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Indices of redistributive effect and reranking: Reinterpretation

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Abstract

The Kakwani decomposition of redistributive effect into vertical and reranking terms is one of the most widely used tools in measurement of income redistribution. However, Urban (2009) argues that the decomposition features some methodological problems and calls for its reinterpretation. This paper builds several different measurement models, constructs new indices of redistributive effect and reranking reinventing the existing ones, and establishes important propositions on the role of reranking in the redistributive process. All this is done to prove that the standard interpretation of the Kakwani decomposition is misleading. New roles are suggested for the well-known indices of redistributive, vertical and reranking effect.

Keywords: Kakwani decomposition, redistributive, reranking and vertical effects.
JEL classifications: D63, H22, H23.

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1 Introduction

The Kakwani (1984) decomposition of redistributive effect into vertical and reranking indices became a cornerstone of the research on income redistribution. This is evidenced by the huge number of empirical studies employing it, and plenty of extensions and upgrades provided by its supporters. The popularity of this decomposition rests on its comprehensiveness (it captures different notions of redistributive justice), simplicity and ease of computation, and its availability for straightforward policy interpretation (redistributive power can be enhanced if horizontal inequity is reduced).

Urban (2009) describes the origins of the Kakwani decomposition and reviews different other methodologies in decomposing the redistributive effect which are rooted in the Kakwani decomposition. This study reveals contrasting opinions of different scholars on the possibility, meaning and interpretation of the decomposition. Atkinson (1980) and Plotnick (1981), who invented the reranking term, implicitly suggested to future users and developers to be cautious about the introduction of reranking into more comprehensive frameworks. Nonetheless, the Kakwani decomposition is exactly such a model capturing vertical equity and reranking.

For Lerman and Yitzhaki (1995) one of the weaknesses of Kakwani approach is the use of pre-tax income rankings: more reranking is regarded as favourable from the policy maker's perspective and the increase in Atkinson-Plotnick reranking index automatically increases the Kakwani vertical effect. On the other hand, for both Kakwani (1984) and Lerman and Yitzhaki (1995), reranking has an active role in determination of the magnitude of the redistributive effect, and this is completely opposed to the views of Atkinson (1980) and Plotnick (1980). Kakwani inspired many followers to claim that elimination of reranking would increase the redistributive effect.

Therefore, the paper by Urban (2009) calls for more thorough research on these important problems. This paper represents such an investigation, attempting to explain the differences in approaches of different scholars and to discover which views are correct and to prove them. The main thesis of this paper is that reranking of income units *cannot* influence the redistributive effect.

To defend this argument, two measurement systems, based on income vector transitions and income and rank distances between units, are carefully built. These approaches are already known in the literature on the Gini coefficient. Here we extend them to other indices and measures of the redistributive effect and reranking. New (and renewed old) concepts of distance narrowing, fiscal deprivation and domination are also presented. Following the derivation of all these measures, we compare them to indices and decompositions existing in the literature, and establish the relationships between them.

The methodological apparatus developed here helps to develop important propositions about the fiscal process, which are then used to prove that Kakwani (1984) and Lerman and Yitzhaki (1995) gave *mistaken interpretations* of the role of reranking in the redistributive process. After obtaining the proof, the two natural questions posed by researchers would be: How to proceed and which measures should be used? The answers are offered in the end of the paper.

The paper is organized as follows. The *second* section prepares a measurement model for subsequent analysis. The *third* section suggests certain arguments concerning the role of reranking in a redistributive process and proves them, using different approaches. It also explains how different measures of the redistributive effect and reranking behave in some specific circumstances. All this analysis leads us to the *fourth* section, in which final propositions and recipes for further research are presented.

2 Measurement model

Gini and concentration coefficients

Variables, vectors and ordering of units

First, let us define the ranking function $r(\mathbf{a})$, which returns a rank for each unit a_k in vector $\mathbf{a}$, such that the smallest element receives rank 1, etc.

Let $\mathbf{y} = [y_1, \ldots, y_s]^{\mathrm{T}}$ be an income vector with s units and mean value $\bar{y}$. Vector $y^y = [y_1^y, \ldots, y_s^y]^{\mathrm{T}}$ contains values $y_i^y \in \mathbf{y}$ such that $i = r(y_k)$. Thus, $\mathbf{y}^y$ has the same values as the original vector $\mathbf{y}$, but sorted in ascending order of $\mathbf{y}$.

Vector $\mathbf{z}=[z_1,\ldots,z_s]^{\mathrm{T}}$ presents another variable. We define $\mathbf{z}^z=[z_1^z,\ldots,z_s^z]^{\mathrm{T}}$, with values $z_i^z\in\mathbf{z}$ such that $i=r(z_k)$. However, we may also define $\mathbf{z}^y=[z_1^y,\ldots,z_s^y]^{\mathrm{T}}$ with values $z_i^y\in\mathbf{z}$ such that $i=r(y_k)$. Observe the following distinction: $\mathbf{z}^y$ has the same values as $\mathbf{z}$, but they are sorted in ascending order of $\mathbf{y}$. Yet another variable, $\mathbf{y}^z$, can be defined analogously.

"Distance from the mean" approach

Equation (1) specifies Gini coefficient (G_y). For each income unit with **y**-rank i and value y_i^y, the distance from the mean value $\bar{y}$ is weighted by $(s-i+\frac{1}{2})$, where s is the highest rank. The weighted distances from mean are then averaged by s^2, and expressed as a share in the mean income.

(1) $$G_y=\frac{2}{s^2\bar{y}}\sum_{i=1}^{n}(s-i+\tfrac{1}{2})(\bar{y}-y_i^y)$$

The concentration coefficient is defined analogously. As equation (2) shows, for each income unit with $\mathbf{z}$-rank i and value y_i^z, the distance from the mean value $\bar{y}$ is weighted. Remember that y_i^z contains values of $\mathbf{y}$ sorted using ranks from the vector $\mathbf{z}$.

(2) $$D_y^z=\frac{2}{s^2\bar{y}}\sum_{i=1}^{s}(s-i+\tfrac{1}{2})(\bar{y}-y_i^z)$$

The Gini coefficient and concentration coefficient can be seen as members of a class of single-parameter or S-Gini coefficients, for which the parameter takes value $\rho=2$. In a discrete case, S-Gini is $G_y=2s^{-2}\bar{y}^{-1}\sum_i^s\omega(i;\rho)(\bar{y}-y_i)$, where $\omega(i;\rho)=\left((s-i+1)^{\rho}-(s-i)^{\rho}\right)/s^{\rho-1}$. In (1) and (2), the term $s-i+\frac{1}{2}$ is actually the weighting scheme $\omega(p_i;\rho)$ obtained for $\rho=2$. Since the number $\frac{1}{2}$ in the term $s-i+\frac{1}{2}$ does not affect the estimate of Gini and concentration coefficients, we will ignore it for simplicity. We will introduce it again in welfare analysis. Thus, we can write:

(3) $$G_y=\frac{2}{s^2\bar{y}}\sum_{i=1}^{s}(s-i)(\bar{y}-y_i^y)$$

(4) $$D_y^z = \frac{2}{s^2 \overline{y}} \sum_{i=1}^{s} (s-i)(\overline{y} - y_i^z)$$

"Distance between units" approach

Another way to calculate Gini and concentration coefficients is based on the differences between income pairs (Lambert, 2001:34) and is even more straightforward. Instead of summing distances from the mean, formula (6) sums *income distances* between units, for all possible pairs (i, j).

(5) $$G_y = \frac{1}{2s^2 \overline{y}} \sum_{i=1}^{s} \sum_{j=1}^{s} \left| y_i - y_j \right|$$

Notice the distinction between the terms of *income difference* and *income distance*. The former is presented by $y_i - y_j$, and can be either positive or negative. The latter term, $\left| y_i - y_j \right|$, is always positive as a result of absolute signs. Now, if instead of values y_i and y_j, we decide to use y_i^y and y_j^y, and if we take only the values such that i is always greater or equal to j, then we can rewrite (5) to obtain Gini coefficient of y, as shown by (6). Analogously, replacing y_i^y and y_j^y in (6) with y_i^z and y_j^z, we obtain the concentration coefficient of y with respect to z, as in (7).

(6) $$G_y = \frac{1}{s^2 \overline{y}} \sum_{i=1}^{s} \sum_{j=1}^{i} (y_i^y - y_j^y)$$

(7) $$D_y^z = \frac{1}{s^2 \overline{y}} \sum_{i=1}^{s} \sum_{j=1}^{i} (y_i^z - y_j^z)$$

For illustrative purposes and easier derivation of other indices later, we draw matrices of the following form: $\mathbf{M}(i, j) = y_i^y - y_j^y$, defined only for $i \geq j$. Because the numbers in these matrices fill only the space on one side of the diagonal, we call them triangular. It is shown in general form by Figure 1. Since diagonal elements are always equal to zero, the presentation of the matrix can be reduced to the form presented in Figure 2.

