ($\alpha = 0.5$)		Atk. ($\alpha = 0$)	
	μ	rank	Θ	rank	Θ	rank	Θ	rank
Guinea Bissau	3.42e+05	4	0.316	1	0.0817	1	0.149	1
Mali	3.78e+05	6	0.332	2	0.0879	2	0.163	2
Chad	3.33e+05	3	0.336	3	0.0918	3	0.169	3
Benin	3.68e+05	5	0.347	4	0.0984	4	0.179	4
Niger	2.63e+05	1	0.35	5	0.103	7	0.182	5
Senegal	5.07e+05	8	0.351	6	0.103	6	0.183	7
Côte d'Ivoire	5.13e+05	9	0.351	7	0.0996	5	0.183	6
Togo	3.85e+05	7	0.381	8	0.119	8	0.214	8
Burkina Faso	3e+05	2	0.386	9	0.124	9	0.216	9

Note: Atk denotes the Atkinson inequality index.

inequality comparisons.

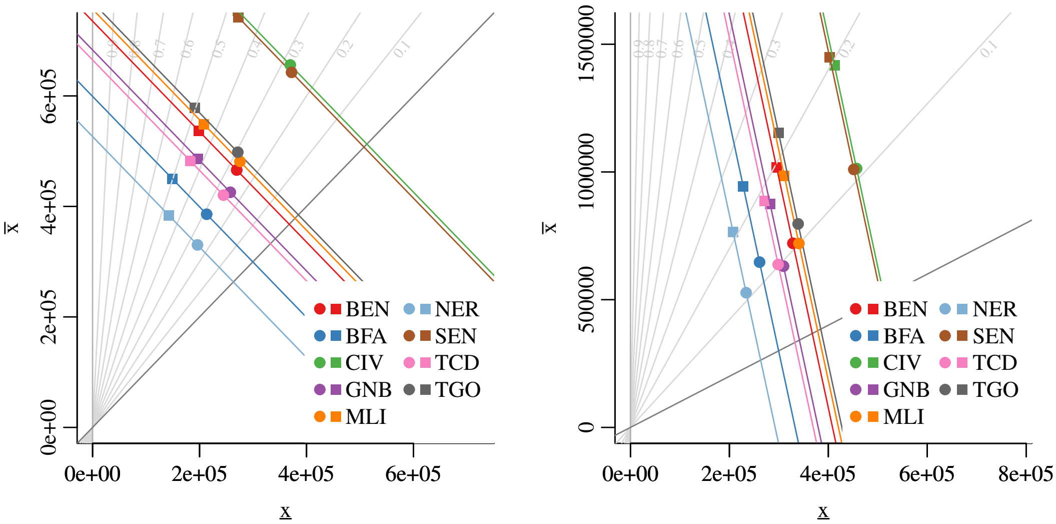
Table 2: Relative bipolarisation levels in selected Sub-Saharan African countries, 2018–2019.

	$\underline{\alpha} = \bar{\alpha} = 1$, $\underline{\delta} = \bar{\delta} = 2$		$\underline{\alpha} = \bar{\alpha} = 1$, $\underline{\delta} = \bar{\delta} = 3$		$\underline{\alpha} = 2; \bar{\alpha} = 0.5$, $\underline{\delta} = \bar{\delta} = 1$		$\underline{\alpha} = 3; \bar{\alpha} = 0$, $\underline{\delta} = \bar{\delta} = 1$	
	$\hat{\Psi}^R$	rank	$\hat{\Psi}^R$	rank	$\hat{\Psi}^R$	rank	$\hat{\Psi}^R$	rank
				$p = 0.5$				
Guinea Bissau	0.246	1	0.176	1	0.39	1	0.361	1
Mali	0.272	6	0.196	6	0.417	4	0.388	4
Chad	0.264	3	0.188	3	0.414	2	0.382	3
Benin	0.267	5	0.189	4	0.421	5	0.388	5
Niger	0.255	2	0.179	2	0.414	3	0.379	2
Senegal	0.267	4	0.19	5	0.422	6	0.388	6
Côte d'Ivoire	0.279	7	0.2	7	0.432	7	0.399	7
Togo	0.294	9	0.209	9	0.457	9	0.419	9
Burkina Faso	0.288	8	0.204	8	0.455	8	0.415	8
				$p = 0.9$				
Guinea Bissau	0.0942	1	0.0679	1	0.15	1	0.132	1
Mali	0.1	2	0.0725	2	0.154	2	0.135	2
Chad	0.102	3	0.0733	3	0.16	3	0.139	3
Benin	0.107	4	0.0769	4	0.168	5	0.146	4
Niger	0.112	7	0.0789	6	0.182	7	0.157	7
Senegal	0.11	6	0.0792	7	0.176	6	0.152	6
Côte d'Ivoire	0.108	5	0.0785	5	0.168	4	0.146	5
Togo	0.119	8	0.0852	8	0.187	8	0.159	8
Burkina Faso	0.129	9	0.0919	9	0.203	9	0.174	9

