Society for the Study of Economic Inequality

Working Paper Series

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ECINEQ WP 2009 – 134

ECINEQ 2009-134
September 2009

www.ecineq.org

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Abstract

An ethnic polarization index is a summary statistic of ethnic diversity in a population. Montalvo and Reynal-Querol (2005,2008) suggested an index of ethnic polarization, the RQ index, and discussed its properties in detail. In this paper we develop some axiomatic characterizations of the RQ index using axioms taken mostly from the earlier literature. A generalized form of the RQ index is also characterized. Finally, we develop an ethnic polarization ordering for ranking alternative ethnic profiles unambiguously.

Keywords: Ethnic polarization, RQ index, axioms, generalization, characterization.
JEL Classification: C43, D63, D74.

* Chakravarty thanks the Yokohama National University, Japan, for support.

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1. Introduction

Polarization has recently attracted a great deal of attention of researchers for several reasons, particularly, because of its role in the analysis of income distribution evolution, social instability and ethnic conflicts. In their highly interesting contributions, Esteban and Ray (1994) and Wolfson (1994) attempted to provide rigorous definitions of polarization. While the Wolfson index is concerned with dispersion of the distribution of income from the median towards the extreme points, Esteban and Ray(1994) developed an axiomatic characterization of a class of polarization indices based on distances between incomes[1].

However, in many important situations there may not be information on a continuous attribute to measure distance across groups or individuals. For instance, in the case of ethnic diversity, on which ethnic polarization relies, the only available information may be whether a person belongs to a particular ethnic group or not. As argued by Duclos et al. (2004), such a dichotomous identification may be necessary in many interesting situations. In such a case the use of a 0-1 indicator function becomes appropriate to signify whether the person belongs to a specific group. In other words, the distance across groups is measured by a 'discrete metric (1-0)' (Montalvo and Reynal-Querol, 2008, p.1838)[2].

A highly ethnically diversified society may generate tensions in the society which ultimately may lead to conflicts[3]. Social instability arising from ethnic diversification may lead to a low economic growth (Easterly and Levine, 1997), high corruption (Mauro, 1995), low social cohesion in the sense of low participation in groups and associations (Alesina and La Ferrara, 2000) and low contribution to local public goods(Alesina et al.,

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[1] Alternatives and variants of the Esteban-Ray and Wolfson indices have been suggested by many authors, including, Gradin(2000),Wang and Tsui (2000),Chakravarty and Majumder(2001),D'Ambrosio (2001), Zhang and Kanbur(2001), Rodriguez and Salas(2003), Duclos et al.(2004) ,Lasso de la Vega and Urrutia(2006), Chakravarty et al.(2007), Esteban et al.(2007), Silber et al.(2007) , Bossert and Schworm(2008), Chakravarty (2009) , Chakravarty and D'Ámbrosio(2009) and Lasso de la Vega et al. (2009).

[2] In his study of cultural fractionalization, Fearon (2003) defined a resemblance factor that takes on the value 1 if the 'two groups speak exactly the same language' and its value is assumed to be zero if the 'two groups languages' come from different families'.

[3] For discussion on the saliency of ethnic conflicts, see Brubaker and Laitin(1998), Fearon and Laitin (2000) and Esteban and Ray (2008a,2009).

1999)[4]. A natural objective of the society should, therefore, be to make ethnic diversity (hence ethnic polarization) as low as possible.

Montalvo and Reynal-Querol (2005) suggested an index of ethnic polarization, which they refer to as the RQ index. They demonstrated empirically that the RQ index can be taken as a significant causal factor for the incidence of civil wars. This makes the RQ index quite attractive from a practical view point.[5]

A discussion on the theoretical properties of a class of discrete polarization indices based on classifications instead of continuous distances has been presented in Montalvo and Reynal-Querol (2008). The Esteban-Ray index forms the basis of this general class. Montalvo and Reynal-Querol (2008) demonstrated rigorously how the choice of a discrete polarization index can be narrowed down to the RQ index under suitable choice of intuitively reasonable postulates.

The objective of this paper is to characterize the RQ index using alternative sets of axioms. Most of our axioms are borrowed from Montalvo and Reynal-Querol (2005, 2008). These characterizations enable us to understand the RQ index in greater detail. None of our first three characterization results begins with assumption of any specific structure, e.g., additivity. From this perspective these results are quite general. More precisely, our characterization reveals how within a general structure we can isolate a set of necessary and sufficient conditions for identifying the RQ index uniquely. In the process we characterize a generalization of the RQ index, which we refer to as the *'Generalized RQ-Index of order* θ*'*. A fourth axiomatization of the RQ index using the additive structure is also developed.

Finally, we develop an ethnic polarization ordering which says under what necessary and sufficient conditions one ethnic group can be regarded as more or less polarized than another by all ethnic polarization indices that satisfy certain desirable criteria. An attractive feature of this ordering is that it is 'consistent' with some properties of an ethnic polarization index considered by Montalvo and Reynal-Querol (2008)(see also Esteban and Ray, 1994).

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[4] See Alesina and La Ferrara (2005) for an extensive discussion.

[5] It may be mentioned that some recent studies on the explanatory variables of genocides and civil wars did not find evidence of the effect of ethnic fractionalization (see, for example, Harff, 2003).

The paper is organized as follows. After discussing the background material in Section 2, we present the characterization theorems in Section 3. The ethnic polarization ordering is presented in Section 4.Finally, Section 5 concludes.

2. The Background

For a population comprising of n ethnic groups E_1 , E_2 ,...., E_n for some $n \in \Gamma = N \setminus \{1,2\}$,where N is the set of natural numbers, let π_i denote the relative frequency of E_i . Consequently, $0 \le \pi_i \le 1$, $1 \le i \le n$ and $\sum_{i=1}^{n} \pi_i = 1$; $n \in \Gamma$ being arbitrary. (The assumption $n \ge 3$ will be necessary for stating some axioms. Therefore, unless specified, throughout the paper we shall assume that $n \in \Gamma$.) This gives rise to a probability distribution $\underline{\pi} = (\pi_1, \pi_2, ..., \pi_n)$, which will be referred to as 'ethnic distribution' occasionally.

We begin with a discussion on the Esteban-Ray index of polarization defined as

$$ER = \mu \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i^{1+\alpha} \pi_j \left| y_i - y_j \right|, \tag{1}$$

where y_i is the representative income, defined in an unambiguous way, of group i , $\mu > 0$ is a constant and $\alpha \in (0, \alpha^*]$, with $\alpha^* \approx 1.6$. For $\alpha = 1$, ER corresponds to the well-known Gini index of inequality. The greater is the value of α , the greater is the divergence from inequality. Hence α may be interpreted as a polarization sensitivity parameter.

For identifying an individual with respect to his ethnicity, it is necessary to check if he ‘belongs to’ or ‘does not belong to’ a particular ethnic group. As noted by Montalvo and Reynal-Querol (2005, 2008), in such a case we should replace the Euclidean metric $d(y_i, y_j) = \left| y_i - y_j \right|$ in (1) by the discrete metric defined as $\delta(y_i, y_j) = 1 \, if \; i \ne j$, $\delta(y_i, y_j) = 0$, otherwise. This gives rise to the class of discrete polarization indices given by

$$DP(\alpha, \mu) = \mu \sum_{i=1}^{n} \sum_{j \ne i} \pi_i^{1+\alpha} \pi_j \; . \tag{2}$$

In order to relate $DP(\alpha,\mu)$ with the RQ index, Montalvo and Reynal-Querol (2008) considered three properties, which are borrowed from Esteban and Ray (1994) and redefined in terms of group's size only. We will also use these properties for our characterizations. The first property is:

Property 1: If there are three groups of sizes *p*, *q* and *r*, and $p > q$ and $q \geq r$, then if we merge the two smallest groups into a new group, $\tilde{q}$, the new distribution is more polarized than the original one. That is, $POL(p,q,r) < POL(p,\tilde{q})$, with $\tilde{q} = (q+r)$.

This property says that of the three groups with relative frequencies *p*, *q* and *r*, if the two smaller groups are merged, then polarization should increase. It corresponds to Axioms 1 and 2 of Esteban and Ray (1994). The next property, which is based on Axiom 3 of Esteban and Ray (1994), demands that, if there are three groups, two of which are of equal size, then polarization should increase under shift of probability mass from the group with unequal size equally to the other two groups. Formally,

Property 2: Assume that there are three groups of sizes *p*, *q* and *p*. Then if we shift mass from the *q* group equally to the other two groups, polarization increases. That is, $POL(p,q,p) < POL(p+x, q-2x, p+x)$.

Montalvo and Reynal-Querol (2008) demonstrated that $DP(\alpha,\mu)$ satisfies Property 1 if and only if $\alpha \geq 1$. The same boundary condition on the value of α can be obtained if we replace Property 1 by Property 1b, whose formulation does not depend upon the assumption that the number of groups is three.

Property1b: Suppose that there are two groups with sizes π_1 and π_2. Take any one group, say π_2 and split it into $m \geq 2$ groups in such a way that $\pi_1 = \tilde{\pi}_1 \geq \tilde{\pi}_i$ for all $i = 2,\ldots,(m+1)$, where $\tilde{\pi}$ is the new vector of population sizes, and clearly $\sum_{i=2}^{m+1} \tilde{\pi}_i = \pi_2$.

Then the polarization under $\tilde{\pi}$ is smaller than that under π.

However, in our present treatise we will make use of a minor variant of Property 1b, which we propose to call Property 1a:

Property1a: Suppose that there are two groups with sizes π_1 and π_2. Take any one group, say π_2 and split it into $m \geq 2$ groups in such a way that $\pi_1 = \tilde{\pi}_1 \geq \tilde{\pi}_i$ for all $i=2,...,(m+1)$, *with strict inequality for at least one i,* where $\tilde{\pi}$ is the new vector of population sizes, and clearly $\sum_{i=2}^{m+1} \tilde{\pi}_i = \pi_2$. Then the polarization under $\tilde{\pi}$ is smaller than that under π.

Montalvo and Reynal-Querol (2008) also showed that the only $DP(\alpha,\mu)$ index that satisfies Property 2 for any distribution is the one with $\alpha=1$. If we fix $\alpha=1$ and choose $\mu=4$, then the resulting index $DP(1,4)$ becomes the RQ index given by

$$RQ=\sum_{i=1}^{n}\sum_{j\neq i}\pi_i^2\pi_j=4\sum_{i=1}^{n}\pi_i^2\left(1-\pi_i\right). \tag{3}$$

If there are only two groups, then RQ equals twice the Ethno-linguistic Fractionalization Index (*FRAC*) which is defined as

$$FRAC=\left(1-\sum_{i=1}^{n}\pi_i^{\,2}\right). \tag{4}$$

However, the equality relationship breaks down if we consider more than two groups. The fractionalization index has a simple interpretation of being the probability that two persons selected at random from the population do not belong to the same ethnic group. Vigdor (2002) considered a model of differential altruism and used *FRAC* to show that estimated fragmentation effects can be regarded as weighted average of within-group affinity in the population.

