Equalizing opportunities from behind a veil of ignorance: A robust approach

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Abstract

This paper provides a robust and operational definition of "opportunity equalization" when applied to alternative societies. Societies are described as collections of distributions of outcomes (i.e., lotteries), one such distribution for every group. We envisage the problem of comparing these societies from the view point of an ethical observer placed behind a veil of ignorance with respect to the group in which he/she could fall if he/she were to be born in this society. Using axioms of choice under ambiguity, we show that the ranking of societies of this ethical observer can be viewed as resulting from the comparison of the expectation of some function of the lottery, assuming an equal probability of falling in every group. Moreover, the function of the lottery can be written as the transformation of some expectation of its consequences under some concave function. We provide a criterion for comparing societies that coincide with the unanimity of all rankings that would command agreement among these ethical observers when they exhibit aversion to inequality of opportunity. The criterion happens to be a conic extension of the zonotope inclusion criterion. We provide various interpretations of this general criterion as well as some illustrations of its possible use, notably in the Indian context where we perform a detailed comparison of the various religious communities of this country in terms of the cross-gender inequality of educational opportunities. We also identifies the elementary transformations that lie behind the criterion in the two-group cases when it is applied to societies with the same average opportunities.

Keywords: equality of opportunities, groups, zonotopes, gender, education.

JEL classification numbers: D63.

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1 Introduction

Equalization of opportunities is considered to be an important social objective by many. In the US, opinion surveys conducted by the Pew research center[1] have consistently found in the last 25 years an agreement by 90% of the respondents on the fact that "our society should do what is necessary to make sure that everyone has an equal opportunity to succeed". A common interpretation of this "equal opportunity to succeed " ideal is through the requirement that the individuals' probabilities (chances) of reaching outcomes of interest be independent from morally irrelevant characteristics such as skin color, gender, national origin, family background, sexual orientation, etc. This of course requires an appropriate identification of what those morally irrelevant characteristics are. But even leaving aside this question, the consensual ideal of an equal opportunity to succeed is a rather poor guide to policy making. For it only indicates what is the destination - equal opportunity - without providing any insight on the way to get there. An example may illustrate this point.

Figure 1 below illustrates the cumulative share of Indian males and females reaching each of the seven ordered level of education reported in the 68th round of the Indian NSSO. Levels are ranked from illiteracy (level 1) up to tertiary education (level 7). An exhaustive description of the data is in section 4. Our focus is on two distinct communities, defined by religion belonging: that of Buddhists and that of Sikhs. In both communities, women and men *do not* have an "equal opportunity to succeed" in education. In particular, women have consistently lower chances of attaining any of the highest educational categories compared to men. As such, women enjoy smaller educational opportunities than men. A form of inequality of educational opportunities prevails in both Sikh and Buddhist communities.

Yet, one may want to go beyond the mere observation that educational opportunities are unequally distributed across genders, in order to make comparative statements on the extent by which gender inequality in educational opportunities differs in the two considered communities. To put it more compactly, one may want to define "opportunity equalization" rather than the mere zero-one "equal/unequal opportunity". In Figure 1 for instance, it could seem that educational opportunities faced by men and women, clearly unequal in both the Sikh and the Buddhist community, are "more unequal" in the latter than in the former. Indeed, the (cumulative) distribution of educational opportunities among Buddhist women is stochastically dominated at the first order by the distribution of educational opportunities faced by Sikh women. Sikh women are therefore less at risk than Buddhist women of not getting an education level above any threshold that one may consider. Moreover an opposite first-order stochastic dominance holds between Buddhist and Sikhs men. While each of these two groups of men faces better educational opportunities then its women counterpart, the Buddhist group has an even better advantage than the Sikh one. Indeed, the educational opportunities faced by Buddhist men dominates at the first order by those faced by the Sikh men. Hence the non-favored group - women - is less favored in the Sikh than in the Buddhist community while the favoured group - men - is less favoured in the Sikh than in the Buddhist community. Since the average - over men and women - distribution of educational

[1] see e.g. https://www.pewresearch.org/2011/03/11/the-elusive-90-solution.

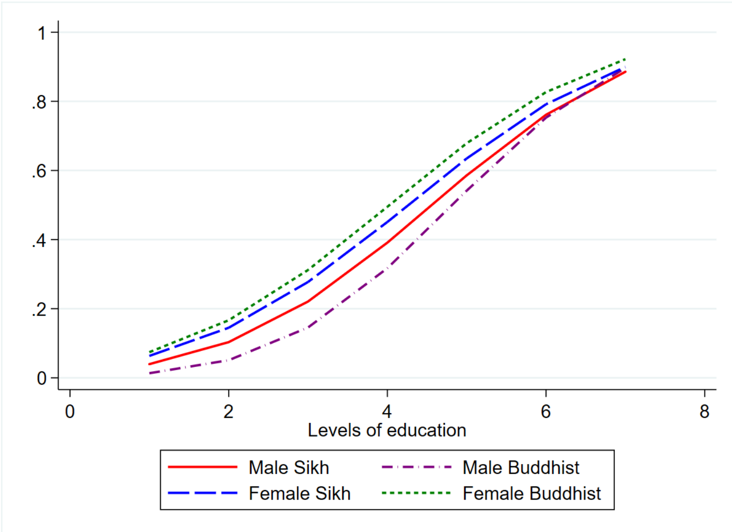


Figure 1: Educational opportunities for men and women in the Buddhist and the Sikh communities, India, 2012 (Source: 68th round of the Indian NSSO).

opportunities in the two communities is quite - albeit not perfectly - similar, it seems tempting indeed to conclude that the inequalities of educational opportunities between men are women are, in India, more important in the Buddhist than in the Sikh community.

The contribution of this paper is to provide a theoretically justified, robust and implementable definition of "opportunity equalization". Our approach rides on the idea that there are many morally arbitrary variables - gender in the above example - that impact unduly on individuals' destiny. Our approach views accordingly an equalization of opportunities as a reduction of the discrepancies in the *probabilities* of achieving whatever outcome of importance - education in the above example - faced by individuals for whom these morally arbitrary variables take different values. The different combinations of values taken by these morally irrelevant variables in the population lead to a partition of this population into groups of individuals for which these variables take the same value. These groups are often referred to as *types* in the economic literature on equality of opportunities surveyed by Roemer and Trannoy (2016). Opportunity equalization hence consists in equalizing the probabilities of achieving the outcomes of relevance to individuals among groups. This of course requires a notion of what it means for two distributions of probabilities to be "more equal" than two others. The current paper provides an operational and reasonably robust such definition.

Before detailing this definition, we find useful to contrast our approach with the abundant economic literature on equality of opportunity surveyed in Roemer and Trannoy (2016). This literature stems from the widely discussed Dworkin (1981) cut between the characteristics that affect the destiny of an individual for which an individual should be held *responsible* and the morally irrelevant ones that determine the various types that the individual can have. The main creed of this literature is that equalization of opportunities should be concerned with *equalizing outcomes* among individuals for whom the "responsibility characteristics" are the same. However, no attempt should be made to equalize outcomes when the differences in those outcomes can be shown to result from the "free" exercise of responsibility. Our approach departs from this equality of opportunity literature based on the Dworkin (1981) cut by taking no stance whatsoever on the question of whether or not individuals are responsible for some of their characteristics. Responsibility plays actually no role whatsoever in our approach, even though one may hold the view that individuals in each group are "responsible" for their success in the life (defined in our approach by their probability of achieving whatever outcome of relevance).

Another important difference between our approach and those of the literature surveyed in Roemer and Trannoy (2016) is that we provide a definition of opportunity equalization, while many contributions to the inequality of opportunity literature are interested in defining -somewhat binarily - either inequality or (perfect) equality of opportunity. Moreover, most of the contributions to the literature that define opportunity equalization either ride heavily on the Dworkin (1981) cut (like for example Peragine (2004)), or do so by means of decomposition of total outcome inequality (measured by some specific index) into within group inequality and between group inequality (see e.g. Ferreira and Gignoux (2011)), with between-group inequality defined, in the tradition of Shorrocks (1984), with respect to the groups' mean outcomes. However focusing on group mean outcome does not account for the possible varying riskiness of these outcomes across groups. The contribution to the literature that appears to be the closest to what is done herein is Andreoli, Havne, and Lefranc (2019), which investigate a robust inequality of opportunity criterion to rank societies. This criterion is shown to be implemented by sequential assessments of an ethical measure of distance between lotteries attributable to different types. As such, the criterion emphasizes inequalities between lotteries whereas neglecting the role of improvements in opportunities. The criterion is robust, in the sense that it requires agreement on a well-defined class of preferences about the extent at which ethical distance between every pair of distributions decreases across configurations, as well as agreement on the ranking of lotteries in each configuration. The criterion we study explicitly considers the trade-off between inequality and improvements in opportunities, it does not impose an ordering of the groups and it refines the pairwise comparison of lotteries by explicitly considering the correlations between types distributions.

Our definition of opportunity equalization stands on the view point of an ethical observer who is behind a (thick) veil of ignorance with respect to the group she/he might fall in if she/he were to born in a given society. How would such an ethical observer compare the various possible societies ? Using results from decision theory under objective ambiguity - and in particular Gravel, Marchant, and Sen (2011) and Gravel, Marchant, and Sen (2012) - one can provide arguments for such an ethical observer to do these comparisons on the basis of a uniform expected utility criterion. Such a criterion evaluates any list of groups distributions of opportunities by a *three-step* procedure. In the first step, a utility level is assigned to every conceivable outcome so that each group becomes identified by an expected utility of achieving those outcomes. In a second step, a utility level is assigned to the expected utility of every group by some utility function. In the third step, a uniform expected utility is calculated for the society under the (uniform) assumption that every group is equally likely. Of course there are many such uniform expected utility ethical observers, as many as there are logically conceivable ways to assign utility levels to each outcome, and to assign (in the second step) utility to expected utility of these outcome. If one makes the additional assumption that an ethical observer dislikes inequality of opportunity in the sense of preferring a society in which the same average distribution of opportunities is observed in every group to one where the average is unevenly distributed among groups, one can obtain the additional restriction that the utility function used to evaluate the expected utility of outcome of every group is a concave function of this expected utility. But this still leaves quite many criteria to consider. The main theoretical contribution of this paper is to provide an empirically operational test that enables one to identify when one distribution of opportunities among groups is better than another for all such ethical observers. The test, explained in detail in the paper, is the inclusion of the quasi-ordering extended zonotopes uniquely associated to the compared societies. The zonotope set of any list of probability distributions is the set of all Minkowski sums of those probability distributions (see e.g. Koshevoy (1995), Koshevoy (1998)). A quasi-ordering extended zonotope is a zonotope set that has been enlarged by a specific collection of translations that capture the assumptions made about the ranking of outcomes faced by members of the groups. Our approach is, indeed, quite general in that respect. If outcomes are completely ordered - as assumed in the education example given above - then the enlargement of the zonotope is made by translations that correspond to all possible ways to generate first order stochastic dominance on the distributions. If at the other extreme, the outcome are not ordered at all, then the zonotope is not enlarged at all and the test amounts to checking for the simple zonotope inclusion. Between these two extremes, our approach handles any incomplete quasi-ordering of the outcomes by enlargements of the zonotope that are specific to the quasi-ordering assumed. While the extended zonotope inclusion test is theoretically implementable with any number of groups, its actual implementation may sometimes be difficult, and may not always lead to an empirical testing criterion. However, we able to provide a finite, feasible test for the general criterion for the specific case in which there are only two groups (for example men and women).

We also put our criterion to work by comparing the religious communities in India in terms of gender inequalities of educational opportunities. We also appraise to what extent the noticeable improvements in educational opportunities offered to Indians in the last thirty years has been associated with a reduction of gender educational opportunities in the different religious communities. For this purpose we use two rounds of data from the Employment-Unemployment survey of the Indian National Sample Survey (NSS) database, corresponding to survey years 1983 and 2011-12

The plan of the remaining of the paper is as follows. The next section describes the general setting in which we define the notion of opportunity equalization and provides a foundation to it though the view point of an ethical observer placed behind the veil of ignorance. Section 3 presents the operational extended zonotope criterion and establishes its equivalence with the ranking of societies made by all opportunity inequality averse uniform expected utility ethical observer. It also indicates how this criterion can be associated with specific elementary transformations and can be easily implemented in the two-group case. Section 4 presents the result of the empirical implementation of the criterion to appraise inequality of educational opportunities between men and women and Section 5 concludes.