Figure 1: "Full" triangular matrix

y_1^y	$y_1^y - y_1^y$						
y_2^y	$y_2^y - y_1^y$	$y_2^y - y_2^y$					
…	…	…	…				
y_p^y	$y_p^y - y_1^y$	$y_p^y - y_2^y$	…	$y_p^y - y_p^y$			
…	…	…	…	…	…		
y_{s-1}^y	$y_{s-1}^y - y_1^y$	$y_{s-1}^y - y_2^y$	…	$y_{s-1}^y - y_p^y$	…	$y_{s-1}^y - y_{s-1}^y$	
y_s^y	$y_s^y - y_1^y$	$y_s^y - y_2^y$	…	$y_s^y - y_p^y$	…	$y_s^y - y_{s-1}^y$	$y_s^y - y_s^y$
	y_1^y	y_2^y	…	y_p^y	…	y_{s-1}^y	y_s^y

Figure 2: Compact triangular matrix

y_2^y	$y_2^y - y_1^y$							
…	…							
y_p^y	$y_p^y - y_1^y$	$y_p^y - y_2^y$	…	$y_p^y - y_{p-1}^y$				
…	…	…	…	…				
y_{s-1}^y	$y_{s-1}^y - y_1^y$	$y_{s-1}^y - y_2^y$	…	$y_{s-1}^y - y_{p-1}^y$	…	$y_{s-1}^y - y_{s-2}^y$		
y_s^y	$y_s^y - y_1^y$	$y_s^y - y_2^y$	…	$y_s^y - y_{p-1}^y$	…	$y_s^y - y_{s-2}^y$	…	$y_s^y - y_{s-1}^y$
	y_1^y	y_2^y	…	y_{p-1}^y	…	y_{s-2}^y		y_{s-1}^y

Equations (6) and (7) can be rewritten in the light of this reduced form of the matrix presentation.

(8)
$$G_y = \frac{1}{s^2 \bar{y}} \sum_{i=2}^{s} \sum_{j=1}^{i-1} \left(y_i^y - y_j^y \right)$$

(9)
$$D_y^z = \frac{1}{s^2 \bar{y}} \sum_{i=2}^{s} \sum_{j=1}^{i-1} \left(y_i^z - y_j^z \right)$$

In all subsequent analysis we will present the formulas in this same form of the matrix presentation. Therefore, it will always be assumed that $i \geq j$. A useful property should be remembered, presented in (10).

(10)
$$\left| y_i - y_j \right| = y_i^y - y_j^y \text{, for all } (i, j) \text{ such that } i \geq j$$

Lorenz and concentration curves approach

As the third way of presenting Gini and concentration coefficients, we mention the original approach that uses Lorenz and the concentration curves. Lorenz curve abscissas are cumulative proportions of units, p_i, and ordinates are cumulative proportions of the variable considered, $L_y(i)$. Equations (9) and (10) are used to obtain them:

(11) $$p_i = \frac{i}{s}$$

(12) $$L_y(p_i) = \frac{1}{s\bar{y}} \sum_{j=1}^{i} y_j^y$$

The Gini coefficient is defined as double the area between the line of absolute equality and Lorenz curve. The line of absolute equality presents a situation in which all values of **y** are equal to $\bar{y}$. Notice that in this case Lorenz curve would be equal to:

(13) $$L_{\bar{y}}(p_i) = \frac{1}{s\bar{y}} \sum_{j=1}^{i} \bar{y} = \frac{1}{s\bar{y}} \bar{y} i = \frac{i}{s} = p_i$$

In the discrete case we deal here with, the Gini can be approximated as double the average of distances between the line of absolute equality and Lorenz curve, $L_y(p_i)$.

(14) $$G_y = \frac{2}{s} \left(\frac{1}{s\bar{y}} \bar{y} i - \frac{1}{s\bar{y}} \sum_{j=1}^{i} y_j^y \right) =$$

$$= \frac{2}{s^2 \bar{y}} \sum_{i=1}^{s} \left(\bar{y} i - \sum_{j=1}^{i} y_j^y \right) = \frac{2}{s} \sum_{i=1}^{s} \left(\frac{i}{s} - \sum_{j=1}^{i} \frac{y_j^y}{s\bar{y}} \right) = \frac{2}{s} \sum_{i=1}^{s} \left(p_i - L_y(p_i) \right)$$

Similarly, the concentration coefficient can be calculated, using the concentration curve $C_y^z(p_i)$ instead of Lorenz curve, as presented in (15).

(15) $$D_y^z = \frac{2}{s} \left(\frac{1}{s\bar{y}} \bar{y} i - \frac{1}{s\bar{y}} \sum_{j=1}^{i} y_j^z \right) =$$

$$= \frac{2}{s^2 \bar{y}} \sum_{i=1}^{s} \left(\bar{y} i - \sum_{j=1}^{i} y_j^z \right) = \frac{2}{s} \sum_{i=1}^{s} \left(\frac{i}{s} - \sum_{j=1}^{i} \frac{y_j^z}{s\bar{y}} \right) = \frac{2}{s} \sum_{i=1}^{s} \left(p_i - C_y^z(p_i) \right)$$

Analysis of income transitions

Income variables

$\mathbf{X}$ and $\mathbf{N}$ are vectors of pre-fiscal and post-fiscal income, respectively; the j th entry of $\mathbf{X}$, X_j, and the j th entry of $\mathbf{N}$, N_j, present income information for the particular income unit j. Vectors $\mathbf{X}^x$, $\mathbf{X}^n$, $\mathbf{N}^n$ and $\mathbf{N}^x$ are different sortings of vectors $\mathbf{X}$ and $\mathbf{N}$, as explained in the previous section.

Table 1 shows a hypothetical population of five and their income vectors $\mathbf{X}$ and $\mathbf{N}$. In these original vectors, the units take either random or alphabetic (perhaps, according to family names) or some other order, independent of incomes. Columns $r(\mathbf{X})$ and $r(\mathbf{N})$ present ranks of units according to pre- and post-fiscal income. We observe they are not identical: indeed each unit changes its rank in the transition from pre- to post-fiscal income.

Table 1 Hypothetical data set

Unit	$\mathbf{X}$	$\mathbf{N}$	$r(\mathbf{X})$	$r(\mathbf{N})$
A	180	80	5	4
B	30	100	3	5
C	70	20	4	1
D	8	40	1	2
E	12	60	2	3

Unit	$\mathbf{X}^x$	$\mathbf{N}^x$
D	8	40
E	12	60
B	30	100
C	70	20
A	180	80

Unit	$\mathbf{X}^n$	$\mathbf{N}^n$
C	70	20
D	8	40
E	12	60
A	180	80
B	30	100

In the second step, we sort units in ascending order of pre-fiscal income and create vectors $\mathbf{X}^x$ and $\mathbf{N}^x$. Notice that the 1st place in $\mathbf{X}^x$ and $\mathbf{N}^x$ is taken by the unit with pre-fiscal rank 1 (unit D), the 2nd place is taken by the unit with pre-fiscal rank 2 (unit E) etc. In the similar way, but using **N**-ranks, we create vectors $\mathbf{X}^n$ and $\mathbf{N}^n$. The 1st place is taken by the unit with post-fiscal rank 1 (C), …, the 5th place is occupied by the unit with post-fiscal rank 5 (B).

We can see from this example that pre-fiscal and post-fiscal rankings of units, represented by $r(\mathbf{X})$ and $r(\mathbf{N})$, are not necessarily identical, and in reality they are certainly not. The difference in them is a consequence of the “process” we will call reranking to which we will devote a great deal of attention in what follows.

Transitions from pre- to post-fiscal income

By means of redistributive effects we will treat various transitions from pre- to post-fiscal income, but also the transitions from pre- to pre- and post- to post-fiscal income. For each transition we derive a specific index of the redistributive effect. Later we will develop further distinct concepts of income distance narrowing, fiscal deprivation / domination, and deprivation / domination due to reranking, and see how these are connected with the redistributive effects.

From pre-fiscal vector $\mathbf{X}$ and post-fiscal vector $\mathbf{N}$, we have derived four ordered vectors: $\mathbf{X}^x$, $\mathbf{X}^n$, $\mathbf{N}^x$ and $\mathbf{N}^n$ which form the basis of the analysis. We will first concentrate on the transitions from pre- to post-fiscal incomes, and leave the transitions from pre- to pre- and post- to post-fiscal income for the next section.

We have four possible transitions from pre- to post-fiscal income:

(a) $\mathbf{X}^x \rightarrow \mathbf{N}^x$ ($X_i^x \rightarrow N_i^x$); (b) $\mathbf{X}^n \rightarrow \mathbf{N}^n$ ($X_i^n \rightarrow N_i^n$)

(c) $\mathbf{X}^x \rightarrow \mathbf{N}^n$ ($X_i^x \rightarrow N_i^n$); (d) $\mathbf{X}^n \rightarrow \mathbf{N}^x$ ($X_i^n \rightarrow N_i^x$).

In transitions $\mathbf{X}^x \rightarrow \mathbf{N}^x$ and $\mathbf{X}^n \rightarrow \mathbf{N}^n$, the pre-fiscal income of one unit is compared to the post-fiscal income of the same unit. In transition $\mathbf{X}^x \rightarrow \mathbf{N}^n$, the pre-fiscal income of the unit with *pre-fiscal* rank i is compared to the post-fiscal income of the unit with *post-fiscal* rank i. In transition $\mathbf{X}^n \rightarrow \mathbf{N}^x$, the pre-fiscal income of the unit with *post-fiscal* rank i is compared to the post-fiscal income of the unit with *pre-fiscal* rank i. In the presence of reranking, these are different units. Thus, for transitions (c) and (d), the link between pre-fiscal and post-fiscal income will not be actual but counterfactual. In the rest of the analysis, we will concentrate on the first three transitions.

These aspects are illustrated in Table 2, based on the hypothetical data set from the previous table. For the first two transitions, the pre-fiscal income of unit D is transformed into the post-fiscal income of unit D (and so for the other four units). However, for the third transition, the pre-fiscal income of D is transformed into the post-fiscal income of unit C; E is translated into D, B into E, etc.

