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Abstract
The paper uses two particular formulations of the Gini index to derive two different relative deprivation measures. We then generalize the formulation of these measures following Donaldson and Weymark (1980) and Berrebi and Silber (1985) and show how these generalizations can be considered as two different classes of indexes. An example illustrates how the use of the two classes of indexes can lead to different results in empirical applications.

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1. Introduction

The concept of relative deprivation may be described as a situation where a person or group of people feel the lack of something in relation to other persons who own that particular something. Such a general definition cannot be traced back to any particular scholar and surely most social scientists in history have dealt in some way with the idea behind the concept of relative deprivation. But the term relative deprivation and its dissemination during the second half of the twentieth century is often attributed to Stouffer et al. (1949) and to Runciman (1966). Stouffer et al. (1949) are credited for having introduced the term relative deprivation in the context of a study on US soldiers during World War Two while Runciman (1966) is credited for having picked up this concept and turned it into a full theory of social justice.

In economics, the concept of relative deprivation has been formalized and simplified by Yitzhaki (1979) who devised a relative deprivation index based on the notion that individuals compare their own income with the incomes of richer individuals so that relative deprivation can be measured as the sum of the distances between one's own income and the incomes of all other richer individuals. Yitzhaki's seminal paper has been followed by numerous contributions that explored further the properties of such index and various extensions, variations and applications (Hey and Lambert, 1980; Kakwani, 1984; Berrebi and Silber, 1985; Chakravarty, 1995, 1997; Podder, 1996; Bossert et al. 2004).

This latter tradition in economics has also explored the relation between measures of inequality and measures of relative deprivation. Yitzhaki was the first to notice in his seminal 1979 article that the relative deprivation measure he constructed was in fact equivalent to the absolute Gini index. The last three decades of the twentieth century have also seen major developments in the study of inequality enlightened by the works of Sen, Atkinson and Kolm among others and these contributions helped to deepen our understanding of the relation between inequality and relative deprivation measures. In particular, Donaldson and Weymark (1980) proposed a generalized formulation of the Gini index building on the Atkinson-Kolm-Sen (AKS) index of inequality and Berrebi and Silber (1985) used Donaldson and Weymark's work to propose a generalization of relative deprivation measures derived from the Gini index.

Building on this tradition, this paper takes a closer look at two particular formulations of the Gini index that have been used in the past to derive relative deprivation measures and illustrates how these formulations can be considered as two different *classes* of relative deprivation measures. We then generalize these formulations following Donaldson and Weymark (1980) and Berrebi and Silber (1985) and show how the two different formulations of relative deprivation proposed correspond to two different normative approaches to the study of relative deprivation and how these two approaches result in different individual deprivation functions that can potentially lead to very different results in empirical applications.

In section two we distinguish between two classes of Gini indexes. In section three we use this distinction to propose two classes of generalized deprivation indexes and in section four we illustrate the two classes of indexes graphically. Section 5 concludes.

2. Relative, Absolute and Generalized Gini indexes

Let y represent income and $y_{\uparrow} = (y_{(1)}, y_{(2)}, ..., _{(n)})$ represent a vector of positive incomes sorted in descending order of income so that $y_{(1)} \geq y_{(2)} \geq ... \geq y_{(n)}$. Let also $\pi = (\pi_1, \pi_2, ..., \pi_n)$ be a vector of positive numbers where π_1 is the number of persons with incomes $y_{(1)}$. An income distribution is then described by the vector $(y_{\uparrow}, \pi)$ and a person with income j (j_{th} individual) is richer than a person with income i (i_{th} individual) if and only if $j < i$ ($i > j$). For simplicity of exposition, we will assume that $\pi_1 = \pi_2 = ... = \pi_n = 1$. Also, $\bar{y}$ will be used to indicate the arithmetic mean of distribution $(y_{\uparrow}, \pi)$.

Following Gini (1912), Kolm (1976a, 1976b), Atkinson (1970) and Sen (1973), the Gini index of relative inequality may be written as

$$RG = 1 - [\frac{1}{n^2 \bar{y}} \sum_{i=1}^{n} (2i - 1) y_{(i)}] \quad (1)$$

The Gini index of absolute inequality is then defined as the relative index multiplied by the mean

$$AG = \bar{y} RG = \bar{y} - [\frac{1}{n^2} \sum_{i=1}^{n} (2i - 1) y_{(i)}] \quad (2)$$