3. The Characterization Theorems

The objective of this section is to characterize the RQ index using alternative sets of axioms. Let Δ_n denote the set of all discrete probability distributions of dimension *n* on *R*, the real line, and $\mathbf{\Delta}$ be the set of all probability distributions on *R*. Obviously, $\Delta=\bigcup_{n\in\Gamma}\Delta_n$.

We begin with the following general definition.

Definition 1: An 'Ethnic Polarization Index' (EPI) is a real-valued function defined on Δ, that is, $P:\Delta\rightarrow R$.

For any $\underline{\pi} \in \Delta$, an EPI simply aggregates its components in an unambiguous way. Given $\underline{\pi} \in \Delta$, the real number $P(\underline{\pi})$ indicates the level of ethnic polarization associated with $\underline{\pi} \in \Delta$.

The following axioms, which have been discussed by Montalvo and Reynal-Querol (2005, 2008) and are satisfied by the RQ index, will also be necessary for our characterizations (see also Esteban and Ray, 1994).

Axiom 1: For all $n \in \Gamma$, $\underline{\pi} \in \Delta_n$, $0 \le P\left(\underline{\pi}\right) \le 1$.

Axiom 2: For all $n \in \Gamma$, $P(\underline{\pi})$ =0 if and only if $\underline{\pi} \in \Delta_n$ is some permutation of (1, 0,…, 0).

Axiom 3: For all $n \in \Gamma$, $P(\underline{\pi})$ =1 if and only if $\underline{\pi} \in \Delta_n$ is some permutation of $\left(\frac{1}{2}, \frac{1}{2}, 0, ..., 0\right)$.

Axiom 1 is a boundedness principle. It says that the EPI is bounded between 0 and 1.The next axiom, which can be referred as a perfect homogeneity principle, says that the EPI achieves its minimum value, zero, if and only if there is complete homogeneity in the sense that all the individuals belong to a particular ethnic group. Finally, axiom 3 is a perfect bipolarity condition, which says that the EPI is maximized if and only if there is an equal splitting of the entire population into two groups. Given the existence of a large ethnic group, if the ethnic minority is not divided into many groups and is large as well, then chances of ethnic conflicts increase (Horowitz, 1985). Since ethnic conflicts are likely to increase with ethnic polarization, it is sensible to assume that ethnic polarization is maximized in the case of a bipolar ethnic distribution (see also Esteban and Ray, 1999,2008b)[6]. Thus, an EPI is an indicator of divergence of the actual ethnic distribution from the extreme distribution $\left(\frac{1}{2},\frac{1}{2},0,....,0\right)$.

The next three axioms impose some minimal conditions on an EPI and are satisfied by the RQ index. In fact they are satisfied by FRAC also.

Axiom 4: For all $n \in \Gamma$, P is continuous on Δ_n.

Axiom 5: For all $n \in \Gamma$, $\underline{\pi} \in \Delta_n$, $P(\underline{\pi}) = P\left(\sigma\left(\underline{\pi}\right)\right)$, where $\sigma\left(\underline{\pi}\right)$ is a permutation of $\underline{\pi}$.

Axiom 6: For all $n \in \Gamma$, $\underline{\pi} \in \Delta$, $P\left(\pi_1,\pi_2,....,\pi_n\right) = P\left(\pi_1,\pi_2,.....,\pi_n,0\right)$.

Axiom 4, which demands continuity of an EPI, ensures that minor changes in π_i's will generate only minor changes in P. Thus, a continuous EPI will not be oversensitive to minor observational errors on π_i's. Axiom 5 is an anonymity or symmetry principle. It means that P remains invariant under any reordering of π_i's. Thus, all characteristics other than π_i's, for example, the names of the individuals belonging to different ethnic groups, are irrelevant to the measurement of ethnic polarization. Given the ethnic groups $E_1, E_2, ..., E_n$ and their relative frequencies , if a new ethnic group is created with zero frequency, then this should not have any impact on the level of polarization. In other words, we say that an EPI satisfies zero frequency independence. Axiom 6 specifies this. This axiom seems quite reasonable if we assume that one aspect of an EPI is dominance of large ethnic groups.

While for our characterizations we are going to use Properties 1, 1a, 2 and axioms 1-6, one additional postulate we wish to use is multiplicative decomposability. The motivation of this axiom relies on a property of the RQ index.

Let $\pi^{(i)} = \left(\pi_1,\pi_2,....,\pi_i,0,...,0\right) \in \Delta_n$, where $1 \le i < n$. For $0 \le \varepsilon \le \pi_i$, define $\underline{\pi}_\varepsilon^{(i)} = \left(\pi_1,\pi_2,....,\pi_i-\varepsilon,\varepsilon,0,...,0\right)$. Thus, $\underline{\pi}_\varepsilon^{(i)}$ is obtained from $\underline{\pi}^{(i)}$ by splitting the

[6] Esteban and Ray (1994) stipulated that a polarization index should achieve its maximum value for the distribution $\left(\frac{1}{2},\frac{1}{2},0,....,0\right)$.

population coming from E_i into two subpopulations, say E_i^1 and E_i^2 with respective masses $\pi_i - \varepsilon$ and ε. Then

$$\frac{1}{4}\left[RQ\left(\underline{\pi}_\varepsilon^{(i)}\right) - RQ\left(\underline{\pi}^{(i)}\right)\right]$$

$$=\left(\pi_i-\varepsilon\right)^2\left(1-\pi_i+\varepsilon\right)+\varepsilon^2\left(1-\varepsilon\right)-\pi_i^{\,2}\left(1-\pi_i\right)$$

$$=\left(\pi_i^2-2\pi_i\varepsilon+\varepsilon^2\right)\left(1-\pi_i+\varepsilon\right)+\varepsilon^2-\varepsilon^3-\pi_i^{\,2}\left(1-\pi_i\right)$$

$$=\pi_i^2\varepsilon-2\pi_i\,\varepsilon+2\pi_i^2\varepsilon-2\pi_i\,\varepsilon^2+\varepsilon^2-\pi_i\,\varepsilon^2+\varepsilon^3+\varepsilon^2-\varepsilon^3$$

$$=3\pi_i^2\varepsilon-3\pi_i\varepsilon^2-2\pi_i\varepsilon+2\varepsilon^2$$

$$=\varepsilon\left(\pi_i-\varepsilon\right)\left(3\pi_i-2\right). \tag{5}$$

Now, note that the probability that two individuals selected at random from E_i in $\underline{\pi}^{(i)}$ belong to two distinct subgroups E_i^1 and E_i^2 in $\underline{\pi}_\varepsilon{}^{(i)}$ is $2\varepsilon(\pi_i-\varepsilon)/\pi_i{}^2$, provided $\pi_i > 0$. Clearly, this has got some connection with our probabilistic interpretation of *FRAC* mentioned earlier. So we propose to call this expression to be the *"Marginal Fractionalization due to transfer of mass* ε *in* $\underline{\pi}$ *from the* i^{th} *to the* $(i+1)^{th}$ *group"* and denote it by $Fr_{\underline{\pi}}\left(i,\varepsilon\right)$. From (5) it therefore follows that

$$\left\lfloor RQ\left(\underline{\pi}_\varepsilon^{(i)}\right) - RQ\left(\underline{\pi}^{(i)}\right)\right\rfloor = Fr_{\underline{\pi}}\left(i,\varepsilon\right)2\pi_i^2\left(3\pi_i-2\right), \tag{6}$$

whenever $\pi_i > 0$.

Equation (6) motivates us to state the following axiom:

Axiom 7: For all $n\in\Gamma$, let $\underline{\pi}^{(i)}$ and $\underline{\pi}_\varepsilon{}^{(i)}$, $1\le i\le n$, be as above. Then

$$\left\lfloor P\left(\underline{\pi}_\varepsilon^{(i)}\right) - P\left(\underline{\pi}^{(i)}\right)\right\rfloor = Fr_{\underline{\pi}}\left(i,\varepsilon\right) f\left(\pi_i\right), \tag{7}$$

where $Fr_{\underline{\pi}}\left(i,\varepsilon\right)$ is the *"Marginal Fractionalization"*, as defined earlier, $\pi_i > 0$ and $f:\left(0,1\right]\to R$ is continuous.

Axiom 7 is a multiplicative decomposability condition, which expresses the difference between ethnic polarization of two populations into two components, the marginal fractionalization term and a continuous function of the population proportion of the group whose population has been split. The formulation is quite general in the sense that it involves the probability that two individuals selected at random will be from two different subgroups and a function f which is assumed to satisfy continuity on its domain. Clearly, if we assume that f is given by $f\left(\pi_i\right)=2\pi_i^2\left(3\pi_i-2\right)$, then our multiplicative decomposability condition specified in Axiom 7 coincides with (6).

We are now in a position to state and prove our characterization theorems. One common feature of the theorems is that we do not make any structural assumptions, for example, additivity, about the EPI. We begin with the following theorem.

Theorem 1: The only EPI $P:\Delta\rightarrow R$ that satisfies Axioms 1-7 and Property 1 is the RQ index.

The proof of Theorem 1 relies on the following lemma.

Lemma 1: An EPI $P:\Delta\rightarrow R$ satisfies Axioms (1) - (7) if and only if it is of the form

$$RQ_\theta=4\sum_{i=1}^{k}\pi_i{}^2\left(1-\pi_i\right)+\theta\sum_{1\le i_1<i_2<i_3\le k}\pi_{i_1}\pi_{i_2}\pi_{i_3}\ , \tag{8}$$

where $\theta\in[0,3)$ is an arbitrary constant, $k\in\Gamma$ and $\underline{\pi}=\left(\pi_1,\pi_2,...,\pi_k\right)\in\Delta_k$ are arbitrary.

Remark 1: If $\theta=0$, RQ_θ coincides with RQ. Therefore, RQ_θ may be regarded as the *'Generalized RQ-Index of order* θ *'*.

Proof of Lemma 1: Suppose P is an EPI that satisfies Axioms (1) – (7). We rewrite (7) as

$$P\left(\underline{\pi}_\varepsilon^{(i)}\right)-P\left(\underline{\pi}^{(i)}\right)=\varepsilon\left(\pi_i-\varepsilon\right)g\left(\pi_i\right), \tag{9}$$

where $g\left(\pi_i\right)=\ \pi_i{}^2 f\left(\pi_i\right)/2$ and $\pi_i>0$. By Axioms 2 and 3, $P(1,0,...,0)$ =0 and $P(1/2,1/2,0,...,0)$ = 1. So, taking i=1 and $\underline{\pi}^{(1)}$ = $(1,0,...,0)$, $\varepsilon=1/2$, (9) gives $\left(1-0\right)=\frac{1}{2}.\frac{1}{2}.g(1)$,which implies that $g(1)=4$.This in turn shows that

$$P\left(\pi_1,1-\pi_1,0\right)=4\pi_1\left(1-\pi_1\right), \tag{10}$$

where $\pi_1\in[0,1]$ is arbitrary.