Table 2: Transitions from pre- to post-fiscal income

$\mathbf{X}^x \rightarrow \mathbf{N}^x$			
Unit	X_i^x	Unit	N_i^x
D	8	D	40
E	12	E	60
B	30	B	100
C	70	C	20
A	180	A	80

$\mathbf{X}^n \rightarrow \mathbf{N}^n$			
Unit	X_i^n	Unit	N_i^n
C	70	C	20
D	8	D	40
E	12	E	60
A	180	A	80
B	30	B	100

$\mathbf{X}^x \rightarrow \mathbf{N}^n$			
Unit	X_i^x	Unit	N_i^n
D	8	C	20
E	12	D	40
B	30	E	60
C	70	A	80
A	180	B	100

Transitions from pre- to pre- and from post- to post-fiscal income

In the previous section we have analyzed transitions from pre- to post-fiscal income. It was indicated that other transitions are also possible: from pre- to pre-fiscal income, and from post- to post-fiscal income. The former occurs between $\mathbf{N}^n$ and $\mathbf{N}^x$, and the latter between $\mathbf{X}^x$ and $\mathbf{X}^n$, as follows:

(a) $\mathbf{N}^n \rightarrow \mathbf{N}^x$ ($N_i^n \rightarrow N_i^x$)

(b) $\mathbf{X}^x \rightarrow \mathbf{X}^n$ ($X_i^x \rightarrow X_i^n$)

In the transition $\mathbf{N}^n \rightarrow \mathbf{N}^x$, the post-fiscal income of the unit with *post-fiscal* rank i is compared to the post-fiscal income of the unit with *pre-fiscal* rank i. In presence of reranking, these are different units. The same relates to the transition $\mathbf{X}^x \rightarrow \mathbf{X}^n$, where the pre-fiscal income of the unit with *pre-fiscal* rank i translates into the pre-fiscal income of the unit with *post-fiscal* rank i.

This is illustrated in Table 3. The post-fiscal income of unit C is transformed into the post-fiscal income of unit D, D is translated into E, E into B, etc. The pre-fiscal income of unit D is translated into the pre-fiscal income of unit C, etc.

Table 3: Transitions from pre- to pre- and from post- to post-fiscal income

$\mathbf{N}^n \rightarrow \mathbf{N}^x$			
Unit	N_i^n	Unit	N_i^x
C	20	D	40
D	40	E	60
E	60	B	100
A	80	C	20
B	100	A	80

$\mathbf{X}^x \rightarrow \mathbf{X}^n$			
Unit	X_i^x	Unit	X_i^n
D	8	C	70
E	12	D	8
B	30	E	12
C	70	A	180
A	180	B	30

Indices of the redistributive effect

All measurement in this study is based upon the concepts of Gini and the concentration coefficients. There are many different ways to calculate them; three methods are used here, which have been explained above. The redistributive effect and other indices, are also formed on the basis of Gini and the concentration coefficients.

Throughout the text, we assume that average post- and pre-fiscal incomes are equal, $\overline{N} = \overline{X}$. This enables easier derivation of the formulas and later we make adaptations to account for the case where $\overline{N} \neq \overline{X}$.

For the first three transitions from pre- to post-fiscal income explained in the previous section, we have the following three indices of the redistributive effect, shown in (16), (17) and (18). For the two other transitions, from pre- to pre- and post- to post-fiscal income, we have two additional indices of the redistributive effect, presented in (19) and (20).

For transition $\mathbf{X}^x \rightarrow \mathbf{N}^x$, the index is RE^x :

(16)
$$RE^x = 2c\sum_{i=1}^{s}(s-i)\left(N_i^x - X_i^x\right) = c\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(\left(X_i^x - X_j^x\right)-\left(N_i^x - N_j^x\right)\right)$$
$$= \frac{2}{s}\sum_{i=1}^{s}\left(C_N^x(p_i) - L_X(p_i)\right) = G_X - D_N^x$$

For transition $\mathbf{X}^n \rightarrow \mathbf{N}^n$, the index is RE^n :

(17)
$$RE^n = 2c\sum_{i=1}^{s}(s-i)\left(N_i^n - X_i^n\right) = c\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(\left(X_i^n - X_j^n\right)-\left(N_i^n - N_j^n\right)\right)$$
$$= \frac{2}{s}\sum_{i=1}^{s}\left(L_N(p_i) - C_X^n(p_i)\right) = D_X^n - G_N$$

For transition $\mathbf{X}^x \rightarrow \mathbf{N}^n$, the index is RE^{xn} :

(18)
$$RE^{xn} = 2c\sum_{i=1}^{s}(s-i)\left(N_i^n - X_i^x\right) = c\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(\left(X_i^x - X_j^x\right)-\left(N_i^n - N_j^n\right)\right)$$
$$= \frac{2}{s}\sum_{i=1}^{s}\left(L_N(p_i) - L_X(p_i)\right) = G_X - G_N$$

For transition $\mathbf{N}^n \rightarrow \mathbf{N}^x$, the index is RE^{rx} :

(19) $$RE^{rx} = 2c\sum_{i=1}^{s}(s-i)\left(N_i^x - N_i^n\right) = c\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(\left(N_i^n - N_j^n\right) - \left(N_i^x - N_j^x\right)\right)$$

$$= \frac{2}{s}\sum_{i=1}^{s}\left(C_N^x(p_i) - L_N(p_i)\right) = G_N - D_N^x$$

For transition $\mathbf{X}^x \rightarrow \mathbf{X}^n$, the index is RE^{rn}:

(20) $$RE^{rn} = 2c\sum_{i=1}^{s}(s-i)\left(X_i^n - X_i^x\right) = c\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(\left(X_i^x - X_j^x\right) - \left(X_i^n - X_j^n\right)\right)$$

$$= \frac{2}{s}\sum_{i=1}^{s}\left(C_X^n(p_i) - L_X(p_i)\right) = G_X - D_X^n$$

where the value of c is equal to $c = 1/s^2\overline{N}$.

Income distance, fiscal deprivation and domination

Fiscal deprivation

At the same time transitions from pre-fiscal to post-fiscal income induce changes in income distances and changes of income ranks. In this section, we will scrutinize the redistributive process at the level of two income units, and afterwards, the relations will be aggregated to the level of the whole population. This will result in new indices of distance narrowing and reranking.

Suppose that two income units have pre-fiscal incomes X_i^x and X_j^x, such that $X_i^x > X_j^x$ and $i > j$. Their respective post-fiscal incomes are N_i^x and N_j^x. First, let us define *distance narrowing* ($\Delta_{i,j}$).

(21) $$\Delta_{i,j} = \left|X_i - X_j\right| - \left|N_i - N_j\right|$$

If the distance between units is narrowed, we have that $\Delta_{i,j} > 0$; if it is widened, there is $\Delta_{i,j} < 0$. Next, we will define the *deprivation due to reranking* ($r_{i,j}^x$) of the unit with pre-fiscal rank i, that may be reranked by the unit with rank j.

(22) $$r_{i,j}^x = \frac{1}{2}\left(\left|N_i^x - N_j^x\right| - \left(N_i^x - N_j^x\right)\right)$$

If $N_i^x > N_j^x$, there is no reranking and $r_{i,j}^x = 0$. However, if $N_i^x < N_j^x$, it means that reranking occurred, and $r_{i,j}^x = N_j^x - N_i^x$.

[1] Finally, let us define the *fiscal deprivation* ($\varsigma_{i,j}^x$) of the unit with pre-fiscal income rank i, over the unit with pre-fiscal income rank j.

(23) $$\varsigma_{i,j}^x = \left(X_i^x - X_j^x\right) - \left(N_i^x - N_j^x\right)$$

The three measures defined above are connected as shown by the following equation.

(24) $$\varsigma_{i,j}^x = \Delta_{i,j} + 2r_{i,j}^x$$

How to interpret all these terms notions intuitively? First, we may say that the difference $X_i^x - X_j^x$ denotes "income supremacy" of i over j. Say that i worked harder, and now enjoys having higher income than j, and $X_i^x - X_j^x$ measures the intensity of this "feeling". However, the fiscal process occurs, and i's "income supremacy" changes to $N_i^x - N_j^x$. Thus, the term $\varsigma_{i,j}^x$ (23) signifies the *change* of income advantage of i over j, in the transition from pre- to post-fiscal income. If i loses a part of this advantage or supremacy ($\varsigma_{i,j}^x > 0$), we say that she is "fiscally deprived", and hence the name for the term. Fiscal deprivation can be divided (24) into two components: distance narrowing ($\Delta_{i,j}$) and reranking ($2r_{i,j}^x$).

Now, assume that the units with ranks i and j are informed that, in order to improve social welfare, the income distance between i and j will be reduced by $\Delta_{i,j}^T \leq \left|X_i - X_j\right|$. What may be the consequences of this action on the "income supremacy" of i, i.e. how large could her

[1] Notice that "deprivation due to reranking" closely resembles the concept of „fiscal looseness" presented by Duclos (2000).

fiscal deprivation be? In the case of no reranking, fiscal deprivation will be equal to $\Delta^{T}_{i,j}$. In the presence of reranking, it increases to $\Delta^{T}_{i,j} + 2r^{x}_{i,j}$.[2]

Assume that "society" agrees that certain distance narrowing is desirable between i and j, i.e. i must sacrifice part of her "income supremacy". However, it is also required that pre-fiscal rankings should not be altered, i.e. i must remain "the rich", and j "the poor". In this light, we may treat the reranking component of fiscal deprivation ($2r^{x}_{i,j}$) as an ***excess** fiscal deprivation* felt by i.

Fiscal domination

Two income units have post-fiscal incomes N^{n}_{i} and N^{n}_{j}, such that $N^{n}_{i} > N^{n}_{j}$ and $i > j$. Their respective pre-fiscal incomes are X^{n}_{i} and X^{n}_{j}. *Distance narrowing* ($\Delta_{i,j}$) is already defined in (21). Here we also define *distance widening* as negative *distance narrowing*.