Note: Countries are ordered in increasing order of the Gini index.

Relative bipolarisation estimates are reported in Table 2. Two-income representations are provided on Figure 3. For the present illustration, we essentially focus the index $\hat{\Psi}^{RG}$, but we also consider other members of $\hat{\Psi}^R$ for sensitivity analysis. More specifically, using the S-Gini weighing scheme, we consider the case $\delta = 3$ for both the bottom and top parts of the distribution, so that, compared to $\hat{\Psi}^{RG}$, $\hat{\Psi}^R$ then shows a larger sensitivity to income changes at the tails than around the income quantile associated with p. We also estimate values of $\hat{\Psi}^R$ with power means ($(\underline{\alpha}, \bar{\alpha}) = c(2, 0.5)$ and $(\underline{\alpha}, \bar{\alpha}) = c(3, 0)$) and

(a) $p = 0.5$.　　　　(b) $p = 0.9$.



Note: Bullets depicts $(\underline{x}, \bar{x})$ while squares are for $(\mu_{\underline{X}}, \mu_{\bar{X}})$.

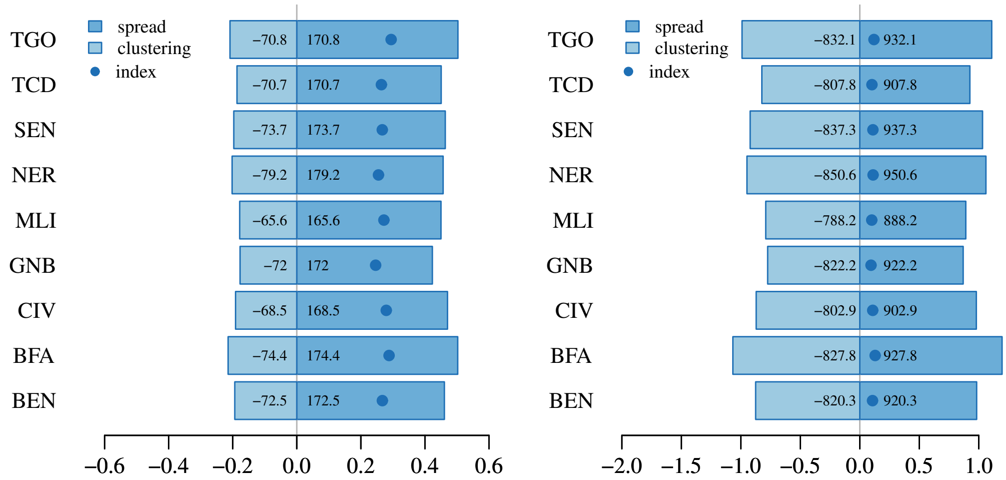
Figure 3: Two-income representations of bipolarisation for selected Sub-Saharan African countries, 2018–2019, $p = \frac{1}{2}$.

w_i only reflecting sampling weights. Since sensitivity to clustering increases as values of $\underline{\alpha}$ and $\bar{\alpha}$ depart from 1, we may expect the level of $\hat{\Psi}^R$ to be lower with the latter set of values in comparison with those obtained with the former set of values for $(\underline{\alpha}, \bar{\alpha})$. The latter choice will also result in a lower sensitivity to income changes at the tails than in the neighborhood of $x(p)$.