Now, fix $\pi_1,\pi_2\in(0,1)$ such that $\left(\pi_1+\pi_2\right)<1$. Then by Axiom 7 and (10) we have $P\left(\pi_1,\pi_2,1-\pi_1-\pi_2\right)=\ P\left(\pi_1,1-\pi_1,0\right)+\pi_2\left(1-\pi_1-\pi_2\right)\ g\left(1-\pi_1\right)$

$$=\ 4\pi_1\left(1-\pi_1\right)+\pi_2\left(1-\pi_1-\pi_2\right)\ g\left(1-\pi_1\right). \tag{11}$$

Now, on the right hand side of (11) use $P(\pi_1, 1-\pi_1, 0) = 4\pi_1(1-\pi_1)$ (by (10)) and $4\pi_1(1-\pi_1)+\pi_2(1-\pi_1-\pi_2)g(1-\pi_1) = 4\pi_2(1-\pi_2) + \pi_1(1-\pi_1-\pi_2)\ g(1-\pi_2)$ (by Axiom 5) and rearrange terms in the resulting expression to get

$$4\{\pi_1(1-\pi_1)-\pi_2(1-\pi_2)\} = (1-\pi_1-\pi_2)\{\pi_1 g(1-\pi_2)-\pi_2 g(1-\pi_1)\}, \quad (12)$$

from which it follows that

$$4(\pi_1-\pi_2)(1-\pi_1-\pi_2) = (1-\pi_1-\pi_2)\{\pi_1 g(1-\pi_2)-\pi_2 g(1-\pi_1)\}. \quad (13)$$

Equation (13), on simplification, becomes

$\{\pi_1 g(1-\pi_2)-\pi_2 g(1-\pi_1)\} = 4(\pi_1-\pi_2)$, from which we get

$$\pi_1\{g(1-\pi_2)-4\} = \pi_2\{g(1-\pi_1)-4\}. \quad (14)$$

Equation (14) implies that

$$\frac{g(1-\pi_1)-4}{\pi_1} = \frac{g(1-\pi_2)-4}{\pi_2}, \quad (15)$$

whenever $\pi_1, \pi_2 \in (0,1)$ and $(\pi_1+\pi_2) < 1$.

Define $h:(0,1)\to R$ by $h(p) = \{g(1-p)-4\}/p$, where $0 < p < 1$. Then from (15) it follows that $h(\pi_1) = h(\pi_2)$ whenever $\pi_1, \pi_2 \in (0,1)$ and $(\pi_1+\pi_2) < 1$. Clearly,

$$h(p) = h(1/2), \quad (16)$$

whenever $0 < p < 1$ and $(p+1/2) < 1$, that is, whenever $0 < p < 1/2$. If $1/2 < p < 1$, then there exists $\delta \in (0,1/2)$ such that $p < (1-\delta)$. So, by (15) we have, $h(p) = h(\delta)$ [$\because (p+\delta) < 1$]. But $h(\delta) = h(1/2)$, by (16). Combining the two results we have $h(p) = h(1/2)$. Thus in all cases, $h(p) = h(1/2) = c$, a constant, whenever $0 < p < 1$. Then $\{g(1-p)-4\} = cp$ for some constant *c*. This implies that $g(x) = \{c(1-x)+4\}$ for some constant *c*, for $0 < x < 1$. By continuity of g this holds for all $x \in [0,1]$.

Now, let us look at the value of $P(\pi_1, \pi_2, ..., \pi_k)$ for k = 3, 4, where $(\pi_1, \pi_2, ..., \pi_k) \in \Delta$. It can be shown that

$P(\pi_1, \pi_2, \pi_3) = 4\sum_{i=1}^{3}\pi_i^2(1-\pi_i) + (c+12)\pi_1\pi_2\pi_3$, where $(\pi_1, \pi_2, \pi_3) \in \Delta$, and

$$P(\pi_1,\pi_2,\pi_3,\pi_4) = 4\sum_{i=1}^{4}\pi_i^2(1-\pi_i)+(c+12)\sum_{1\le i<j<l\le 4}\pi_i\pi_j\pi_l \text{ , where}(\pi_1,\pi_2,\pi_3,\pi_4)\in\Delta .$$

So, as a natural follow-up we conjecture the following hypothesis, which we prove subsequently using induction: for all $k\in\Gamma$ and for all $(\pi_1,\pi_2,....,\pi_k)\in\Delta$, we have,

$$P(\pi_1,\pi_2,....,\pi_k) = 4\sum_{i=1}^{k}\pi_i^2(1-\pi_i)+\theta\sum_{1\le i<j<l\le k}\pi_i\pi_j\pi_l \text{ ,} \tag{17}$$

where $\theta = c+12$.

Assume the validity of the assertion for $k=n\in\Gamma$. Then considering $(\pi_1,\pi_2,....,\pi_{n+1})\in\Delta$, by Axiom 7 we get,

$$P(\pi_1,\pi_2,....,\pi_{n+1})$$

$$= P\left(\pi_1,\pi_2,....,\pi_n,1-\sum_{i=1}^{n}\pi_i\right)$$

$$= P\left(\pi_1,\pi_2,....,\pi_{n-1},1-\sum_{i=1}^{n-1}\pi_i,0\right)+\pi_n\left(1-\sum_{i=1}^{n}\pi_i\right)\left\{c\sum_{i=1}^{n-1}\pi_i+4\right\}\text{, which in view of Axiom 6}$$

becomes

$$P\left(\pi_1,\pi_2,....,\pi_{n-1},1-\sum_{i=1}^{n-1}\pi_i\right)+\pi_n\left(1-\sum_{i=1}^{n}\pi_i\right)\left\{c\sum_{i=1}^{n-1}\pi_i+4\right\}$$

$$= 4\sum_{i=1}^{n-1}\pi_i^2(1-\pi_i) + 4\left(1-\sum_{i=1}^{n-1}\pi_i\right)^2\left(\sum_{i=1}^{n-1}\pi_i\right) + \pi_n\pi_{n+1}\left\{c\sum_{i=1}^{n-1}\pi_i+4\right\} +$$

$$(c+12)\left[\sum_{1\le i<j<l\le(n-1)}\pi_i\pi_j\pi_l+\sum_{1\le i<j\le(n-1)}\pi_i\pi_j(\pi_n+\pi_{n+1})\right] \text{ (by induction hypothesis)}$$

$$= 4\sum_{i=1}^{n-1}\pi_i^2(1-\pi_i) + 4(\pi_n+\pi_{n+1})^2\left(\sum_{i=1}^{n-1}\pi_i\right) + \pi_n\pi_{n+1}\left\{c\sum_{i=1}^{n-1}\pi_i\right\} + 4\pi_n\pi_{n+1} +$$

$$(c+12)\left[\sum_{1\le i<j<l\le(n-1)}\pi_i\pi_j\pi_l+\sum_{1\le i<j\le(n-1)}\pi_i\pi_j(\pi_n+\pi_{n+1})\right]$$

$$= 4\sum_{i=1}^{n-1}\pi_i^2(1-\pi_i)+4\pi_n^2\left(\sum_{i=1}^{n-1}\pi_i+\pi_{n+1}\right)+4\pi_{n+1}^2\left(\sum_{i=1}^{n}\pi_i\right)+8\pi_n\pi_{n+1}\left(\sum_{i=1}^{n-1}\pi_i\right)+\pi_n\pi_{n+1}\left\{c\sum_{i=1}^{n-1}\pi_i\right\}$$

$$-4\pi_n^2\pi_{n+1}-4\pi_{n+1}^2\pi_n+4\pi_n\pi_{n+1}+(c+12)\left[\sum_{1\le i<j<l\le(n-1)}\pi_i\pi_j\pi_l+\sum_{1\le i<j\le(n-1)}\pi_i\pi_j(\pi_n+\pi_{n+1})\right]$$

$$= 4\sum_{i=1}^{n+1} \pi_i^{\,2}\left(1-\pi_i\right) + \left(c+12\right)\left\lfloor \sum_{1 \le i < j < l \le (n+1)} \pi_i \pi_j \pi_l \right\rfloor,$$

since $\left(-\pi_n^{\,2}\pi_{n+1} - \pi_{n+1}^{\,2}\pi_n + \pi_n\pi_{n+1}\right) = \pi_n\pi_{n+1}\left\{\sum_{i=1}^{n-1}\pi_i\right\}$.

This completes the proof of our induction hypothesis. So P must be of the form RQ_θ given by (8).

Now to find the required bounds on θ in (8), we note that for any fixed $n \in \Gamma$, for the vector $\underline{\gamma_n} = \left(\frac{1}{n}, \frac{1}{n}, \ldots., \frac{1}{n}\right) \in \Delta$ (the term $\frac{1}{n}$ within parentheses being repeated n times), $P\left(\underline{\gamma_n}\right) = \left\{4\frac{1}{n^2}\left(1-\frac{1}{n}\right)n\right\} + \left\{\theta\frac{1}{n^3}\binom{n}{3}\right\} = 4\frac{(n-1)}{n^2} + \theta\frac{(n-1)(n-2)}{6n^2}$. By Axiom 1 we have, $0 \le P\left(\underline{\gamma_n}\right) \le 1$, which implies that $0 \le \left\{4\frac{(n-1)}{n^2} + \theta\frac{(n-1)(n-2)}{6n^2}\right\} \le 1$. From the last inequality we get $\left\{4 + \frac{\theta(n-2)}{6}\right\} \ge 0$ and $\frac{\theta(n-2)}{6} \le \left\{\frac{n^2}{(n-1)} - 4\right\}$, from which it follows that $\theta \ge \frac{-24}{(n-2)}$ and $\theta \le \frac{6(n-2)}{(n-1)}$ .This holds for all $n \in \Gamma$. So, we must have, $0 \le \theta \le 3$. But note that the value $\theta = 3$ is not admissible since in that case, $P\left(1/3, 1/3, 1/3, 0, \ldots., 0\right) =$ 1, contrary to Axiom 2 .Thus, $\theta \in [0,3)$. Therefore, we are through with one half of the proposition.

Now, we proceed for a proof of the converse. It is trivial to verify that RQ_θ satisfies Axioms 2,4,5 and 6. Next, to demonstrate that RQ_θ obeys Axiom 7, take $\underline{\pi}^{(i)}$ and $\underline{\pi}_\varepsilon^{(i)}$ as before. Then $RQ_\theta\left(\underline{\pi}_\varepsilon^{(i)}\right) - RQ_\theta\left(\underline{\pi}^{(i)}\right) =$

$$4\varepsilon\left(\pi_i - \varepsilon\right)\left(3\pi_i - 2\right) + \left(c+12\right)\left\lfloor \sum_{1 \le i_1 \le i_2 \le i} \pi_{i_1}\pi_{i_2}\left\{\left(\pi_i - \varepsilon\right) + \varepsilon - \pi_i\right\} + \sum_{1 \le l \le i-1} \pi_l\left\{\left(\pi_i - \varepsilon\right)\varepsilon\right\}\right\rfloor =$$

$4\varepsilon\left(\pi_i - \varepsilon\right)\left(3\pi_i - 2\right) + \left(c+12\right)\varepsilon\left(\pi_i - \varepsilon\right)\left(1-\pi_i\right) = \varepsilon\left(\pi_i - \varepsilon\right)\left(c + 4 - c\pi_i\right)$. Hence Axiom 7 follows.