2 A general framework appraising equalization of opportunities.

We are interested in comparing societies on the basis of their performances in *equalizing opportunities* among some exogenously given - but possibly variable across societies - groups of individuals. These groups can be based on religion, race, gender, family background, etc. Our approach to appraising equality of opportunities does not enquire about the origin of these groups. We neither assume that the number of such groups is the same across societies. For instance, we may consider societies formed by one group only. Our approach would then view such one-group societies as achieving (trivially) perfect equality of opportunities. The opportunities offered to a group in a society are described by the *ex ante probability of achieving* any relevant outcome faced by a member of the group. We assume specifically that there are k such outcomes. We hence in general depict a society $\mathbf{p}$ as an $n(\mathbf{p}) \times k$ row-stochastic matrix:

$$\mathbf{p} = \begin{bmatrix} p_{11} & \cdots & p_{1k} \\ \cdots & \cdots & \cdots \\ p_{n(\mathbf{p})1} & \cdots & p_{n(\mathbf{p})k} \end{bmatrix}$$

where p_{ij}, for $i = 1, ..., n(\mathbf{p})$ and $j = 1, ..., k$ denotes the probability that an individual from group i achieves outcome j in society $\mathbf{p}$ and $n(\mathbf{p})$ denotes the number of groups in $\mathbf{p}$. For any society $\mathbf{p}$, we denote by p_i the distribution of probabilities (opportunities) associated to group i in such a society and by $\overline{p}$ its (symmetric) average distribution of opportunities defined by:

$$\overline{p} = \frac{1}{n(\mathbf{p})} \sum_{i=1}^{n(\mathbf{p})} p_i.$$

The set of all conceivable societies, in which there are k possible outcomes, is denoted by $\mathbf{S}$:

$$\mathbf{S} := \bigcup_{n \geq 1} \left(\Delta^{k-1}\right)^n, \text{ where } \Delta^{k-1} := \left\{ \mathbf{x} \in \mathbb{R}^k : \ x_j \geq 0 \ \forall j, \ \sum_{j=1}^{k} x_j = 1 \right\}.$$

We can view the outcomes $\{1, ..., k\}$ as anything that individuals have reason to value and that are observable somehow. Examples would include income categories, or education levels. They may also be combinations of, say, health and education levels. Hence, the general approach that we propose does not require the outcomes to be completely ordered. We may even take the extreme point of view that they are not ordered at all. For example, we may care about the distribution, among say males and females, of the opportunities of entering in the army. In such an case, there would be only two outcomes (joining the army and not joining the army) which are not ordered in an obvious way. Formally we suppose that there exists an quasi-ordering[2] $\geq_{QO}$ of the set of outcomes $\{1, ..., k\}$ with the interpretation that $j \geq_{QO} h$ for any two distinct outcomes h and j in $\{1, ..., k\}$ if and only if j is "better" for an agent than h. An extreme form of incomplete ordering would be the case, discussed earlier, where *none* of the outcomes can be compared with one another. Let us denote by $\geq_{\varnothing}$ this empty quasi-ordering. At the other extreme, one could of course have the case, very much discussed in the equality of opportunity literature surveyed in Roemer and Trannoy (2016), of a complete ordering of outcomes (based for example on income), that we denote by $\geq_C$.[3] But intermediate cases between these two extremes are certainly possible. For example, outcomes could be combinations of two binary (e.g. taking values 0 or 1) indicators of well-being such as health (variable 1) and education (variable 2).[4]

Alternative societies are to be compared by an *ethical observer*, agreeing with the quasi-ordering $\geq_{QO}$, and who is placed behind a "veil of ignorance" as to the group to which he (she) would belong if he (she) was to live in the considered societies. We assume that such ethical observer uses the ordering $\succsim$, with asymmetric and symmetric factors $\succ$ and $\sim$ respectively to compare these societies. We interpret the statement $\mathbf{p} \succsim \mathbf{q}$ as meaning "The ethical observer would weakly prefer being born in society $\mathbf{p}$ than in society $\mathbf{q}$". A similar interpretation is given to the statements $\mathbf{p} \succ \mathbf{q}$ (strict preference) and $\mathbf{p} \sim \mathbf{q}$ (indifference). Since the ordering $\succsim$ is defined on the whole set $\mathbf{S}$, it is in particular defined on the set Δ^{k-1} of all conceivable one-group societies and, therefore, of all probability distributions over the k outcomes.

We focus on ethical observers evaluating societies using an ordering $\succsim$ that can be represented as follows: there exists a function $\Psi : \Delta^{k-1} \to \mathbb{R}$ such that, for all societies $\mathbf{p}$ and $\mathbf{q}$ in $\mathbf{S}$, one has:

$$\mathbf{q} \succsim \mathbf{p} \Longleftrightarrow \sum_{i=1}^{n(\mathbf{q})} \frac{\Psi(q_i)}{n(\mathbf{q})} \geq \sum_{i=1}^{n(\mathbf{p})} \frac{\Psi(p_i)}{n(\mathbf{p})}. \tag{1}$$

An ordering satisfying this property could therefore be thought of as resulting from the comparisons of the *average evaluation* of the lotteries offered by two compared societies for some evaluation function, under the assumption that the ethical observer is equally likely to fall in any group. Notice that formula (1) defines a *family* of social criteria, with as many members as there are logically conceivable functions Ψ. Following Gravel, Marchant, and Sen (2012), we refer to any ranking that satisfies (1) for some function Ψ as to a *Uniform Expected Utility* (UEU) ranking of societies. This name comes from the decision under ignorance context in which this family was studied. Indeed, any ranking of societies that is numerically represented by (1) for some function Ψ can be though of as resulting from the comparison of the expected utility of the various lotteries offered by the societies, under the (uniform) assumption that the ethical

[2] A *quasi-ordering* R on X is a *reflexive* (i.e. xRx) and *transitive* (i.e. xRy and yRz imply that xRz) binary relation on X. It is called an *ordering* if it is also *complete* (i.e. xRy or yRx for any $x, y \in X$).

[3] It will be (without loss of generality modulo a permutation) uniquely defined by $k \geq_C k-1 \geq_C .... \geq_C 2 \geq_C 1$ (outcomes are ranked in increasing order in the set $\{1, ..., k\}$).

[4] In such a setting, where the outcomes would be $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$, a plausible quasi-ordering $\geq_{QO}$ could be $(1,1) \geq_{QO} (0,1) \geq_{QO} (0,0)$ and $(1,1) \geq_{QO} (1,0) \geq_{QO} (0,0)$ (leaving the outcomes $(0,1)$ and $(1,0)$ incomparable).

observer assigns an equal probability to belonging to every group. Proposition 5 in appendix (Section 6.1) recalls a result of Gravel, Marchant, and Sen (2012), stating that the subset of orderings which can be represented by such utility functions is characterized by a series of classical axioms, namely *anonimity*, *continuity*, *averaging* and *independence* (see Axioms 1-4 in Section 6.1).
This proposition does not restrict much the function Ψ. A somewhat natural restriction would be to require the ranking of one-group societies - for which the issue of disparities of opportunities among groups vanishes - to obey the well-known VNM axiom (see Axiom 5 in Section 6.1). It can be shown (see Proposition 6 in Section 6.1) that, if we additionally assume this axiom holds, then the function Ψ in (1) can be written as follows:

$$\Psi(\pi_1, ..., \pi_k) = \Phi\left(\sum_{h=1}^{k} \pi_h u_h\right) \tag{2}$$

for some real numbers $u_1, ..., u_k$ and some function $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$. In this specification, the real numbers $u_1, ..., u_k$ are interpreted as the utility evaluations, made by the ethical observer, of the various outcomes. Hence the expression in brackets, constructed with these numbers, can be seen as the *expected utility* associated to the lottery $(\pi_1, ..., \pi_k)$, and the function Φ can be seen as a transformation of this expected utility into some magnitude, which reflects the attitude of the ethical observer with respect to ambiguity. Since the quasi-ordering $\geq_{QO}$ of the set of outcomes $\{1, ..., k\}$ is universally accepted, it must be the case that any list of utility numbers $(u_1, ..., u_k)$ ranks the set $\{1, ..., k\}$ of outcomes in a way *compatible with* $\geq_{QO}$. In more technical terms, any list of utility numbers must belong to the set $\mathcal{U}^{\geq_{QO}} \subset \mathbb{R}^k$, where

$$\mathcal{U}^{\geq_{QO}} = \{(u_1, ..., u_k) \in \mathbb{R}^k : \ j \geq_{QO} h \Longrightarrow u_j \geq u_h, \ \forall j, h \in \{1, ..., k\}\} \tag{3}$$

For instance, the quasi-order $\geq_{\varnothing}$ does not restrict whatsoever the familiy of lists of utility numbers whiile, at the other extreme, the quasi-order $\geq_C$ significantly restricts this family by limiting the lists of utility numbers $(u_1, ..., u_k)$ that appear in Expression (2) to those who are weakly increasing with respect to the ordered outcomes.[5]

Ethical observers who rank societies behind a veil of ignorance may be distinguished according to what could be called "aversion to inequality of opportunities". Intuitively, aversion to inequality of opportunities would correspond to a preference for societies who exhibit no disparity of opportunities - say because they are made of one single group - over societies who exhibit some disparity of opportunities among their different groups. This suggests the following notion of comparative aversion to inequality of opportunities among ethical observers.

Definition 1 *Given two orderings* $\succsim_1$ *and* $\succsim_2$ *on* $\mathbf{S}$ *, we say that* $\succsim_1$ *exhibits* at least as much aversion to inequality of opportunity *as* $\succsim_2$ *if, for every lottery* $\rho \in \Delta^{k-1}$ *and society* $\mathbf{p} \in \mathbf{S}$*, we have* $\rho \succsim_2 \mathbf{p} \Longrightarrow \rho \succsim_1 \mathbf{p}$*.* [6]

[5] The "extreme" sets $\mathcal{U}^{\geq_{\varnothing}}$ and $\mathcal{U}^{\geq_C}$ are given by $\mathcal{U}^{\geq_{\varnothing}} = \mathbb{R}^k$ and $\mathcal{U}^{\geq_C} = \{(u_1, ..., u_k) \in \mathbb{R}^k : u_1 \leq u_2 \leq ... \leq u_k\}$.

[6] If ρ in Δ^{k-1}, we abuse notation by denoting also by ρ the *one group society* in which all members face the distribution ρ of opportunities.

In words, an ethical observer who compares societies by means of the binary relation $\succsim_1$ exhibits at least as much aversion to inequality of opportunities as another who bases his/her comparisons on $\succsim_2$ if any preference that the later will have for a society with no inequality of opportunities (as compared to any reference society) would also be endorsed by the former. It is not difficult to see that this notion of "comparative aversion to opportunity inequality" can translate, when expressed for UEU rankings, into a statement of "comparative concavity" applied to the function Ψ of Expression (1). Specifically, the following proposition can be established (see Gravel, Marchant, and Sen (2012) for a proof).

Proposition 1 *Let $\succsim_1$ and $\succsim_2$ be two orderings on* $\mathbf{S}$ *which can be represented as per (1) for some functions Ψ^1 and Ψ^2 respectively. Then $\succsim_1$exhibits at least as much aversion to inequality of opportunity as $\succsim_2$ if and only if there exists some increasing and concave real function Φ such that, for every $p \in \Delta^{k-1}$, one has $\Psi^1(p) = \Phi\left(\Psi^2(p)\right)$.*

Hence, for comparisons of societies made by a UEU criterion, the statement "has more aversion to opportunity inequality" as can be translated into "has a more concave evaluation function as". While this is reminiscent of standard definition in the context of standard inequality measurement, there is an important difference. In the usual income inequality setting, there is a (natural) benchmark to define "neutrality to income equality". An ethical observer concerned about distributions of incomes is usually considered as being neutral vis-à-vis income equality if it considers as equivalent all income distributions that have the same per capita income. Given this benchmark, it is standard to define someone has exhibiting aversion to inequality "in the absolute" if this person exhibits more aversion to income inequality than a person who is neutral to inequality. In the current setting, we are not aware of the existence of a well-accepted standard of neutrality toward equality of opportunities. One such benchmark could be to consider as equivalent all societies which distribute among their groups the same (symmetric) average probability distribution over outcomes. If one agrees with this standard of neutrality with respect to equality of opportunities, then one could define an ethical observer as exhibiting aversion to inequality of opportunities whenever the observer has more aversion to inequality of opportunities than an observer who exhibits neutrality with respect to equality of opportunities. Formally, this would amount to define neutrality and aversion with respect to equality of opportunities as follows.

Definition 2 *Le $\succsim$ be an ordering on* $\mathbf{S}$.

(*i*) $\succsim$ *is said to exhibit* neutrality with respect to equality of opportunities *if for any two societies* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbf{S}$ *such that $\overline{p} = \overline{q}$, one has* $\mathbf{p} \sim \mathbf{q}$.

(*ii*) $\succsim$ *is said to exhibit* aversion to inequality of opportunity *if there exists some ordering $\succsim_0$, exhibiting neutrality to inequality of opportunity, such that $\succsim$ exhibits at least as much aversion to inequality of opportunity as $\succsim_0$.*

It can be proved that an ordering exhibits neutrality to inequality of opportunity if and only if it is represented by a multilinear function (see Proposition 7 in appendix). Combining this observation with Proposition 1, we obtain the following

Proposition 2 *An ordering* $\succsim$ *on* $\mathbf{S}$ *exhibits* aversion to inequality of opportunity *if and only if can be represented as per (2), with a concave function* Φ.

In the light of proposition 2, we now introduce the dominance criterion between pairs of societies, capturing the fact that all utilitarian ethical observers exhibiting aversion to inequality of opportunity would unanimously rather be born in one than in the other.

Definition 3 *Given a quasi-order on outcomes* $\geq_{QO}$*, we say that* $\mathbf{q}$ *dominates* $\mathbf{p}$ *for the* $\geq_{QO}$-UEU inequality averse dominance*, denoted by* $\mathbf{q} \succsim_{UEU}^{QO} \mathbf{p}$*, if*

$$\frac{1}{n(\mathbf{q})}\sum_{i=1}^{n(\mathbf{q})}\Phi\left(\sum_{h=1}^{k}q_{ih}u_h\right) \geq \frac{1}{n(\mathbf{p})}\sum_{i=1}^{n(\mathbf{p})}\Phi\left(\sum_{h=1}^{k}p_{ih}u_h\right) \tag{4}$$

for all increasing and concave functions $\Phi : \mathbb{R} \to \mathbb{R}$ *and all list of numbers* $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$.