When the top and bottom parts of the distribution are defined with respect to the median, bipolarisation is the lowest for Guinea Bissau and the largest for Togo. Visual inspection shows that the bipolarisation ordering obtained with $\hat{\Psi}^{RG}$ is close to the inequality ordering observed with the Gini index. Rank correlation is not perfect—Spearman's rank correlation coefficient is 0.73—, but significantly different from 0 at the 5% confidence level. This is indicative of distributions that, when $p = 0.5$, essentially differ with respect to their spread component. Nevertheless, it is worth pointing that Mali ranked 2nd with respect to inequality but only 6th with respect to bipolarisation, while Niger performs relatively better regarding bipolarisation in comparison with inequality. Figure 4a shows the result of decomposition (8). It can then be checked that, for our sample of Sub-Saharan African countries, the most important part of the variance of $\hat{\Psi}^{RG}$ is due to the heterogeneity of the spread component. As this element is common to $\hat{\Psi}^{RG}$ and G, it explains why the two orderings are so close. Considering other members of $\hat{\Psi}^R$ and other inequality indices does not affect this general result. Visual inspection of Figure 3a also show that bipolarisation differences among the nine countries would be much larger in the absence of within-group inequalities.

We also consider bipolarisation when the consumption distributions are split at the last population decile ($p = 0.9$). While Guinea Bissau is still the least bipolarized country (bottom panel of Table 2), splitting distributions at the top decile results in Burkina Faso

(a) $p = 0.5$. (b) $p = 0.9$.



Note: Figures within bar indicates the size, in percentage, of the clustering and spread components relative to the value of the bipolarisation index.

Figure 4: Spread and clustering components of $\hat{\Psi}^{RG}$ for selected Sub-Saharan African countries, 2018–2019.

being the most bipolarised country among our sample. Figure 4b now shows that the size of the clustering component amounts, in absolute terms, to more than 90% of the spread component, hence resulting in lower values of $\hat{\Psi}^{RG}$ when compared with the case $p = 0.5$.

Figure 5 plots the value of $\hat{\Psi}^{RG}$ for values of p ranging from 0.02 to 0.98. We first note that in all nine cases, we observe a systematic decrease in $\hat{\Psi}^{RG}$ as p increases although nothing prevents $\hat{\Psi}^{RG}$ to increase with p. Considering other members of $\hat{\Psi}^{R}$ does not change this result.[15] Surprisingly, we also observe that, for some pairs of countries, the bipolarisation ordering is not sensitive to the choice of p. For instance we observe that Guinea Bissau systematically shows less bipolarisation than Niger and Mali, and the value of $\hat{\Psi}^{RG}$ is always inferior for Mali when compared with Côte d'Ivoire, Burkina Faso, and Togo. Such robust orderings suggest that these countries essentially differ with respect to the relative spread between tails. On the other hand, we also observe multiple crossings when comparing for instance Benin and Mali. Bipolarisation is larger in Benin for $p \in [.02, .29] \cup [.72, .98]$ and larger in Mali between these two intervals.

7 Concluding remarks

This paper introduces a new theoretical approach to bipolarisation measurement. Assuming an equivalence between any observed income distribution and a counterfactual perfectly bimodal distribution with the same mean thanks to possible tradeoffs between within- and between-group progressive transfers, it suggests absolute and relative indices that show many appealing features. Firstly, they make it possible to overcome the usual

[15]Figures are not provided here but are available upon request.