To prove Axiom 1, observe that the non-negativity of RQ_θ is quite clear. To show that $RQ_\theta \le 1$, we employ the method of Lagrange Multipliers to find the extreme values for RQ_θ at the interior of Δ_k for various values of $k \ge 3$. Note that for a given $(\pi_1, \pi_2, \pi_3, ..., \pi_k) \in \Delta_k$, RQ_θ is an increasing function of θ. This implies that for all $\theta \in [0,3)$, $RQ_\theta \le RQ_3$. Therefore, to show that $RQ_\theta \le 1$ for all $\theta \in [0,3)$ it is enough to show that $RQ_3 \le 1$.In the process we establish the validity of Axiom 3 as well.

Case 1: k=3: Here $RQ_3(\pi_1, \pi_2, \pi_3) = 4\sum_{i=1}^{3} \pi_i^2 (1-\pi_i) + 3\pi_1\pi_2\pi_3$ whenever $(\pi_1, \pi_2, \pi_3) \in \Delta_3$ .Consider $u(\pi_1, \pi_2, \pi_3) = 4\sum_{i=1}^{3} \pi_i^2 (1-\pi_i) + 3\pi_1\pi_2\pi_3$ $+\lambda(\pi_1 + \pi_2 + \pi_3 - 1)$, where λ is the Lagrange multiplier and π_1, π_2, π_3 are assumed to be independent variables. (Therefore, π_1, π_2, π_3 are not connected by the relation $\sum_{i=1}^{3} \pi_i = 1$ and so the partial derivatives of u with respect to each $\pi_i, 1 \le i \le 3$, exists.) Then the extreme points of RQ_3 are given by solving the equations $\frac{\partial u}{\partial \pi_1} = 0$, $\frac{\partial u}{\partial \pi_2} = 0$ and $\frac{\partial u}{\partial \pi_3} = 0$. So, if (p_1, p_2, p_3) is an extreme point of RQ_3 in Δ_3, then we have,

$$4\left(2p_1 - 3p_1^2\right) + 3p_2p_3 + \lambda = 0, \tag{18.1}$$

$$4\left(2p_2 - 3p_2^2\right) + 3p_3p_1 + \lambda = 0, \tag{18.2}$$

$$4\left(2p_3 - 3p_3^2\right) + 3p_1p_2 + \lambda = 0. \tag{18.3}$$

Solving (18.1), (18.2) and (18.3) and using the restriction $p_1 + p_2 + p_3 = 1$, we get $(p_1, p_2, p_3) = \left(1/3, 1/3, 1/3\right)$ or a permutation of $\left(4/9, 4/9, 1/9\right)$. Now, $RQ_3\left(1/3, 1/3, 1/3\right)$ $= 1$ and $RQ_3\left(4/9, 4/9, 1/9\right) = 4\left[2.\left(\frac{4}{9}\right)^2.\frac{5}{9} + \left(\frac{1}{9}\right)^2.\frac{8}{9}\right] + 3.\left(\frac{4}{9}\right)^2.\frac{1}{9} = \frac{80}{81} < 1$. Hence, for all (π_1, π_2, π_3) in the interior of Δ_3, we have, $RQ_3(\pi_1, \pi_2, \pi_3) \le 1$ and consequently, for all $\theta \in [0,3)$, $RQ_\theta(\pi_1, \pi_2, \pi_3) < 1 = RQ_\theta\left(1/2, 1/2, 0\right)$.

But Δ_3 being closed and bounded is compact. Since RQ_θ is continuous on Δ_3, it attains its global maximum at some point inside Δ_3 (Rudin, 1987, p.89). But it is clear from above that the point is not in the interior of Δ_3. Therefore, it must lie somewhere on the boundary $\partial\Delta_3$ of Δ_3, which is given by $\partial\Delta_3 = \bigcup_{i=1}^{3}\left\{\left(\pi_1,\pi_2,\pi_3\right)\in\Delta_3 : \pi_i = 0\right\}$. So, it is easily seen that $\max_{(\pi_1,\pi_2,\pi_3)\in\Delta_3} RQ_\theta\left(\pi_1,\pi_2,\pi_3\right)$ = 4. $1/4$ = 1, the maximum being attained only at those points which are permutations of $\left(1/2, 1/2, 0\right)$. This shows that both Axioms 1 and 3 hold for RQ_θ in this case.

Case 2: k $\geq$ 4: In this case we have, $RQ_3\left(\pi_1,\pi_2,\pi_3,....,\pi_k\right)$ = $4\sum_{i=1}^{k}\pi_i^{\,2}\left(1-\pi_i\right)+3\sum_{1\le i_1<i_2<i_3\le k}\pi_{i_1}\pi_{i_2}\pi_{i_3}$ whenever $\left(\pi_1,\pi_2,\pi_3,...,\pi_k\right)\in\Delta_k$. As before, consider $u\left(\pi_1,\pi_2,\pi_3,...,\pi_k\right)$ = $4\sum_{i=1}^{3}\pi_i^{\,2}\left(1-\pi_i\right)$ + $3\sum_{1\le i_1<i_2<i_3\le k}\pi_{i_1}\pi_{i_2}\pi_{i_3}$ + $\lambda\left(\sum_{i=1}^{k}\pi_i - 1\right)$, where $\pi_1,\pi_2,\pi_3,...,\pi_k$ are assumed to be independent variables. Extreme points of RQ_3 are given by solving the equations $\frac{\partial u}{\partial \pi_1}$ = 0, $\frac{\partial u}{\partial \pi_2}$ = 0, ……, $\frac{\partial u}{\partial \pi_k}$ = 0.

But $\frac{\partial u}{\partial \pi_l}$ = $4\left(2\pi_l - 3\pi_l^{\,2}\right)+3\sum_{\substack{i<j\\ i,j\ne l}}\pi_i\pi_j+\lambda$ for all $l = 1,2,...,k.$ So, if $\left(p_1,p_2,p_3,...,p_k\right)$ is an extreme point of RQ_3 in Δ_k, then we have the following sequence of equations,

$$4\left(2p_1 - 3p_1^{\,2}\right)+3\sum_{\substack{i<j\\ i,j\ne 1}}p_i p_j+\lambda = 0, \tag{19.1}$$

$$4\left(2p_2 - 3p_2^{\,2}\right)+3\sum_{\substack{i<j\\ i,j\ne 2}}p_i p_j+\lambda = 0, \tag{19.2}$$

. . .

$$4\left(2p_k - 3p_k^{\,2}\right) + 3\sum_{\substack{i<j\\ i,j\neq k}} p_i p_j + \lambda = 0\,. \qquad (19.k)$$

Subtraction of (19.2) from (19.1) gives $\left(p_1 - p_2\right)\left\{8 - 12\left(p_1 + p_2\right) - 3\sum_{i=3}^{k} p_i\right\} = 0$ from which it follows that $p_1 = p_2$ or $8 - 12\left(p_1 + p_2\right) = 3 - 3\left(p_1 + p_2\right)$, that is, $\left(p_1 + p_2\right) = 5/9$. Clearly, p_1, p_2 in the above relation can be replaced by p_i, p_j for arbitrary *i, j* with $i \neq j$. In view of this, the set $\{p_1, p_2, p_3, ..., p_k\}$ can be partitioned into subsets with equal entries. But whenever $p_i \neq p_j$ and $p_i \neq p_l$ we have, $\left(p_i + p_j\right) = 5/9$ $= \left(p_i + p_l\right)$, which implies that $p_j = p_l$. So, there can be at most two such subsets in any such partition. Take them to be $\{p_1, p_2, ..., p_m\}$ and $\{p_{m+1}, p_{m+2}, ..., p_k\}$.

Assume without loss of generality that the number of entries in the second set is greater or equal to that of the first, that is, $(k - m) \geq m$. We claim that $m \leq 1$. To see this, note that $mp_1 + (k-m)p_k$ =1 and $p_1 + p_k$ = 5/9 solving which we get, $\left(k - 2m\right)p_k = 1 - \left(5m/9\right) < 0$ whenever $m \geq 2$. This leads to a contradiction since the left hand side is non-negative. So, $m \leq 1$.

Thus we are left with two choices only: either (*a*) all the p_i's are equal; (the common value being $1/k$) or, (*b*) all but one of the p_i's are equal. In case (*b*) holds, by Axiom 5, we can take

$$\left(p_1, p_2, p_3, ..., p_k\right) = \left(\frac{4}{9(k-2)}, \frac{4}{9(k-2)}, ..., \frac{4}{9(k-2)}, 1 - \frac{4(k-1)}{9(k-2)}\right). \qquad (20)$$

Now, for the case (a), $RQ_3\left(1/k, 1/k, ..., 1/k\right) = \left\{4\frac{1}{k^2}\left(1 - \frac{1}{k}\right)k\right\} + \left\{3\frac{1}{k^3}\binom{k}{3}\right\} =$ $4\frac{(k-1)}{k^2} + \frac{(k-1)(k-2)}{2k^2} = \frac{(k-1)(k+6)}{2k^2} < 1$ since $2k^2 - (k-1)(k+6) = \left(k^2 - 5k + 6\right) =$ $(k-2)(k-3) > 0$ $[\because k > 3]$.

For the case (b),

$$RQ_3\left(\frac{4}{9(k-2)},\frac{4}{9(k-2)},......,\frac{4}{9(k-2)},1-\frac{4(k-1)}{9(k-2)}\right)$$

$$= 4\left[\frac{16}{81(k-2)^2}\left\{1-\frac{4}{9(k-2)}\right\}(k-1)+\left\{1-\frac{4(k-1)}{9(k-2)}\right\}^2\frac{4(k-1)}{9(k-2)}\right]+\frac{3.64}{729(k-2)^3}\binom{k-1}{3} +$$

$$\frac{3.16}{81(k-2)^2}\left\{1-\frac{4(k-1)}{9(k-2)}\right\}\binom{k-1}{2}$$

$$=\frac{16(k-1)}{729(k-2)^3}\left[4\{9(k-2)-4\}+(5k-14)^2+2(k-2)(k-3)+\frac{3}{2}(k-2)(5k-14)\right]$$

$$=\frac{16(k-1)}{729(k-2)^3}\left[25k^2-104k+108+2k^2-10k+12+\frac{15}{2}k^2-36k+42\right]$$

$$=\frac{16(k-1)}{729(k-2)^3}\left[\frac{69}{2}k^2-150k+162\right]=\frac{8(k-1)}{729(k-2)^3}\left[69k^2-300k+324\right]. \tag{21}$$

To see that the expression in (21) is less than l for all $k \geq 4,$ define

$$\phi(k) = 729(k-2)^3-8(k-1)(69k^2-300k+324)=(177k^3-1422k^2+3756k-3240), \tag{22}$$

where $k \geq 3$ is any real number. Then $\phi(4)$ = 360 > 0. Also, $\phi'(k)$ = $(531k^2-2844k+3756)$ so that $\phi'(4)=876>0$[7]. Since $\phi''(k)$ = $(1062k-2844)>0$ for all $k \geq 3$, it follows that ϕ' is increasing on $[3,\infty)$. Consequently, $\phi'(k) \geq \phi'(4)>0$ for all $k \geq 4$. So, ϕ is increasing on $[4,\infty)$, which implies that $\phi(k)>0$ for all $k \geq 4$, thus proving that the expression in (21) is less than 1.