3 An operational definition of opportunities' equalization

3.1 The criterion in the general case

In the rest of the paper we assume that the societies we compare contain the same number of groups: $n(\mathbf{p}) = n(\mathbf{q}) = n$. The family of functionals for which inequality (4) must be checked in order to establish whether or not $\mathbf{q} \succsim_{UEU}^{QO} \mathbf{p}$ is large. Given any two societies, it would therefore be a very exhausting (if not impossible) task of verifying whether one is better than the other for all such UEU criteria. In this section, we identify an operational criterion that enables this verification.

From a mathematical point of view, it can be checked that $\mathcal{U}^{\geq_{QO}}$ is a non-empty closed convex cone for any antisymmetric quasi-ordering $\geq_{QO}$ of the set $\{1, ..., k\}$. The *dual cone*[7] relative $\mathcal{U}^{\geq_{QO}}$, which is denoted by $\mathcal{U}_*^{\geq_{QO}}$, is defined by:

$$\mathcal{U}_*^{\geq_{QO}} = \{(v_1, ..., v_k) \in \mathbb{R}^k : \sum_{j=1}^{k} v_j u_j \geq 0 \text{ for all } (u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}\} \tag{5}$$

We observe that $\mathcal{U}_*^{\geq_{QO}} = \{\mathbf{0}^k\}$ if and only if $\geq_{QO} = \geq_{\varnothing}$. If outcomes cannot be compared, then the only vector v that is dual to the set of all logically conceivable lists of k numbers - that is $\mathbb{R}^k$ - is the zero vector. Another observation that can be made about the dual cone $\mathcal{U}_*^{\geq_{QO}}$ is that all the $k-$tuples $(v_1, ..., v_k)$ that it contains have their components that sum to 0. We state this formally as follows.

Remark 1 *Let* $(v_1, ..., v_k) \in \mathcal{U}_*^{\geq_{QO}}$ *for some quasi-ordering of* $\geq_{QO}$ $\{1, ..., k\}$*. Then* $v_1 + ... + v_k = 0$.

[7] which is the negative of what Rockafellar (1970) p. 121 calls the polar of $\mathcal{U}^{\geq_{QO}}$

The dual cone associated to $\mathcal{U}^{\geq_{QO}}$ has an intuitive interpretation. It is the set of all changes in the probability distribution over outcomes that increase expected utility for all utility functions compatible with the underlying quasi-ordering. In plain English, it is the set of all clear improvements in the opportunities of achieving the outcomes as (possibly) incompletely ordered by the quasi-ordering. This interpretation is supported by the fact that the sum of these changes is zero and, as a result, they produce a new probability distribution over outcomes which cumulates to 1, just like the initial distribution. What exactly these changes in the distribution are depends of course of the precise definition of the quasi-ordering.

The operational definition of opportunity equalization that we propose makes an important use of the Zonotope set $\mathbf{Z}(\mathbf{p}) \subset \mathbb{R}^k_+$ associated to any society $\mathbf{p} \in \mathbf{S}$, and defined by:

$$Z(\mathbf{p}) = \left\{ \mathbf{z} = (z_1, \dots, z_k) : \mathbf{z} = \sum_{i=1}^{n(\mathbf{p})} \theta_i p_i, \ \theta_i \in [0,1] \ \forall i = 1, \dots, n \right\} \tag{6}$$

A closely related set has been used by Koshevoy (1995) (see also Koshevoy and Mosler (1996)) to define a criterion called by this author *Lorenz majorization.* We use this zonotope set to define what we call Quasi-Ordering Extended Zonotope (QOEZ) dominance between two societies as follows.

Definition 4 *We say that* $\mathbf{q}$ *dominates* $\mathbf{p}$ *for the* $\geq_{QO}$*- extended Zonotope dominance criterion, which we write as* $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ *, if and only if*

$$Z(\mathbf{q}) + \mathcal{U}_*^{\geq_{QO}} \subseteq Z(\mathbf{p}) + \mathcal{U}_*^{\geq_{QO}}.$$

While QOEZ dominance may be difficult to verify in general; we will soon provide an empirical finite test for it in the important case where there are only two groups. Koshevoy and Mosler (2007) have also proposed, in a different context, a somewhat similar test based on the inclusion of suitably extended Zonotope sets.

We now establish the following main equivalence between the ranking of two societies as per QOEZ dominance and the ranking of those societies agreed upon by all opportunity-inequality averse UEU ethical observers who compare outcomes by means of the quasi-ordering $\geq_{QO}$.

Theorem 1 *The two following statements are equivalent:*

(i) $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$*;*

(ii) $\mathbf{q} \succsim_{UEU}^{QO} \mathbf{p}$*.*

As mentioned above, this theorem may be considered too general for practical purposes. For one thing, it rides on a quasi-ordering $\geq_{QO}$ of outcomes on which little is known *a priori.* We will give below a more ready-to-use version of the theorem in the two natural (but extreme) cases, namely empty and complete orderings. An additional difficulty raised by Theorem 1 is the uncountably infinite size of the set $\mathcal{U}^{\geq_{QO}}$ of lists $(u_1, ..., u_k)$ of utility numbers compatible with $\geq$ with respect to which the "dual cone" $\mathcal{U}_*^{\geq_{QO}}$ of changes $(v_1, ..., v_k)$ in the distribution - that must be added to the Zonotope sets before checking for inclusion - is defined. How can one identify in practice the dual cone of an uncountably infinite set ? In the following proposition, we alleviate this difficulty by showing that for any uncountably infinite set $\mathcal{U}^{\geq_{QO}}$ of lists $(u_1, ..., u_k)$ of utility numbers compatible with $\geq_{QO}$, there is a finite set of lists of utility numbers (each actually taken in the pair $\{0, 1\}$) that generates exactly the same dual cone $\mathcal{U}_*^{\geq_{QO}}$. Hence, this proposition simplifies the computational problem of finding the appropriate dual cone that is relevant for the implementation of the criterion. The proposition that we establish is the following.

Proposition 3 *We have*

$$\mathcal{U}_*^{\geq_{QO}} = \left\{ \mathbf{v} \in \mathbb{R}^k : \sum_{j=1}^{k} v_j u_j \geq 0 \ \forall (u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}} \cap \{0, 1\}^k \right\}.$$

Another simple, but interesting, implication of the dominance of one society by another in terms of the QOEZ dominance criterion is the dominance of the average distribution of opportunities of the dominating society over that of the dominated one by all list of utility numbers compatible with $\geq_{QO}$. In effect,

Remark 2 *Suppose that* $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$. *Then* $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq_{QO}}$.

As mentioned above, it may be useful to interpret Theorem 1 in the two extreme cases where no outcomes are comparable, and where all outcomes are ordered as per their rank in the set $\{1, ..., k\}$.

We start with the first case. Combining standard results on one dimensional inequality measurement and Theorem 3.1 in Koshevoy and Mosler (1996), we establish the following result.

Proposition 4 *Suppose that* $\overline{p} = \overline{q}$. *Then the two following statements are equivalent:*

(*i*) $Z(\mathbf{q}) \subset Z(\mathbf{p})$;

(*ii*) $\mathbf{q} \succsim_{UEU}^{\emptyset} \mathbf{p}$.

We now turn to the case, typically considered in the equality of opportunity measurement literature, where all outcomes are ordered from the worst (1) to the best (k). In that case, the lists of utility numbers $(u_1, ..., u_k) \in \mathbb{R}^k$ over which a unanimity is looked for are those lists that satisfy $u_1 \leq u_2 \leq ... \leq u_k$. Exploiting the result of Proposition 3, we can limit our attention to those lists of numbers lying in the set $\{0, 1\}$ that satisfy these inequalities. The dual cone of the set of those lists of 0 and 1 bears a close connection with the notion of *first order stochastic dominance* applied to the distributions of outcomes.[8]

We now observe, thanks to Proposition 3, that the dual cone of the set $\mathcal{U}^{\geq_C}$ can be taken to be the set of changes $(v_1, ..., v_k)$ in the distributions of opportunities that produce first order stochastic improvements over the distributions of opportunities to which they are applied. Specifically, using Proposition 3, one can observe the following.

[8]For any two distributions p and $q \in \Delta^{k-1}$, we say that q first order stochastically dominates p, denoted $q \succsim^{1st} p$, if and only if one has: $\sum_{h=j}^{k} q_h \geq \sum_{h=j}^{k} p_h$, for any $j = 1, ..., k$

Remark 3 $\mathcal{U}_*^{\geq_C} = \{v \in \mathbb{R}^k : \sum_{j=1}^k v_j = 0,\ \sum_{g=h}^k v_g \geq 0 \text{ for } h = 2, ..., k\}$.

The connection between $\succsim_Z^C$ and 1st-order stochastic dominance is not surprising from an intuitive point of view. Any ethical observer who agrees on the complete ranking of the outcomes also agrees on the fact that a relation of stochastic dominance between two groups indicates that the dominating group has better opportunities than the dominated group (a similar observation is made in Andreoli, Havne, and Lefranc (2019)). As a result, any such ethical observer - at least when he or she dislikes opportunity inequalities - would like to reduce the dispersion between those two distributions provided that the reduction of the dispersion does not modify the average distribution of opportunities in the two groups. Observe also that introducing a complete ordering of the outcomes immediately introduces a trade-off between opportunity equalization and overall opportunity improvement (through first-order stochastic dominance). In effect, any distribution of opportunities between n groups whatsoever, no matter how unequal it is, would be considered better by the $\geq_C$- extended Zonotope dominance criterion than the perfectly equal (but abysmal) distribution of opportunities in which every individual in every group is sure to end up in the worst outcome (1). In the other direction, any distribution of opportunities would be considered worse than the egalitarian ideal distribution of opportunities in which everyone in every group is sure to end in the best possible outcome (k).

3.2 Elementary operations

An alternative understanding of the $\geq_{QO}$- extended Zonotope dominance criteria (for various specifications of $\geq_{QO}$) can be obtained from the identification of the *elementary transformations* in the distributions of opportunities among groups that underlie them. While we do not identify exactly all these elementary transformations in the general n-groups case - see however the results of the next subsection concerning two-group societies - we can at least identify some of them. We start with the following one, also identified by Kolm (1977) in the more general setting of multidimensional inequality measurement.

Definition 5 (Uniform averaging) *We say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through a* uniform averaging *operation if there exists an* $n \times n$ *bistochastic*[9] *matrix* $\mathbf{b}$ *such that* $\mathbf{q} = \mathbf{b}.\mathbf{p}$

This operation consists in *uniformly averaging* the various distributions of outcomes of the different groups. Specifically, if $\mathbf{q}$ is obtained from $\mathbf{p}$ through a *uniform averaging* operation, then for every group i, the probability q_{ih} that someone from that group achieves outcome h is a weighted average of the probabilities that people from the different groups in $\mathbf{p}$ achieve that outcome. This averaging is "uniform" in the sense that, for any group i, the weights used in the calculation of the average do not depend upon the outcome. To illustrate this point, consider the societies $\mathbf{p}$, $\mathbf{p}'$ and $\mathbf{p}''$ that distribute opportunities of

[9] A *bistochastic* matrix is a nonnegative matrix where all rows and all columns sum to one.

achieving three outcomes between two groups as follows:

$\mathbf{p} =$

	outcome 1	outcome 2	outcome 3
group 1	1/4	1/12	2/3
group 2	2/3	1/4	1/12

$\mathbf{p}' =$

	outcome 1	outcome 2	outcome 3
group 1	17/48	1/8	25/48
group 2	9/16	5/24	11/48

$\mathbf{p}'' =$

	outcome 1	outcome 2	outcome 3
group 1	11/24	1/8	5/12
group 2	11/24	5/24	1/3

Observe that the average probability of achieving the three outcomes in the three societies is the same (namely $(11/24, 1/6, 9/24)$. In both societies $\mathbf{p}'$ and $\mathbf{p}''$, it can be observed that the probability that a member of a group will achieve a given outcome is a weighted average, over the two groups in society $\mathbf{p}$, of the probabilities that the same outcome will be achieved. Hence, both $\mathbf{p}'$ and $\mathbf{p}''$ are obtained from $\mathbf{p}$ as a result of an *averaging* operation. However only society $\mathbf{p}'$ results from $\mathbf{p}$ out of a *uniform* averaging operation that uses the same weights - namely 3/4 and 1/4 for group 1 and 1/4 and 3/4 for group 2 - for determining the probability of achieving any outcome. The property of uniform averaging has been shown by Kolm (1977) - in the context considered by this author where the objects distributed among groups are consumption bundles rather than probability distributions over outcomes - to be equivalent to the ranking of distributions of consumption bundles that would be made by summing all Schur-concave functions. Since the function G defined by $G(\mathbf{p}) = \sum_{i=1}^{n} \Phi \left(\sum_{h=1}^{k} p_{ih} u_h \right)$ is concave and symmetric (across groups) if Φ is concave, it is therefore Schur-concave.[10] Hence, thanks to the result by Kolm (1977), and irrespective of the ordering of outcomes, any uniform averaging operation would be considered worth doing by any ethical observer considered in this paper.

The second elementary operation that we consider is what we call an *bilateral equalizing transfer*. Contrary to uniform averaging - which does not use information on the ranking of the outcomes - the operation of bilateral equalizing transfer rides heavily on such an information. The formal definition of such a transfer is as follows.