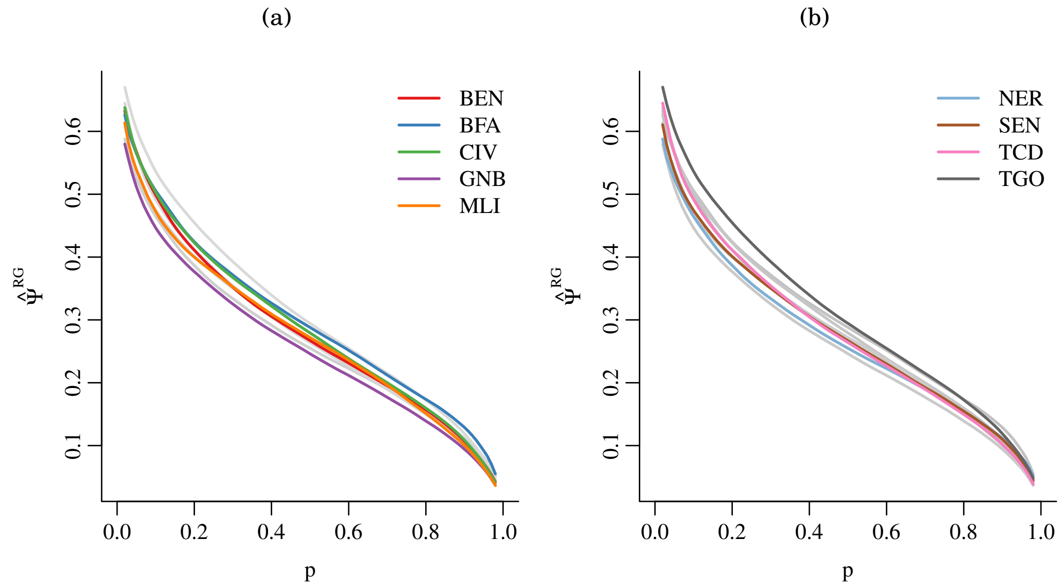


Figure 5: Sensitivity of $\hat{\Psi}^{RG}$ to the choice of p for selected Sub-Saharan African countries, 2018–2019.

grouping of the population into two halves by considering any partition into two non-overlapping groups of different sizes. Secondly, they can easily be interpreted either as i) the representative gap between the two groups or ii) the representative relative excess income share of the richest group. Thirdly, they can additively be decomposed into components respectively associated with the difference in average income between the two groups (spread component) and with the dispersion of income within each group (clustering component). Finally, some of these indices can be expressed as functions of average income and widely used inequality indices estimated for both the bottom and the top income groups. This property eases both the estimation of the indices and the study of their relationship with overall inequality.

Like other ethical indices in distributional analysis, our proposed bipolarisation indices do not provide estimates of the true economic costs for bringing the economy from its current state to a non-polarised situation. Indeed they should be regarded as assessments of the social cost of bipolarisation. However, since they are not explicitly built on social welfare functions, a decisive contribution would be to unveil the social preferences upon which our classes of bipolarisation indices could be constructed. If the EDE approach has failed concerning this objective, we hope our 'two-income' approach could be more fruitful.

The paper also provides an illustration on consumption bipolarisation for a sample of nine Sub-Saharan African countries. Unsurprisingly, it shows that more unequal countries tend to exhibit more bipolarised consumption distributions. The imperfect correlation between relative inequality and bipolarisation indices notably shows that even if the spread component is a major driver of both inequality and bipolarisation, differences in clustering empirically result in reverse orderings. Another remarkable result is the possibility of observing bipolarisation orderings that do not depend on the choice of the population percentile used to split the population into two parts, hence motivating the search of dominance criteria that, unlike the bipolarisation curves proposed by Foster and Wolfson (2010), do not depend on a specific cutoff for sharing income distributions into a bottom and a top part.