Following Donaldson and Weymark (1980), call

$$E_2 = \frac{1}{n^2}\sum_{i=1}^{n}(2i-1)y_{(i)} = \frac{\sum_{i=1}^{n}[i^2-(i-1)^2]y_{(i)}}{n^2} = \sum_{i=1}^{n}[(\frac{i}{n})^2-(\frac{i-1}{n})^2]y_{(i)} \tag{3}$$

where $\frac{i}{n}$ is the cumulative population or the proportion of the population earning at least as individual i and $\frac{(i-1)}{n}$ is the proportion of the population earning more than individual i. We may then write the two Gini indexes as

$$RG = 1 - \frac{E_2}{\bar{y}} \tag{4}$$

and

$$AG = \bar{y} - E_2 \tag{5}$$

Donaldson and Weymark (1980) have shown that the two indexes can be generalized to a class of single parameter indexes (S-gini) by substituting to E_2

$$E_\alpha = \sum_{i=1}^{n}[(\frac{i}{n})^\alpha-(\frac{i-1}{n})^\alpha]y_{(i)} \tag{6}$$

with $\alpha \geq 2$ representing an inequality aversion parameter (equal to two for the Gini). The greater is α, the greater is the weight attributed to lower values of incomes. The generalized (relative) S-gini index can then be described as

$$RG = 1 - [\sum_{i=1}^{n}[(\frac{i}{n})^\alpha-(\frac{i-1}{n})^\alpha]\frac{y_{(i)}}{\bar{y}}] \tag{7}$$

With $a_i = (\frac{i}{n})^\alpha - (\frac{i-1}{n})^\alpha$, and given that $a_{(1)} \leq a_{(2)}.... \leq a_{(i)}... \leq a_{(n)}$ and $\sum_{i=1}^{n} a_{(i)} = 1$, we can also express the relative S-gini as

$$RG = \sum_{i=1}^{n} a_{(i)} - \sum_{i=1}^{n} a_{(i)}\frac{y_{(i)}}{\bar{y}} = \sum_{i=1}^{n} a_{(i)}\frac{\bar{y}-y_{(i)}}{\bar{y}} \tag{8}$$

and this is can be generalized to

$$RG_a = \sum_{i=1}^{n}[(\frac{i}{n})^\alpha-(\frac{i-1}{n})^\alpha]\frac{\bar{y}-y_{(i)}}{\bar{y}} \tag{9}$$

$$RG_a = \frac{1}{n^\alpha} \sum_{i=1}^{n} [(i)^\alpha - (i-1)^\alpha] \frac{\bar{y} - y_{(i)}}{\bar{y}} \tag{10}$$

where $\frac{i}{n}$ represents the cumulated population up to i, $\frac{(i-1)}{n}$ represents the population proportion richer than i and where individuals are ranked in decreasing order or $y/\bar{y}$. We call these indexes the *a-form* Gini indexes (indicated by the subscript a). The value expressed as differences of rank in squared parentheses can be considered as the weight (determined by rank) attached to each income difference. This weight will be greater the lower is income (the greater is rank) and will increase exponentially with increasing values of α.

Following Berrebi and Silber (1985) the relative Gini index of equation (1) can also be written as

$$RG = \sum_{i=1}^{n} [(\frac{n-i}{n}) - (\frac{i-1}{n})] \frac{y_{(i)}}{n\bar{y}} \tag{11}$$

and this can also be generalized to[1]

$$RG_b = \sum_{i=1}^{n} [(\frac{n-i}{n})^{\alpha-1} - (\frac{i-1}{n})^{\alpha-1}] \frac{y_{(i)}}{n\bar{y}} \tag{12}$$

$$RG_b = \frac{1}{n^\alpha} \sum_{i=1}^{n} [(n-i)^{\alpha-1} - (i-1)^{\alpha-1}] \frac{y_{(i)}}{\bar{y}} \tag{13}$$

where $\frac{(n-i)}{n}$ represents the population proportion earning less than individual i and $\frac{(i-1)}{n}$ represents the population proportion earning more as in equation (9). We call these indexes the *b-form* Gini indexes (indicated by the subscript b).

3. Two Classes of Generalized Deprivation indexes

The two forms of Gini proposed in the previous section ($RG_a and RG_b$) can be used to introduce two different classes of deprivation indexes. Consider first the *a-form* Gini in relative terms. This index can be regarded as a relative deprivation index where 'deprivation' is defined as the distance between mean income and individual income normalized

[1]Note that in equation (12) the exponential is correctly specified as $\alpha - 1$ to provide a comparable formula to equation (9).

by the mean ($\frac{\bar{y}-y_i}{\bar{y}}$) and weighted by a function of rank. Thus, a generalized measure of relative deprivation in the *a-form* is defined as

$$RD_a = \frac{1}{n^\alpha}\sum_{i=1}^{n}[(i)^\alpha - (i-1)^\alpha]\frac{\bar{y}-y_{(i)}}{\bar{y}} \tag{14}$$

While the formula for relative deprivation in the *a-form* is the same as the Gini index in the *a-form*, the interpretation of the index at the social and individual level is different. At the social (aggregate) level the Gini and relative deprivation indexes provide the same value and the interpretation of the index depends simply on whether we consider the index a measure of inequality or deprivation. As already mentioned in the introduction, considering inequality and deprivation as similar concepts is not new in the literature. Yitzhaki (1979), for example, has shown how his relative deprivation index is equal to the absolute Gini index and, as shown in the previous section, Berrebi and Silber (1985) used a particular formulation of the Gini index to devise their generalized index of deprivation.