Hence $\max\limits_{(\pi_1,\pi_2,....,\pi_k)\in \mathrm{int}\Delta_k} RQ_3(\pi_1,\pi_2,....,\pi_k)<1$, where ‘int A’ denotes the interior of the set A. Since for any $\theta \in [0,3)$ and $(\pi_1,\pi_2,\pi_3,....,\pi_k)\in \Delta_k$ we have, $RQ_\theta(\pi_1,\pi_2,\pi_3,....,\pi_k) \leq RQ_3(\pi_1,\pi_2,\pi_3,....,\pi_k)$, it follows that $\max\limits_{(\pi_1,\pi_2,\pi_3,....,\pi_k)\in\Delta_k} RQ_\theta(\pi_1,\pi_2,\pi_3,....,\pi_k)$,which is $\geq RQ_\theta\left(\tfrac{1}{2},\tfrac{1}{2},0,....,0\right)$ =1, cannot be attained at any interior point of Δ_k. But since Δ_k is compact and RQ_θ is continuous on it,

[7] We denote the first and second derivatives of the function ϕ by ϕ' and ϕ'' respectively.

using the same argument as in case 1, the maximum has to be attained at some point on the boundary $\partial\Delta_k$. It is easily seen that $\partial\Delta_k = \bigcup_{i=1}^{k}\{(\pi_1,\pi_2,\pi_3,...,\pi_k)\in\Delta_k : \pi_i = 0\}$. So finding $\max RQ_\theta$ in $\partial\Delta_k$ amounts to finding the maximum in Δ_{k-1}. Repeating the same argument, this maximum can again be found only in the boundary $\partial\Delta_{k-1}$.

Descending thus, finally we come down to Δ_3 wherein, by case 1, the maximum value 1 is attained only at those points which are permutations of $\left(\frac{1}{2},\frac{1}{2},0\right)$. Hence, $\max\limits_{(\pi_1,\pi_2,...,\pi_k)\in\Delta_k} RQ_\theta(\pi_1,\pi_2,...,\pi_k)$ will be attained only at those points which are permutations of $\left(\frac{1}{2},\frac{1}{2},\underbrace{0,....,0}_{(k-2)times}\right)$. This establishes the fact that RQ_θ satisfies both Axioms 1 and 3. This therefore completes the proof of the lemma.

Proof of Theorem 1: By Lemma 1 we know that RQ_θ given by (8) is the only EPI that satisfies Axioms1-7. We now show that the only value of θ, $0\le\theta<3$, for which RQ_θ satisfies Property 1 is $\theta=0$. Note that $RQ_\theta(p,q,r)-RQ_\theta(p,\tilde{q},0)=$ $qr\{c+4-c(q+r)\}<0$ if and only if $(c+4)<c(q+r)$. The last inequality holds if and only if $(\theta-8)<(\theta-12)(q+r)$, that is, if and only if $(q+r)<\left(\frac{8-\theta}{12-\theta}\right)$, where $\theta=c+12$ and $0\le\theta<3$.

This should hold for all $q,r\ge 0$ satisfying $q\ge r$ and $\{1-(q+r)\}>q$. Let $\rho\in(0,2/3)$ be fixed. Choosing $p=\left(\frac{1}{3}+\rho\right)$, $q=r=\left(\frac{1}{3}-\frac{\rho}{2}\right)$, we require $\left(\frac{2}{3}-\rho\right)<\left(\frac{8-\theta}{12-\theta}\right)$. Now letting $\rho\to 0$, we demand that $\left(\frac{8-\theta}{12-\theta}\right)\ge\frac{2}{3}$, which implies that $\theta\le 0$. So the only possibility is $\theta=0$, in which case RQ_θ coincides with RQ.

In order to complete the proof we need to verify that RQ satisfies Property 1. As seen earlier, $RQ\left(\underline{\pi}_\varepsilon^{(i)}\right)-RQ\left(\underline{\pi}^{(i)}\right) = 4\varepsilon(\pi_i-\varepsilon)(3\pi_i-2)<0$ whenever $\pi_i<\frac{2}{3}$. In the present context, in the distribution $(p, q+r, 0)$, the second group (of size $\tilde{q}=(q+r)$) can be thought of being split into two subgroups with relative frequencies q and r. We claim then that $(q+r)<\frac{2}{3}$. For, if $(q+r)\geq\frac{2}{3}$, then $p\leq\frac{1}{3}$ $\left[\because (p+q+r)=1\right]$ and $q\geq\frac{1}{3}$ $\left[\because q\geq r\right]$, thereby contradicting the assumption that $p>q$. So, $(q+r)<\frac{2}{3}$ which implies that $RQ(p,q,r)<RQ(p,\tilde{q})$.This completes the proof of the theorem.

The next characterization of the RQ index is based on Property 1a.

Theorem 2: The only EPI $P:\Delta\to R$ that satisfies Axioms 1-7 and Property 1a is the RQ index.

Proof: Lemma 1 shows that RQ_θ given by (8) is the only EPI that satisfies Axioms 1-7. Now, fix $n\geq 5$ and let $0<\varepsilon<\frac{1}{n}$. Consider $p^{(n)}=\left(1-\frac{1}{n},\frac{1}{n},0\right)\in\Delta_3$ an $p^{(n,\varepsilon)}=\left(\frac{1}{n}+\varepsilon,\frac{1}{n},\frac{1}{n},...,\frac{1}{n},\frac{1}{n}-\varepsilon\right)\in\Delta_n$. Then $RQ_\theta\left(p^{(n)}\right)=4\frac{1}{n}\left(1-\frac{1}{n}\right)$ and

$$RQ_\theta\left(p^{(n,\varepsilon)}\right)=4\left[\frac{1}{n^2}\left(1-\frac{1}{n}\right)(n-2)+\left(\frac{1}{n}+\varepsilon\right)^2\left(1-\frac{1}{n}-\varepsilon\right)+\left(\frac{1}{n}-\varepsilon\right)^2\left(1-\frac{1}{n}+\varepsilon\right)\right]+$$

$$\theta\left[\frac{1}{n^3}\binom{n-2}{3}+\frac{1}{n^2}\left(\frac{1}{n}+\varepsilon\right)\binom{n-2}{2}+\frac{1}{n^2}\left(\frac{1}{n}-\varepsilon\right)\binom{n-2}{2}+\left(\frac{1}{n}+\varepsilon\right)\left(\frac{1}{n}-\varepsilon\right)\frac{1}{n}(n-2)\right].$$

We require that $RQ_\theta\left(p^{(n,\varepsilon)}\right)<RQ_\theta\left(p^{(n)}\right)$ for all permissible values of $\varepsilon>0$ .Letting $\varepsilon\to 0$, we require that $4\frac{1}{n}\left(1-\frac{1}{n}\right)\geq 4\frac{1}{n}\left(1-\frac{1}{n}\right)+\theta\frac{1}{n^3}(n-2)\left\{\frac{(n-3)(n-4)}{6}+(n-3)+1\right\}$.

This is possible only when $\theta=0$, that is , only for the index RQ.(It is easy to see that for $n=3$ and 4, $RQ_\theta\left(p^{(n,\varepsilon)}\right)<RQ_\theta\left(p^{(n)}\right)$ holds for all permissible values of $\varepsilon>0$ only if $\theta=0$.)

To show that RQ satisfies Property 1a, it is enough to show that $RQ\left(p_1,\sum_{i=2}^{k}p_i\right)>RQ(p_1,p_2,...,p_k)$ whenever $(p_1,p_2,...,p_k)\in\Delta_k$ and $p_1\geq p_i$ for all $i\geq 2$, with strict inequality for at least one i. But

$$\frac{1}{4}\left\{RQ\left(p_1,\sum_{i=2}^{k}p_i\right)-RQ\left(p_1,p_2,...,p_k\right)\right\}$$

$$= p_1\left(1-p_1\right)-p_1^{\,2}\left(1-p_1\right)-\sum_{i=2}^{k}p_i^{\,2}\left(1-p_i\right)$$

$$= p_1\left(1-p_1\right)^2-\sum_{i=2}^{k}p_i^{\,2}\left(1-p_i\right).$$

Thus, it is enough to show that $\sum_{i=2}^{k}p_i^{\,2}\left(1-p_i\right)<p_1\left(1-p_1\right)^2$. The right hand side of this inequality is $p_1\left(\sum_{i=2}^{k}p_i\right)^2 = p_1\left(\sum_{i=2}^{k}p_i^{\,2}+2\sum_{2\le i<j\le k}p_ip_j\right)$. So, the required inequality follows if we demonstrate that $\sum_{i=2}^{k}p_i^{\,2}\left(1-p_i-p_1\right)<2p_1\sum_{2\le i<j\le k}p_ip_j$ .Since the left hand side is $\sum_{2\le i<j\le k}p_ip_j\left(p_i+p_j\right)$, this inequality is obvious because $\left(p_i+p_j\right)\le 2p_1$ for all $i, j\ge 2$, with strict inequality in at least one case [$\because$ $p_1\ge p_i$ for all $i\ge 2$, with strict inequality for at least one i].This completes the proof of the theorem.

Finally, we show how property 2 can be employed to characterize the RQ index.

Theorem 3: The only EPI $P:\Delta\to R$ that satisfies Axioms 1-7 and Property 2 is the RQ index.

Proof: Let $\pi^1 = \left(p,p,1-2p\right)$ and $\pi^2=\left(p+x,p+x,1-2p-2x\right)$, where $0<p<1/2$ and $0<\left(p+x\right)<1/2$, $x>0$. Now, let RQ_θ satisfy Property 2. Then $RQ_\theta\left(\pi^1\right) = 8p\left(1-3p+3p^2\right)+\theta p^2\left(1-2p\right)$. So, we require $\frac{\partial}{\partial p}RQ_\theta\left(\pi_1\right)\ge 0$, that is, $\left\{\left(72p^2-48p+8\right)+\theta\left(2p-6p^2\right)\right\}\ge 0.$ Equivalently, we require E = $\left\{\left(36-3\theta\right)p^2-\left(24-\theta\right)p+4\right\}\ge 0$ for all $p\in\left(0,\frac{1}{2}\right)$. But discriminant of E $=\left(24-\theta\right)^2-16\left(36-3\theta\right)=\theta^2\ge 0$. So the only possibility is $\theta^2 = 0$, that is, $\theta=0$.This generates the RQ index.

To demonstrate the converse, note that $RQ\left(\pi^1\right)$ = $4\left\{2p^2\left(1-p\right)+\left(1-2p\right)^2 2p\right\} = 8p\left\{p(1-p)+(1-2p)^2\right\} = 8p\left(1-3p+3p^2\right)$ so that $\frac{1}{8}\left\{RQ\left(\pi^2\right)-RQ\left(\pi^1\right)\right\}$ = $\left(p+x\right)\left\{1-3\left(p+x\right)+3\left(p+x\right)^2\right\}-p\left(1-3p+3p^2\right)$ = $\left\lfloor p\left(1-3x+6px+3x^2\right)+x\left\{1-3\left(p+x\right)+3\left(p+x\right)^2\right\}\right\rfloor > 0 \quad \left[\because p>0, x>0\right]$. Thus, $RQ\left(\pi^1\right)$ is increasing in *p*. So, *RQ* satisfies Property 2. These two demonstrations along with Lemma 1 complete the proof of the theorem.