A Proofs

A.1 Proof of Proposition 1 to 4

The sufficiency of the continuity restriction on $\underline{\tau}$ and $\bar{\tau}$ for CON to hold is straightforward. The same is true regarding ANO and the symmetry restriction for $\underline{\tau}$ and $\bar{\tau}$. It can also easily be checked that replication homogeneity makes sure POP is satisfied. Indeed:

$$\hat{\Psi}^A\big(d_\lambda(X),p\big) = \bar{\tau}_{d_\lambda(\bar{X})} - \underline{\tau}_{d_\lambda(\underline{X})} = \bar{\tau}_{\bar{X}} - \underline{\tau}_{\underline{X}} = \hat{\Psi}^A\big(X,p\big), \tag{24}$$

$$\hat{\Psi}^R\big(d_\lambda(X),p\big) = \frac{(1-p)\big(\bar{\tau}_{d_\lambda(\bar{X})} - \underline{\tau}_{d_\lambda(\underline{X})}\big)}{p\underline{\tau}_{d_\lambda(\underline{X})} + (1-p)\bar{\tau}_{d_\lambda(\bar{X})}} = \frac{(1-p)(\bar{\tau}_{\bar{X}} - \underline{\tau}_{\underline{X}})}{p\underline{\tau}_{\underline{X}} + (1-p)\bar{\tau}_{\bar{X}}} = \hat{\Psi}^R\big(X,p\big). \tag{25}$$

In the case of $\hat{\Psi}^A$, TRI results from $\underline{\tau}^A$ and $\bar{\tau}^A$ being unit-translatable. Considering $\hat{\Psi}^R$, the satisfaction of SCI is a direct consequence of the homogeneity of degree 1 of $\underline{\tau}^R$ and $\underline{\tau}^R$ with respect to income vectors.

Regarding SPR and CLU, we first note that:

$$\frac{\partial \hat{\Psi}^A}{\partial \underline{\tau}} = -1 \leq 0, \tag{26}$$

$$\frac{\partial \hat{\Psi}^A}{\partial \bar{\tau}} = 1 \geq 0. \tag{27}$$

$$\frac{\partial \hat{\Psi}^R}{\partial \underline{\tau}} = -\frac{(1-p)\bar{\tau}}{(p\underline{\tau} + (1-p)\bar{\tau})^2} \leq 0, \tag{28}$$

$$\frac{\partial \hat{\Psi}^R}{\partial \bar{\tau}} = \frac{(1-p)\underline{\tau}}{(p\underline{\tau} + (1-p)\bar{\tau})^2} \geq 0. \tag{29}$$

Since $\underline{\tau}$ and $\bar{\tau}$ are increasing functions of the elements of their income vectors, if changes in $\underline{\tau}$ and $\bar{\tau}$ are the results of income increments, we can easily see from equations (26)-(29) that SPR is satisfied. If we observe a progressive transfer within $\underline{X}$, then, by S-convexity, $\underline{\tau}$ decreases and (26) and (28) indicates this will result in an increase respectively in $\hat{\Psi}^A$ and $\hat{\Psi}^R$. In the same manner, a progressive transfer within $\bar{X}$ yields an increase in $\bar{\tau}$ and consequently a larger value for $\hat{\Psi}^A$ and $\hat{\Psi}^R$ since (27) and (29) are positive. Hence, CLU is satisfied.

New, let's consider ENO. We have $\underline{\tau}_{\underline{X}} = \bar{\tau}_{\bar{X}}$ if $\bar{X} = \underline{X}$. Since $\underline{\tau}$ and $\bar{\tau}$ are replication-invariant functions, we necessarily have $\underline{\tau}_{\underline{X}} = \bar{\tau}_{\bar{X}}$,$\forall \{\underline{X}, \bar{X}\} \subset \mathcal{D}$ and $p \in]0,1[$, such that $d_\lambda(\bar{X}) = d_{\lambda'}(\underline{X})$ with $\lambda = (1-p)n$ and $\lambda' = pn$. As a consequence, $\forall X \in \mathcal{D}$ and $p \in]0,1[$, we have $\Psi^A = \Psi^R = 0$. ENO is thus satisfied. Regarding MNO, the inspection of (13) shows that $\underline{\tau}^R = 0 \ \forall \underline{X} \in \mathcal{O}$. In this case, we necessarily have $\hat{\Psi}^R = \frac{(1-p)\bar{\tau}}{(1-p)\bar{\tau}} = 1$, hence the satisfaction of MNO.