At the individual level, the individual scores for the relative Gini and deprivation measures are also the same but the meaning is clearly different. The individual Gini score does not indicate individual inequality (which is meaningless) or the individual contribution to total inequality. In fact, the Gini individual values cannot be aggregated by sub-groups so as to indicate the contribution of each group to total inequality and the Gini cannot be decomposed by sub-groups in this manner (although it can be decomposed by population subgroups in within and between inequality). On the contrary, individual deprivation defined as above is a measure of individual deprivation and can be aggregated into sub-groups to represent sub-group deprivation. Individual deprivation in this case is defined as the shortfall from mean income (relative to the mean) weighted by a function of rank. These properties are very useful in empirical applications if one wants to decompose deprivation by population subgroups or use deprivation as an individual variable in econometric models. For example, the vector of deprivation scores could be used as a variable to explain a wide range of phenomena such as life satisfaction or social unrest.

A second and different class of deprivation indexes can be defined using the *b-form* Gini indexes as first suggested by Berrebi and Silber (1985). Individual deprivation can be regarded as the difference between the share of people above and the share of people below in the income distribution, which amounts to reverting the order of the two components $(n-i)$ and $(i-1)$ in equation (11). The greater this difference, the greater is deprivation. Moreover, as shown by Berrebi and Silber (1985), the index can be generalized in the same way as Donaldson and Weymark (1980) generalized the Gini index so that the second class of generalized deprivation indexes (the *b-class*) can be defined as

$$RD_b = \frac{1}{n^\alpha}\sum_{i=1}^{n}[(i-1)^{\alpha-1} - (n-i)^{\alpha-1}]\frac{y_{(i)}}{\bar{y}} \tag{15}$$

In this case, we can say that deprivation is determined by rank (the difference between the shares of people above and below in the income distribution) weighted by relative income (relative to the mean).

The two forms of indexes proposed present some relevant similarities as well as dissimilarities. Both forms are expressed as sums across the population of individual values and they are both normalized by a population factor n^α. Also, both forms lead to the same aggregated value (although the *b-form* carries a negative sign). Instead and as already remarked, the *a-form* index defines deprivation in terms of relative income distances from the mean and weighs these distances with rank. Vice-versa, the *b-form* index defines deprivation in terms of rank and weighs rank by relative income. We could argue that the *a-form* is income defined and rank weighted and that the *b-form* is rank defined and income weighted. As a consequence, the individual measures of deprivation are different and represent two different perspectives on the measurement of deprivation.

4. A graphical illustration

To illustrate the deprivation indexes proposed, we use a reduced sample from the 2000 British Household Panel Survey (BHPS) restricting the population to employees in age 41-50 and using as a measure of welfare income net of taxes. We also removed extreme values of the distribution to better appreciate the shapes of the distributions. The remaining sample includes 709 observations.[2]

[2]The BHPS sample used is extracted from the Consortium of Household Panels for European Socio-economic Research (CHER), an harmonized data set of panel surveys managed by CEPS/INSTEAD in Luxembourg. More information on the data set can be found on:

In Figure 1, we plot the individual functions of the two indexes and their respective rank and income components separately. All components in Figure 1 are normalized by population size and the population is plotted on the x-axis. We also repeat this exercise using two different reference incomes and two different values of α. As reference incomes we use mean income and a relative poverty line equal to 50% of the median value (a standard choice of poverty line in the European Union). As values of α, we use $\alpha = 2$ and $\alpha = 3$. Both types of indexes are sorted in descending order of income.[3]

For both the a and b types of indexes, the rank component is always monotonic positive. The rank component increases with rank and this is true for both indexes. With increasing values of the α index the rank component takes an exponential shape attributing more weight to higher ranked (poorer) individuals. With the *a-form* index, the income component is always monotonic positive while with the *b-form* is always monotonic negative as evident from the algebra of these components and the sorting of incomes in descending order. These particular combinations result in the *a-form* deprivation functions to be convex and in the *b-form* deprivation functions to be concave. With the *a-form* index, the rank component, which we considered as the 'weight' of the index, turns the individual deprivation function U-shaped. In substance, the two indexes attribute different values of deprivation to the same individuals and represent two different normative views of relative deprivation at the individual level.