Definition 6 (Equalizing transfer) *We say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through a bilateral equalizing transfer if there exist indices* i_1, i_2, i_1' *and* i_2' $\in \{1, ..., n\}$ *and* $v \in \mathcal{U}_*^{\geq_{QO}}$ *such that:*

$$q_{i_1'} = p_{i_1} + v,\ q_{i_2'} = p_{i_2} - v,\ p_{i_2} - p_{i_1} - v \in \mathcal{U}_*^{\geq_{QO}}$$

and $p_j = q_j$ *for all* $j \notin \{i_1, i_2, i_1', i_2'\}$.

In words, a bilateral equalizing transfer is an operation that improves (through some change v) a distribution of opportunity in a group and that deteriorates

[10] A function $h : A \subset \mathbb{R}^k \to \mathbb{R}$ is *Schur-concave* if for every $a \in A$, and every bistochastic matrix $\mathbf{b} \in \mathbb{R}^{k \times k}$, $h(\mathbf{b}.a) \geq h(a)$

(through the same v applied in opposite direction) a distribution of opportunities in another group in the case where the distribution of opportunities in the latter group is unambiguously better than that of the other group from the view point of the quasi-ordering $\geq_C$ or equivalently, thanks to Donaldson and Weymark (1998), of all complete rankings of outcomes whose intersection is $\geq_C$. It is intuitively clear that such a reduction in the "expected-utility gap" between the two distributions - provided that it is done in a way that does not affect the average distribution of opportunities in the society - would be recorded favorably by an opportunity-averse ethical observer who evaluates those expected utilities through a (uniform) expectation of a concave function. We observe that such a transformation only concerns two distributions of opportunities in each of the two societies (and leaves the other distributions faced by the other groups unchanged). Hence, by comparison with the uniform averaging operation which concerns the *totality* of the matrix, a bilateral equalizing transfer, as its name suggests,is a *local operation* that concerns only two rows of each of the matrices under comparison.

In order to illustrate this transformation in the case of an incomplete ranking of the outcomes, consider the binary health-education example given earlier where the outcomes are $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ and the quasi-ordering $\geq_{QO}$ is $(1,1) \geq_{QO} (0,1) \geq_{QO} (0,0)$ and $(1,1) \geq_{QO} (1,0) \geq_{QO} (0,0)$ ($(0,1)$ and $(1,0)$ being incomparable). Assume that there are only two groups, and consider the two distributions:

$\mathbf{p} =$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
group 1	1/2	1/6	1/6	1/6
group 2	1/4	1/4	1/4	1/4

and:

$\mathbf{q} =$

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
group 1	7/16	19/86	1/6	19/86
group 2	5/16	7/32	1/4	7/32

The first observation is that the distribution of probabilities of achieving the four outcomes in group 1 provides a lower expected utility than that of group 2 in society $\mathbf{p}$. This can be seen by the fact that, for any of the two complete rankings of the four outcomes that are consistent with $\geq_{QO}$, the distribution of outcome in group 2 first order stochastically dominate that in group 2. The second observation is that the move from $\mathbf{p}$ to $\mathbf{q}$ has been done by improving the probability distribution of group 1 by the vector $v = (-1/16, 1/32, 0, 1/32)$ and deteriorating the probability distribution by the corresponding vector $-v = (1/16, -1/32, 0, 1/32)$. Observe that these two "balanced" offsetting changes in the distributions of outcomes have preserved the dominance of group 2 over group 1 from . Yet, the spread of that difference has shrunk, and this carefully constructed shrinking is appraised favorably by an opportunity-inequality averse UEU ethical observer.

The last elementary operation that we discuss is not related to reducing inequalities of opportunities. It is rather concerned with improving those opportunities for some, or all, of the groups (up to a permutation of some of the groups, thanks to the anonymity principle). Specifically, we define as follows the notion of an anonymous and unanimous expected utility improvement.

Definition 7 (Expected Utility Improvement) *We say that* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through an* anonymous and unanimous expected utility improvement *if there exists a one-to-one function* $\pi : \{1, ..., n\} \longrightarrow \{1, ..., n\}$ *such that for every* $i \in \{1, ..., n\}$*, there exists* $v_i \in \mathcal{U}_*^{\geq_{QO}}$ *for which one has:*

$$q_{\pi(i)} = p_{\pi(i)} + v_i$$

In words, $\mathbf{q}$ is obtained from $\mathbf{p}$ through an anonymous and unanimous expected utility improvement if one can finds a permutation of the groups such that every thus permuted group in $\mathbf{q}$ as a weakly better distribution of opportunities than the corresponding group in $\mathbf{p}$ when appraised by the expectation over any list of utility numbers compatible with the underlying quasi-ordering. An anonymous and unanimous expected utility improvement reduces to an anonymous and unanimous first-order stochastic increment in the case where the ranking of outcomes is complete. Any such anonymous and unanimous expected utility improvement will be appraised favorably by any ethical observer considered herein. In the following lemma, we establish formally that performing a favorable transfer *or* a uniform averaging are also elementary operation that are considered worth doing by those same ethical observers.

Lemma 1 *If* $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through either a* uniform averaging *or a* favorable transfer *then* $\mathbf{q} \succsim_{UEU}^{QO} \mathbf{p}$.

3.3 Equalizing opportunities between two groups

The criteria of opportunity equalization discussed in the preceding sub-section are general and work for any number of groups. However, the operational criterion of QOEZ dominance may be considered somewhat difficult to interpret and to use in practical applications, even thought it can be implemented through some computer algorithm. As it turns out, the difficulty is significantly alleviated when there are only two groups (for example females and males) among which opportunities are equalized. Indeed, in this case, we can implement QOEZ dominance by the finite, and somewhat simple, procedure of "majorization", by each of the two distribution of opportunities of the dominating society, of some weighted average of the two distributions of the dominated society. To understand this procedure, consider the family $\mathcal{F}^{\geq_{QO}}$ of sets whose elements form a chain with respect to the quasi-ordering $\geq$. This family is formally defined by:

$$\mathcal{F}^{\geq_{QO}} = \{J \subset \{1, ..., k\} : h \in J \text{ and } j \geq^{QO} h \Longrightarrow j \in J\}$$

This family is closely related to the dual cone $\mathcal{U}_*^{\geq^{QO}}$ of the quasi-ordering the $\geq$ which can indeed be defined, thanks to Proposition 3, by:

$$\mathcal{U}_*^{\geq_{QO}} = \left\{ v \in \mathbb{R}^k : \sum_{h=1}^{k} v_h = 0 \text{ and } \sum_{h \in J} v_h \geq 0 \text{ for all } J \in \mathcal{F}^{\geq_{QO}} \right\} \tag{7}$$

The family $\mathcal{F}^{\geq_{QO}}$ is important because it provides the complete (and finite) list of sets of outcomes whose increases in the likelihood are indisputably perceived as improving opportunities. For example, if the quasi-ordering $\geq$ is taken to be $\geq_C$(the case where the outcomes are completely ordered), the family

$\mathcal{F}^{\geq_{QO}}$would consist in the (anti) cumulated lists of outcome $\{k\}$, $\{k-1,k\}$,...,$\{1,2,...,k\}$ used to check for first order stochastic dominance. For any probability distribution $p \in \Delta^{k-1}$, and any $J \in \mathcal{F}^{\geq_{QO}}$, we let $p(J)$ denote the "cumulated" probability of achieving an outcome in that set defined by $p(J) = \sum_{j \in J} p_j$. The majorization procedure that we propose as a test for $\geq$- extended Zonotope dominance between two-group societies works as follows. For any two such societies, one first checks if the average distribution of opportunities is better in one society than in the other for all expected utility criteria compatible with the quasi-ordering of outcomes. If no such dominance is observed, then we know from Proposition 1 that the two societies cannot be compared by QOEZ dominance and the test is over. If, on the other hand, such a dominance is observed, then the society with the dominating average is a candidate for being a dominating society as per our criterion. To verify that it is indeed so, one looks, for *each* of the two distributions of opportunities in the (possibly) dominating society, at the mixtures of the two distributions of opportunities in the (possibly) dominated society that yield the same probability of reaching outcomes in some members of $\mathcal{F}^{\geq_{QO}}$. There may not be any such mixture in which case one concludes in the absence of dominance. If there are, however, such mixtures, then the verdict of dominance would be obtained if each of the two distributions of opportunities in the (possibly) dominating society dominates at least one such mixture of the two distributions in the dominated society.
The following theorem describes this procedure and shows its equivalence to QOEZ dominance

Theorem 2 *Suppose that $n = 2$. Let Λ_i (for $i = 1, 2$) be defined by*[11]

$$\Lambda_i = \{1\} \cup \{\lambda \in [0,1] : \exists J \in \mathcal{F}^{\geq_{QO}} \ \ s.t. \ \ q_i(J) = \lambda p_1(J) + (1-\lambda) p_2(J)\}.$$

Then $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ if and only if $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq_{QO}}$ and there are $\lambda_i \in \Lambda_i$ (for $i = 1, 2$) such that $q_1 - (\lambda_1 p_1 + (1-\lambda_1) p_2) \in \mathcal{U}_*^{\geq_{QO}}$ and $q_2 - (\lambda_2 p_1 + (1-\lambda_2) p_2) \in \mathcal{U}_*^{\geq_{QO}}$.

It may be useful to appreciate the simplicity of the procedure described by this theorem through an example of two societies made of two groups where the dominance of one society over the other is not immediately apparent.

Example 1 *Consider the two following societies:*

$\mathbf{p} =$

	1	2	3	4
group 1	16/36	4/36	6/36	10/36
group 2	13/36	3/36	12/36	8/36

and

$\mathbf{q} =$

	1	2	3	4
group 1	16/36	2/36	8/36	10/36
group 2	13/36	5/36	9/36	9/36

[11] Adding value 1 to this set might seem arbitrary there; the reason for doing so will be made clear in the proof of Theorem 2. One could alternatively decide to add 0 instead of 1, and modify the proof accordingly. Note that these two sets are then finite and non-empty.

Assume that the quasi-ordering of outcomes is the complete ordering $\geq_C$. Observe that

$$\overline{q} - \overline{p} = \frac{1}{72}(0, 0, -1, 1) \in \mathcal{U}_*^{\geq_{QO}},$$

which means that $q_2 + q_1 = p_2 + p_1 + v$, where $v = \frac{1}{36}(0, 0, -1, 1)$. In (almost) plain English, the distribution $\overline{q}$ stochastically dominates the distribution $\overline{p}$. Hence **q** *is possibly a society that dominates society* **p** *for the criterion* $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$. *Let us use the procedure described in Theorem 2 to verify that this is indeed the case. The family $\mathcal{F}^{\geq_C}$ here is defined by $\mathcal{F}^{\geq_C} = \{\{1, 2, 3, 4\}, \{2, 3, 4\}, \{3, 4\}, \{4\}\}$. The sets Λ_1 and Λ_2 are therefore respectively defined as the union of the singleton $\{1\}$ and the sets of solutions, in the $[0, 1]$ interval, of the following equations:*

$$10/36 = \lambda_{11}10/36 + (1 - \lambda_{11})8/36 \Rightarrow \lambda_{11} = 1$$
$$18/36 = \lambda_{12}16/36 + (1 - \lambda_{12})20/36 \Rightarrow \lambda_{12} = 1/2$$
$$20/36 = \lambda_{13}20/36 + (1 - \lambda_{13})23/36 \Rightarrow \lambda_{13} = 1$$

for Λ_1 and of the equations:

$$9/36 = \lambda_{21}10/36 + (1 - \lambda_{21})8/36 \Rightarrow \lambda_{21} = 1/2$$
$$18/36 = \lambda_{22}16/36 + (1 - \lambda_{22})20/36 \Rightarrow \lambda_{22} = 1/2$$
$$23/36 = \lambda_{23}20/36 + (1 - \lambda_{23})23/36 \Rightarrow \lambda_{23} = 0$$

for Λ_2. We thus have $\Lambda_1 = \{1/2, 1\}$ and $\Lambda_2 = \{0, 1/2, 1\}$. Since q_1 1st-order stochastically dominates p_1, we have $q_1 - (\lambda p_1 + (1 - \lambda)p_2 \in \mathcal{U}_*^{\geq_{QO}}$ for $\lambda = 1 \in \Lambda_1$. One can also observe that

$$q_2 = \left(\frac{26}{72}, \frac{10}{72}, \frac{18}{72}, \frac{18}{72}\right)$$

1st order stochastically dominates the mixture of p_1 and p_2 given by:

$$\frac{p_1}{2} + \frac{p_2}{2} = \left(\frac{29}{72}, \frac{7}{72}, \frac{18}{72}, \frac{18}{72}\right)$$

Hence $\mathbf{q} \succsim_Z^C \mathbf{p}$.

Remark 4 *Interestingly, this example also provides insights concerning the elementary operations discussed in the preceding section. We can use it to prove that the three aforementionned operations are not the only ones that underlie the notion of opportunity equalization discussed herein. Indeed, it is not possible to go from* **p** *to* **q** *by a finite sequence of Uniform averaging, bilateral equalizing transfers and Anonymous expected utility improvements. That no equalizing transfers can be performed to go from* **p** *to* **q** *is clear since none of the two distributions of opportunities p_1 and p_2 first order stochastically dominate the other. One can also see that no uniform averaging operation, however small, can be done. Indeed, for any $\lambda \in [0, 1[$, $q_1 - (\lambda p_1 + (1 - \lambda)p_2) \notin \mathcal{U}_*^{\geq_C}$. This is so because the probability of achieving the worst outcome for the first group in society* **q** *is strictly larger than any mixture of the probabilities of achieving that worst outcome the two groups in society* **p** *($q_{11} = 16/36 > \lambda 16/36 + (1 - \lambda)13/36$ for all $0 \leq \lambda < 1$). Finally, we can show that there is no margin to perform* *an anonymous and unanimous utility improvement however small on the initial society* $\mathbf{p}$ *in a way that preserve dominance of* $\mathbf{q}$ *over the transformed* $\mathbf{p}$. *See the appendix for the proof.*

Hence, while the three elementary transformations of the distributions of opportunities discussed in the preceding are considered worth doing by the criteria considered here, they are not the complete list of such transformations.