Finally, let $\underline{\tau}_0$ ($\underline{\tau}_1$) and $\bar{\tau}_0$ ($\bar{\tau}_1$), be the two income levels associated with $X_0 \in \mathcal{E}^n$ ($X_1 \in$

$\mathcal{B}_p^n$). It can be noted that by definition $\bar{X}_0 = \frac{1}{1-p}\bar{X}_1$. Since $\bar{\tau}^R$ is homogeneous of degree 1, we consequently have $\bar{\tau}_1 = \bar{\tau}^R_{\bar{X}_1} = \bar{\tau}^R\left(\frac{\bar{X}_0}{1-p}\right) = \frac{1}{1-p}\bar{\tau}^R(X_0) = \frac{1}{1-p}\bar{\tau}_0$. We have:

$$
\begin{aligned}
\alpha\hat{\Psi}^R(X_0) + (1-\alpha)\hat{\Psi}^R(X_1) &= \alpha\frac{(1-p)(\bar{\tau}_0 - \underline{\tau}_0)}{p\underline{\tau}_0 + (1-p)\bar{\tau}_0} + (1-\alpha)\frac{(1-p)(\bar{\tau}_1 - \underline{\tau}_1)}{p\underline{\tau}_1 + (1-p)\bar{\tau}_1}, && (30)\\
&= \alpha\frac{(1-p)(\bar{\tau}_0 - \underline{\tau}_0)}{\bar{\tau}_0} + (1-\alpha)\frac{(1-p)(\bar{\tau}_1 - \underline{\tau}_1)}{(1-p)\frac{1}{1-p}\bar{\tau}_0}, && (31)\\
&= \alpha\frac{(1-p)(\bar{\tau}_0 - \underline{\tau}_0)}{\bar{\tau}_0} + (1-\alpha)\frac{(1-p)(\bar{\tau}_1 - \underline{\tau}_1)}{\bar{\tau}_0}, && (32)\\
&= \frac{(1-p)\big(\alpha\bar{\tau}_0 - \alpha\underline{\tau}_0 + (1-\alpha)\bar{\tau}_1 - (1-\alpha)\underline{\tau}_1\big)}{\bar{\tau}_0}, && (33)\\
&= \frac{(1-p)\big(\alpha\bar{\tau}_0 - \alpha\underline{\tau}_0 + (1-\alpha)\bar{\tau}_1 - (1-\alpha)\underline{\tau}_1\big)}{\alpha\bar{\tau}_0 + (1-\alpha)\bar{\tau}_0}, && (34)\\
&= \frac{(1-p)\big(\alpha\bar{\tau}_0 - \alpha\underline{\tau}_0 + (1-\alpha)\bar{\tau}_1 - (1-\alpha)\underline{\tau}_1\big)}{\alpha\big(p\bar{\tau}_0 + (1-p)\bar{\tau}_0\big) + (1-\alpha)\big(p\underline{\tau}_1 + (1-p)\bar{\tau}_1\big)}, && (35)\\
&= \frac{(1-p)\big(\alpha\bar{\tau}_0 - \alpha\underline{\tau}_0 + (1-\alpha)\bar{\tau}_1 - (1-\alpha)\underline{\tau}_1\big)}{\alpha\big(p\underline{\tau}_0 + (1-p)\bar{\tau}_0\big) + (1-\alpha)\big(p\underline{\tau}_1 + (1-p)\bar{\tau}_1\big)}, && (36)\\
&= \frac{(1-p)\big(\alpha\bar{\tau}_0 - \alpha\underline{\tau}_0 + (1-\alpha)\bar{\tau}_1 - (1-\alpha)\underline{\tau}_1\big)}{p\big(\alpha\underline{\tau}_0 + (1-\alpha)\underline{\tau}_1\big) + (1-p)\big(\alpha\bar{\tau}_0 + (1-\alpha)\bar{\tau}_1\big)}, && (37)\\
&= \hat{\Psi}^R\big(\alpha X_0 + (1-\alpha)X_1\big). && (38)
\end{aligned}
$$