(25) $$\nabla_{i,j} = -\Delta_{i,j}$$

Let us define the *domination due to reranking* ($r^{n}_{i,j}$) of the unit with post-fiscal rank i, that might have reranked the unit with post-fiscal rank j.

(26) $$r^{n}_{i,j} = \frac{1}{2}\left(\left|X^{n}_{i} - X^{n}_{j}\right| - \left(X^{n}_{i} - X^{n}_{j}\right)\right)$$

If $X^{n}_{i} \geq X^{n}_{j}$, there was no reranking and $r^{n}_{i,j} = 0$. However, if $X^{n}_{i} < X^{n}_{j}$, it means that reranking occurred, whereby $r^{n}_{i,j} = X^{n}_{j} - X^{n}_{i}$. Finally, we will define the *fiscal domination* ($\varsigma^{n}_{i,j}$) of the unit with post-fiscal income rank i, over the unit with post-fiscal income rank j, as in (27).

(27) $$\varsigma^{n}_{i,j} = \left(N^{n}_{i} - N^{n}_{j}\right) - \left(X^{n}_{i} - X^{n}_{j}\right)$$

---

[2] Is it "just" that i must sacrifice $\Delta^{T}_{i,j}$ of her income supremacy? For a "libertarian", the only permissible situation is $\Delta^{T}_{i,j} = 0$. For an "equalitarian", the perfect situation would be that $\Delta^{T}_{i,j} = \left|X_{i} - X_{j}\right|$, so that $N_{i} = N_{j}$. Usually, we would say that it is "just" that $\Delta^{T}_{i,j} > 0$, but the allowed magnitude of $\Delta^{T}_{i,j}$ would vary.

The relationship between the measures is represented by the following equation.

$$\varsigma_{i,j}^{n} = -\Delta_{i,j} + 2r_{i,j}^{n} = \nabla_{i,j} + 2r_{i,j}^{n} \tag{28}$$

The difference $N_i^n - N_j^n$ denotes the post-fiscal "income supremacy" of i over j. The former unit enjoys higher income, either because she worked harder (whereby earning higher pre-fiscal income), or as a consequence of the fiscal process. Fiscal domination ($\varsigma_{i,j}^{n}$) measures a change of i's "income supremacy" in the transition from pre- to post-fiscal income. This change can be arrived at through two channels: distance widening ($\nabla_{i,j}$) and reranking ($2r_{i,j}^{n}$). On the other hand, distance narrowing ($\Delta_{i,j} = -\nabla_{i,j}$) reduces fiscal domination.

The decomposition (28) also tells us that, for given $\nabla_{i,j}$, fiscal domination will be larger, the higher reranking is . Therefore, we may treat $2r_{i,j}^{n}$ as an ***augmented** fiscal domination* of the unit with post-fiscal rank i.

Comparison of the approaches

Compare the role of reranking in this and the previous section: it *increases* both fiscal domination (28) and fiscal deprivation (24). Since domination and deprivation are opposite terms, it means that reranking plays a reverse role in the two approaches: it is "bad" when causing excess fiscal deprivation (24), but it is "good" when it enhances fiscal domination (28).

The concept of fiscal domination is somewhat odd because it favours (assuming that a positive value of a measure means "good") both distance widening and reranking, two concepts that are usually disapproved of.

Indices of change in distance narrowing, fiscal deprivation and domination

In the previous two sections, we have defined exactly five new terms related to distances and ranks of income units. All these terms were defined for pairs of units (i, j). Fortunately, we can easily aggregate them to obtain indices that reflect these concepts for the whole population of units.

The index of *distance narrowing*, Δ, is derived from (21). By rule (10), we have that $\left|X_i - X_j\right| = \left(X_i^x - X_j^x\right)$ and $\left|N_i - N_j\right| = \left(N_i^n - N_j^n\right)$ for all (i, j) where $i \geq j$.

(29) $$\Delta = c\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(\left|X_i - X_j\right| - \left|N_i - N_j\right|\right) = c\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(\left(X_i^x - X_j^x\right) - \left(N_i^n - N_j^n\right)\right)$$

The index of *deprivation due to reranking*, R^x, is derived from (22).

(30) $$R^x = \frac{c}{2}\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(\left|N_i^x - N_j^x\right| - \left(N_i^x - N_j^x\right)\right)$$

The *fiscal deprivation* index, V^x, is derived from (23).

(31) $$V^x = c\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(\left(X_i^x - X_j^x\right) - \left(N_i^x - N_j^x\right)\right)$$

The index of *domination due to reranking*, R^n, is derived from (26).

(32) $$R^n = \frac{c}{2}\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(\left|X_i^n - X_j^n\right| - \left(X_i^n - X_j^n\right)\right)$$

The *reverse fiscal domination* index, V^n, is derived from (27). The "true" index of *fiscal domination* would be $-V^n$, but this reversal was done for easier alignment with other indices as will be witnessed later.

(33) $$V^n = -c\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(\left(N_i^n - N_j^n\right) - \left(X_i^n - X_j^n\right)\right)$$

Relationships between new and existing indices

We have already defined a number of different concepts, terms and indices above. At this point we have to reveal the relationships between them, and with measures already present in the literature. As we will see, all the new indices have their traditional correspondents. Urban (2009) provides detailed overview of the latter indices, and here we briefly summarize them. The "classical" or standard index of the redistributive effect (RE), the Kakwani (1977; 1984) index of vertical effect (V^K), the Lerman and Yitzhaki (1995) index of "gap narrowing" (V^{LY}), Atkinson (1980) and Plotnick (1981) index of reranking (R^{AP}), and Lerman and Yitzhaki (1995) index of reranking (R^{LY}), are respectively defined in equations (34) through (38).

(34) $$RE = G_X - G_N$$

(35) $$V^K = G_X - D_N^x$$

(36) $$V^{LY} = D_X^n - G_N$$

(37) $$R^{AP} = G_N - D_N^x$$

(38) $$R^{LY} = G_X - D_X^n$$

The index of the redistributive effect RE^{xn} in (18), has the same content as the index of distance narrowing Δ in (29), and is identical to the standard redistributive effect RE in (34).

(39) $$RE^{xn} = \Delta = RE$$

The index of the redistributive effect RE^x in (16), and the fiscal deprivation index V^x in (31), correspond to Kakwani index of vertical effect V^K in (35).

(40) $$RE^x \equiv V^x \equiv V^K$$

The index of the redistributive effect RE^n in (17) and the reverse fiscal domination index V^n in (33) correspond to Lerman-Yitzhaki index of "gap narrowing" V^{LY} in (36).

(41) $$RE^n \equiv V^n \equiv V^{LY}$$

The index of the redistributive effect RE^{rx} in (19) equals twice the index of deprivation due to reranking R^x in (30), and is identical to Atkinson-Plotnick index of reranking R^{AP} in (37).

(42) $$RE^{rx} = 2R^x = R^{AP}$$

The index of the redistributive effect RE^{rn} in (20) is equal to twice the index of domination due to reranking R^n in (32), and has the same content as Lerman-Yitzhaki index of reranking R^{LY} in (38).

(43) $$RE^{rn} = 2R^n = R^{LY}$$

3 Properties of the redistributive effects, distance narrowing and reranking

The arguments

Above we have defined indices of the redistributive effect, and of distance narrowing and reranking. Now we reveal their interrelatedness and present several important properties. The sections that follow aim to explain and prove the following three arguments. They are important for deriving the main conclusions about the Kakwani and Lerman-Yitzhaki decompositions in the section 4.

(1) *Distance narrowing and reranking are independent*

(2) *Elimination of reranking cannot change the extent of distance narrowing*

(3) *Redistributive effects can be presented as combinations of distance narrowing and reranking*

Distance narrowing and reranking as separate effects

In this section, we prove the Argument 1, that distance narrowing and reranking are distinct and independent concepts. Imagine that we have two lottery boxes, one with balls representing pre-fiscal and the other post-fiscal incomes. We draw the balls one by one randomly and simultaneously from both boxes, and write the *combination* on the board, creating vectors $\mathbf{X}$ and $\mathbf{N}$, with pairs (X_i, N_i), as in Table 1.

Now, observe the formula (29) for distance narrowing and imagine that we repeat the lottery, obtaining many combinations. The fact is that, for each combination, the index Δ will be the same. The distance narrowing index does not depend on the order in which the units are drawn (sorted, ranked). Recall now the two formulas for deprivation / domination due to reranking, (30) and (32). The situation is quite different for reranking: each combination will result in different values of the indices RE^{rx} and RE^{rn}.

For given vectors $\mathbf{X}$ and $\mathbf{N}$, imagine a process of income swapping within any pair of units, so that the first unit obtains the post-fiscal income of the other, and vice versa. Referring to the above, we conclude that such swapping will affect reranking, but not distance narrowing.

We have seen that identity exists between indices RE^{xn} and Δ. We may conclude that the redistributive effect RE^{xn}, except that depicting the transition $\mathbf{X}^x \rightarrow \mathbf{N}^n$, is also a true measure of distance narrowing. This has some important implications: RE^{xn}, in the same way as Δ, is not sensitive to income ranks. Given the elements of the vectors $\mathbf{X}$ and $\mathbf{N}$, for any actual permutation of $\mathbf{N}^x$ and $\mathbf{X}^n$, the indices Δ and RE^{xn} will have the same values.