There is, however, a particular - but theoretically important - case where some of those elementary transformations coincide with the QOEZ dominance criterion. This case is when the two societies offer the same average opportunities to the two groups, and only differ in the inequality with which this common average opportunity is split between the two groups. In this case, and when of course there are two groups, then QOEZ dominance actually coincides with the possibility of going from the dominated to the dominating distribution by a finite sequence of equalizing transfers and uniform averaging operations. The following theorem establishes that fact. specifically prove in this subsection the following theorem.

Theorem 3 *Suppose that* $n = 2$ *and* $\overline{p} = \overline{q}$. *The three following statements are equivalent:*

1. $\mathbf{q}$ *is obtained from* $\mathbf{p}$ *through a uniform averaging or an equalizing transfer;*
2. $\mathbf{q} \succsim_{UEU}^{QO} \mathbf{p}$;
3. $\mathbf{q} \succsim_{Z}^{QO} \mathbf{p}$.

The equivalence established in Theorem 3 *between* the domination of a two-group society by another in terms of the $\geq$- extended Zonotope criterion *and* the possibility of going from the dominated to the dominating society by either an equalizing transfer or a uniform averaging operation when the average distribution of opportunities is the same provides a simple way to check for dominance in that case. This is at least so if one focuses on the case where the outcomes are completely ordered and where, as a result, the dual cone of the set of lists of utility numbers $(u_1, ..., u_k)$ that are increasing with respect to outcomes is the set of changes v that generates a first-order dominance between distributions. In that case, one can observe the following (obtained as an immediate consequence of Theorem 3)

Remark 5 *Suppose that* $\overline{p} = \overline{q}$ *and* $n = 2$. *Assume that either* $p_1 \succsim^{1st} p_2$ *or* $p_2 \succsim^{1st} p_1$. *Consider the indexing* i_1 *and* i_2 *of the two groups such that* $p_{i_2} \succsim^{1st} p_{i_1}$. *Then* $\mathbf{q} \succsim_{Z}^{C} \mathbf{p}$ *if and only if* $p_{i_2} \succsim^{1st} q_{i_1} \succsim^{1st} p_{i_1}$ *and* $p_{i_2} \succsim^{1st} q_{i_2} \succsim^{1st} p_{i_1}$.

This remark leads itself to a very simple test of opportunity equalization in the two-group case, at least when the average distribution of opportunities is the same, and when one group in one society is stochastically dominated by the other. The test amounts to verifying if, in the other society, the distributions of opportunities of two groups lie "in between" those of the two groups in terms of first order stochastic dominance.

In the next section, we shall use this test to analyze the inequality of educational opportunities in India between men and women both across religions, cast groups and over time.

Category	Education level	Equivalent years of education
1	Illiterate	0
2	Literate without formal schooling	Not Applicable
3	Below primary	less than 4
4	Primary	4
5	Middle but below secondary	5 to 10
6	Secondary but not graduate	11 to 15
7	Graduate and above	16 or more

Table 1: Rough equivalence between education levels and years of education

4 Empirical Illustration

India is characterized by a significant and quite astonishing by Western standards - cross-gender inequalities in educational attainment. Females, as compared to males, are more likely to be over-represented at the bottom of the education distribution and under-represented at the top. However these inequalities between genders also happen to vary across different social strata based on religion to which Indians belong. In this section, we put the criteria developed in the previous section to appraise the inequality of educational opportunity between men and women considering two religious communities - Sikh and Buddhist. We also investigate, over a span of three decades, to what extent the improvement in average educational opportunity in both these communities has been accompanied by a reduction in gender opportunity gaps.

Although neither Buddhism nor Sikhism are followed by majority of the Indians, they are two of the ancient religions in India. According to the census of 2011 India is home to over 20 million Sikhs and 8 million Buddhists. To illustrate the educational opportunity gap among genders across religious societies in India we have taken data from two rounds of National Sample Survey (NSS) database. In particular we consider the earliest and the latest available rounds of the Employment-Unemployment schedule of NSS that corresponds to the survey years 1983 (the 38th round) and 2011-12 (the 68th round), respectively. Information on education is recorded for every member of the household in the above mentioned surveyed. We however limit our illustration to all Indian adults aged between 20 to 50 years so as to focus on the prime working age population of the country. For the sake of comparability across the two rounds of data we have regrouped the given education levels in 7 mutually exclusive and exhaustive groups, where illiteracy is considered as the worst possible level and having a graduate degree or above is considered as the best one. To give an idea of the educational categories in India we provide in Table 1, the (roughly) equivalent years of education that is common to the pan-Indian education system[12].

Figure 2 draws the cumulative density of the average educational opportunities over genders, separately for the Buddhists and the Sikhs for the time span of

[12] However notice that we are unable to provide this assessment for the second category - 'literate without formal schooling'. In addition to formal schooling, various literacy programs are common in India, which we want to keep as a separate category so as to distinguish 'literacy' from 'illiteracy'.

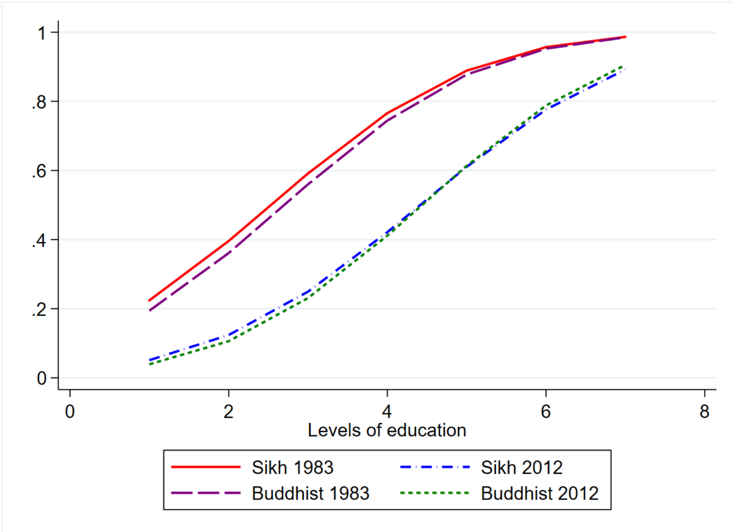


Figure 2: Improvement in education among Buddhists and Sikhs, India (1983-2012)

1983-2012. Two things are immediately noticeable from this figure. First that the overall educational opportunity is significantly better in 2012 than that of 1983 for both the religions. Second, educational opportunities of Buddhists and Sikhs nevertheless remain close to each other over this three decades. Indeed the two-sample t-test rejects any statistically significant difference in mean educational levels of the two religions for either time period.

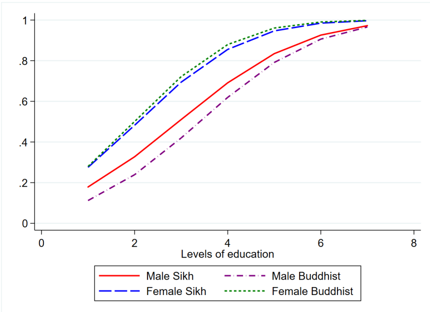
The top row of Table 2 reports the population share of females and males, separately for the two religious communities and for the two different time frames. India is yet to achieve a balanced share of males and females in either religious groups even in 2012. But Table 2 further reveals that the educational profile of either religion is not independent of the sexual identity of a person. While a significant share of both Sikh and Buddhist population was illiterate in 1983, the share of illiterate females outweigh that of males in either religions. This discrimination across genders is still present in 2012 as well, with particularly striking difference among the Buddhists. Indeed in 2012, while 16% of total Buddhist population are illiterate female Buddhists, only 9% are illiterate male Buddhists. On the other hand the share of graduates in either religion is abysmally low even in 2012, but a bit more worse for females. While it reflects on the presence of opportunity gap in education between genders in either religion, we want further to see in which religion this gap is more severe. We aim to answer this question applying the theory developed in this paper, in particular, testing the statement as proposed in *Remark 5*.

Figure 3 plots the cumulative density function of the levels of education separately for the two genders of the two religions (hence a total of four *types* in the parlance of the related literature), where the left-hand side panel corresponds to the older survey round of 1983 and the right-hand side panel draws the same for

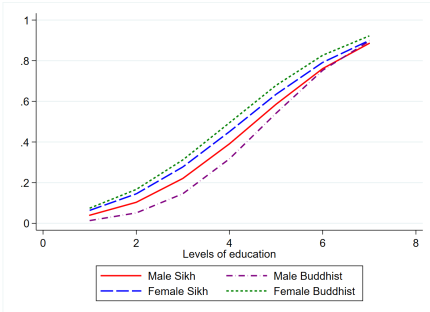
	Sikh				Buddhist			
	1983		2011-12		1983		2011-12	
	Female	Male	Female	Male	Female	Male	Female	Male
Population share $\rightarrow$	*0.47*	*0.53*	*0.48*	*0.52*	*0.49*	*0.51*	*0.46*	*0.54*
Illiterate	0.32	0.28	0.15	0.13	0.34	0.21	0.16	0.09
Primary	0.0.05	0.0.07	0.08	0.08	0.05	0.11	0.07	0.10
Secondary	0.02	0.04	0.10	0.14	0.01	0.04	0.09	0.14
Graduate and above	0.005	0.009	0.02	0.03	0.003	0.01	0.02	0.03

Table 2: Sample summary[a]

[a]The table reads as - During 1983, 47% (53%) of the total Sikh population was female (male) and 32% (28%) of the total Sikh population are illiterate females (males).



(a) India 1983



(b) India 2012

Figure 3: Educational opportunity gap between genders across Buddhists and Sikhs

2012. Apparently from the visuals we can see that the opportunity gap between men and women are present in either religious communities over the entire span of the study (1983-2012). But in either of the survey years, the pair of educational distributions (for the two genders) in the Sikh community are contained within the pair of distributions of the Buddhist community. The formal test of first order stochastic dominance as proposed by Davidson and Duclos (2000) indeed confirms the dominance of Buddhist men over the Sikh men at order one in both survey years and for all levels of educations described above. Whereas on the other hand the null of first order stochastic dominance of Sikh women over Buddhist women can not be rejected.

Table 3 presents the first order stochastic dominance results for comparing educational opportunities between each different pairs of the four types, separately for the two survey years[13]. Some features are noteworthy from this schematic ta-

[13]The null of first order stochastic dominance of males over females is tested by the difference in their respective empirical distributions at all levels of education. The test statistic for education level j and groups (r, r') is - $\frac{F(\hat{j,r}) - F(\hat{j,r'})}{\sqrt{\frac{V(F(\hat{j,r}))}{N_r} + \frac{V(F(\hat{j,r'}))}{N_{r'}}}}$. The statistical inference is drawn on the basis of the union-intersection criteria proposed by BishFornThist92 that rejects the dominance of group-r over group-r' if at least one of the test statistics are significantly

ble. First of all, females never dominate their male peers, but Buddhist females had the worst educational opportunity. On the other hand Buddhist males are the most advantageous group among the four discussed here and remain so over a span of three decades. Secondly, denoting first order stochastic dominance by $\succ$ we can rank the four types as follows - Buddhist males $\succ$ Sikh males $\succ$ Sikh females $\succ$ Buddhist females. Therefore following the statement of *Remark 5*, we can say that the educational opportunity is more 'equalized' among the Sikhs than among the Buddhists. Third and probably the most policy-relevant observation is that, although the average educational opportunity (over men and women) improved significantly from 1983 to 2012 for both religions, the above mentioned ranking did not change, which indicates that the gender gap in educational achievement remain a more severe issue for the Buddhists in spite of the overall development in education.

max width=

		Buddhist Male	Buddhist Female	Sikh Male	Sikh Female
5*1983	Buddhist Male	-	$\succ$	$\succ$	
	Buddhist Female	$\nsucc$	-	$\nsucc$	$\nsucc$
	Sikh Male	$\nsucc$	$\succ$	-	$\succ$
	Sikh Female	$\nsucc$	$\succ$	$\nsucc$	-
5*2012	Buddhist Male	-	$\succ$	$\succ$	
	Buddhist Female	$\nsucc$	-	$\nsucc$	$\nsucc$
	Sikh Male	$\nsucc$	$\succ$	-	$\succ$
	Sikh Female	$\nsucc$	$\succ$	$\nsucc$	-

Table 3: Dominance table (order one)[a]

[a]Where $\succ$ denotes first order stochastic dominance of row over column and $\nsucc$ denotes that row does not dominate column at order one

5 Conclusion

TO BE PROVIDED

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6 Appendix

6.1 Axiomatic foundations

We provide the missing details on the axioms characterizing the set of orderings that can be represented as in (2). We start by identifying the four axioms of $\succsim$ that characterize the UEU family of social rankings. In line with the discussion above, the first axiom requires the social ranking to be *anonymous*. That is, the names or the natures of the groups do not matter for appraising the disparity of opportunities between groups in a society. Hence all societies offering the same distributions of outcomes among the same number of groups are normatively equivalent. We state formally this axiom as follows.