Since Theorems 1, 2 and 3 characterize the same functional form of the polarization index using alternative sets of axioms; the following theorem can be stated:

Theorem 4: Let $P:\Delta \to R$ be an EPI. Then the following statements are equivalent.

(i) P satisfies Axioms 1-7 and Property 1.

(ii) P satisfies Axioms 1-7 and Property 1a.

(iii) P satisfies Axioms 1-7 and Property 2.

(iv) P coincides with the RQ index given by (3).

We have noted the relationship between the RQ index and FRAC. It may be worthwhile to use this relation to axiomatize the RQ index in an alternative structure. For this purpose we assume that the domain of EPI is given by $\Delta' = \bigcup_{n\in\Omega} \Delta_n$, where $\Omega = N \setminus \{1\}$. We also assume additivity of the EPI P, that is, $P:\Delta' \to R$ is of the following form:

$$P\left(\pi_1, \pi_2, ..., \pi_k\right) = \sum_{i=1}^{k} \psi\left(\pi_i\right), \tag{23}$$

where $\psi:[0,1] \to R$ is continuous. Here $\psi\left(\pi_i\right)$ may be assumed to represent the impact of group i on overall polarization.

The next axiom specifies a simple relationship between the impact factors when there are only two groups.

Axiom 8: If there are two ethnic groups with non-zero group-sizes *p* and (1-*p*), then the ratio of group-impacts equals the ratio of group-sizes, that is, $\frac{\psi(1-p)}{\psi(p)} = \left(\frac{1-p}{p}\right)$.

The following normalization axiom will be necessary for the characterization.

Axiom 9: Assume that there are two ethnic groups. Then the EPI $P:\Delta_2 \to R$ is a positive multiple of *FRAC*.

The following theorem can now be stated.

Theorem 5: The only EPI $P:\Delta' \to R$ of the form (23) that satisfies Axioms 3, 8 and 9 is the RQ index.

Proof: Note that Axiom 9 gives

$$\psi(\pi_1)+\psi(1-\pi_1)=c\pi_1(1-\pi_1), \tag{24}$$

for some constant $c>0$ whenever $\pi_1 \in [0,1]$.

Now, let $0<p<1$. Consider an ethnic distribution with sizes p and (1-p). By Axiom 8, the ratio of group-impacts is given by $\left(\frac{1-p}{p}\right)=\frac{\psi(1-p)}{\psi(p)}$. Consequently, $\psi(1-p)=\left(\frac{1-p}{p}\right)\psi(p)$. But from (24) we have, $\psi(p)+\left(\frac{1-p}{p}\right)\psi(p)=cp(1-p)$, which implies that $\psi(p)=cp^2(1-p)$. This is true for all $p\in(0,1)$ and hence, by continuity of ψ, $\psi(p)=cp^2(1-p)$ for all $p\in[0,1]$. This gives $\psi(0)=0$. But using the specific structure (23) of P and Axiom 3 we have $2\psi\left(\frac{1}{2}\right)+\psi(0)=1$, which, in view of $\psi(0)=0$, implies that $\psi\left(\frac{1}{2}\right)=\frac{1}{2}$. Putting $\pi_1=\frac{1}{2}$ in (24) we get $c=2.2=4$.

Consequently, $\psi(p)=4p^2(1-p)$. Substituting this form of ψ in (23) we get, P = *RQ*. This demonstrates the necessity part of the theorem. The sufficiency part can be verified easily.

We conclude this section with a proof of the following result which drops out as a corollary to Lemma 1.

Corollary 1: Consider an EPI $P:\Delta \to R$ satisfying Axioms (1) - (7). If an ethnic group E_i (with relative frequency π_i) be such that there is at least one more group with relative frequency not less than π_i and if E_i is split into two subgroups, then new ethnic distribution becomes less polarized.

Proof: By Lemma 1, the EPI must be of the form RQ_θ with $0 \le \theta < 3$. Recall that $\theta = (c+12)$. So, $-12 \le c < -9$ implies that $\frac{-1}{9} < \frac{1}{c} \le \frac{-1}{12}$, from which we get $\frac{5}{9} < \left(1 + \frac{4}{c}\right) \le \frac{2}{3}$. Now let's go back to the proof of the 'converse' part of the lemma. We have, $RQ_\theta\left(\underline{\pi}_\varepsilon^{(i)}\right) - RQ_\theta\left(\underline{\pi}^{(i)}\right) = \varepsilon(\pi_i - \varepsilon)(c + 4 - c\pi_i)$. So, whenever $RQ_\theta\left(\underline{\pi}_\varepsilon^{(i)}\right) \ge RQ_\theta\left(\underline{\pi}^{(i)}\right)$ we must have $\pi_i \ge \left(1 + \frac{4}{c}\right) > \frac{5}{9}$. But we are given that there is an ethnic group E_l, with $i \ne l$, such that $\pi_i \le \pi_l$, so that $\pi_i \le \frac{5}{9}$, for otherwise, $\sum_{j=1}^{k} \pi_j \ge 2\pi_i \ge \frac{10}{9}$, a contradiction. Consequently, $RQ_\theta\left(\underline{\pi}_\varepsilon^{(i)}\right) < RQ_\theta\left(\underline{\pi}^{(i)}\right)$. In other words, $\underline{\pi}_\varepsilon^{(i)}$ is less polarized than $\underline{\pi}^{(i)}$.This completes the proof of the corollary.

Thus our requirement on E_i is that if the underlying ethnic distribution is unimodal, then E_i should be a non-modal group while if the ethnic distribution is multimodal, then E_i can be any group. While Property1 claims that polarization increases under merger of two groups, Corollary1 indicates reduction in polarization if a group is broken down into two subgroups. Hence the essential idea underlying Property1 and Corollary1 is the same.

4. An Ethnic Polarization Ordering

Given a population with k ethnic groups $E_1, E_2, ..., E_k$ for some $k \in \Gamma$ and π_i = relative frequency of E_i, define, as before, $\underline{\pi} = (\pi_1, \pi_2, ..., \pi_k)$ and let $\Delta_k^* = \{(p_1, p_2, ..., p_k) \in \Delta : p_1 \ge p_2 \ge .... \ge p_k\}$ .The set of all ethnic distributions in this case is $\Delta^* = \bigcup_{k \in \Gamma} \Delta_k^*$. For all $k \in \Gamma$, we denote the corresponding extreme ethnic distributions $(1/2, 1/2, 0, ..., 0), (1, 0, 0, ..., 0) \in \Delta_k^*$ by $\underline{\upsilon}$ and $\underline{\beta}$ respectively.

While discussing on the necessity of introducing the *RQ*-index for ethnic polarization, Montalvo and Reynal Querol (2008, p.1838) mentioned that "the original purpose of this (*RQ*) index was to capture how far the distribution of the ethnic groups is from the (1/2, 0,....,0, 1/2) distribution (bipolar), which represents the highest level of polarization". Thus, as the distance of an ethnic profile from $\underline{\upsilon}$, which we refer to as the $\underline{\upsilon}$-based distance, decreases, ethnic polarization should increase. Since an EPI has been assumed to take its minimum value (0) at all those points which are permutations of $\underline{\beta}$, one may assume that the "distance of an ethnic distribution from $\underline{\beta}$", which we refer to as the $\underline{\beta}$-based distance, is increasingly related to ethnic polarization (that is, with greater Euclidean distance from $\underline{\beta}$, polarization should increase). We combine these two views together and look for all the EPI's which are decreasing in the distance from $\underline{\upsilon}$ and increasing in the distance from $\underline{\beta}$. Our notion of polarization ordering relies on such distances. For simplicity of exposition we assume that the distance is measured by the Euclidean distance. Given $\underline{\pi},\underline{\sigma} \in \Delta_k^*$, $k \in \Gamma$, let $d\left(\underline{\pi},\underline{\sigma}\right)$ be the Euclidean distance between them[8].

Before we proceed for definition of the ordering based on the distances, we investigate some properties of the distance measures.

Theorem 6: The $\underline{\upsilon}$-based distance function satisfies Properties 1 and 1a.

Proof: (i) **Property 1**: Let's have three ethnic groups of sizes *p*, *q* and *r* and $p > q$ and $q \geq r$ and let $\tilde{q} = \left(q + r\right)$. Consider $\underline{\upsilon}$ = $\left(1/2, 1/2, 0\right)$ .First let $p \geq \left(q + r\right)$. Then $\underline{\pi}^1, \underline{\pi}^2 \in \Delta_3^*$, where $\underline{\pi}^1$ = $\left(p, \tilde{q}, 0\right)$ and $\underline{\pi}^2 = \left(p, q, r\right)$. So, $d^2\left(\underline{\pi}^1, \underline{\upsilon}\right) - d^2\left(\underline{\pi}^2, \underline{\upsilon}\right) =$ $\left(q + r - \frac{1}{2}\right)^2 - \left(q - \frac{1}{2}\right)^2 - r^2$ = $2r\left(q - \frac{1}{2}\right)$ <0. [Since $p > q,$ $q < 1/2$, for otherwise $\left(p + q\right) > 1$.] Consequently, $d\left(\underline{\pi}^1, \underline{\upsilon}\right) < d\left(\underline{\pi}^2, \underline{\upsilon}\right)$, that is, $\underline{\pi}^1$ is more polarized than $\underline{\pi}^2$. This is in agreement with Property 1.If, however, $p < \left(q + r\right)$, then considering $\underline{\pi}^1{}_0$ = $\left(\tilde{q}, p, 0\right) \in \Delta_3^*$ we can conclude, as above, that $d\left(\underline{\pi}^1{}_0, \underline{\upsilon}\right) < d\left(\underline{\pi}^2, \underline{\upsilon}\right)$ which again shows that $\underline{\pi}^1$ is more polarized than $\underline{\pi}^2$.

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[8] In fact, the Euclidean distance can be replaced by any other distance generated by any other norm on R^k, the k-dimensional Euclidean space, since any two norms on R^k are equivalent.