Thus, RE^{xn} fully registers the distance narrowing effect induced by the fiscal system. At the same time, it is completely indifferent about rank changes of the units in the transition from pre-fiscal to post-fiscal income. These are important messages to users of the index, having a normative significance that should not be neglected. Thus, if we use RE^{xn} as our *sole* measure of the redistributive effect, it means the following:

(a) We do not care about the reranking of units in the transition from pre- to post-fiscal income;

(b) Any final or post-fiscal ranking of units is equally good;

(c) Reranking is neither good nor bad: it does not improve nor does it weaken inequality reduction.

To illustrate the meaning of these conclusions, imagine a case of three units A, B and C, with pre-fiscal incomes 10, 20 and 60. An "equalitarian" would like to see the following post-fiscal incomes: 30, 30 and 30. In this case, maximum distance narrowing Δ and the redistributive effect RE^{xn} would be achieved, $\Delta = RE^{xn} = G_X$.

In an alternative setting, let the post-fiscal incomes of A, B and C be the following: 60, 20 and 10. Thus, C transferred 50 money units to A, and became the "new poor" member of society, while A became the "new rich". In this case we have that $RE^{xn} = 0$, and obviously, everything that one would conclude solely through inspecting RE^{xn} is that the fiscal system did not change the inequality. On the other hand, quite a lot of redistribution has occurred, probably beyond what many observers would regard as acceptable or sustainable. But, RE^{xn} is completely silent about reranking between A and C. How do the other two redistributive effects react?

Impact of a reranking-eliminating transfer

This section and the next one aim to prove Argument 2, regarding the following question: If reranking is somehow eliminated, what would be the impact of that change on the redistributive effect? To answer the question, we must first determine how the reranking could be eliminated. Unfortunately, we are not offered the recipe. However, there is one very intuitive way to achieve this: through transfer of post-fiscal income from the outranking unit to the unit that was outranked. Let us see how such transfer would affect different measures of income redistribution: vertical effect, reranking and the redistributive effect.

A and B are units with pre-fiscal incomes X_u^x and X_v^x and pre-fiscal ranks u and v, such that $X_u^x < X_v^x$ and consequently $u < v$. The fiscal process has resulted in reranking, and A has higher post-fiscal income than B: $N_{u,0}^x > N_{v,0}^x$. The post-fiscal ranks of units A and B are y and z, where $y > z$, because of reranking. Their pre-fiscal incomes are $X_{y,0}^n$ and $X_{z,0}^n$, $X_{y,0}^n < X_{z,0}^n$.

Assume that we want to eliminate reranking between these two units through a transfer of post-fiscal income from A to B equal to $\tau = N_{u,0}^x - N_{v,0}^x$. After the transfer we have new post-fiscal incomes $N_{u,1}^x = N_{v,0}^x$ and $N_{v,1}^x = N_{u,0}^x$, and new pre-fiscal incomes $X_{y,1}^n = X_{z,0}^n$ and $X_{z,1}^n = X_{y,0}^n$.

Proposition 1

A transfer $\tau = N_{u,0}^x - N_{v,0}^x$ of post-fiscal income *from* unit A with pre-fiscal (post-fiscal) rank u (y) to unit B with pre-fiscal (post-fiscal) rank v (z) induces a change of:

(a) The Kakwani vertical effect V^K and Atkinson-Plotnick reranking effect R^{AP} by $2c\tau(u-v)$.

(b) The Lerman-Yitzhaki vertical effect V^{LY} by $2c(z-y)(X_{y,0}^n - X_{z,0}^n)$ and the Lerman-Yitzhaki reranking effect R^{LY} by $-2c(z-y)(X_{y,0}^n - X_{z,0}^n)$.

Proof.

(a) First, observe that reranking-inducing transfer of post-fiscal income does not change the order of units in N_i^x. The changes in post-fiscal incomes are equal to:

$$\Delta N_u^x = N_{u,1}^x - N_{u,0}^x = N_{v,0}^x - N_{u,0}^x = -\tau ;$$

$$\Delta N_v^x = N_{v,1}^x - N_{v,0}^x = N_{u,0}^x - N_{v,0}^x = +\tau .$$

Recall formulas (16) for $RE^x = V^K$, and (19) for $RE^{rx} = R^{AP}$. We may abstract from all the fixed elements and concentrate only on the changes ΔN_u^x and ΔN_v^x. The changes of V^K and R^{AP} are then equal to:

$$\Delta V^K = 2c\big((s-u)(-\tau) + (s-v)\tau\big) = 2c\tau(u-v) .$$

$$\Delta R^{AP} = 2c\big((s-u)(-\tau) + (s-v)\tau\big) = 2c\tau(u-v) .\square$$

(b) Notice that the order of units in vector X_i^n changes because of the reranking-inducing transfer of post-fiscal income. The changes in pre-fiscal income are:

$$\Delta X_y^n = X_{y,1}^n - X_{y,0}^n = X_{z,0}^n - X_{y,0}^n ;$$

$$\Delta X_z^n = X_{z,1}^n - X_{z,0}^n = X_{y,0}^n - X_{z,0}^n .$$

For easier presentation, define the counterfactual transfer $\tilde{\tau} = X_{y,0}^n - X_{z,0}^n$. Recall formulas (17) for $RE^n = V^{LY}$, and (20) for $RE^{rn} = R^{LY}$. The changes of V^{LY} and R^{LY} are as follows:

$$\Delta V^{LY} = 2c\big((s-y)(-\Delta X_y^n) + (s-z)(-\Delta X_z^n)\big) = 2c\big((s-y)(-(-\tilde{\tau})) + (s-z)(-\tilde{\tau})\big) =$$

$$\Delta V^{LY} = 2c(z-y)\tilde{\tau} = 2c(z-y)(X_{y,0}^n - X_{z,0}^n) .$$

$$\Delta R^{LY} = 2c\big((s-y)(\Delta X_y^n) + (s-z)(\Delta X_z^n)\big) = 2c\big((s-y)(-\tilde{\tau}) + (s-z)\tilde{\tau}\big) =$$

$$\Delta R^{LY} = -2c(z-y)\tilde{\tau} = -2c(z-y)(X_{y,0}^n - X_{z,0}^n) .\square$$

From Proposition 1 we conclude that this transfer of post-fiscal income between the two units, which is equal to the difference between their post-fiscal incomes, does not affect the redistributive effect. Let us see how:

(a) The change of Kakwani vertical effect is identical to the change of Atkinson-Plotnick reranking index: $\Delta V^K = \Delta R^{AP} = 2c\tau(u-v)$. Therefore $\Delta RE = \Delta V^K - \Delta R^{AP} = 0$.

(b) The change of Lerman-Yitzhaki vertical effect is the same in absolute amount, but of opposite sign from, the change in the reranking effect:

$\Delta V^{LY} = -\Delta R^{LY} = 2c(w-v)(X^{n}_{v,0} - X^{n}_{w,0})$. Therefore $\Delta RE = \Delta V^{LY} + \Delta R^{LY} = 0$.

Now, imagine a series of reranking-eliminating transfers τ between different units in the population. Each transfer has impact on vertical and reranking indices as shown above, and the total effect is equal to the sum of single impacts. If the transfer process is guided in a specific way, full values of reranking indices can be restored.

Robin Hood regards the current post-fiscal situation, presented in Table 4, as unacceptable, because there is too much reranking. Pre-fiscal income is already earned and cannot be changed or influenced (this is a usual assumption in the analysis of income redistribution). Also, assume that at the moment additional taxes cannot be collected and neither do there exist some reserve funds from which cash benefits could be paid. In this situation, in order to fix the problem, Robin Hood must rely on transfers of post-fiscal income between reranked units: to take from the undeservingly rich and give to the harmed poor.

Table 4 presents incomes of five hypothetical units from Table 1. According to Robin Hood's report, the harmed units are C, who had pre-fiscal rank $i = 4$ and post-fiscal rank of only $k = 1$, and A, with pre-fiscal rank $i = 5$ and post-fiscal rank $k = 4$. Three units (D, E and B) outranked C, while A was outranked by one unit (B).

A series of transfers occurred in four steps described in Table 5 and Table 6. We will concentrate on the former table, while for the latter, the interpretation is analogous. As can be seen in column 2 of Table 5, in the first step a transfer of $\tau_t = 20$ goes from D to C, enlarging the income of C by 20, and decreasing the income of D by the same amount. The consequence is a decrease of R^{AP} by $\Delta R^{AP} = -2c \cdot 60$ (observe that incomes of units participating in transfers are in bold letters).

During the first three steps, C's income has grown to 100, 20 more than he 'deserves'. Thus, in the fourth step, a transfer of 20 goes from C to A, and in column 6 we see the final vector of post-fiscal incomes. We reveal what was Robin Hood's idea: to achieve that pre-fiscal rankings are preserved in the final state. Summing the values in the last row of Table 5, we can see that during the transfer process the index R^{AP} fell by $\Delta R^{AP} = -2c \cdot 160$ in total, which is exactly the starting value of R^{AP}: in the end there is no reranking.

Notice also that by Proposition 1(a), V^K must have also been changed by the same amount of $\Delta V^K = -2c \cdot 160$, leaving the redistributive effect RE unchanged. The Lerman-Yitzhaki index of reranking has changed by $\Delta R^{LY} = -2c \cdot 310$, as shown in the bottom row of Table 6, while according to Proposition 1(b), the vertical effect increased by $\Delta V^{LY} = +2c \cdot 310$.