Axiom 1 *(Anonymity) For every society* $\mathbf{p} \in \mathbf{S}$ *and all* $n(\mathbf{p}) \times n(\mathbf{p})$ *permutation matrix* π*, one has* $\pi.\mathbf{p} \sim \mathbf{p}$*.*

The second axiom imposed on $\succsim$ is a continuity condition that concerns the comparison of a one-group society vis-à-vis any other. This axiom requires that the strict ranking of a single lottery associated to a one-group society *vis-à-vis* any other society should be robust to "small" changes in the probabilities of achieving any given outcome. Its formal statement is as follows.

Axiom 2 *(Continuity) For every society* $\mathbf{p} \in \mathbf{S}$*, the sets* $B(\mathbf{p}) = \{\rho \in \Delta^{k-1} : \rho \succsim \mathbf{p}\}$*,* $W(p) = \{\rho \in \Delta^{k-1} : \mathbf{p} \succeq \rho\}$ *are both closed in* $\mathbb{R}^k$*.*

The next axiom is called *averaging* in Gravel, Marchant, and Sen (2012). In the current context, the axiom evaluates what happens to the disparity of opportunities in a given society when the number of groups is enlarged. It says that if the disparity of opportunities in the added groups is better (worse) than what they are in the initial society, then the addition of those groups improves (deteriorates) the disparity of opportunities. It says also, conversely, that if a society loses (gains) from identifying new groups with specific distributions of outcome among their members, then this can only be because the distribution of outcomes within those groups is worse (better) than that already present in the original society. This axiom is formally stated as follows.

Axiom 3 *(Averaging) For all societies* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbf{S}$*, we have* $\mathbf{p} \succsim \mathbf{q} \Leftrightarrow \mathbf{p} \succsim (\mathbf{p}, \mathbf{q}) \Leftrightarrow (\mathbf{p}, \mathbf{q}) \succsim \mathbf{q}$*.* [14]

When applied to an ordering, the Averaging axiom implies several other properties. One of them is the axiom called "replication equivalence" by Blackorby, Bossert, and Donaldson (2005) (p. 197) in the somewhat different context of population ethics. This axiom states that, for societies where every group faces the same opportunities, the *number* of those groups does not matter. This property is rather natural in the context of equalizing opportunities. If all groups in a society were offering the same opportunities, then the number of those groups would be irrelevant. We state formally this property as follows.

[14] If $\mathbf{p}$ is a society in $(\Delta^{k-1})^m$ and $\mathbf{q}$ is a society in $(\Delta^{k-1})^n$, we denote by $(\mathbf{p}, \mathbf{q})$ the society in $(\Delta^{k-1})^{m+n}$ where the m first groups face the opportunities associated to the matrix $\mathbf{p}$ and the n last groups face the opportunities associated to $\mathbf{q}$ (in the corresponding order).

Condition 1 *(Irrelevance of the number of groups in case of equal opportunities) For every lottery* $\rho \in \Delta^{k-1}$ *and every society* $\mathbf{p} \in \mathbf{S}$ *such that* $p_i = \rho$ *for all* $i = 1, ..., n(p)$*, one has* $\mathbf{p} \sim \rho$.

This condition is implied by averaging if $\succsim$ is reflexive. The proof of this claim is left to the reader.

The next, and last, axiom that requires the ranking of any two societies with the same number of groups to be robust to the addition, to both societies, of a common distribution of opportunities. That is to say, the ranking of any two societies with the same number of group should be independent from any group that they have in common. Formally, this axiom is stated as follows.

Axiom 4 *(Same number group independence) For all societies* $\mathbf{p}$, $\mathbf{p}'$ *and* $\mathbf{p}''$ *in* $\mathbf{S}$ *such that* $n(\mathbf{p}) = n(\mathbf{p}')$, $(\mathbf{p}, \mathbf{p}'') \succsim (\mathbf{p}', \mathbf{p}'')$ *if and only if* $\mathbf{p} \succsim \mathbf{p}'$.

It can be checked that any UEU ranking satisfies anonymity, continuity, averaging and Same Number Group Independence. Gravel, Marchant, and Sen (2012) (see also Gravel, Marchant, and Sen (2011)) have established the converse implication. Hence, one has:

Proposition 5 *Let* $\succsim$ *be an ordering on* $\mathbf{S}$ *satisfying anonymity, continuity, averaging and same number expansion consistency. Then* $\succsim$ *is a UEU social ordering. Furthermore, the function* Ψ *of Expression (1) is unique up to a positive affine transformation, and is continuous.*

Axiom 5 *(VNM for One-Group societies) For every lotteries* p, p' *and* p" $\in \Delta^{k-1}$ *and every number* $\lambda \in [0, 1]$, $p \succsim p'$ *if and only if* $\lambda p + (1-\lambda)p'' \succsim \lambda p' + (1-\lambda)p''$.

It is then immediate to obtain the following result (see e.g. Proposition 6 in Gravel, Marchant, and Sen (2012)).

Proposition 6 *Let* $\succsim$ *be an ordering on* $\mathbf{S}$ *satisfying anonymity, Continuity, Averaging, Same Number Group Independence and VNM for One-Group Societies. Then* $\succsim$ *is a UEU social ordering and the function* Ψ *of Expression (1) can be written as per Expression (2) for some function* $\Phi : \mathbb{R} \to \mathbb{R}$ *and some list of* k *real numbers* $u_1, ..., u_k$.

When applied to a UEU ranking of societies, the definition of neutrality with respect to equality of opportunities (Definition 2 (i)) implies that the function Ψ that represents such a ranking as per Expression (1) is linear. We state this formally as follows.

Proposition 7 *Let* $\succsim$ *be an ordering on* $\mathbf{S}$ *that can be represented as per (1) for some functions* Ψ*. Then* $\succsim$ *exhibits neutrality with respect to equality of opportunities if and only if, for every lottery* $p \in \Delta^{k-1}$*, one has*

$$\Psi(p) = \sum_{j=1}^{k} \beta_j p_j, \text{ for some real numbers } \beta_1, ..., \beta_k.$$

6.2 Proofs

Proof of Remark 1. Let C be the vector sub-space of $\mathbb{R}^k$, generated by the vector $(1, ..., 1)$. Observe that C is a convex cone, and is contained in $\mathcal{U}^{\geq QO}$. Thus, by standard results, $\mathcal{U}_*^{\geq QO} \subset C_* = \left\{(v_1, ..., v_k) \in \mathbb{R}^k : \sum_{j=1}^k v_k = 0\right\}$. ■

Proof of Theorem 1. We first show that Statement 2 implies Statement 1 and, therefore, that $Z(\mathbf{q})+ \mathcal{U}_*^{\geq QO} \subseteq Z(\mathbf{p})+ \mathcal{U}_*^{\geq QO}$. For this sake, it is sufficient to show that, for any $\theta \in \{0, 1\}^n$, there exists $v \in \mathcal{U}_*^{\geq QO}$ and $\lambda \in [0, 1]^n$ such that:

$$\sum_{i=1}^n \theta_i q_i = \sum_{i=1}^n \lambda_i p_i + v. \tag{8}$$

Note that since $\sum_{j=1}^k v_j = 0$ (by Remark 1) and p_i and q_i both belong to Δ^{k-1}, we necessarily have $\sum_{i=1}^n \lambda_i = m$, where $m = \#\{i : \theta_i = 1\}$. Hence, by re-indexing the distributions q_i (for $i = 1, ..., n$) in such a way that $\theta_i = 1$ for $i = 1, ..., m$, Expression (8) can equivalently be written as:

$$\frac{1}{m}\sum_{i=1}^m q_i = \sum_{i=1}^n \frac{\lambda_i}{m}\mathbf{p}_i + \frac{1}{m}v.$$

Let the set $\mathbf{D}$ be defined by:

$$\mathbf{D} := \overline{q} - Co\{p_1, ..., p_n\}$$

What we need to show is that $\mathbf{D}\cap \mathcal{U}_*^{\geq QO} \neq \varnothing$. Suppose by contradiction that $\mathbf{D}\cap \mathcal{U}_*^{\geq QO} = \varnothing$. Since $\mathbf{D}$ is a polytope (Rockafellar (1970), p. 12) and $\mathcal{U}_*^{\geq QO}$ is a closed convex cone, one can conclude from Theorem 2 at p. 80 of Berge (1959) that there are vectors $(d_1^*, ...d_k^*) \in \mathbf{D}$ and $(v_1^*, ...v_k^*) \in \mathcal{U}_*^{\geq QO}$such that:

$$\left(\sum_{h=1}^k (d_h^* - v_h^*)^2\right)^{1/2} = \min_{(d_1,...,d_k)\in D,(v_1,...,v_k)\in\mathcal{U}_*^{\geq QO}} \left(\sum_{h=1}^k (d_h - v_h)^2\right)^{1/2}$$

by continuity of the euclidian norm, and using the fact that the set $\mathbf{D} \times \mathcal{U}_*^{\geq QO}$ on which it is minimized can be made compact by taking a suitable intersection of $\mathcal{U}_*^{\geq QO}$ with some closed ball in $\mathbb{R}^k$. Define the vector $(\widehat{v}_1, ..., \widehat{v}_k)$ by $\widehat{v}_h = v_h^* - d_h^*$ for $h = 1, ..., k$. Then the hyperplane passing through $(v_1^*, ...v_k^*)$ and orthogonal to $(\widehat{v}_1, ..., \widehat{v}_k)$ strongly separates $\mathbf{D}$ and $\mathcal{U}_*^{\geq QO}$ in the sense that:

$$\inf_{(v_1,...,v_k)\in\mathcal{U}_*^{\geq QO}} \sum_{h=1}^k v_h\widehat{v}_h \geq \sum_{h=1}^k v_h^*\widehat{v}_h > \sup_{(d_1,...,d_k)\in D} \sum_{h=1}^k d_h\widehat{v}_h \tag{9}$$

Since $(0, ..., 0) \in \mathcal{U}_*^{\geq QO}$ one must have that:

$$0 \geq \sum_{h=1}^k v_h^*\widehat{v}_h$$

Moreover since $(\lambda v_1^*, ..., \lambda v_k^*) \in \mathcal{U}_*^{\geq QO}$ for every number $\lambda > 0$, one must also have that:

$$\sum_{h=1}^{k} v_h^* \widehat{v}_h \geq 0$$

Indeed, assuming $\sum_{h=1}^{k} v_h^* \widehat{v}_h < 0$ would be contradictory, after taking a suitably large λ, with the strict inequality (9). These two last inequalities enable therefore one to rewrite Inequality (9) more precisely as:

$$\inf_{(v_1,...,v_k)\in\mathcal{U}_*^{\geq QO}} \sum_{h=1}^{k} v_h \widehat{v}_h \geq 0 > \sup_{(d_1,...,d_k)\in D} \sum_{h=1}^{k} d_h \widehat{v}_h \tag{10}$$

By the first of these two inequalities, we conclude that $(\widehat{v}_1, ..., \widehat{v}_k)$ belongs to the dual cone of the set $\mathcal{U}_*^{\geq QO}$, which is itself the dual cone of the set $\mathcal{U}^{\geq_{QO}}$. By the bipolar theorem for convex cones (see for example Theorem 14.1 in Rockafellar (1970)), it therefore follows that the dual cone of $\mathcal{U}_*^{\geq QO}$ is $\mathcal{U}^{\geq_{QO}}$ so that $(\widehat{v}_1, ..., \widehat{v}_k) \in \mathcal{U}^{\geq_{QO}}$. Now since Statement 2 of the theorem holds, we know that the inequality

$$\sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} q_{ih} u_h\right) \geq \sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} p_{ih} u_h\right)$$

holds for all concave Φ and all lists of real numbers $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$. By the Hardy-Littlewood-Polya theorem (see for example Berge (1959), p. 191), this is equivalent to the requirement that the list of n numbers

$$\left(\sum_{h=1}^{k} q_{1h} u_h, ..., \sum_{h=1}^{k} q_{nh} u_h\right)$$

Lorenz dominates[15] the list of n numbers

$$\left(\sum_{h=1}^{k} p_{1h} u_h, ..., \sum_{h=1}^{k} p_{nh} u_h\right)$$

for all list of real numbers $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$. In particular this is true for $(\hat{u}_1, ..., \hat{u}_k)$, and thus there exists an indexing $i_1(\widehat{u}), ..., i_n(\widehat{u})$[16] such that

$$\sum_{h=1}^{k} p_{i_1(\widehat{u})h} \widehat{u}_h \leq \sum_{h=1}^{k} p_{i_2(\widehat{u})h} \widehat{u}_h \leq ... \leq \sum_{h=1}^{k} p_{i_n(\widehat{u})h} \widehat{u}_h$$

and

$$\sum_{i=1}^{m} \sum_{h=1}^{k} q_{ih} \widehat{u}_h \geq \sum_{j=1}^{m} \sum_{h=1}^{k} p_{i_j(\widehat{u})h} \widehat{u}_h. \tag{11}$$

[15] Given two vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^k$, we say that $\mathbf{b}$ Lorenz dominates $\mathbf{a}$ if $\sum_{g=h}^{k} b_{\pi_b(g)} \geq \sum_{g=h}^{k} a_{\pi_a(g)}$, where $\pi : \{1, ..., k\} \to \{1, ..., k\}$ is bijective and such that $a_{\pi_{\mathbf{a}}(1)} \leq ... \leq a_{\pi_{\mathbf{a}}(k)}$ and $b_{\pi_{\mathbf{b}}(1)} \leq ... \leq b_{\pi_{\mathbf{b}}(k)}$.