5. Conclusion

Following Donaldson and Weymark (1980) and Berrebi and Silber (1985), the paper has reconsidered two generalized formulations of the Gini index and noted that these formulations can be used to derive two generalized indexes of relative deprivation. We then argued that these indexes can be considered as two separate classes of indexes. We showed how the two forms of indexes are structurally different, with the first based on an income component weighted by a rank component (the *a-form*) and with the second based on a rank component weighted by an income component (the *b-form*). This

http://ec.europa.eu/research/social-sciences/projects/.
Detailed information on the BHPS can be found on:
http://www.iser.essex.ac.uk/survey/bhps.

[3]Note that if y_r is not a constant but a variable such as predicted incomes, the *b-form* of indexes should be sorted by y/y_r.

different structure implies a different normative view of relative deprivation and different distributions of individual relative deprivation values that can potentially lead to very different results in empirical applications.

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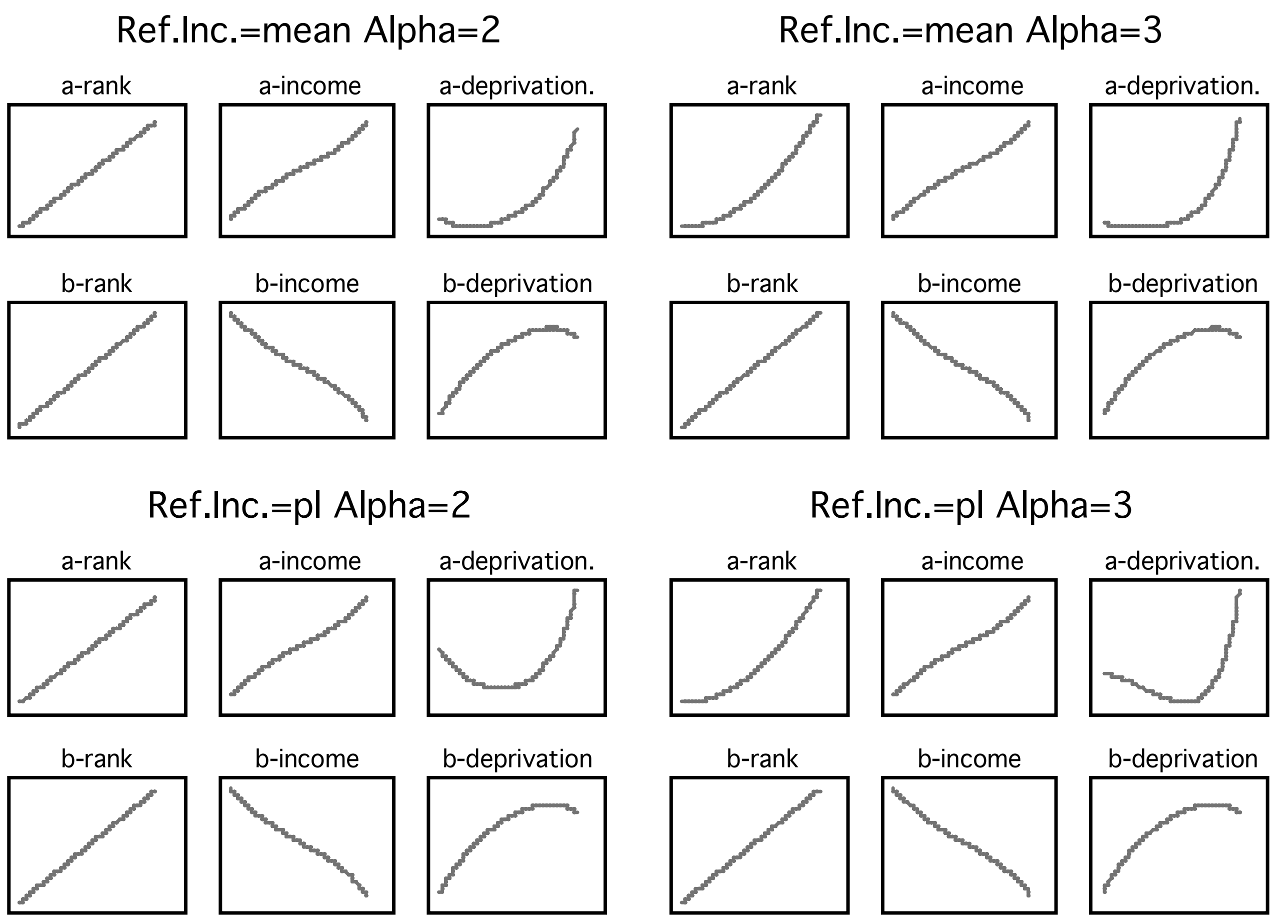
Figure 1: Individual Deprivation Functions



Note: In all figures the title represents what is plotted on the y-axis (rank component, income component or total deprivation scores) while the population is plotted on the x-axis. Therefore the range of population values is identical for the x-axis in all plots while the values on the y-axis range between the minimum and maximum values of rank, income or deprivation scores. “Ref.Inc.” stands for reference income and “povline” stands for poverty line.