Consequently, LIN is satisfied for $\hat{\Psi}^R$. In the case of $\hat{\Psi}^A$, we simply have:

$$
\begin{aligned}
\alpha\hat{\Psi}^A(X_0) + (1-\alpha)\hat{\Psi}^A(X_1) &= \alpha(\bar{\tau}_0 - \underline{\tau}_0) + (1-\alpha))(\bar{\tau}_1 - \underline{\tau}_1), && (39)\\
&= \big(\alpha\bar{\tau}_0 + (1-\alpha)\bar{\tau}_1\big) - \big(\alpha\underline{\tau}_0 + (1-\alpha)\underline{\tau}_1\big), && (40)\\
&= \hat{\Psi}^A\big(\alpha X_0 + (1-\alpha)X_1\big). && (41)
\end{aligned}
$$

So, LIN is also satisfied with $\hat{\Psi}^A$.

A.2 Additional tables and figures

Table 3: Lorenz dominance tests for selected Sub-Saharan African countries, 2018–2019.

	BEN	BFA	CIV	GNB	MLI	NER	SEN	TCD
BFA	$\prec_L$							
CIV	$\emptyset$	$\emptyset$						
GNB	$\succ_L$	$\succ_L$	$\succ_L$					
MLI	$\succ_L$	$\succ_L$	$\succ_L$	$\emptyset$				
NER	$\emptyset$	$\succ_L$	$\emptyset$	$\prec_L$	$\emptyset$			
SEN	$\emptyset$	$\succ_L$	$\emptyset$	$\prec_L$	$\prec_L$	$\emptyset$		
TCD	$\emptyset$	$\succ_L$	$\succ_L$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
TGO	$\prec_L$	$\emptyset$	$\prec_L$	$\prec_L$	$\prec_L$	$\emptyset$	$\prec_L$	$\prec_L$

Note: $\succ_L$ ($\prec_L$) means that the first distribution Lorenz dominates (is dominated by) the second distribution; $\emptyset$ is for no-dominance relationships.

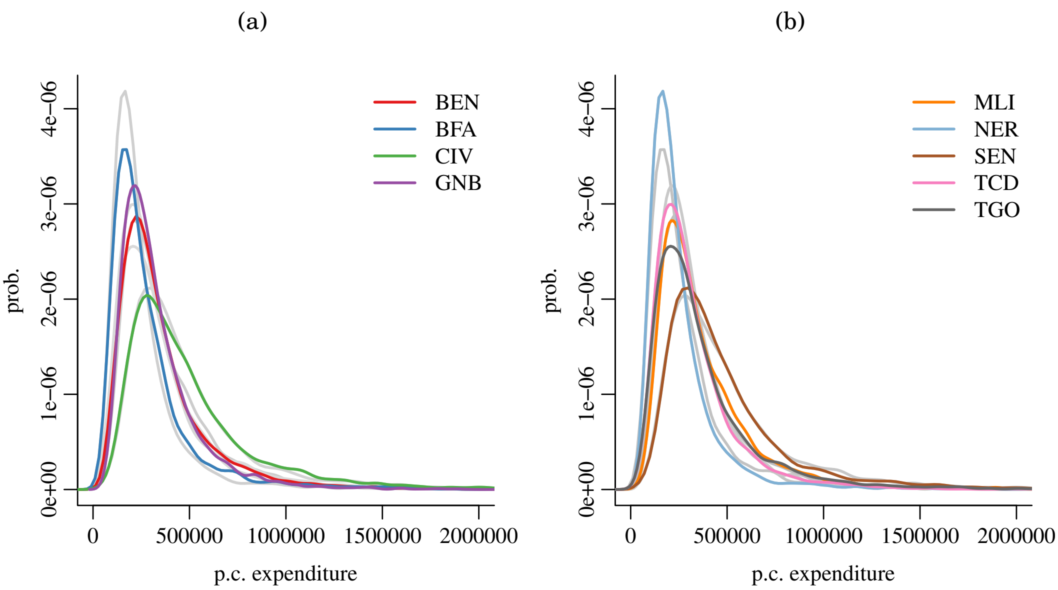


Figure 6: Income distribution in selected Sub-Saharan African countries, kernel estimates, 2018–2019.