[16] (which depends of course upon the k-tuple $(\widehat{u}_1, ..., \widehat{u}_k)$)

However, by the second inequality of Expression (10),we have (remembering the definition of $\mathbf{D}$):

$$0 > \sum_{h=1}^{k} \overline{q}_h \widehat{u}_h - \sum_{h=1}^{k} p_{ih} \widehat{u}_h \tag{12}$$

for all $i = 1, ..., n$. It follows therefore from Inequalities (11) and (12) that:

$$\sum_{h=1}^{k} p_{ih} \widehat{u}_h > \sum_{h=1}^{k} \overline{q}_h \widehat{u}_h \geq \frac{1}{m} \sum_{j=1}^{m} \sum_{h=1}^{k} p_{i_j(\widehat{u})h} \widehat{u}_h, \text{ for } i = 1, ..., n,$$

which is not possible. This concludes the proof of the first implication.

Let us now prove the reverse implication. Suppose that Statement 1 of the Theorem holds and pick any $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$. We must show, using again the Hardy-Littlewood-Polya theorem, that the list of n numbers

$$\left(\sum_{h=1}^{k} q_{1h} u_h, ..., \sum_{h=1}^{k} q_{nh} u_h \right)$$

Lorenz dominates the list of n numbers

$$\left(\sum_{h=1}^{k} p_{1h} u_h, ..., \sum_{h=1}^{k} p_{nh} u_h \right).$$

Without loss of generality (since the ranking of societies induced $\succsim_Z^{QO}$ is anonymous), we can write the indices of the rows of the two matrices $\mathbf{q}$ and $\mathbf{p}$ in such a way that the two lists are increasingly ordered so that

$$\sum_{h=1}^{k} q_{1h} u_h \leq ... \leq \sum_{h=1}^{k} q_{nh} u_h \text{ and } \sum_{h=1}^{k} p_{1h} u_h \leq ... \leq \sum_{h=1}^{k} p_{nh} u_h.$$

Hence, we need to show that for any $n_0 \leq n - 1$,

$$\sum_{i=1}^{n_0} \sum_{h=1}^{k} p_{1h} u_h \leq \sum_{i=1}^{n_0} \sum_{h=1}^{k} q_{1h} u_h$$

Since statement 1 of the theorem holds, we know that there exists $v \in \mathcal{U}_*^{\geq_{\mathbf{QO}}}$ and $\theta_1, ..., \theta_n \in [0, 1]$ which can be in such a way that $\sum_{i=1}^{n} \theta_i = n_0 \leq n$ such that:

$$\sum_{i=1}^{n_0} q_i = \sum_{l=1}^{n} \theta_l p_l + v$$

It thus follows that:

$$\begin{aligned}
\sum_{i=1}^{n_0}\sum_{h=1}^{k} q_{1h}u_h &= \sum_{j=1}^{n}\theta_h\sum_{h=1}^{k} p_{jh}u_h + \sum_{h=1}^{k} v_h u_h \\
&\geq \sum_{j=1}^{n}\theta_h\sum_{h=1}^{k} p_{jh}u_h \text{ (since } v \in \mathcal{U}_*^{\geq QO}\text{)} \\
&\geq \sum_{j=1}^{n_0}\theta_j\sum_{h=1}^{k} p_{jh}u_h + \sum_{g=n_0+1}^{n}\theta_g\sum_{h=1}^{k} p_{n_0 h}u_h \text{ (rows are ordered)} \\
&= \sum_{j=1}^{n_0}\theta_j\sum_{h=1}^{k} p_{jh}u_h + \sum_{j=1}^{n_0}[1-\theta_j]\sum_{h=1}^{k} p_{n_0 h}u_h \text{ (since } \sum_{j=1}^{n}\theta_i = n_0\text{)} \\
&\geq \sum_{j=1}^{n_0}\theta_j\sum_{h=1}^{k} p_{jh}u_h
\end{aligned}$$

as required. ■

Proof of Proposition 3. The fact that

$$\mathcal{U}_*^{\geq QO} \subseteq \left\{ (v_1, ..., v_k) \in \mathbb{R}^k : \sum_{j=1}^{k} v_j u_j \geq 0 \ \forall (u_1, ..., u_k) \in \mathcal{U}^{\geq QO} \cap \{0,1\}^k \right\}$$

directly follows from the fact that $\mathcal{U}^{\geq QO} \cap \{0,1\}^k \subset \mathcal{U}^{\geq QO}$.

We now prove the reverse inclusion. Consider any $(v_1, ..., v_k)$ satisfying $\sum_{h=1}^{k} v_h = 0$ and $\sum_{j=1}^{k} v_j u_j \geq 0$ for all $(u_1, ..., u_k) \in \mathcal{U}^{\geq QO} \cap \{0,1\}^k$. We must show that it satisfies also $\sum_{j=1}^{k} v_j u_j \geq 0$ for any $(u_1, ..., u_k) \in \mathcal{U}^{\geq QO}$. Consider therefore any such $(u_1, ..., u_k) \in \mathcal{U}^{\geq QO}$. By continuity of the map $(u_1, ..., u_k) \mapsto \sum_{j=1}^{k} v_j u_j$, we may assume without loss of generality that $u_h \neq u_i$ for any two distinct h and i in $\{1, ..., k\}$. Let $j : \{1, ..., k\} \longrightarrow \{1, ..., k\}$ be a one-to-one function such that such that $u_{j(1)} < u_{j(2)} < ... < u_{j(k)}$. We have:

$$\sum_{j=1}^{k} v_j u_j = \sum_{h=1}^{k} v_{j(h)} u_{j(h)} = \sum_{h=2}^{k} v_{j(h)}(u_{j(h)} - u_{j(1)}) \ , \tag{13}$$

since $\sum_{h=1}^{k} v_h = 0$. Using Abel decomposition formula, one can alternatively write this equality as:

$$\sum_{j=1}^{k} v_j u_j = \sum_{h=2}^{k} (u_{j(h)} - u_{j(h-1)}) \sum_{g=h}^{k} v_{j(g)}$$

Now, for any $h = 2, ..., k$, let $w^h \in \{0,1\}^k$ be defined by:

$$\begin{aligned} w^h_{j(g)} &:= 0 \text{ if } g < h \text{ and,} \\ &:= 1 \text{ if } g \geq h \end{aligned}$$

We observe that, for any $h \in \{2, ..., k\}$, $(w^h_{j(1)}, ..., w^h_{j(k)}) \in \mathcal{U}^{\geq_{QO}}$. Indeed, if $l >^{QO} g$ for two distinct outcomes g and l in $\{1, ..., k\}$, then $u_l > u_g$ by definition of $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$. Given h, three cases are possible:
(i) $l < h$. In this case, one has $w^h_{j(g)} = 0 = w^h_{j(l)}$ from the definition of w^h.
(ii) $g < h \leq l$. In this case, $w^h_{j(g)} = 0 < 1 = w^h_{j(l)}$ holds from the definition of w^h and the required weak inequality $w^h_{j(g)} \leq w^h_{j(l)}$ is also satisfied.
(iii) $h \leq g < l$. In this case $w^h_{j(g)} = 1 = w^h_{j(l)}$ holds from the definition of w^h.
Hence, in all the three cases, the required weak inequality $w^h_{j(g)} \leq w^h_{j(l)}$ is satisfied. Since $(w^h_{j(1)}, ..., w^h_{j(k)}) \in \mathcal{U}^{\geq} \cap \{0, 1\}^k$ for any $h = 2, ..., k$, we have $\sum_{g=1}^k v_{j(g)} w^h_{j(g)} = \sum_{g=h}^k v_{j(g)} \geq 0$ for any such h. But this implies that $\sum_{h=2}^k v_{j(h)}(u_{j(h)} - u_{j(1)}) \geq 0$ for any such h which, thanks to Equality (13), establishes the result. ■

Proof of Remark 2. Suppose that $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ and, as a result, that $Z(\mathbf{q}) + \mathcal{U}_*^{\geq_{QO}} \subset Z(\mathbf{p}) + \mathcal{U}_*^{\geq_{QO}}$. Since in particular $\sum_{i=1}^n q_i \in Z(\mathbf{q}) + \mathcal{U}_*^{\geq_{QO}}$, there is a collection of n numbers $\theta_1, ..., \theta_n$ in the $[0, 1]$ interval and a vector $v \in \mathcal{U}_*^{\geq_{QO}}$ such that:

$$\sum_{i=1}^n q_i = \sum_{i=1}^n \theta_i p_i + v.$$

or, writing this equality for outcome j :

$$\sum_{i=1}^n q_{ij} = \sum_{i=1}^n \theta_i p_{ij} + v_j.$$

Summing over all outcomes, and exploiting the fact that $\sum_{j=1}^k v_j = 0$ (Remark 1) and $\sum_{j=1}^k p_{ij} = \sum_{j=1}^k q_{ij} = 1$ for any i) one has:

$$\sum_{j=1}^n \sum_{i=1}^n q_{ij} = n = \sum_{i=1}^n \theta_i \sum_{j=1}^n p_{ij} + \sum_{j=1}^n v_j = \sum_{i=1}^n \theta_i$$

which implies that $\theta_i = 1$ for all i. Hence:

$$\sum_{i=1}^n q_i = \sum_{i=1}^n p_i + v$$

and

$$\sum_{i=1}^n q_i - \sum_{i=1}^n p_i = v \in \mathcal{U}_*^{\geq_{QO}}$$

as required. ■

Proof of Proposition 4. Observing that the inequality

$$\sum_{i=1}^n \Phi \left(\sum_{h=1}^k q_{ih} u_h \right) \geq \sum_{i=1}^n \Phi \left(\sum_{h=1}^k p_{ih} u_h \right)$$

for all concave Φ and all lists of real numbers $u_1, ..., u_k$ is equivalent, thanks to the Hardy-Littlewood-Polya theorem (see for example Berge (1959) p. 191), to the requirement that the list of n numbers $\left(\sum_{h=1}^{k} q_{1h}u_h, ..., \sum_{h=1}^{k} q_{nh}u_h\right)$ Lorenz dominates the list of n numbers $\left(\sum_{h=1}^{k} p_{1h}u_h, ..., \sum_{h=1}^{k} p_{nh}u_h\right)$ for all list of real numbers $u_1, ..., u_k$. This latter requirement is in turn equivalent to the requirement that the matrix $\mathbf{q}$ price majorizes (using Kolm (1977) terminology) the matrix $\mathbf{p}$ for all "price" vectors $(u_1, ..., u_k)$. Koshevoy (1995) (Theorem 1) proves that the fact for a matrix $\mathbf{q} \in \mathbb{R}^{nd}$ to price majorize a matrix $\mathbf{p} \in \mathbb{R}^{nd}$ is equivalent to observing:

$$\begin{aligned}
\overline{Z}(\mathbf{q}) &= \left\{ \mathbf{z} \in \mathbb{R}^{k+1} : \ \mathbf{z} = \sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, q_{i1}, ..., q_{ik}\right), \ \theta_i \in [0,1] \ \forall i = 1, \dots, n \right\} \\
&\subseteq \left\{ \mathbf{z} \in \mathbb{R}^{k+1} : \ \mathbf{z} = \sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, p_{i1}, ..., p_{ik}\right), \ \theta_i \in [0,1] \ \forall i = 1, \dots, n \right\} \\
&= \overline{Z}(\mathbf{p})
\end{aligned}$$

Observe that the set $\overline{Z}(\mathbf{a})$ (for any matrix $\mathbf{a} \in \mathbb{R}^{nd}$) defined in Koshevoy (1995) is somewhat similar to the set defined in Equation 6 above, with the exception that it takes the Minkowski sums over the "population share extended" vectors $(1/n, p_{i1}, ..., p_{ik})$ rather than over the vectors $(p_{i1}, ..., p_{ik})$ themselves. Hence we only need to prove that $\overline{Z}(\mathbf{q}) \subseteq \overline{Z}(\mathbf{p})$ is equivalent to $Z(\mathbf{q}) \subseteq Z(\mathbf{p})$ to complete the argument. The fact that $\overline{Z}(\mathbf{q}) \subseteq \overline{Z}(\mathbf{p})$ implies $Z(\mathbf{q}) \subseteq Z(\mathbf{p})$ is obvious. To establish the other direction assume that $Z(\mathbf{q}) \subseteq Z(\mathbf{p})$. This means that for any list of numbers $\theta_1, ..., \theta_n$ in the $[0,1]$ interval, one can find a list of numbers $\theta'_1, ..., \theta'_n$ in the $[0,1]$ interval such that

$$\sum_{i=1}^{n} \theta_i q_i = \sum_{i=1}^{n} \theta'_i p_i$$

Observe that this equality implies that for any $j = 1, ..., k$ one has:

$$\sum_{i=1}^{n} \theta_i q_{ij} = \sum_{i=1}^{n} \theta'_i p_{ij}$$

Summing these equalities over all j yields (exploiting the fact that the probability distributions lie in Δ^{k-1}):

$$\sum_{i=1}^{n} \theta_i \sum_{j=1}^{k} q_{ij} = \sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} \theta'_i \sum_{j=1}^{k} q_{ij} = \sum_{i=1}^{n} \theta'_i$$