(ii) **Property 1a:** Consider two ethnic groups with sizes p and (1-p). Let the second group be split into n subgroups $p_1, p_2, ..., p_n$ such that $p \geq p_1 \geq p_2 \geq .... \geq p_n > 0$ (with strict inequality in at least one place) and $\sum_{i=1}^{n} p_i = (1-p)$. Note that $2p_1 \leq (p+p_1) \leq 1$, which implies that $(1-2p_1) \geq 0$. First, let $p \geq 1/2$. Let $\underline{\pi}^1$ = $\left(p, 1-p, \underbrace{0, ..., 0}_{(n-1) times} \right)$ and $\underline{\pi}^2$ = $(p, p_1, p_2, ..., p_n)$. Then $\underline{\pi}^1, \underline{\pi}^2 \in \Delta_{n+1}^*$, $d^2\left(\underline{\pi}^1, \underline{\upsilon}\right)$ = $2\left(p-\frac{1}{2}\right)^2$ and $d^2\left(\underline{\pi}^2, \underline{\upsilon}\right)$ = $\left(p-\frac{1}{2}\right)^2 + \left(p_1-\frac{1}{2}\right)^2 + \sum_{i=1}^{n} p_i^2$ so that

$$
\begin{aligned}
d^2\left(\underline{\pi}^1, \underline{\upsilon}\right) - d^2\left(\underline{\pi}^2, \underline{\upsilon}\right) &= \left(p-\frac{1}{2}\right)^2 - \left(p_1-\frac{1}{2}\right)^2 - \sum_{i=1}^{n} p_i^2 \\
&= \left(p_1 - \frac{1}{2} + \sum_{i=2}^{n} p_i\right)^2 - \left(p_1-\frac{1}{2}\right)^2 - \sum_{i=1}^{n} p_i^2 \quad \left[\because \sum_{i=1}^{n} p_i = (1-p)\right] \\
&= \left(\sum_{i=2}^{n} p_i\right)^2 + 2\left(p_1-\frac{1}{2}\right)\left(\sum_{i=2}^{n} p_i\right) - \sum_{i=1}^{n} p_i^2 \\
&< \left(\sum_{i=2}^{n} p_i\right)\left(\sum_{i=2}^{n} p_i + 2p_1 - 1\right) \\
&= \left(\sum_{i=2}^{n} p_i\right)\left(\sum_{i=1}^{n} p_i + p_1 - 1\right) \\
&= \left(\sum_{i=2}^{n} p_i\right)(1-p+p_1-1) \quad \left[\because \sum_{i=1}^{n} p_i = (1-p)\right] \\
&\leq 0 \; [\because p_1 \leq p.].
\end{aligned}
$$

Thus , $d\left(\underline{\pi}^1, \underline{\upsilon}\right) < d\left(\underline{\pi}^2, \underline{\upsilon}\right)$.

Next, considering $p < 1/2$, we can replace $\underline{\pi}^1$ by $\underline{\pi}^1{}_0 = \left(1-p, p, \underbrace{0, ..., 0}_{(n-1) times}\right)$ and then repeat the same argument to get $d\left(\underline{\pi}^1, \underline{\upsilon}\right) < d\left(\underline{\pi}^2, \underline{\upsilon}\right)$. Thus in all cases, polarization of $\underline{\pi}^1$ is greater than that of $\underline{\pi}^2$.This completes the proof of the theorem.

The next theorem shows that $\underline{\upsilon}$-based distance function satisfies Property 2 also, in a weaker form:

Property $2'$**:** Assume that there are three groups of sizes *p*, *q*, *p* *with* $p \geq q$. Then if we shift mass from the *q* group equally to the other two groups, polarization increases. That is, $POL(p,q,p) < POL(p+x, q-2x, p+x)$.

The only difference between Properties 2 and $2'$ is that in Property $2'$**,** *p* is at least as large as *q*, whereas there is no such restriction in Property 2.

Theorem 7: The $\underline{\upsilon}$-based distance function satisfies Property $2'$.

Proof: Let $\underline{\pi}^1 = (p,p,q) = (p,p,1-2p)$, $\underline{\pi}^2 = (p',p',q') = (p',p',1-2p')$, where $p \geq q$ and $p' > p$ so that $p' > q'$ and hence $\underline{\pi}^1, \underline{\pi}^2 \in \Delta_3^*$.Obviously, $(1-2p) \geq 0$ which implies that $p \leq 1/2$. Similarly, $p' \leq 1/2$.But $p < p'$ so that $(p+p') < 1$. Then $d^2(\underline{\pi}^1, \underline{\upsilon}) = 2\left(p - \frac{1}{2}\right)^2 + (1-2p)^2 = \left(6p^2 - 6p + 3/2\right)$ and $d^2(\underline{\pi}^2, \underline{\upsilon}) = \left(6p'^2 - 6p' + 3/2\right)$.Therefore, $d^2(\underline{\pi}^1, \underline{\upsilon}) - d^2(\underline{\pi}^2, \underline{\upsilon}) = (6p^2 - 6p) - (6p'^2 - 6p') = 6(p-p')(p+p'-1)$ >0. [Since $p < p'$ and $(p+p') < 1$.] Thus, whenever $p < p'$ we have, $d(\underline{\pi}^1, \underline{\upsilon}) > d(\underline{\pi}^2, \underline{\upsilon})$, that is, $\underline{\pi}^2$ is more polarized than $\underline{\pi}^1$, which is what we wanted to demonstrate.

Remark 2: However, the $\underline{\upsilon}$-based distance function obeys Property 2 even when $p \leq q$, provided $(p+x) \leq 1/3$ and $(2p+x) < 1/2$. To see this, consider, as before, $\underline{\pi}^1 = (q,p,p) = (1-2p,p,p)$, $\underline{\pi}^2 = (q',p',p') = (1-2p',p',p')$ with $p' = (p+x) \leq 1/3$ so that $p' \leq (1-2p')$ and $p \leq 1/3$ .This implies that $p \leq (1-2p)$. Hence $\underline{\pi}^1, \underline{\pi}^2 \in \Delta_3^*$. Therefore, $d^2(\underline{\pi}^1, \underline{\upsilon}) = \left(2p - \frac{1}{2}\right)^2 + \left(p - \frac{1}{2}\right)^2 + p^2 = \left(6p^2 - 3p + 1/2\right)$ and $d^2(\underline{\pi}^2, \underline{\upsilon}) = \left(6p'^2 - 3p' + 1/2\right)$ .Hence $d^2(\underline{\pi}^1, \underline{\upsilon}) - d^2(\underline{\pi}^2, \underline{\upsilon}) = (6p^2 - 3p) - (6p'^2 - 3p') = 3(p-p')\{2(p+p')-1\}$. Now note that $\{2(p+p')-1\} < 0$ since

$(p+p')=(2p+x)<1/2$. So, $d(\underline{\pi}^1,\underline{\upsilon}) > d(\underline{\pi}^2,\underline{\upsilon})$, which means that $\underline{\pi}^2$ is more polarized than $\underline{\pi}^1$.

Remark 3: Using arguments employed for proving Theorem 6 it can be proven that the $\underline{\beta}$-based distance function satisfies Properties 1 and 1a under the additional assumption that the relative frequency of the largest ethnic group is at least $1/2$. But one can show that this function is a violator of Property 2 and its weaker form Property $2'$**.** However, one can check that it satisfies the following restricted version of Property 2.

Property $2''$: Assume that there are three groups of sizes *p, q, p* with $p \le q$. Then if we shift mass $2x$ from the q group equally to the other two groups such that $(p+x) \le (q-2x)$, then polarization increases. That is, $POL(p,q,p) < POL(p+x,q-2x,p+x)$.

Since the two distance functions are consistent with variants of Properties 1,1a and 2, it becomes worthwhile to use them in the measurement of ethnic polarization. The following definition is based on this argument.

Definition 2: An EPI $P:\Delta^* \to R$ is called distance–based if for all $k \in \Gamma$, $\underline{\pi} \in \in \Delta_k^*$, $P(\underline{\pi})$ can be expressed $P(\underline{\pi}) = G\left(d(\underline{\pi},\underline{\upsilon}), d(\underline{\pi},\underline{\beta})\right)$,where the real-valued continuous function $G:R_+^2 \to R$ is decreasing in its first and increasing in its second argument , R_+^2 being the non-negative part of the 2-dimensional Euclidean space.

It may be verified that the RQ index cannot be expressed as a distance-based index. To see this, consider $\underline{\pi}^1,\underline{\pi}^2 \in \Delta_3^*$, where $\underline{\pi}^1 = (1/3,1/3,1/3)$, $\underline{\pi}^2 = (1/2,1/4,1/4)$. Then $RQ(\underline{\pi}^1) = \frac{8}{9}$ and $RQ(\underline{\pi}^2) = = \frac{7}{8}$ so that $RQ(\underline{\pi}^1) > RQ(\underline{\pi}^2)$. But $d^2(\underline{\pi}^1,\underline{\upsilon}) = \frac{1}{6}$ and $d^2(\underline{\pi}^2,\underline{\upsilon}) = \frac{1}{8}$. So, $d(\underline{\pi}^2,\underline{\upsilon}) < d(\underline{\pi}^1,\underline{\upsilon})$. Thus, we note that the RQ index and the $\underline{\upsilon}$-based distance rank the two ethnic profiles in the same direction and hence one of the defining conditions in Definition 2 is violated. However, this should not be taken as a shortcoming of the RQ index. This is because one possible way of choosing an EPI is to define properties which an EPI should satisfy. Certainly, an EPI satisfying them is not meant to supplant an index, which may not fulfill some of them, because a particular index may be generated using a given concept and a specific property may not be relevant there.

For all $k \in \Gamma$, for all $\underline{\pi}^1, \underline{\pi}^2 \in \Delta_k^*$, we say that $\underline{\pi}^1$ is more ethnically polarized than $\underline{\pi}^2$, what we write $\underline{\pi}^1 >_P \underline{\pi}^2$, if and only if $P(\underline{\pi}^1) > P(\underline{\pi}^2)$ for all distance-based EPIs $P : \Delta^* \to R$.

The following theorem describes an easily implementable equivalent condition of the ordering $>_P$.

Theorem 8: For arbitrary $k \in \Gamma$, let $\underline{\pi}^1, \underline{\pi}^2 \in \Delta_k^*$ be arbitrary. Then the following statements are equivalent:

(i) $\underline{\pi}^1 >_P \underline{\pi}^2$.

(ii) $d\left(\underline{\pi}^1, \underline{\upsilon}\right) \leq d\left(\underline{\pi}^2, \underline{\upsilon}\right)$ and $d\left(\underline{\pi}^1, \underline{\beta}\right) \geq d\left(\underline{\pi}^2, \underline{\beta}\right)$, with strict inequality in at least one case.

Proof: $(ii) \Rightarrow (i)$: Follows from the definition.

$(i) \Rightarrow (ii)$: Fix $\varepsilon > 0$ and let $P_\varepsilon(\underline{\pi}) = d(\underline{\pi}, \underline{\beta}) - \varepsilon\, d(\underline{\pi}, \underline{\upsilon})$ and $P'_\varepsilon(\underline{\pi}) = \varepsilon d(\underline{\pi}, \underline{\beta}) - d(\underline{\pi}, \underline{\upsilon})$.

Then it is easy to see that $P_\varepsilon, P'_\varepsilon$ are distance based EPI's. So, $P_\varepsilon(\underline{\pi}^1) > P_\varepsilon(\underline{\pi}^2)$ implies that $\left\{d(\underline{\pi}^1, \underline{\beta}) - d(\underline{\pi}^2, \underline{\beta})\right\} > \varepsilon\left\{d(\underline{\pi}^1, \underline{\upsilon}) - d(\underline{\pi}^2, \underline{\upsilon})\right\}$. Letting $\varepsilon \to 0$ in this inequality we have $d\left(\underline{\pi}^1, \underline{\beta}\right) \geq d\left(\underline{\pi}^2, \underline{\beta}\right)$. Similarly $P'_\varepsilon(\underline{\pi}^1) > P'_\varepsilon(\underline{\pi}^2)$ implies that $\varepsilon\left\{d(\underline{\pi}^1, \underline{\beta}) - d(\underline{\pi}^2, \underline{\beta})\right\} > \left\{d(\underline{\pi}^1, \underline{\upsilon}) - d(\underline{\pi}^2, \underline{\upsilon})\right\}$ and as $\varepsilon \to 0$, we get $d\left(\underline{\pi}^1, \underline{\upsilon}\right) \leq d\left(\underline{\pi}^2, \underline{\upsilon}\right)$.