Table 4: Hypothetical case

Unit	i	X_i^x	N_i^x
D	1	8	40
E	2	12	60
B	3	30	100
C	4	70	20
A	5	180	80

Unit	k	X_k^n	N_k^n
C	1	70	20
D	2	8	40
E	3	12	60
A	4	180	80
B	5	30	100

Table 5: A series of transfers and a change in Atkinson-Plotnick reranking

i	$N_{i,1}^x = N_i^x$	$N_{i,2}^x$	$N_{i,3}^x$	$N_{i,4}^x$	$N_{i,1}^x = N_i^n$
1	2	3	4	5	6
1	**40**	20	20	20	20
2	60	**60**	40	40	40
3	100	100	**100**	60	60
4	**20**	**40**	**60**	**100**	80
5	80	80	80	**80**	100
v_t	1	2	3	4	
$N_{v,t}^x$	40	60	100	100	
w_t	4	4	4	5	
$N_{w,t}^x$	20	40	60	80	
τ_t	20	20	40	20	
$(v-w)\tau$	–60	–40	–40	–20	$= -160$

Table 6: A series of transfers and a change in Lerman-Yitzhaki reranking

k	$X^n_{k,1} = X^n_k$	$X^n_{k,2}$	$X^n_{k,3}$	$X^n_{k,4}$	$X^n_{k,5} = X^x_k$
1	2	3	4	5	6
1	**70**	8	8	8	8
2	**8**	**70**	12	12	12
3	12	**12**	**70**	30	30
4	180	180	180	**180**	70
5	30	30	**30**	**70**	180
$\widetilde{v}_t$	1	2	3	4	
$X^n_{v,t}$	70	70	70	180	
$\widetilde{w}_t$	2	3	5	5	
$X^n_{w,t}$	8	12	30	70	
$\widetilde{\tau}$	62	58	40	110	
$-(\widetilde{w}-\widetilde{v})\widetilde{\tau}$	–62	–58	–80	–110	$=-310$

However, one may wonder: is there any other model of change in the income distribution that would show something different? We can experiment with the following option: A and B are units with pre-fiscal incomes $X^x_a < X^x_{a+1}$, ranks a and $a+1$, and post-fiscal incomes $N^n_a > N^n_{a+1}$. One way of eliminating reranking between them would be to transfer $\tau_0 = (N^n_a - N^n_{a+1})/2$ from A to B, in which case they would have the same incomes. It can be shown that this process would decrease R^{AP} by $4c\tau_0$, while the decrease of V^K would be only $2c\tau_0$, with the final consequence: a rise in RE by $2c\tau_0$!

However, a careful analysis reveals that the above process can be divided into two parts:

(1) A transfer of $\tau_1 = N^n_a - N^n_{a+1} = 2\tau_0$ from A to B that eliminates reranking and reduces both R^{AP} and V^K by $2c\tau_1 = 4c\tau_0$ (thus, $\Delta RE = 0$), and

(2) An additional transfer of $\tau_2 = (N^n_a - N^n_{a+1})/2 = \tau_0$ from B to A, that equalizes their incomes, and increases both V^K and RE by $2c\tau_2 = 2c\tau_0$.

The crucial point is that the increase of the redistributive effect caused by transfer τ_0 is not a consequence of reranking elimination, but of income equalization or distance narrowing between units A and B.

Decompositions of the redistributive effects

This section is devoted to Argument 3, which claimed that the redistributive effects can be presented as combinations of distance narrowing and reranking. First, we deal with the redistributive effect RE^x and after that with RE^n. We also establish a relationship between these and other indices presented earlier in the text.

The redistributive effect RE^x depicts the transition $\mathbf{X}^x \to \mathbf{N}^x$. The same superscript x in both $\mathbf{X}^x$ and $\mathbf{N}^x$ symbolizes that the transition preserves pre-fiscal income ranks. Let us break this transition into two sub-transitions:

(44) $$\mathbf{X}^x \to \mathbf{N}^x \Leftrightarrow \left(\mathbf{X}^x \to \mathbf{N}^n\right) \to \left(\mathbf{N}^n \to \mathbf{N}^x\right)$$

The first sub-transition, $\mathbf{X}^x \to \mathbf{N}^n$, ascribes to each unit with pre-fiscal income rank i and pre-fiscal income X_i^x its counterfactual post-fiscal income N_i^n; N_i^n is a post-fiscal income of the unit with rank i on the post-fiscal ranking scale. Thus, the sub-transition $\mathbf{X}^x \to \mathbf{N}^n$ breaks the ranking link. Another sub-transition, $\mathbf{N}^n \to \mathbf{N}^x$, restores the ranking link between pre- and post-fiscal income.

We can write: $X_i^x - N_i^x = \left(X_i^x - N_i^n\right) + \left(N_i^n - N_i^x\right)$. Summing over (i, j) and multiplying by c we obtain:

(45) $$c\left(\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(X_i^x - X_j^x\right) - \sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(N_i^x - N_j^x\right)\right) =$$

$$c\left(\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(X_i^x - X_j^x\right) - \sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(N_i^n - N_j^n\right)\right) +$$

$$+c\left(\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(N_i^n - N_j^n\right) - \sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(N_i^x - N_j^x\right)\right)$$

Comparing (45) with (16), (18), (19), (29), (30) and (40) we reach several conclusions. First, the redistributive effect RE^x can be decomposed into a sum of the redistributive effects RE^{xn} and RE^{rx}.

(46) $$RE^x = RE^{xn} + RE^{rx}$$

Second, the redistributive effect RE^x, which corresponds to the fiscal deprivation index V^x, can be decomposed into distance narrowing and deprivation due to reranking effects.

(47) $$RE^x(=V^x) = \Delta + 2R^x$$

Third, when (46) or (47) is translated into terms of traditional indices, we obtain that the Kakwani vertical effect V^K $(= RE^x = V^x)$ is the sum of the redistributive effect RE $(= RE^{xn} = \Delta)$ and the Atkinson-Plotnick index of reranking R^{AP} $(= RE^{rx} = 2R^x)$.

(48) $$V^K = RE + R^{AP}$$

We conclude that V^K is composed of distance narrowing and reranking. The identification of V^K with V^x results in further interesting conclusions. $V^K(=V^x)$ now also represents total fiscal deprivation, and should be compared to total reduction of income distance Δ. The difference between these two is the *excess* fiscal deprivation ($R^{AP} = V^K - RE$), the part of total V^K not necessary to achieve actual distance narrowing Δ.

The redistributive effect RE^n explains the transition $\mathbf{X}^n \to \mathbf{N}^n$. The superscript n in both vectors means that the transition preserves post-fiscal ranking. As in the previous section, we break this transition into two sub-transitions. The decomposition is slightly more complicated, with minus signs meaning the transition goes in the opposite direction.

(49) $$\mathbf{X}^n \to \mathbf{N}^n \Leftrightarrow -\left(\mathbf{N}^n \to \mathbf{X}^n\right)$$

$$\Leftrightarrow -\left(\left(\mathbf{N}^n \to \mathbf{X}^x\right) + \left(\mathbf{X}^x \to \mathbf{X}^n\right)\right) \Leftrightarrow -\left(-\left(\mathbf{X}^x \to \mathbf{N}^n\right) + \left(\mathbf{X}^x \to \mathbf{X}^n\right)\right)$$

$$\Leftrightarrow \left(\mathbf{X}^x \to \mathbf{N}^n\right) - \left(\mathbf{X}^x \to \mathbf{X}^n\right)$$

The first sub-transition, $\mathbf{X}^x \to \mathbf{N}^n$, is distance narrowing and breaks the ranking link. However, another sub-transition, $\mathbf{X}^x \to \mathbf{X}^n$, restores it. We can write: $X_i^n - N_i^n = \left(X_i^x - N_i^n\right) - \left(X_i^x - X_i^n\right)$. Summing over (i, j) and multiplying by c we obtain:

(50) $$c\left(\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(X_i^n - X_j^n\right) - \sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(N_i^n - N_j^n\right)\right) =$$

$$c\left(\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(X_i^x - X_j^x\right) - \sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(N_i^n - N_j^n\right)\right) -$$

$$-c\left(\sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(X_i^x - X_j^x\right) - \sum_{i=2}^{s}\sum_{j=1}^{i-1}\left(X_i^n - X_j^n\right)\right)$$

Comparing (50) with (17), (19), (20), (32), (33) and (41) we may reach several conclusions. Firstly, the redistributive effect RE^n can be decomposed into difference of RE^{xn} and RE^{rx}.

(51) $$RE^n = RE^{xn} - RE^{rn}$$

Secondly, the redistributive effect RE^n, which is identical to the reverse fiscal domination index V^n, can be decomposed into effects of distance narrowing and domination due to reranking.

(52) $$RE^n(=V^n) = \Delta - 2R^n$$

Finally, "translating" (51) and (52), we obtain a decomposition of Lerman-Yitzhaki index of "progressivity" V^{LY} $(= RE^n = V^n)$ into the redistributive effect RE $(= RE^{xn} = \Delta)$ and the Lerman-Yitzhaki index of reranking R^{LY} $(= RE^{rn} = 2R^n)$.

(53) $$V^{LY} = RE - R^{LY}$$

It can be seen that V^{LY}, just as V^K, can be decomposed into distance narrowing and reranking.

Analysis: a series of small transfers between two units

This section again relies on an experiment with transfers, but this time we deal with a series of small transfers. Up to now, we have not considered the meaning of the weights in the Gini index, $\omega(p_i;2) = s - i$, described earlier. Interpretation is straightforward: the units with lower positions i receive larger weights, and vice versa. It can be shown that a small transfer σ

from the unit with rank v to the unit with rank $w < v$ will decrease the Gini coefficient by $2c\sigma(v-w)$.

A *small* transfer from the rich to the poor decreases inequality and increases welfare because the sacrifice felt by the rich is valued as less important than the marginal benefit to the poor. We must stress that the terms "poor" and "rich" correspond to the relative positions of persons involved, *before* and *after* the transfer.