But this implies that for any for any list of numbers $\theta_1, ..., \theta_n$ in the $[0,1]$ interval, one can find a list of numbers $\theta'_1, ..., \theta'_n$ in the $[0,1]$ interval such that;

$$\sum_{i=1}^{n} \theta_i \left(\frac{1}{n}, q_{i1}, ..., q_{ik}\right) = \sum_{i=1}^{n} \theta'_i \left(\frac{1}{n}, p_{i1}, ..., p_{ik}\right)$$

That is, this implies that $\overline{Z}(\mathbf{q}) \subseteq \overline{Z}(\mathbf{p})$ holds, as required. ■

Proof of Lemma 1. For uniform averaging, we simply observe that the function $\Psi : \Delta^{k-1} \to \mathbb{R}$ defined, for every $(s_1, ..., s_k) \in \Delta^{k-1}$ by:

$$\Psi(s_1, ..., s_k) = \Phi\left(\sum_{h=1}^{k} s_h u_h\right)$$

is concave if Φ is concave irrespective of what the real numbers $(u_1, ..., u_k)$ are. Hence, by virtue of Theorem 3 in Kolm (1977),

$$\sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} q_{ih} u_h\right) \geq \sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} p_{ih} u_h\right)$$

if there exists a bistochastic matrix $n \times n$ bistochastic matrix $\mathbf{b}$ such that $\mathbf{q} = \mathbf{b}.\mathbf{p}$.
Assume now that $(u_1, ..., u_k) \in \mathcal{U}^{\geq_{QO}}$ and that $\mathbf{q}$ results from from $\mathbf{p}$ through a favorable transfer as per Definition 6. We must show that

$$\sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} q_{ih} u_h\right) \geq \sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} p_{ih} u_h\right).$$

Since all rows others than i_1 and i_2 in the matrix $\mathbf{p}$ and others than i'_1 and i'_2 in the matrix $\mathbf{q}$ are unaffected by the change, we have:

$$\sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} q_{ih} u_h\right) \geq \sum_{i=1}^{n} \Phi\left(\sum_{h=1}^{k} p_{ih} u_h\right)$$

$$\Longleftrightarrow$$

$$\Phi\left(\sum_{h=1}^{k} q_{i',h} u_h\right) + \Phi\left(\sum_{h=1}^{k} q_{i'_2 h} u_h\right) \geq \Phi\left(\sum_{h=1}^{k} p_{i,h} u_h\right) + \Phi\left(\sum_{h=1}^{k} p_{i_2 h} u_h\right) \quad (14)$$

We now observe that the vector $\left(\sum_{h=1}^{k} q_{i',h} u_h, \sum_{h=1}^{k} q_{i'_2 h} u_h\right)$ Lorenz-dominates the vector $\left(\sum_{h=1}^{k} p_{i,h} u_h), \sum_{h=1}^{k} p_{i_2 h} u_h\right)$. Indeed, one has:

$$\sum_{h=1}^{k} p_{i,h} u_h \leq \sum_{h=1}^{k} p_{i_2 h} u_h - \sum_{h=1}^{k} v_h u_h = \sum_{h=1}^{k} q_{i'_2 h} u_h \leq \sum_{h=1}^{k} p_{i_2 h} u_h$$

and:

$$\sum_{h=1}^{k} p_{i,h} u_h \leq \sum_{h=1}^{k} p_{i_1 h} u_h + \sum_{h=1}^{k} v_h u_h = \sum_{h=1}^{k} q_{i'_1 h} u_h \leq \sum_{h=1}^{k} p_{i_2 h} u_h$$

Inequality (14) then follows from the Hardy-Littlewood-Polya Theorem. ■

Proof of Theorem 2. Using the reasoning in the proof of Proposition 2, one can observe that if $n(\mathbf{p}) = n(\mathbf{q}) = 2$, the statement $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ is equivalent to the requirement that $\overline{q} - \overline{p} \in \mathcal{U}_*^{\geq_{QO}}$ and that there exist θ_1 and $\theta_2 \in [0, 1]$ such

that $q_1 - (\theta_1 p_1 + (1-\theta_1)p_2) \in \mathcal{U}_*^{\geq QO}$ and $q_2 - (\theta_2 p_1 + (1-\theta_2)p_2)$. Since these θ_1 and θ_2 may belong respectively to Λ_1 and Λ_2, this establishes one direction of the implication.
For the other direction, it is sufficient to prove that the statement $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ implies the existence of $\lambda_1 \in \Lambda_1$ such that $q_1 - (\lambda_1 p_1 + (1-\lambda_1)p_2) \in \mathcal{U}_*^{\geq QO}$ (the argument being similar for λ_2). If $q_1 - p_1 \in \mathcal{U}_*^{\geq QO}$, then one selects $\lambda_1 = 1 \in \Lambda_1$ and the proof is over. If $q_1 - p_1 \notin \mathcal{U}_*^{\geq QO}$, then we know that since $q_1 - (\theta_1 p_1 + (1-\theta_1)p_2) \in \mathcal{U}_*^{\geq QO}$ for some $\theta_1 \in [0,1]$, there exists some $v_1 \in \mathcal{U}_*^{\geq QO}$ such that:

$$q_1 = \theta_1 p_1 + (1-\theta_1)p_2 + v_1$$

Let $\mathbf{D}(q_1)$ denote the (compact) set of distributions of opportunities that are weakly dominated by q_1, with respect to the quasi-order, defined by:

$$\mathbf{D}(q_1) = \{x \in \Delta^{k-1} : q_1 - x \in \mathcal{U}_*^{\geq QO}\}$$

Consider the continuous map $x : [0,1] \to [0,1]$ defined by:

$$x(t) = tp_1 + (1-t)p_2$$

Since $q_1 - p_1 \notin \mathcal{U}_*^{\geq QO}$ one has that $x(1) \notin \mathbf{D}(q_1)$ while $x(\theta_1) \in \mathbf{D}(q_1)$. Let $\overline{\theta}_1$ be defined by:

$$\overline{\theta}_1 = \max\{t \geq \theta_1 : x(t) \in \mathbf{D}(q_1)\} \tag{15}$$

We then have $\overline{\theta}_1 \in [\theta_1, 1[$ and $x(\overline{\theta}_1) \in \mathbf{D}(q_1)$. We therefore have:

$$q_1 = \overline{\theta}_1 p_1 + (1-\overline{\theta}_1)p_2 + \overline{v}_1$$

for some $\overline{v}_1 \in \mathcal{U}_*^{\geq QO}$. Also observe that $\overline{v}_1$ must be such that $\sum_{j \in J} \overline{v}_{1j} = 0$ for some $J \in \mathcal{F}^{\geq QO}$. Indeed, using Expression (7), assuming that $\sum_{j \in J} \overline{v}_{1j} > 0$ for all $J \in \mathcal{F}^{\geq QO}$ would imply the possibility of increasing a bit the t above $\overline{\theta}_1$ while maintaining $x(t)$ in the set $\mathbf{D}(q_1)$ in the maximization described by Expression (15), and will therefore be contradictory. Hence for the set J where $\sum_{j \in J} \overline{v}_{1j} = 0$, one has $q_1(J) = \overline{\theta}_1 p_1(J) + (1 - \overline{\theta}_1 p_2(J)$ and this completes the proof. ■

Proof of Remark 4. Let the transformed society $\mathbf{p}'$ be defined by $p_1' = p_1 + w_1$ and $p_2' = p_2 + w_2$ for some $w_1, w_2 \in \mathcal{U}_*^{\geq C}$. We claim that if $\mathbf{q} \succsim_Z^C \mathbf{p}'$ then $w_1 + w_2 = 0$. Suppose indeed that:

$$q_1 - (\theta_1 p_1' + (1-\theta_1)p_2') \in \mathcal{U}_*^{\geq C}, \; q_2 - (\theta_2 p_1' + (1-\theta_2)p_2') \in \mathcal{U}_*^{\geq C}$$

and

$$q_2 + q_1 - (p_1' + p_2') \in \mathcal{U}_*^{\geq QO_C}.$$

Then it follows that $\theta_1 = 1$, as we have seen in the argument that we just made about the impossibility of performing a uniform averaging. This implies that $q_1 - p_1 - w_1 \in \mathcal{U}_*^{\geq C}$, that is $\frac{1}{36}(0,-2,2,0) - w_1 \in \mathcal{U}_*^{\geq C}$. Secondly $q_2 - (\theta_2 p_1 + (1-\theta_2)p_2) = \frac{1}{36}(-3\theta_2, 2-\theta_2, -3+6\theta_2, 1-2\theta_2)$. This vector belongs to $\mathcal{U}_*^{\geq C}$ if and only if $\theta_2 = 1/2$ and it is then equal to $\frac{1}{72}(-3,3,0,0)$. To sum up we have:

$$\frac{1}{36}(0,-2,2,0) - w_1 \in \mathcal{U}_*^{\geq C}, \ \ \frac{1}{72}(-3,3,0,0) - \frac{1}{2}(w_1 + w_2) \in \mathcal{U}_*^{\geq C}$$

and

$$\frac{1}{36}(0,0,-1,1) - (w_1 + w_2) \in \mathcal{U}_*^{\geq C}.$$

Now $w_1 + w_2 = (a,b,c,d)$ is by assumption an element of $\mathcal{U}_*^{\geq C}$. The condition $\frac{1}{36}(-3,3,0,0) - (w_1 + w_2) \in \mathcal{U}_*^{\geq C}$ implies that $c = d = 0$. On the other hand the condition $\frac{1}{36}(0,0,-1,1) - (w_1 + w_2) \in \mathcal{U}_*^{\geq C}$ implies that $a = b = 0$. Thus $w_1 + w_2 = 0$ and, actually, $w_1 = w_2 = 0$. ■

Proof of Theorem 3. The fact that Statement 1 implies Statement 2 has been proved (for any number of groups) by Lemma 1 while the implication of Statement 3 by Statement 1 has been established by Theorem 1. We therefore only need to prove that Statement 3 implies Statement 1. Suppose therefore that $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$.

First consider the case where $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$.[17] Then

$$Z(\mathbf{p}) + \mathcal{U}_*^{\geq QO} \subseteq \{\theta p_1 + v : \ \theta \in [0,2], \ v \in \mathcal{U}_*^{\geq QO}\}.$$

Since $\mathbf{q} \succsim_Z^{QO} \mathbf{p}$ we have:

$$q_1 = \theta_1 p_1 + v_1; \ \ q_2 = \theta_2 p_1 + v_2,$$

where $\theta_1, \theta_2 \in [0,2]$ and $v_1, v_2 \in \mathcal{U}_*^{\geq QO}$. Now, q_1 and q_2 being both in Δ^{k-1} and v_1 and v_2 having both their components summing to zero, we must have $\theta_1 = \theta_2 = 1$. As a result $q_1 = p_1 + v_1$ and $q_2 = p_1 + v_2$. Since $p_1 + p_2 = q_1 + q_2$ we have $p_2 = p_1 + v_1 + v_2$. Hence:

$$q_1 = p_1 + v_1, \ q_2 = p_2 - v_1 \text{ and } p_2 - p_1 - v_1 = v_2 \in \mathcal{U}_*^{\geq QO}$$

which means that $\mathbf{q}$ has been obtained from $\mathbf{p}$ through a favorable transfer.

Consider now the case where neither $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$ nor $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$. Since $Z(\mathbf{q}) + \mathcal{U}_*^{\geq QO} \subseteq Z(\mathbf{p}) + \mathcal{U}_*^{\geq QO}$ and both q_1 and $q_2 \in Z(\mathbf{q}) + \mathcal{U}_*^{\geq QO}$, there are numbers θ_1^1, θ_1^1, θ_1^2 and $\theta_2^2 \in [0,1]$ satisfying $\theta_1^1 + \theta_2^1 = \theta_1^2 + \theta_2^2 = 1$ such that:

$$q_1 = \theta_1^1 p_1 + \theta_2^1 p_2 + v_1 \text{ and } q_2 = \theta_1^2 p_1 + \theta_2^2 p_2 + v_2,$$

for some v_1 and $v_2 \in \mathcal{U}_*^{\geq QO}$. Since $p_1 + p_2 = q_1 + q_2$ we then have:

$$\begin{aligned} v_1 + v_2 &= q_1 - \theta_1^1 p_1 - \theta_2^1 p_2 + q_2 - \theta_1^2 p_1 - \theta_2^2 p_2 \\ &= p_1 + p_2 - \theta_1^1 p_1 - \theta_2^1 p_2 - \theta_1^2 p_1 - \theta_2^2 p_2 \\ &= (1 - \theta_1^1 - \theta_1^2)(p_1 - p_2). \end{aligned} \tag{16}$$

Now, since neither $p_2 - p_1 \in \mathcal{U}_*^{\geq QO}$ nor $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$ while $v_1 + v_2 \in \mathcal{U}_*^{\geq QO}$,

[17] The case where $p_1 - p_2 \in \mathcal{U}_*^{\geq QO}$ is similar.

the only way by which Equality (16) can hold is if $(1-\theta_1^1-\theta_1^2)=0$ and, as a result, $v_1+v_2=0$. Setting in that case $\theta_1=\theta_1^1=\theta_1^2$, we must therefore have:

$$q_1=\theta_1 p_1+(1-\theta_1)p_2;\;\; q_2=(1-\theta_1)p_1+\theta_1 p_2$$

so that $\mathbf{q}=\mathbf{m}.\mathbf{p}$ for the bistochastic matrix $\mathbf{m}=\begin{bmatrix}\theta_1 & 1-\theta_1\\ 1-\theta_1 & \theta_1\end{bmatrix}$. Hence $\mathbf{q}$ can be obtained from $\mathbf{p}$ through a uniform averaging operation in that case. ■