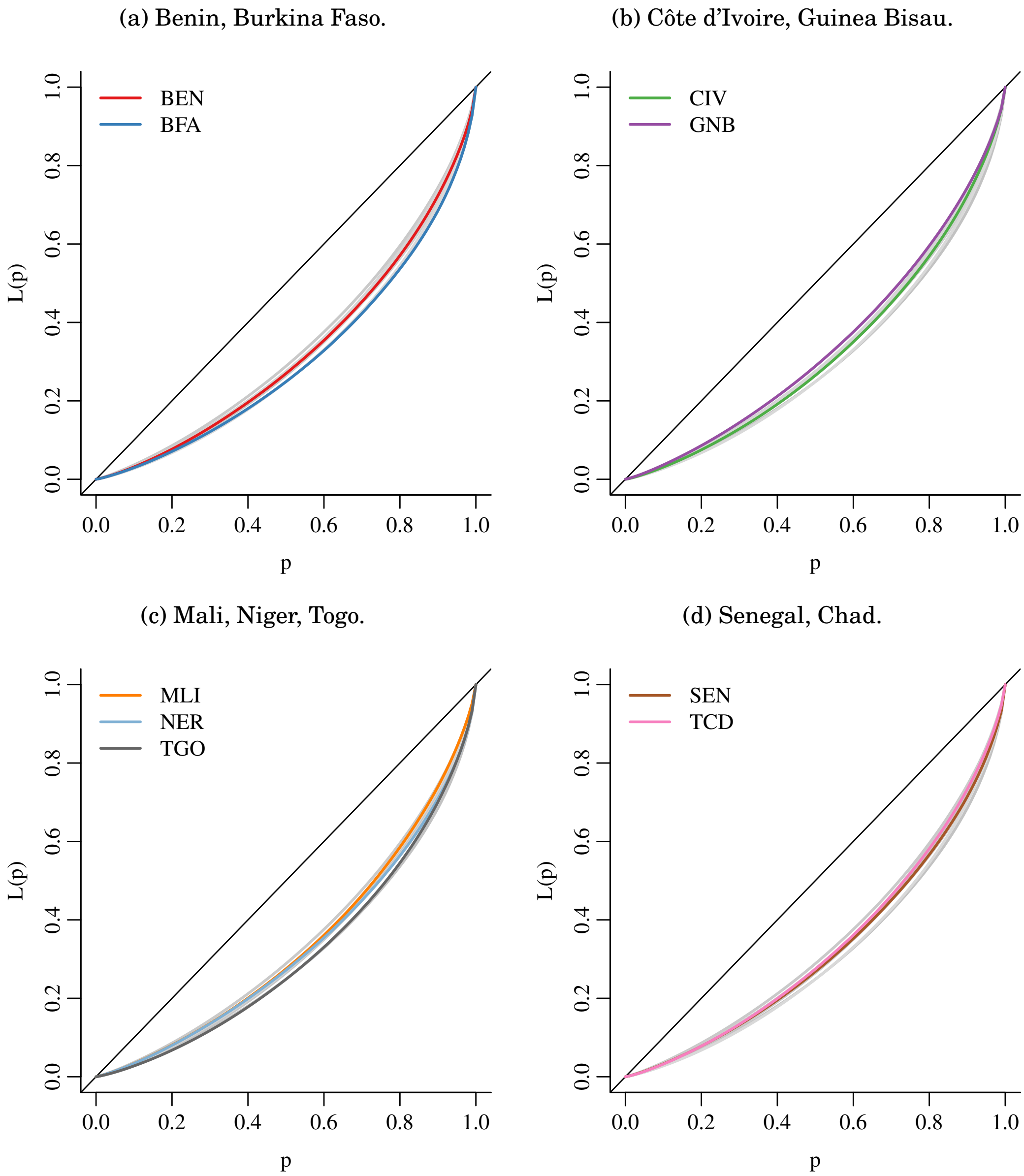


Figure 7: Lorenz curves for selected Sub-Saharan African countries, 2018–2019.

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