Now, imagine a *series of small transfers* from the rich B to the poor A. Obviously, after each of these transfers B will be becoming less rich and A will be getting less poor: the income distance between them will be narrowing and the income supremacy of B will be falling. In one moment, these persons' incomes will be equalized. After that point, the next small transfer from B to A will reverse the situation: the "poor" A will become the rich one, and the "rich" B will become the poor. Reranking occurs. Suppose that the transfers continue to the point where B and A completely swap their incomes. How do the measurement concepts analyzed in this study respond to the challenge? We analyze the changes of our indices during a series of small transfers between two hypothetical units in the following example.

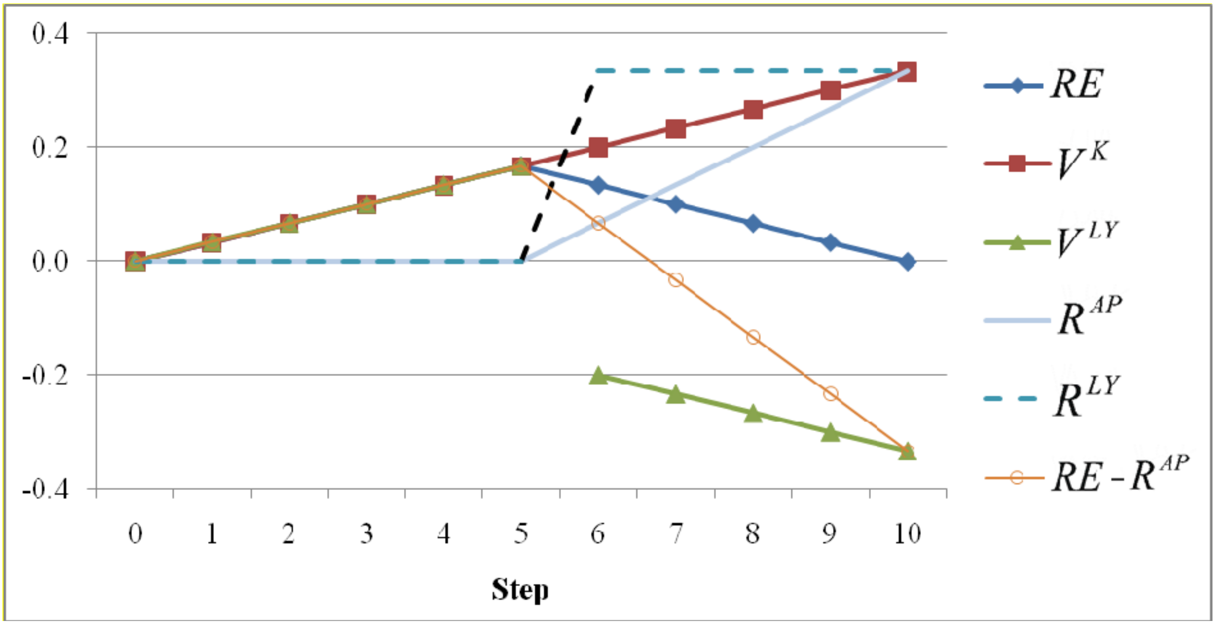
The transfer process is presented both in Table 7 and Figure 3. Unit A starts with income of 10 and ends with 20, while it is the opposite for B. There are ten steps, each presenting a small transfer of 1 monetary unit (not all steps are shown in the table, for better visibility). Indices of the redistributive effect and reranking for each step are all calculated with respect to step 0. Thus, for example, indices in step 7 are based on pre-fiscal incomes 10 and 20, and post-fiscal incomes 17 and 13, for A and B respectively.

Table 7: Small transfers and indices of the redistributive effect

Step	0	1	3	5	6	7	9	10
Income of A	10	11	13	15	16	17	19	20
Income of B	20	19	17	15	14	13	11	10
RE	0	0.033	0.100	0.167	0.133	0.100	0.033	0
V^K	0	0.033	0.100	0.167	0.200	0.233	0.300	0.333
R^{AP}	0	0	0	0	0.067	0.133	0.267	0.333
V^{LY}	0	0.033	0.100	0.167	-0.200	-0.233	-0.300	-0.333
R^{LY}	0	0	0	0	0.333	0.333	0.333	0.333
$RE-R^{AP}$	0	0.033	0.100	0.167	0.067	-0.033	-0.233	-0.333

In the first 5 steps, there is no reranking ($R^{AP} = R^{LY} = 0$), and therefore the three redistributive effects are identical: $RE = V^K = V^{LY}$. In the 5th step, the incomes are equalized and the distance narrowing ($RE = \Delta$) reaches its maximum of 0.167. After this point, the redistributive effects completely diverge: (a) RE falls back toward zero; (b) V^K continues to grow; (c) V^{LY} has a breaking point at the 5th step, when it drops significantly and continues to fall in later steps. Reranking effects also behave differently. R^{LY} is equal for all steps after the 5th, while R^{AP} grows toward the value of V^K in the 10th step.

Figure 3: Small transfers and indices of the redistributive effect



As a potpourri to the discussion of the above hypothetical results, we cite a lucid argument delivered by Lerman and Yitzhaki (1995), in their critique of Kakwani vertical effect:

> *Imagine a rich taxpayer who becomes poor because of heavy taxation. According to before-tax rankings, the taxpayer will continue to be considered as rich even if the tax causes him to become poor. Reliance on the before-tax ranking may lead the analyst to recommend increasing a tax on progressivity grounds even though the additional tax will be paid by the poor.*

And this is exactly what we can conclude observing the development of V^K in our example, after the reranking has occurred in the 5th step. In subsequent steps, unit B, who was rich, now becomes poorer and poorer, but V^K increases yet further. Thus the measure V^K "rewards" reranking, which looks contradictory since we know that it is based on pre-fiscal ranks, and given that fact, it should “protect” the pre-fiscally richer. To our surprise, this *is* achieved by V^{LY}, the measure based on post-fiscal ranks. V^{LY} falls as we go to the right from the 5th step, thus “penalizing” reranking.

Our example confirms that Lerman and Yitzhaki were right when saying that dependence on pre-fiscal ranks would lead the analyst to recommend more redistribution even when reranking has occurred and the formerly rich became the poor. V^{K} continues to rise even when the “rich” person is left with zero or negative income. This was one of the reasons which caused them to propose their index V^{LY}, which is attractive, but also has a deficiency. Observe in the example that between steps 5 and 6 there is only a small difference, but the index falls drastically, from $+0.167$ to -0.200. The reason for such a plunge lies in R^{LY}, which appears as a deducting element in $V^{LY} = RE - R^{LY}$. Recall that R^{LY} is based exclusively on pre-fiscal incomes, which do not change in our experiment and are the same all the way, once reranking has occurred.

One intuitive choice, although not based on algebraic facts, was to draw a curve that also deducts reranking from the redistributive effect, but using R^{AP} instead of R^{LY}. We obtained a measure $RE - R^{AP}$ (recall that $V^{K} = RE + R^{AP}$), which does have a quality of falling when the outranked person further loses her income, but there is no break in the turning point at the 5th step. The latter is due to the fact that R^{AP} is based on post-fiscal incomes.

4 Setting the new context for existing indices

Problems with Kakwani and Lerman-Yitzhaki decompositions

Kakwani (1984) and Lerman and Yitzhaki (1995) derived two different, but conceptually related decompositions of the redistributive effect (RE) into vertical and reranking effects. The former became one of the most widely used tools in the analysis of the redistributive effect, while the latter aimed to replace it, but without success. Urban (2009) thoroughly describes their origins and debates on certain unsolved issues, which are dealt with extensively in the present paper. The decompositions are respectively represented by the following two equations.

(54) $$RE = V^{K} - R^{AP}$$

(55) $$RE = V^{LY} + R^{LY}$$

Kakwani decomposes RE into a *difference* between vertical and reranking effects, while Lerman and Yitzhaki decompose RE into a *sum* of vertical and reranking effects. By construction, the reranking effects, R^{AP} and R^{LY}, are always positive, while vertical effects may be either positive or negative.

Based on the algebraic constructions of the formulas, the authors respectively concluded that R^{AP} contributes negatively, while R^{LY} contributes positively to the redistributive effect RE. For them, reranking plays a distinctive role in the determining the magnitude of RE. For Kakwani, reranking deteriorates RE, while for Lerman-Yitzhaki it improves RE. For both Kakwani and Lerman-Yitzhaki, the respective vertical effects V^K and V^{LY} are also standalone concepts, completely independent of reranking. Kakwani (1984) identifies V^K with *potential* redistributive effect, interpreted as the amount of RE that would be achieved in the absence of reranking. Thus, RE could be increased through *elimination* of reranking, while at the same time V^K would remain unchanged. Lerman and Yitzhaki (1995) follow this interpretation, but in their version, RE could be enlarged through *enhancement* of reranking, while V^{LY} would stay the same.

In the foregoing sections, we have provided a lot of material to answer the problem with these interpretations of indices. The principal concern is a specific connection between vertical and reranking effects. Each attempt to decrease (increase) overall reranking R^{AP} (R^{LY}), automatically leads to a decrease (decrease) of vertical effect V^K (V^{LY}). The consequence is that RE remains unchanged.

The most illustrative proof of this contention was the analysis of the impact of a series of transfers between population units which eliminate reranking. Further evidence about the relation between reranking and vertical effect is that V^K (V^{LY}) is a sum (difference) of distance narrowing and reranking, as shown by equations (48) and (53). Recall that it was proven that distance narrowing and reranking are separate and independent concepts.