Now note that if $d(\underline{\pi}^1, \upsilon) = d(\underline{\pi}^2, \upsilon)$ and $d(\underline{\pi}^1, \underline{\beta}) = d(\underline{\pi}^2, \underline{\beta})$, then $P(\underline{\pi}^1) = P(\underline{\pi}^2)$ for all distance-based EPIs $P : \Delta^* \to R$. So, strict inequality has to hold in at least one case. This completes the proof of the theorem.

Theorem 8 says that if condition (ii) holds then we can conclude that the ethnic profile $\underline{\pi}^1$ is more polarized than the profile $\underline{\pi}^2$ for all distance-based polarization indices. Thus, if this condition holds for two ethnic profiles we do not need to calculate the numerical values of the distance-based indices for polarization ranking of the profiles.

Although Theorem 8 has taken all the distance-based EPIs into consideration, it is shown in the next theorem that it is enough if condition (i) holds for a much smaller class.

Theorem 9: For arbitrary $k \in \Gamma$, let $\underline{\pi}^1, \underline{\pi}^2 \in \Delta_k^*$ be arbitrary. Then the following statements are equivalent:

(i) $P(\underline{\pi}^1) > P(\underline{\pi}^2)$ for all distance-based EPIs satisfying Axioms (1)-(6).
(ii) $d\left(\underline{\pi}^1, \underline{\upsilon}\right) \le d\left(\underline{\pi}^2, \underline{\upsilon}\right)$ and $d\left(\underline{\pi}^1, \underline{\beta}\right) \ge d\left(\underline{\pi}^2, \underline{\beta}\right)$ with strict inequality in at least one case.
Proof: $(ii) \Rightarrow (i)$: Follows from Theorem 8.

$(i) \Rightarrow (ii)$: Fix $m, n \in N$ and define $P_{m,n}\left(\underline{\pi}\right) = \dfrac{\left\{d\left(\underline{\pi}, \underline{\beta}\right)\right\}^m}{\left\{d\left(\underline{\pi}, \underline{\beta}\right)\right\}^m + \left\{d\left(\underline{\pi}, \underline{\upsilon}\right)\right\}^n}$. Note that the denominator is always positive (since $d\left(\underline{x}, \underline{y}\right) = 0$ if and only if $\underline{x} = \underline{y}$). Hence $P_{m,n}\left(\underline{\pi}\right)$ is well-defined. Moreover, $P_{m,n}\left(\underline{\pi}\right)$ is increasing in $d(\underline{\pi}, \underline{\beta})$ and decreasing in $d(\underline{\pi}, \underline{\upsilon})$. Also the following facts are evident: (a) $0 \le P_{m,n}\left(\underline{\pi}\right) \le 1$ for all $\underline{\pi} \in \Delta_k^*$; (b) $P_{m,n}\left(\underline{\pi}\right) = 0$ if and only if $d\left(\underline{\pi}, \underline{\beta}\right) = 0$, which is same as the condition that $\underline{\pi} = \underline{\beta}$, and (c) $P_{m,n}\left(\underline{\pi}\right) = 1$ if and only if $d\left(\underline{\pi}, \underline{\upsilon}\right) = 0$, which is equivalent to the requirement that $\underline{\pi} = \underline{\upsilon}$. Thus, for all $m, n \in N$, $P_{m,n}$ is a distance-based EPI satisfying Axioms (1) - (3). It is easy to verify that $P_{m,n}$ fulfills Axioms (4)-(6). So, for all $m, n \in N$, $P_{m,n}(\underline{\pi}^1) > P_{m,n}(\underline{\pi}^2)$. This implies that for all $m, n \in N$, $\left(\dfrac{y_1{}^m}{y_1{}^m + x_1{}^n}\right) > \left(\dfrac{y_2{}^m}{y_2{}^m + x_2{}^n}\right)$, where $y_1 = d\left(\underline{\pi}^1, \underline{\beta}\right)$, $x_1 = d\left(\underline{\pi}^1, \underline{\upsilon}\right)$, $y_2 = d\left(\underline{\pi}^2, \underline{\beta}\right)$, $x_2 = d\left(\underline{\pi}^2, \underline{\upsilon}\right)$. From the last inequality it follows that

$$y_1^{\,m} x_2^{\,n} > y_2^{\,m} x_1^{\,n} \geq 0. \tag{25}$$

Obviously then $y_1 > 0$ and $x_2 > 0$.

Various cases come under consideration.

Case I: Suppose $y_2 > 0$ and $x_1 > 0$. (i.e., $\underline{\pi}^1 \neq \underline{\upsilon}$, $\underline{\pi}^2 \neq \underline{\beta}$). We rewrite the inequality in (25) as $\left(\frac{y_1}{y_2}\right)^m > \left(\frac{x_1}{x_2}\right)^n$, which is true for all $m, n \in N$. If possible, let $x_1 > x_2$. Take $m = 1$ and let $n \to \infty$ to get $\frac{y_1}{y_2} \to \infty$, a contradiction. So, $x_1 \leq x_2$. Similarly, if $y_1 < y_2$, then taking $n = 1$ and letting $m \to \infty$ we get $\frac{x_2}{x_1} \to \infty$, which is again a contradiction. So, $y_1 \geq y_2$.

Case II: Suppose $y_2 = 0$ and $x_1 > 0$. Then obviously, $y_1 \geq y_2$ and $\underline{\pi}^2 = \underline{\beta}$. Consider the sequence $\left\{\underline{\beta}^{(p)}\right\}$ in Δ_k^* defined by $\underline{\beta}^{(p)} = \left(1 - \frac{1}{p}, \frac{1}{p}, 0, \ldots, 0\right)$ for all $p \in N$. Then $d^2\left(\underline{\beta}^{(p)}, \underline{\beta}\right) = \frac{2}{p^2} \to 0$ as $p \to \infty$ so that $\left\{\underline{\beta}^{(p)}\right\}$ converges to $\underline{\beta}$. Let P be a distance-based EPI satisfying Axioms (1)-(6). P being continuous we have, $P\left(\underline{\beta}^{(p)}\right) \to P\left(\underline{\beta}\right)$. Now, by (i), $P\left(\underline{\pi}^1\right) > P\left(\underline{\beta}\right)$ which implies that there is a subsequence $\left\{\underline{\beta}^{(p_j)}\right\}$ of $\left\{\underline{\beta}^{(p)}\right\}$ such that $P\left(\underline{\pi}^1\right) > P\left(\underline{\beta}^{(p_j)}\right)$ for all $j \in N$ (Rudin, 1987, p.56). Note that $\underline{\beta}^{(p_j)} \neq \underline{\beta}$ for all j. So by Case I, $x_1 = d\left(\underline{\pi}^1, \underline{\upsilon}\right) \leq d\left(\underline{\beta}^{(p_j)}, \underline{\upsilon}\right) \to d\left(\underline{\beta}, \underline{\upsilon}\right)$, as $j \to \infty$. Thus, $x_1 \leq x_2$.

Case III: Suppose $x_1 = 0$ and $y_2 > 0$. Then clearly $x_1 \leq x_2$ and $\underline{\pi}^1 = \underline{\upsilon}$. Proceeding exactly the same way as in Case II, we can show that $y_1 = d\left(\underline{\upsilon}, \underline{\beta}\right) \geq d\left(\underline{\pi}^2, \underline{\beta}\right) = y_2$.

Case IV: Suppose $x_1 = 0$ and $y_2 = 0$. Then obviously, $x_1 < x_2$ and $y_2 < y_1$. Thus, in all cases $x_1 \leq x_2$ and $y_1 \geq y_2$. But applying the same argument as in Theorem 8 we conclude that $x_1 = x_2$ and $y_1 = y_2$ cannot happen simultaneously. So, at least one inequality has to be strict. This completes the proof.

In view of Theorems 8 and 9, we can now state the following:

Theorem 10: For arbitrary $k \in \Gamma$, let $\underline{\pi}^1, \underline{\pi}^2 \in \Delta_k^*$ be arbitrary. Then the following statements are equivalent:

(i) $P(\underline{\pi}^1) > P(\underline{\pi}^2)$ for all distance-based EPIs.

(ii) $P(\underline{\pi}^1) > P(\underline{\pi}^2)$ for all distance-based EPIs satisfying Axioms (1)-(6).

(iii) $d\left(\underline{\pi}^1, \underline{\upsilon}\right) \leq d\left(\underline{\pi}^2, \underline{\upsilon}\right)$ and $d\left(\underline{\pi}^1, \underline{\beta}\right) \geq d\left(\underline{\pi}^2, \underline{\beta}\right)$, with strict inequality in at least one case. .

Remark 4: From Theorem 6 and Remark 3 it follows that depending on the distance function considered, the ordering $>_P$ becomes consistent Properties 1 and 1a or their variants.

Remark 5: Although the ordering $>_P$ is reflexive and transitive, it is not complete. To see this, consider two distributions $\underline{\eta} = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$ and $\underline{\xi} = \left(\frac{3}{4}, \frac{1}{4}, 0\right)$. Then $d^2\left(\underline{\eta}, \underline{\upsilon}\right)$ $= \frac{1}{18}$ and $d^2\left(\underline{\xi}, \underline{\upsilon}\right) = \frac{1}{8}$, whereas $d^2\left(\underline{\eta}, \underline{\beta}\right) = \frac{8}{9}$ and $d^2\left(\underline{\xi}, \underline{\beta}\right) = \frac{9}{8}$. Thus, $d\left(\underline{\eta}, \underline{\upsilon}\right) < d\left(\underline{\xi}, \underline{\upsilon}\right)$ but $d\left(\underline{\eta}, \underline{\beta}\right) < d\left(\underline{\xi}, \underline{\beta}\right)$. Hence, none of the inclusions $\underline{\eta} >_P \underline{\xi}$ and $\underline{\xi} >_P \underline{\eta}$ is valid.

5. Conclusion

An indicator of ethnic polarization is an aggregated form of ethnic diversity in a population. Montalvo and Reynal-Querol (2005, 2008) introduced an index of ethnic polarization, which they refer to as the RQ index, investigated its properties in detail and explained its role empirically as an explanatory variable for incidence of civil wars. In this paper we characterize the RQ index using alternative sets of axioms. Our characterizations rely mostly on some axioms of polarization suggested by Esteban and Ray (1994) and Montalvo and Reynal-Querol (2005, 2008). We also characterize a generalized form of the RQ index. Finally, an ethnic polarization ordering based on a specific class of ethnic polarization indices has been presented and analyzed.

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