These conclusions support Atkinson's (1980) views that "changes in the ranking of observations as a result of taxation do not in themselves affect the degree of inequality in the post-tax distribution". In other words, since the distribution of pre-tax income is also assumed to be unchanged by taxation, Atkinson claimed that reranking does not influence the redistributive effect (RE). However, the suggestion was ignored in the subsequent work of both Kakwani and Lerman and Yitzhaki.

We have demonstrated another problem with the Kakwani decomposition, advanced by Lerman and Yitzhaki (1995), using an appealing example of taxation which makes a rich person poor. The Kakwani vertical effect (V^K) rewards reranking, “asking for” an ever larger take from the formerly rich, now poor, and giving to the formerly poor, now the rich. At the same time, proponents of the Kakwani decomposition blame reranking for this trouble. If reranking were eliminated, the redistributive effect would increase to V^K . But, as we have already seen, there is no practicable scheme that would tell us how to achieve this.

Which indices to use?

After a thorough discussion of the existing methodologies and criticism of their contemporary interpretations, a course for future research should be provided. A straight answer to the question posed by this section title will perhaps sound surprising: the same indices we used before; however, with an important distinction: they must be interpreted properly. In this section we discuss acceptable interpretations for each of these indices.

Recall that we analyzed properties of the indices (of redistributive, vertical and reranking effects) using different approaches (vector transitions and income units’ “feelings”). Each of them revealed a certain interesting aspect of the measure the researchers should have in mind when clarifying the meaning of their estimated indicators. In Table 8 we summarize these aspects for five indices and two approaches, and then explain how each index should be treated.

Table 8: Interpretation of indices

	Vector transitions	Income units’ “feelings”
RE	$\mathbf{X}^x \rightarrow \mathbf{N}^n$; breaks the link between pre- and post-fiscal incomes	distance narrowing; Δ
V^K	$\mathbf{X}^x \rightarrow \mathbf{N}^x$; preserves the link between pre- and post-fiscal incomes. Decomposable into $\mathbf{X}^x \rightarrow \mathbf{N}^n$ and $\mathbf{N}^n \rightarrow \mathbf{N}^x$	fiscal deprivation; V^x
R^{AP}	$\mathbf{N}^n \rightarrow \mathbf{N}^x$, reranks post-fiscal incomes	deprivation due to reranking; R^x
V^{LY}	$\mathbf{X}^n \rightarrow \mathbf{N}^n$; preserves the link between pre- and post-fiscal incomes. Decomposable into $\mathbf{X}^x \rightarrow \mathbf{N}^n$ and $\mathbf{X}^x \rightarrow \mathbf{X}^n$	reverse fiscal domination; V^n
R^{LY}	$\mathbf{X}^x \rightarrow \mathbf{X}^n$; reranks pre-fiscal incomes	domination due to reranking; R^n

The redistributive effect (RE). This will remain the main indicator of the redistributive effect. RE is synonymous with distance narrowing and is indifferent about rank changes. For two systems with equal distance narrowing ($\Delta = RE$) and different amounts of reranking, RE will be identical. Thus, analysts who do think that reranking has a negative or positive normative significance, will consider the indices below as a supplement to RE.

Kakwani vertical effect V^K. We have seen different problems with the index itself, and also with its contemporary interpretation . Should we completely avoid the use of V^K? In one of its forms, the index can still be interesting: as a measure of fiscal deprivation ($V^x = V^K$).

Take an analyst who holds that the fiscal system should preserve *differences in incomes*. In other words, this principle says that everybody should pay (receive) *equal* amounts of taxes (benefits). Then, the index V^x measures the violation of this principle: positive fiscal deprivation means that the richer lost their income advantages over the poorer. Additionally, in case of reranking, the richer people not only use their income supremacy, but end up poorer, and this notion is captured by V^x as compared to RE.

Atkinson-Plotnick reranking effect R^{AP}. In the context of fiscal deprivation, $R^{AP} = 2R^x$ is titled excess fiscal deprivation. It is a part of total fiscal deprivation (V^x), that stands above the fraction of fiscal deprivation that is necessary to achieve actual distance narrowing (Δ).

This is perhaps an opportunity to divorce R^{AP} from V^K, with whom it was unhappily married during the last 25 years. Unlike the other term, R^{AP} remains what it was since its appearance: an index measuring the extent of reranking caused by the fiscal process. It is a perfect complement of RE in judging the redistributive performance of the fiscal system.

Lerman-Yitzhaki vertical effect V^{LY} and reranking effect R^{LY}. Lerman and Yitzhaki (1995) called V^{LY} the index of “gap-narrowing”, assuming that it quantifies a process that is independent of reranking. We have seen that the contention was wrong: V^{LY} decreases with the increase of reranking. In this paper, the similar term “distance narrowing” is used for a truly independent concept, measured by RE.

Nevertheless, V^{LY} can be an interesting choice for the analyst who appreciates distance narrowing, but believes that pre-fiscal rankings should be preserved. V^{LY} is a single measure that combines both of these notions and is suitable for a comparison of performance of different fiscal systems. It is higher the larger the distance narrowing and the lower the reranking.

R^{LY} can be used as a measure of reranking in the same way as R^{AP}. Remember that the difference between the two lies in the income vector on which they are built: in the former case it is pre-fiscal income, and in the latter, post-fiscal income , which makes it slightly more intuitive.

Analogously to V^K (V^x) and R^{AP} ($2R^x$), there are alternative interpretations for V^{LY} and R^{LY}, in terms of fiscal domination. The reverse fiscal domination index $V^n = V^{LY}$ is a counterpart to the index of fiscal deprivation, and suitable for analysts who consider that the fiscal process should insist on reranking of units, disrespecting pre-fiscal ranks.

5 References

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6 Appendix

EXAMPLE 1

Table 9: Index of distance narrowing

$\mathbf{M}_1(i,j) = |X_i - X_j|$

B	30	150			
C	70	110	40		
D	8	172	22	62	
E	12	168	18	58	4
		180	30	70	8
		A	B	C	D

$\mathbf{M}_2(i,j) = |N_i - N_j|$

B	100	20			
C	20	60	80		
D	40	40	60	20	
E	60	20	40	40	20
		80	100	20	40
		A	B	C	D

The index of distance narrowing is obtained by (29) as a difference between the sums of values within triangles $\mathbf{M}_1$ and $\mathbf{M}_2$ of Table 9, multiplied by c.

$$\Delta = (5^{-2} \cdot 60^{-1}) \cdot (804 - 400) = 0.2693$$

EXAMPLE 2

Table 10: Indices of deprivation and domination due to reranking

$\mathbf{M}_1(i,j) = r^x_{i,j}$

E	60	0			
B	100	0	0		
C	20	20	40	80	
A	80	0	0	20	0
		40	60	100	20
		D	E	B	C

$\mathbf{M}_2(i,j) = r^n_{i,j}$

D	8	62			
E	12	58	0		
A	180	0	0	0	
B	30	40	0	0	150
		70	8	12	180
		C	D	E	A

The triangle $\mathbf{M}_1$ in Table 10 contains values of $r^x_{i,j}$ obtained by (22). There are four non-zero values indicating the cases of reranking, for the following pairs of units (C,D), (C,E), (C,B) and (A,B). For these pairs we have that $N^x_j > N^x_i$: units with pre-fiscal ranks i were outranked or deprived by those with ranks j. Total deprivation due to reranking of the unit C with the pre-fiscal rank $i = 4$ is equal to $(N^x_1 - N^x_4) + (N^x_2 - N^x_4) + (N^x_3 - N^x_4)$=140, and deprivation of the unit A, with the pre-fiscal rank $i = 5$, is $(N^x_4 - N^x_5)$=20.

The triangle $\mathbf{M}_2$ in Table 10 contains values of $r^n_{i,j}$ obtained by (26). Again, there are four non-zero values, for the pairs (D,C), (E,C), (B,C) and (B,D). Notice that these are the same pairs as above, because we are using the same hypothetical example. For these pairs we have that $X^n_j > X^n_i$: the units with post-fiscal rankings i outranked or *dominated* those with rankings j. Total domination due to reranking over the unit C with the post-fiscal rank $j=1$ is $(X^n_1 - X^n_2)$ $+(X^n_1 - X^n_3) + (X^n_1 - X^n_5)$=160, and of the unit A, with the post-fiscal rank $j=4$, is $(X^n_4 - X^n_5)$=150.

The indices of deprivation and domination due to reranking are calculated according to (30) and (32), respectively, summing the values inside $\mathbf{M}_1$ and $\mathbf{M}_2$, and multiplying them by c.

$$R^x = (5^{-2} \cdot 60^{-1}) \cdot 160 = 0.1067$$

$$R^n = (5^{-2} \cdot 60^{-1}) \cdot 310 = 0.2067$$

EXAMPLE 3

Table 11: Fiscal deprivation and domination indices

$\mathbf{M}_1(i,j) = \varsigma^x_{i,j}$

	D	E	B	C
E	-16			
B	-38	-22		
C	82	98	120	
A	132	148	170	50

$\mathbf{M}_2(i,j) = \varsigma^n_{i,j}$

	C	D	E	A
D	82			
E	98	16		
A	-50	-132	-148	
B	120	38	22	170

The triangle $\mathbf{M}_1$ in Table 11 contains values of $\varsigma^x_{i,j}$ obtained by (23). The triangle $\mathbf{M}_2$ contains values of $\varsigma^n_{i,j}$ obtained by (27). The indices of fiscal deprivation and domination can be easily calculated using (31) and (33), respectively, summing the values in $\mathbf{M}_1$ and $\mathbf{M}_2$ multiplying them with c.

$$V^x = (5^{-2} \cdot 60^{-1}) \cdot 724 = 0.4827$$

$$V^n = -(5^{-2} \cdot 60^{-1}) \cdot 216 = -0.1440$$